Nonlinear Analysis: Modelling and Control, Vol. 28, No. 5, 825–840
https://doi.org/10.15388/namc.2023.28.32142

Weak Wardowski contractive multivalued mappings and solvability of generalized φ-Caputo fractional snap boundary inclusions

Hossein Jafari[a,b,c], **Babak Mohammadi**[d], **Vahid Parvaneh**[e], **Mohammad Mursaleen**[f,1]

[a]Department of Applied Mathematics,
University of Mazandaran,
Babolsar, Iran
jafari.usern@gmail.com

[b]Department of Mathematical Sciences,
University of South Africa,
UNISA0003, Pretoria, South Africa

[c]Department of Medical Research,
China Medical University Hospital,
China Medical University (Taiwan),
Taichung, Taiwan

[d]Department of Mathematics, Marand Branch,
Islamic Azad University,
Marand, Iran
bmohammadi@marandiau.ac.ir

[e]Department of Mathematics, Gilan-E-Gharb Branch,
Islamic Azad University,
Gilan-E-Gharb, Iran
zam.dalahoo@gmail.com

[f]Department of Mathematics,
Aligarh Muslim University,
Aligarh 202002, India
mursaleenm@gmail.com

Received: October 31, 2022 / **Revised:** April 11, 2023 / **Published online:** June 30, 2023

Abstract. In this paper, we introduce the notion of weak Wardowski contractive multivalued mappings and investigate the solvability of generalized φ-Caputo snap boundary fractional differential inclusions. Our results utilize some existing results regarding snap boundary fractional differential inclusions. An example is given to illustrate the applicability of our main results.

Keywords: weak Wardowski contractions, multivalued mapping, contractive mapping, φ-Caputo integral, fractional derivative, snap boundary inclusions.

[1]Corresponding author.

1 Introduction

Fractional integro-differential operators have many applications to investigate the mathematical modeling of physical phenomenon in many technological fields, namely, mechanics and physics. To see several published papers in this regard, the reader is referred to [2, 7, 8, 13]. Among these works, the Riemann–Liouville and Caputo integro-differential operators are the most fractional operators, which have been used. Recently, a new fractional integro-differential operator, namely, φ-Caputo fractional derivative, which means that the fractional derivative is defined with respect to another strictly increasing differentiable function, was introduced in [14] and used in [6]. Then some researchers used this operator in different subjects (see, for example, [1, 3, 5, 11, 15, 23]).

In [9], da C. Sousa and de Oliveira introduced a fractional derivative with respect to another function, the so-called ψ-Hilfer fractional derivative, and discussed some properties and important results of the fractional calculus. In this sense, they presented some results involving uniformly convergent sequence of function, uniformly continuous function, and examples including the Mittag-Leffler function with one parameter. Finally, they presented a wide class of integrals and fractional derivatives by means of the fractional integral with respect to another function and the ψ-Hilfer fractional derivative.

In [10], da C. Sousa and de Oliveira studied a Leibniz-type rule for the ψ-Hilfer fractional differential operator in two forms. They also presented some specific cases of Leibniz-type rule for this operator. In [19], these authors also presented a differential operator of arbitrary order defined by means of a Caputo-type modification of the generalized fractional derivative. As an application, they proved the fundamental theorem of fractional calculus associated with this differential operator.

Recently, Samei et al. [22] investigated the following φ-Caputo fractional differential inclusion:

$$\begin{aligned}
&{}^cD_{a^+}^{k;\varphi}\big({}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big)\big)\\
&\qquad\in f\big(t, z(t), {}^cD_{a^+}^{q;\varphi}z(t), {}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big), {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big)\big),\\
&z(a) = z_0, \qquad {}^cD_{a^+}^{q;\varphi}z(a) = z_1,\\
&{}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(a)\big) = z_2, \qquad {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(a)\big)\big) = z_3,
\end{aligned}\tag{1}$$

where ${}^cD_{a^+}^{\eta;\varphi}$ is the φ-Caputo fractional-order derivative introduced by Jarad et al. [14], $f : [a,b] \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$ is a multivalued function, η belong to $\{q, p, r, k\}$ such that $0 < q, p, r, k \leqslant 1$, the increasing function $\varphi \in C^1([a,b])$ is such that $\varphi'(t) \neq 0$, $t \in [a,b]$, and $z_0, z_1, z_2, z_3 \in \mathbb{R}$.

The authors investigated the solvability of the above-mentioned problem by using the α–ψ-contractive multivalued mappings defined in [17].

In this paper, we pursue two goals: In Section 3, we introduce a new multivalued contraction named the weak Wardowski multivalued contraction and prove the existence of a fixed point for such mappings. In Section 4, we use our new contraction to show that the above φ-Caputo fractional differential inclusion (1) is solvable when the right-hand function $f : [a,b] \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$ does not always involve α–ψ-contraction for multivalued mappings.

2 Preliminaries and auxiliary notions

Let $(\mathfrak{X}, d)$ is a metric space. Following [18], denote by $\mathcal{P}_{\mathrm{cb}}(\mathfrak{X})$ the class of all nonempty closed bounded subsets of $\mathfrak{X}$. Let $\mathcal{H}$ be the Hausdorff–Pompeiu metric on $\mathcal{P}_{\mathrm{cb}}(\mathfrak{X})$ induced by the metric d given as

$$\mathcal{H}(\varUpsilon_1, \varUpsilon_2) = \max\Bigl\{\sup_{\varsigma_1 \in \varUpsilon_1} d(\varsigma_1, \varUpsilon_2),\ \sup_{\varsigma_2 \in \varUpsilon_2} d(\varsigma_2, \varUpsilon_1)\Bigr\}$$

for every $\varUpsilon_1, \varUpsilon_2 \in \mathcal{P}_{\mathrm{cb}}(\mathfrak{X})$.

An element $\theta \in \mathfrak{X}$ is said to be a fixed point of a multivalued mapping $T : \mathfrak{X} \to \mathcal{P}(\mathfrak{X})$ if $\theta \in T\theta$.

Recently, Parvaneh and Farajzadeh [20] introduced and obtained the weak Wardowski contractions and obtained some fixed point theorems for this contractions via the notion of measure of noncompactness.

Denote by $\varXi$ the set of all functions $\mathfrak{F} : [0, \infty] \to [-\infty, \infty]$ so that:

(δ_1) $\mathfrak{F}$ is increasing and continuous;
(δ_2) $\mathfrak{F}(s) = 0 \iff s = 1$.

As examples of elements of $\varXi$:

$$\text{(i)}\ \ \mathfrak{F}_1(t) = \begin{cases} \ln(t), & t \in (0, \infty), \\ -\infty, & t = 0, \\ \infty, & t = \infty, \end{cases} \qquad \text{(iii)}\ \ \mathfrak{F}_3(t) = \begin{cases} -\frac{1}{\sqrt{t}} + 1, & t \in (0, \infty), \\ -\infty, & t = 0, \\ 1, & t = \infty, \end{cases}$$

$$\text{(ii)}\ \ \mathfrak{F}_2(t) = \begin{cases} \ln(t) + t, & t \in (0, \infty), \\ -\infty, & t = 0, \\ \infty, & t = \infty, \end{cases} \qquad \text{(iv)}\ \ \mathfrak{F}_4(t) = \begin{cases} -\frac{1}{t} + 1, & t \in (0, \infty), \\ -\infty, & t = 0, \\ 1, & t = \infty. \end{cases}$$

Denote by $\varTheta'$ the collection of all functions $\vartheta : \mathbb{R} \to (0, \infty)$ such that ϑ is continuous.

As examples of elements of $\varTheta'$:

$$\text{(i)}\ \ \vartheta_1(s) = \tau, \quad \tau > 0, \qquad \text{(iii)}\ \ \vartheta_3(s) = \tau + \mathrm{e}^{-s}, \quad \tau > 0,$$
$$\text{(ii)}\ \ \vartheta_2(s) = \tau\mathrm{e}^{-s}, \quad \tau > 0, \qquad \text{(iv)}\ \ \vartheta_4(s) = \tau + s^2, \quad \tau > 0,$$
$$\text{(v)}\ \ \vartheta_5(s) = s^2 + s + 1.$$

Now, let us recall some introductive definitions of fractional differential equations [16, 21]. For a continuous function $f : [a, b] \to \mathbb{R}$, the Riemann–Liouville integral of fractional order α is defined by

$$\mathcal{I}_a^\alpha f(t) = \frac{1}{\Gamma(r)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau)\, \mathrm{d}\tau. \tag{2}$$

The Caputo derivative of fractional order α is defined by

$$ {}^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau \tag{3}$$

for $n-1 < \alpha < n$, $n = [\alpha]+1$. Here the Riemann–Liouville fractional derivative of order α is defined by

$$ D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{4}$$

for $n-1 < \alpha < n$, $n = [\alpha]+1$.

Definition 1. (See [3].) Let φ is an increasing map so that $\varphi'(s) > 0$ for any $s \in [a,b]$. Then the φ-Riemann–Liouville integral of order r of a integrable function $f : [a,b] \to \mathbb{R}$ with respect to φ is defined as

$$ \mathcal{I}_{a^+}^{r;\varphi} f(t) = \frac{1}{\Gamma(r)} \int_a^t \varphi'(\xi)\big(\varphi(t)-\varphi(\xi)\big)^{r-1} f(\xi)\,\mathrm{d}\xi, \tag{5}$$

provided that the right-hand side of equality is finite-valued.

It should be noted that if $\varphi(t) = t$, then, clearly, the φ-Riemann–Liouville integral (5) reduces to the standard Riemann–Liouville integral (2).

Definition 2. (See [14].) Let $n = [r]+1$. For a real mapping $f \in C([a,b],\mathbb{R})$, the φ-Riemann–Liouville derivative of order r is formulated as

$$ D_{a^+}^{r;\varphi} f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{1}{\varphi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \varphi'(\xi)\big(\varphi(t)-\varphi(\xi)\big)^{n-r-1} f(\xi)\,\mathrm{d}\xi, \tag{6}$$

provided that the right-hand side of equality is finite-valued.

In the similar manner, if $\varphi(t) = t$, then it is obvious that the φ-Riemann–Liouville derivative (6) reduces to the standard Riemann–Liouville derivative (4). Inspired by these operators, Almeida presented a new φ-version of the Caputo derivative in the following formulation.

Definition 3. (See [3].) Let $n = [r]+1$, and let $f \in AC^n([a,b],\mathbb{R})$ be an increasing map with $\varphi'(s) > 0$ for any $s \in [a,b]$. The φ-Caputo derivative of order r of f with respect to φ is

$$ {}^{c}D_{a^+}^{r;\varphi} f(t) = \frac{1}{\Gamma(n-r)} \int_a^t \varphi'(\xi)\big(\varphi(t)-\varphi(\xi)\big)^{n-r-1} \left(\frac{1}{\varphi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n f(\xi)\,\mathrm{d}\xi, \tag{7}$$

provided that the right-hand side of equality possesses values finitely.

It should be noted that if $\varphi(s) = s$, then it is obvious that the φ-Caputo derivative of order r in formula (7) reduces to the standard Caputo derivative of order r in (3). In the following, some useful specifications of the φ-Caputo and φ-Riemann–Liouville integro-derivative operators can be seen. Let $AC([a,b],\mathbb{R})$ stand for the set of absolutely continuous functions from $[a,b]$ into $\mathbb{R}$. Define $AC^n_\varphi([a,b],\mathbb{R}$ by

$$AC^n_\varphi\bigl([a,b],\mathbb{R}\bigr) = \biggl\{w : [a,b] \to \mathbb{R} | \delta^{n-1}_\varphi w \in AC\bigl([a,b],\mathbb{R}\bigr),\ \delta_\varphi = \frac{1}{\varphi'(y)}\frac{\mathrm{d}}{\mathrm{d}y}\biggr\}.$$

Lemma 1. (*See* [14].) *Let* $n = [r] + 1$. *For a real mapping* $f \in AC^n([a,b],\mathbb{R})$,

$$\mathcal{I}^{r;\varphi}_{a^+}{}^c D^{r;\varphi}_{a^+} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(\delta^k_\varphi f)(a)}{k!}\bigl(f(t) - f(a)\bigr)^k,$$

where $\delta^k_\varphi = \delta_\varphi \delta_\varphi \cdots \delta_\varphi$.

Lemma 2. (*See* [4].) *Let* $n = [\alpha] + 1$, $\alpha, \beta > 0$. *For a real mapping* $f \in C([a,b],\mathbb{R})$, *we have*:

$$\begin{gathered}
\mathcal{I}^{\alpha;\varphi}_{a^+}\mathcal{I}^{\beta;\varphi}_{a^+} f(t) = \mathcal{I}^{\alpha+\beta;\varphi}_{a^+} f(t), \qquad {}^c D^{\alpha;\varphi}_{a^+}\mathcal{I}^{\alpha;\varphi}_{a^+} f(t) = f(t),\\
{}^c D^{\alpha;\varphi}_{a^+}\bigl(\varphi(t) - \varphi(a)\bigr)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\bigl(\varphi(t) - \varphi(a)\bigr)^{\beta-\alpha-1},\\
\mathcal{I}^{\alpha;\varphi}_{a^+}\bigl(\varphi(t) - \varphi(a)\bigr)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\bigl(\varphi(t) - \varphi(a)\bigr)^{\beta+\alpha-1},\\
{}^c D^{\alpha;\varphi}_{a^+}\bigl(\varphi(t) - \varphi(a)\bigr)^k = 0, \quad k = 0, 1, 2, \dots, n-1.
\end{gathered}$$

3 Main results

In 2005, Echenique [12] started combining fixed point theory and graph theory. Consider a directed graph G on a metric space (X,d) such that the set of its vertices $V(G)$ coincides with X (i.e., $V(G) = X$), and the set of its edges $E(G)$ is such that $E(G) \supseteq \Delta$, where $\Delta = \{(x,x)\colon x \in X\}$. Let us also assume that G has no parallel edges. We can identify G with the pair $(V(G), E(G))$. The graph G is called a (C)-graph if for any sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Now, we are ready to state and prove the main results of this study.

Definition 4. Let (X,d) be a metric space. Assume that G is a directed graph on X. Let $T : X \to \mathcal{P}_{\mathrm{cb}}(X)$ be a multivalued mapping. We say that T is a weak Wardowski multivalued contraction if there exist $\mathfrak{F} \in \varXi$ and $\vartheta \in \varTheta'$ such that

$$\mathfrak{F}\bigl(\mathcal{H}(Tx,Ty)\bigr) \leqslant \mathfrak{F}\bigl(M(x,y)\bigr) - \vartheta\bigl(\mathfrak{F}\bigl(M(x,y)\bigr)\bigr)$$

for all $x, y \in X$, where

$$M(x,y) = \max\biggl\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\biggr\}.$$

Theorem 1. *Let (X, d) be a complete metric space, and let G be a directed graph on X. Assume that $T : X \to \mathcal{P}_{\mathrm{cb}}(X)$ is a weak Wardowski multivalued contraction satisfying comparable approximate valued property. If G is a c-graph, then T has a fixed point.*

Proof. Choose a fixed element $\varsigma_0 \in X$. If $\varsigma_0 \in T\varsigma_0$, then we have nothing to prove. Suppose that $\varsigma_0 \notin T\varsigma_0$. Since T has comparable approximative valued property, there exists $\varsigma_1 \in T\varsigma_0$ such that $(\varsigma_0, \varsigma_1) \in E(G)$ and $d(\varsigma_0, T\varsigma_0) = d(\varsigma_0, \varsigma_1)$. It is clear that $\varsigma_1 \neq \varsigma_0$. If $\varsigma_1 \in T\varsigma_1$, then ς_1 is a fixed point of T. Suppose that $\varsigma_1 \notin T\varsigma_1$. Then there exists $\varsigma_2 \in T\varsigma_1$ such that $(\varsigma_1, \varsigma_2) \in E(G)$ and $d(\varsigma_1, T\varsigma_1) = d(\varsigma_1, \varsigma_2)$. It is clear that $\varsigma_2 \neq \varsigma_1$. By continuing this process we obtain a sequence $\{\varsigma_n\}$ in X such that $\varsigma_n \in T\varsigma_{n-1}$, $(\varsigma_{n-1}, \varsigma_n) \in E(G)$, $\varsigma_n \neq \varsigma_{n-1}$, and $d(\varsigma_{n-1}, \varsigma_n) = d(\varsigma_{n-1}, T\varsigma_{n-1})$ for all $n \in \mathbb{N}$. In view of (3), we obtain that

$$\begin{aligned}\mathfrak{F}\big(d(\varsigma_{n+1}, \varsigma_{n+2})\big) &= \mathfrak{F}\big(d(\varsigma_{n+1}, T\varsigma_{n+1})\big) = \mathfrak{F}\big(\mathcal{H}(T\varsigma_n, T\varsigma_{n+1})\big)\\ &\leqslant \mathfrak{F}\big(M(\varsigma_n, \varsigma_{n+1})\big) - \vartheta\big(\mathfrak{F}\big(M(\varsigma_n, \varsigma_{n+1})\big)\big),\end{aligned}$$

where

$$\begin{aligned}&\max\big\{d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_{n+1}, \varsigma_{n+2})\big\}\\ &\quad\leqslant M(\varsigma_n, \varsigma_{n+1})\\ &\quad= \max\left\{d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_n, T\varsigma_n), d(\varsigma_{n+1}, T\varsigma_{n+1}), \frac{1}{2}\big[d(\varsigma_n, T\varsigma_{n+1}) + d(\varsigma_{n+1}, T\varsigma_n)\big]\right\}\\ &\quad\leqslant \max\big\{d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_{n+1}, \varsigma_{n+2})\big\}.\end{aligned}$$

Thus, $M(\varsigma_n, \varsigma_{n+1}) = \max\{d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_{n+1}, \varsigma_{n+2})\}$. If

$$\max\big\{d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_{n+1}, \varsigma_{n+2})\big\} = d(\varsigma_{n+1}, \varsigma_{n+2}),$$

then

$$\mathfrak{F}\big(d(\varsigma_{n+1}, \varsigma_{n+2})\big) \leqslant \mathfrak{F}\big(d(\varsigma_{n+1}, \varsigma_{n+2})\big) - \vartheta\big(\mathfrak{F}\big(d(\varsigma_{n+1}, \varsigma_{n+2})\big)\big) < \mathfrak{F}\big(d(\varsigma_{n+1}, \varsigma_{n+2})\big),$$

which is a contradiction. Thus,

$$\max\big\{d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_{n+1}, \varsigma_{n+2})\big\} = d(\varsigma_n, \varsigma_{n+1}).$$

Therefore,

$$\mathfrak{F}\big(d(\varsigma_{n+1}, \varsigma_{n+2})\big) \leqslant \mathfrak{F}\big(d(\varsigma_n, \varsigma_{n+1})\big) - \vartheta\big(\mathfrak{F}\big(d(\varsigma_n, \varsigma_{n+1})\big)\big) \tag{8}$$

for each $n \geqslant 0$. Put $t_n := d(\varsigma_n, \varsigma_{n+1})$. From (8) we have

$$\mathfrak{F}(t_{n+1}) \leqslant \mathfrak{F}(t_n) - \vartheta\big(\mathfrak{F}(t_n)\big) < \mathfrak{F}(t_n) \quad \forall n \geqslant 0. \tag{9}$$

Since $\mathfrak{F}$ is increasing, we get $t_{n+1} < t_n$, and so there is $r \geqslant 0$ such that $t_n \to r^+$. Now, we show that $r = 0$. Suppose to the contrary $r > 0$. Passing to the limit through (9), $\mathfrak{F}(r) \leqslant \mathfrak{F}(r) - \vartheta(\mathfrak{F}(r)) < \mathfrak{F}(r)$, which is a contradiction. So $\lim_{n\to\infty} t_n = r = 0$. We claim that $\{\varsigma_n\}$ is Cauchy. If $\{\varsigma_n\}$ is not Cauchy, then there are $\varepsilon > 0$ and subsequences $\{\varsigma_{m_i}\}$ and $\{\varsigma_{n_i}\}$ of $\{\varsigma_n\}$ so that

$$n_i > m_i > i, \quad d(\varsigma_{m_i}, \varsigma_{n_i}) \geqslant \varepsilon \tag{10}$$

and

$$d(\varsigma_{m_i}, \varsigma_{n_i-1}) < \varepsilon.$$

Using (10), we get

$$\varepsilon \leqslant d(\varsigma_{m_i}, \varsigma_{n_i}) \leqslant d(\varsigma_{m_i}, \varsigma_{n_i-1}) + d(\varsigma_{n_i-1}, \varsigma_{n_i}) < \varepsilon + d(\varsigma_{n_i-1}, \varsigma_{n_i}).$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} d(\varsigma_{m_i}, \varsigma_{n_i}) = \varepsilon.$$

Also, we have

$$\begin{aligned} &d(\varsigma_{m_i}, \varsigma_{n_i}) - d(\varsigma_{m_i}, \varsigma_{m_i+1}) - d(\varsigma_{n_i}, \varsigma_{n_i+1}) \\ &\quad \leqslant d(\varsigma_{m_i+1}, \varsigma_{n_i+1}) \leqslant d(\varsigma_{m_i}, \varsigma_{m_i+1}) + d(\varsigma_{m_i}, \varsigma_{n_i}) + d(\varsigma_{n_i}, \varsigma_{n_i+1}). \end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} d(\varsigma_{m_i+1}, \varsigma_{n_i+1}) = \varepsilon.$$

Also,

$$d(\varsigma_{m_i+1}, \varsigma_{n_i+1}) \leqslant d(\varsigma_{m_i+1}, T\varsigma_{m_i}) + \mathcal{H}(T\varsigma_{m_i}, T\varsigma_{n_i}) = \mathcal{H}(T\varsigma_{m_i}, T\varsigma_{n_i}).$$

By (3) we find

$$\begin{aligned} \mathfrak{F}\big(d(\varsigma_{m_i+1}, \varsigma_{n_i+1})\big) &\leqslant \mathfrak{F}\big(\mathcal{H}(T\varsigma_{m_i}, T\varsigma_{n_i})\big) \\ &\leqslant \mathfrak{F}\big(M(\varsigma_{m_i}, \varsigma_{n_i})\big) - \vartheta\big(\mathfrak{F}\big(M(\varsigma_{m_i}, \varsigma_{n_i})\big)\big). \end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned} &d(\varsigma_{m_i}, \varsigma_{n_i}) \\ &\quad \leqslant M(\varsigma_{m_i}, \varsigma_{n_i}) \\ &\quad \leqslant \max\left\{d(\varsigma_{m_i}, \varsigma_{n_i}), d(\varsigma_{m_i}, \varsigma_{m_i+1}), d(\varsigma_{n_i}, \varsigma_{n_i+1}), \frac{1}{2}\big[d(\varsigma_{n_i}, \varsigma_{m_i+1}) + d(\varsigma_{m_i}, \varsigma_{n_i+1})\big]\right\} \\ &\quad \leqslant d(\varsigma_{m_i}, \varsigma_{n_i}) + d(\varsigma_{m_i}, \varsigma_{m_i+1}) + d(\varsigma_{n_i}, \varsigma_{n_i+1}). \end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} M(\varsigma_{m_i}, \varsigma_{n_i}) = \varepsilon.$$

Taking limit in both sides of (11),

$$\mathfrak{F}(\varepsilon) \leqslant \mathfrak{F}(\varepsilon) - \vartheta\big(\mathfrak{F}(\varepsilon)\big) < \mathfrak{F}(\varepsilon),$$

a contradiction.

Thus, $\{\varsigma_n\}$ is a Cauchy sequence in the complete metric space (X,d), hence there is $z \in X$ so that

$$\lim_{n\to\infty} \varsigma_n = z.$$

We claim that $d(z,Tz)=0$. Suppose to the contrary $d(z,Tz)\neq 0$.

Since G is a (C)-graph, there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k},x)\in E(G)$ for all $k\in\mathbb{N}$.

We have

$$\begin{aligned}\mathfrak{F}\bigl(d(\varsigma_{n_k+1},Tz)\bigr) &\leqslant \mathfrak{F}\bigl(\mathcal{H}(T\varsigma_{n_k},Tz)\bigr)\\ &\leqslant \mathfrak{F}\bigl(M(\varsigma_{n_k},z)\bigr)-\vartheta\bigl(\mathfrak{F}\bigl(M(\varsigma_{n_k},z)\bigr)\bigr).\end{aligned} \tag{12}$$

Also,

$$\begin{aligned}&\lim_{n\to\infty} M(\varsigma_{n_k},z)\\ &\quad=\lim_{n\to\infty}\max\biggl\{d(\varsigma_{n_k},z),d(\varsigma_{n_k},\varsigma_{n_k+1}),d(z,Tz),\frac{1}{2}\bigl[d(\varsigma_{n_k+1},z)+d(\varsigma_{n_k},Tz)\bigr]\biggr\}\\ &\quad=d(z,Tz).\end{aligned}$$

Passing to the limit through (12), we obtain $\mathfrak{F}(d(z,Tz))<\mathfrak{F}(d(z,Tz))$, which is a contradiction. Thus, $d(z,Tz)=0$. Now, since G has comparable approximate valued property, there exists $u\in X$ such that $u\in Tz$, $(z,u)\in E(G)$, and $d(z,u)=d(z,Tz)$. Consequently, $d(z,u)=0$ and so $z=u\in Tz$. The proof is completed. □

Denote by $\mathcal{P}_{cp}(X)$ the family of all nonempty compact subsets of X.

Corollary 1. *Let (X,d) be a complete metric space, and let G be a directed graph on X. Assume that $T: X\to\mathcal{P}_{cp}(X)$ is a weak Wardowski multivalued contraction. Moreover, assume that $Graph(T)=\{(x,y)\colon\ y\in Tx\}\subseteq E(G)$. If G is a c-graph, then T has a fixed point.*

4 Application to fractional differential equations

From now on, assume that $X=C([a,b],\mathbb{R})$ is the Banach space of continuous functions $z:[a,b]\to\mathbb{R}$ endowed with the norm

$$\begin{aligned}\|z\| = \sup_{t\in[a,b]}\bigl|z(t)\bigr| + \sup_{t\in[a,b]}\bigl|{}^cD^{q;\varphi}_{a^+}z(t)\bigr| + \sup_{t\in[a,b]}\bigl|{}^cD^{p;\varphi}_{a^+}\bigl({}^cD^{q;\varphi}_{a^+}z(t)\bigr)\bigr|\\ + \sup_{t\in[a,b]}\bigl|{}^cD^{r;\varphi}_{a^+}\bigl({}^cD^{p;\varphi}_{a^+}\bigl({}^cD^{q;\varphi}_{a^+}z(t)\bigr)\bigr)\bigr| \quad \forall z\in C\bigl([a,b],\mathbb{R}\bigr).\end{aligned}$$

Define $d(z_1,z_2)=\|z_1-z_2\|$ for all $z_1,z_2\in C([a,b],\mathbb{R})$. Then (X,d) is a complete metric space. From [22] we know that the function $z\in\mathcal{C}:=C([a,b],\mathbb{R})$ is a solution of system (1) if it satisfies the boundary conditions and there is $\mathfrak{z}\in L^1([a,b])$ such that $\mathfrak{z}(t)\in\hat{f}_z(t)$ for almost all $t\in[a,b]$, where

$$\hat{f}_z(t)=f\bigl(t,z(t),{}^cD^{q;\varphi}_{a^+}z(t),{}^cD^{p;\varphi}_{a^+}\bigl({}^cD^{q;\varphi}_{a^+}z(t)\bigr),{}^cD^{r;\varphi}_{a^+}\bigl({}^cD^{p;\varphi}_{a^+}\bigl({}^cD^{q;\varphi}_{a^+}z(t)\bigr)\bigr)\bigr)$$

and

$$z(t) = z_0 + z_1 \frac{(\varphi(t) - \varphi(a))^q}{\Gamma(q+1)} + z_2 \frac{(\varphi(t) - \varphi(a))^{q+p}}{\Gamma(q+p+1)} + z_3 \frac{(\varphi(t) - \varphi(a))^{q+p+r}}{\Gamma(q+p+r+1)} + \int_a^t \varphi'(\tau) \frac{(\varphi(t) - \varphi(\tau))^{q+p+r+k-1}}{\Gamma(q+p+r+k)} \mathfrak{z}(\tau)\,\mathrm{d}\tau$$

for all $t \in [a, b]$. For each $z \in \mathcal{C}$, we define the set of selections of the operator f as follows:

$$\mathfrak{S}_{f,z} = \bigl\{\mathfrak{z} \in L^1\bigl([a,b]\bigr)\colon \mathfrak{z}(t) \in \hat{f}_z(t)\ \forall t \in [a,b]\bigr\}.$$

Define the operator $\mathfrak{U} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$\mathfrak{U}(z) = \bigl\{\mathfrak{p} \in \mathcal{C}\colon \text{there exists } \mathfrak{z} \in \mathfrak{S}_{f,z} \text{ such that } \mathfrak{p}(t) = \Upsilon(t)\ \forall t \in [a,b]\bigr\}, \tag{13}$$

where

$$\Upsilon(t) = z_0 + z_1 \frac{(\varphi(t) - \varphi(a))^q}{\Gamma(q+1)} + z_2 \frac{(\varphi(t) - \varphi(a))^{q+p}}{\Gamma(q+p+1)} + z_3 \frac{(\varphi(t) - \varphi(a))^{q+p+r}}{\Gamma(q+p+r+1)} + \int_a^t \varphi'(\tau) \frac{(\varphi(t) - \varphi(\tau))^{q+p+r+k-1}}{\Gamma(q+p+r+k)} \mathfrak{z}(\tau)\,\mathrm{d}\tau.$$

From now on, we assume that for the pair of functions $(\mathfrak{F}, \vartheta)$, $\mathfrak{F}^{-1}\{\mathfrak{F}(\cdot) - \vartheta(\mathfrak{F}(\cdot))\}$ is a nondecreasing function.

Theorem 2. *Let $f : [a,b] \times \mathbb{R}^4 \to \mathcal{P}_{cp}(\mathbb{R})$ be a multivalued mapping. Suppose that the following conditions are satisfied*:

(i) *The multivalued mapping f is integrable, and $f(\cdot, v_1, v_2, v_3, v_4) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for $v_1, v_2, v_3, v_4 \in \mathbb{R}$;*
(ii) *There exist $\mathfrak{F} \in \Xi$ and $\vartheta \in \Theta'$ such that*

$$\mathcal{H}\bigl(f(t, v_1, v_2, v_3, v_4), f(t, v_1', v_2', v_3', v_4')\bigr) \leqslant \mathcal{O}^* \mathfrak{F}^{-1}\left(\mathfrak{F}\left(\sum_{i=1}^4 |v_i - v_i'|\right) - \vartheta\left(\mathfrak{F}\left(\sum_{i=1}^4 |v_i - v_i'|\right)\right)\right) \tag{14}$$

for all $t \in [a,b]$ and $v_1, v_2, v_3, v_4, v_1', v_2', v_3', v_4' \in \mathbb{R}$, where $\mathcal{O}^* = \mathcal{O}^{-1}$ and

$$\mathcal{O} = \frac{(\varphi(b) - \varphi(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\varphi(b) - \varphi(a))^{p+r+k}}{\Gamma(p+r+k+1)} + \frac{(\varphi(b) - \varphi(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\varphi(b) - \varphi(a))^{k}}{\Gamma(k+1)}.$$

Then the inclusion problem (1) *has at least one solution.*

Proof. We shall show that the multivalued mapping $\mathfrak{U}$ defined in (13) has a fixed point. Let $z, z' \in \mathcal{C}$ and $\hbar_1^* \in \mathfrak{U}(z')$ and choose $\mathfrak{z}_1 \in \mathfrak{S}_{f,z'}$ such that

$$\hbar_1^*(t) = z_0 + z_1 \frac{(\varphi(t) - \varphi(a))^q}{\Gamma(q+1)} + z_2 \frac{(\varphi(t) - \varphi(a))^{q+p}}{\Gamma(q+p+1)} + z_3 \frac{(\varphi(t) - \varphi(a))^{q+p+r}}{\Gamma(q+p+r+1)} + \int_a^t \varphi'(\tau) \frac{(\varphi(t) - \varphi(\tau))^{q+p+r+k-1}}{\Gamma(q+p+r+k)} \mathfrak{z}_1(\tau)\,\mathrm{d}\tau$$

for all $t \in [a, b]$. From (14) we have

$$\begin{aligned}
\mathcal{H}\big(\hat{f}_z(t), \hat{f}_{z'}(t)\big) \leqslant{}& \mathcal{O}^* \mathfrak{F}^{-1} \big\{ \mathfrak{F}\big(\big|z(t) - z'(t)\big| + \big| {}^c D_{a^+}^{q;\varphi} z(t) - {}^c D_{a^+}^{q;\varphi} z'(t) \big| \\
&+ \big| {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) - {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big| \\
&+ \big| {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) \big) - {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big) \big| \big) \\
&- \vartheta \big(\mathfrak{F}\big(\big|z(t) - z'(t)\big| + \big| {}^c D_{a^+}^{q;\varphi} z(t) - {}^c D_{a^+}^{q;\varphi} z'(t) \big| \\
&+ \big| {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) - {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big| \\
&+ \big| {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) \big) - {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big) \big| \big) \big) \big\}
\end{aligned}$$

for all $z, z' \in \mathcal{C}$. Thus, there exists $\Upsilon \in \hat{f}_z$ such that

$$\begin{aligned}
\big| \mathfrak{z}_1(t) - \Upsilon(t) \big| \leqslant{}& \mathcal{O}^* \mathfrak{F}^{-1} \big\{ \mathfrak{F}\big(\big|z(t) - z'(t)\big| + \big| {}^c D_{a^+}^{q;\varphi} z(t) - {}^c D_{a^+}^{q;\varphi} z'(t) \big| \\
&+ \big| {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) - {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big| \\
&+ \big| {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) \big) - {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big) \big| \big) \\
&- \vartheta \big(\mathfrak{F}\big(\big|z(t) - z'(t)\big| + \big| {}^c D_{a^+}^{q;\varphi} z(t) - {}^c D_{a^+}^{q;\varphi} z'(t) \big| \\
&+ \big| {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) - {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big| \\
&+ \big| {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) \big) - {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big) \big| \big) \big) \big\}.
\end{aligned}$$

Now, define a multivalued mapping $\mathfrak{N} : [a, b] \to \mathcal{P}(\mathcal{C})$ as

$$\mathfrak{N}(t) = \big\{ \Upsilon \in \mathcal{C}\colon\ \big| \mathfrak{z}_1(t) - \Upsilon(t) \big| \leqslant \zeta(t) \big\}$$

for all $t \in [a, b]$, where

$$\begin{aligned}
\zeta(t) ={}& \mathcal{O}^* \mathfrak{F}^{-1} \big\{ \mathfrak{F}\big(\big|z(t) - z'(t)\big| + \big| {}^c D_{a^+}^{q;\varphi} z(t) - {}^c D_{a^+}^{q;\varphi} z'(t) \big| \\
&+ \big| {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) - {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big| \\
&+ \big| {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) \big) - {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big) \big| \big) \\
&- \vartheta \big(\mathfrak{F}\big(\big|z(t) - z'(t)\big| + \big| {}^c D_{a^+}^{q;\varphi} z(t) - {}^c D_{a^+}^{q;\varphi} z'(t) \big| \\
&+ \big| {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) - {}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big| \\
&+ \big| {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z(t) \big) \big) - {}^c D_{a^+}^{r;\varphi} \big({}^c D_{a^+}^{p;\varphi} \big({}^c D_{a^+}^{q;\varphi} z'(t) \big) \big) \big| \big) \big) \big\}.
\end{aligned}$$

As $\mathfrak{z}_1$ and $\zeta(t)$ are measurable, so there is $\mathfrak{N}(\cdot) \cap \hat{f}_z(\cdot)$. Now, let $\mathfrak{z}_2 \in \hat{f}_z(t)$ be such that

$$\begin{aligned}\bigl|\mathfrak{z}_1(t) - \mathfrak{z}_2(t)\bigr| \leqslant{} & \mathcal{O}^* \mathfrak{F}^{-1}\bigl\{\mathfrak{F}\bigl(\bigl|z(t) - z'(t)\bigr| + \bigl|{}^cD_{a^+}^{q;\varphi} z(t) - {}^cD_{a^+}^{q;\varphi} z'(t)\bigr| \\ & + \bigl|{}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr) - {}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr| \\ & + \bigl|{}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr)\bigr) - {}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr)\bigr|\bigr) \\ & - \vartheta\bigl(\mathfrak{F}\bigl(\bigl|z(t) - z'(t)\bigr| + \bigl|{}^cD_{a^+}^{q;\varphi} z(t) - {}^cD_{a^+}^{q;\varphi} z'(t)\bigr| \\ & + \bigl|{}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr) - {}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr| \\ & + \bigl|{}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr)\bigr) - {}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr)\bigr|\bigr)\bigr)\bigr\}.\end{aligned}$$

Now, we define $\hbar_2^* \in \mathfrak{U}(z)$ as

$$\begin{aligned}\hbar_2^*(t) ={} & z_0 + z_1 \frac{(\varphi(t) - \varphi(a))^q}{\Gamma(q+1)} + z_2 \frac{(\varphi(t) - \varphi(a))^{q+p}}{\Gamma(q+p+1)} \\ & + z_3 \frac{(\varphi(t) - \varphi(a))^{q+p+r}}{\Gamma(q+p+r+1)} + \int_a^t \varphi'(\tau) \frac{(\varphi(t) - \varphi(\tau))^{q+p+r+k-1}}{\Gamma(q+p+r+k)} \mathfrak{z}_2(\tau)\, \mathrm{d}\tau\end{aligned}$$

for all $t \in [a, b]$. Then

$$\begin{aligned}\bigl|\hbar_1^*(t) - \hbar_2^*(t)\bigr| \leqslant{} & \mathcal{I}_{a^+}^{r+p+q+k;\varphi}\bigl|\mathfrak{z}_1(t) - \mathfrak{z}_2(t)\bigr| \\ \leqslant{} & \frac{(\varphi(b) - \varphi(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} \mathcal{O}^* \mathfrak{F}^{-1}\bigl\{\mathfrak{F}\bigl(\bigl|z(t) - z'(t)\bigr| \\ & + \bigl|{}^cD_{a^+}^{q;\varphi} z(t) - {}^cD_{a^+}^{q;\varphi} z'(t)\bigr| \\ & + \bigl|{}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr) - {}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr| \\ & + \bigl|{}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr)\bigr) - {}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr)\bigr|\bigr) \\ & - \vartheta\bigl(\mathfrak{F}\bigl(\bigl|z(t) - z'(t)\bigr| + \bigl|{}^cD_{a^+}^{q;\varphi} z(t) - {}^cD_{a^+}^{q;\varphi} z'(t)\bigr| \\ & + \bigl|{}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr) - {}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr| \\ & + \bigl|{}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr)\bigr) - {}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr)\bigr|\bigr)\bigr)\bigr\} \\ \leqslant{} & \frac{(\varphi(b) - \varphi(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} \mathcal{O}^* \mathfrak{F}^{-1}\bigl\{\mathfrak{F}\bigl(\|z - z'\|\bigr) - \vartheta\bigl(\mathfrak{F}\bigl(\|z - z'\|\bigr)\bigr)\bigr\},\end{aligned}$$

$$\begin{aligned}& \bigl|{}^cD_{a^+}^{q;\varphi} \hbar_1^*(t) - {}^cD_{a^+}^{q;\varphi} \hbar_2^*(t)\bigr| \\ & \quad \leqslant \mathcal{I}_{a^+}^{r+p+k;\varphi}\bigl|\mathfrak{z}_1(t) - \mathfrak{z}_2(t)\bigr| \\ & \quad \leqslant \frac{(\varphi(b) - \varphi(a))^{p+r+k}}{\Gamma(p+r+k+1)} \mathcal{O}^* \mathfrak{F}^{-1}\bigl\{\mathfrak{F}\bigl(\bigl|z(t) - z'(t)\bigr| + \bigl|{}^cD_{a^+}^{q;\varphi} z(t) - {}^cD_{a^+}^{q;\varphi} z'(t)\bigr| \\ & \qquad + \bigl|{}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr) - {}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr| \\ & \qquad + \bigl|{}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr)\bigr) - {}^cD_{a^+}^{r;\varphi}\bigl({}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr)\bigr|\bigr) \\ & \qquad - \vartheta\bigl(\mathfrak{F}\bigl(\bigl|z(t) - z'(t)\bigr| + \bigl|{}^cD_{a^+}^{q;\varphi} z(t) - {}^cD_{a^+}^{q;\varphi} z'(t)\bigr| \\ & \qquad + \bigl|{}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z(t)\bigr) - {}^cD_{a^+}^{p;\varphi}\bigl({}^cD_{a^+}^{q;\varphi} z'(t)\bigr)\bigr|\end{aligned}$$

$$
\begin{aligned}
&+ \big|{}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big) - {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big)\big|\big)\big)\big\} \\
&\leqslant \frac{(\varphi(b)-\varphi(a))^{p+r+k}}{\Gamma(p+r+k+1)}\mathcal{O}^*\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big) - \vartheta(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\},
\end{aligned}
$$

$$
\begin{aligned}
&\big|{}^cD_{a^+}^{p+q;\varphi}\hbar_1^*(t) - {}^cD_{a^+}^{p+q;\varphi}\hbar_2^*(t)\big| \\
&\leqslant \mathcal{I}_{a^+}^{r+k;\varphi}\big|\mathfrak{z}_1(t) - \mathfrak{z}_2(t)\big| \\
&\leqslant \frac{(\varphi(b)-\varphi(a))^{r+k}}{\Gamma(r+k+1)}\mathcal{O}^*\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(|z(t)-z'(t)| + \big|{}^cD_{a^+}^{q;\varphi}z(t) - {}^cD_{a^+}^{q;\varphi}z'(t)\big| \\
&+ \big|{}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big) - {}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big| \\
&+ \big|{}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big) - {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big)\big|\big) \\
&- \vartheta\big(\mathfrak{F}\big(|z(t)-z'(t)| + \big|{}^cD_{a^+}^{q;\varphi}z(t) - {}^cD_{a^+}^{q;\varphi}z'(t)\big| \\
&+ \big|{}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big) - {}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big| \\
&+ \big|{}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big) - {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big)\big|\big)\big)\big\} \\
&\leqslant \frac{(\varphi(b)-\varphi(a))^{r+k}}{\Gamma(r+k+1)}\mathcal{O}^*\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big) - \vartheta\big(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|{}^cD_{a^+}^{r+p+q;\varphi}\hbar_1^*(t) - {}^cD_{a^+}^{r+p+q;\varphi}\hbar_2^*(t)\big| \\
&\leqslant \mathcal{I}_{a^+}^{k;\varphi}\big|\mathfrak{z}_1(t) - \mathfrak{z}_2(t)\big| \\
&\leqslant \frac{(\varphi(b)-\varphi(a))^{k}}{\Gamma(k+1)}\mathcal{O}^*\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(|z(t)-z'(t)| + \big|{}^cD_{a^+}^{q;\varphi}z(t) - {}^cD_{a^+}^{q;\varphi}z'(t)\big| \\
&+ \big|{}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big) - {}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big| \\
&+ \big|{}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big) - {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big)\big|\big) \\
&- \vartheta\big(\mathfrak{F}\big(|z(t)-z'(t)| + \big|{}^cD_{a^+}^{q;\varphi}z(t) - {}^cD_{a^+}^{q;\varphi}z'(t)\big| \\
&+ \big|{}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big) - {}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big| \\
&+ \big|{}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z(t)\big)\big) - {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}z'(t)\big)\big)\big|\big)\big)\big\} \\
&\leqslant \frac{(\varphi(b)-\varphi(a))^{k}}{\Gamma(k+1)}\mathcal{O}^*\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big) - \vartheta(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\hbar_1^* - \hbar_2^*\| &= \sup_{t\in[a,b]}\big|\hbar_1^*(t) - \hbar_2^*(t)\big| + \sup_{t\in[a,b]}\big|{}^cD_{a^+}^{q;\varphi}\hbar_1^*(t) - {}^cD_{a^+}^{q;\varphi}\hbar_2^*(t)\big| \\
&+ \sup_{t\in[a,b]}\big|{}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}\hbar_1^*(t)\big) - {}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}\hbar_2^*(t)\big)\big| \\
&+ \sup_{t\in[a,b]}\big|{}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}\hbar_1^*(t)\big)\big) - {}^cD_{a^+}^{r;\varphi}\big({}^cD_{a^+}^{p;\varphi}\big({}^cD_{a^+}^{q;\varphi}\hbar_2^*(t)\big)\big)\big| \\
&\leqslant \bigg(\frac{(\varphi(b)-\varphi(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\varphi(b)-\varphi(a))^{p+r+k}}{\Gamma(p+r+k+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\varphi(b)-\varphi(a))^{r+k}}{\Gamma(r+k+1)}+\frac{(\varphi(b)-\varphi(a))^{k}}{\Gamma(k+1)}\Big)\\
&\times\mathcal{O}^{*}\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big)-\vartheta\big(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\}\\
=\;&\mathcal{O}\mathcal{O}^{*}\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big)-\vartheta\big(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\}\\
=\;&\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big)-\vartheta\big(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathcal{H}\big(\mathfrak{U}(z),\mathfrak{U}(z')\big)&\leqslant\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(\|z-z'\|\big)-\vartheta\big(\mathfrak{F}\big(\|z-z'\|\big)\big)\big\}\\
&\leqslant\mathfrak{F}^{-1}\big\{\mathfrak{F}\big(M(z,z')\big)-\vartheta\big(\mathfrak{F}\big(M(z,z')\big)\big)\big\},
\end{aligned}
$$

and so

$$
\mathfrak{F}\big(\mathcal{H}\big(\mathfrak{U}(z),\mathfrak{U}(z')\big)\big)\leqslant\mathfrak{F}\big(M(z,z')\big)-\vartheta\big(\mathfrak{F}\big(M(z,z')\big)\big).
$$

Now, taking a graph G on $\mathcal{C}$ such that $E(G)=\mathcal{C}\times\mathcal{C}$, all the conditions of Corollary 1 are satisfied. Thus, $\mathfrak{U}$ has a fixed point, and so the problem (1) has a solution. □

Example 1. Consider the fractional differential inclusion

$$
\begin{aligned}
&{}^{c}D_{0^+}^{1/2;\varphi}\big({}^{c}D_{0^+}^{1/3;\varphi}\big({}^{c}D_{0^+}^{1/4;\varphi}\big({}^{c}D_{0^+}^{1/5;\varphi}z(t)\big)\big)\big)\\
&\quad\in\left[0,\ \mathrm{e}^{t}+\frac{1}{\Pi}\cdot\frac{3|z(t)+{}^{c}D_{0^+}^{1/5;\varphi}z(t)+{}^{c}D_{0^+}^{1/4+1/5;\varphi}z(t)+{}^{c}D_{0^+}^{1/3+1/4+1/5;\varphi}z(t)|}{3+|z(t)+{}^{c}D_{0^+}^{1/5;\varphi}z(t)+{}^{c}D_{0^+}^{1/4+1/5;\varphi}z(t)+{}^{c}D_{0^+}^{1/3+1/4+1/5;\varphi}z(t)|}\right],\\
&\quad t\in[0,1],\ \varphi(t)=2t,\\
&z(0)=1,\qquad {}^{c}D_{0^+}^{1/5;\varphi}z(0)=2,\\
&{}^{c}D_{0^+}^{1/4;\varphi}\big({}^{c}D_{0^+}^{1/5;\varphi}z(0)\big)=3,\qquad {}^{c}D_{0^+}^{1/3;\varphi}\big({}^{c}D_{0^+}^{1/4;\varphi}\big({}^{c}D_{0^+}^{1/5;\varphi}z(0)\big)\big)=4,
\end{aligned}
$$

where

$$
\Pi=\frac{2^{1/5+1/4+1/3+1/2}}{\Gamma(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1)}+\frac{2^{1/4+1/3+1/2}}{\Gamma(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1)}+\frac{2^{1/3+1/2}}{\Gamma(\frac{1}{3}+\frac{1}{2}+1)}+\frac{2^{1/2}}{\Gamma(\frac{1}{2}+1)}.
$$

Note that

$$
f(t,v_1,v_2,v_3,v_4)=\left[0,\ \mathrm{e}^{t}+\frac{1}{\Pi}\frac{3|v_1+v_2+v_3+v_4|}{3+|v_1+v_2+v_3+v_4|}\right].
$$

Obviously, f is continuous. Here $q=1/5$, $p=1/4$, $r=1/3$, $k=1/2$, $a=0$, $b=1$, $z_i=i+1$, $i=0,1,2,3$.

Thus,

$$
\begin{aligned}
\mathcal{O}&=\frac{(\varphi(b)-\varphi(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}+\frac{(\varphi(b)-\varphi(a))^{p+r+k}}{\Gamma(p+r+k+1)}\\
&\quad+\frac{(\varphi(b)-\varphi(a))^{r+k}}{\Gamma(r+k+1)}+\frac{(\varphi(b)-\varphi(a))^{k}}{\Gamma(k+1)}\\
&=\Pi.
\end{aligned}
$$

Therefore, $\mathcal{O}^{*}=\mathcal{O}^{-1}=1/\Pi$.

Take $\vartheta(t) = 1/3$ and $\mathfrak{F}(t) = -1/t + 1$. Then we have

$$\begin{aligned}
&\mathcal{H}\big(f(t, v_1, v_2, v_3, v_4), f(t, v_1', v_2', v_3', v_4')\big)\\
&\quad\leqslant \mathcal{O}^{-1}\left|\frac{3|v_1 + v_2 + v_3 + v_4|}{3 + |v_1 + v_2 + v_3 + v_4|} - \frac{3|v_1' + v_2' + v_3' + v_4'|}{3 + |v_1' + v_2' + v_3' + v_4'|}\right|\\
&\quad= \mathcal{O}^{-1}\left|\frac{|v_1 + v_2 + v_3 + v_4|}{1 + \frac{1}{3}|v_1 + v_2 + v_3 + v_4|} - \frac{|v_1' + v_2' + v_3' + v_4'|}{1 + \frac{1}{3}|v_1' + v_2' + v_3' + v_4'|}\right|\\
&\quad= \mathcal{O}^{-1}\frac{||v_1 + v_2 + v_3 + v_4| - |v_1' + v_2' + v_3' + v_4'||}{(1 + \frac{1}{3}|v_1 + v_2 + v_3 + v_4|)(1 + \frac{1}{3}|v_1' + v_2' + v_3' + v_4'|}\\
&\quad\leqslant \mathcal{O}^{-1}\frac{||v_1 + v_2 + v_3 + v_4| - |v_1' + v_2' + v_3' + v_4'||}{1 + \frac{1}{3}||v_1 + v_2 + v_3 + v_4| - |v_1' + v_2' + v_3' + v_4'||}\\
&\quad\leqslant \mathcal{O}^{-1}\frac{\sum_{i=1}^{4}|v_i - v_i'|}{1 + \frac{1}{3}\sum_{i=1}^{4}|v_i - v_i'|}\\
&\quad= \mathcal{O}^{*}\mathfrak{F}^{-1}\left(\mathfrak{F}\left(\sum_{i=1}^{4}|v_i - v_i'|\right) - \vartheta\left(\mathfrak{F}\left(\sum_{i=1}^{4}|v_i - v_i'|\right)\right)\right).
\end{aligned}$$

Thus, condition (ii) in Theorem (2) is satisfied. Now, all conditions of Theorem (2) are satisfied. Thus, by this theorem the problem (1) has a solution.

5 Conclusion

In this paper, we first introduce a new multivalued contraction called weak Wardowski multivalued contraction and show that such mappings have fixed points. Second, we use our new contraction to show that the φ-Caputo fractional differential inclusion (1) is solvable when the right-hand function $f : [a, b] \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$ does not require the α–ψ-contractive condition for multivalued mappings. An example is given to show the usability of our new results. We intend to develop a coupled fixed point theorem for two variable multivalued mappings satisfying a weak Wardowski-type multivalued contraction in the future. Then we propose to investigate the solvability of the φ-Caputo fractional differential systems of inclusions (1) when the right-hand functions satisfy a weak Wardowski multivalued contraction.

References

1. T. Abdeljawad, F. Madjidi, F. Jarad, N. Sene, On dynamic systems in the frame of singular function dependent kernel fractional derivatives, *Mathematics*, **7**(10):946, 2019, https://doi.org/10.3390/math7100946.
2. R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.*, **109**(3):973–1033, 2010, https://doi.org/https://doi.org/10.1007/s10440-008-9356-6.

3. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, **44**:460–481, 2017, https://doi.org/https://doi.org/10.1016/j.cnsns.2016.09.006.
4. R. Almeida, A.B. Malinowska, M.T.T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Math. Methods Appl. Sci.*, **41**(1):336–352, 2018, https://doi.org/https://doi.org/10.1002/mma.4617.
5. R. Ameen, F. Jarad, T. Abdeljawad, Ulam stability for delay fractional differential equations with a generalized Caputo derivative, *Filomat*, **32**(15):5265–5274, 2018, https://doi.org/https://doi.org/10.2298/FIL1815265A.
6. S. Belmor, F. Jarad, T. Abdeljawad, M.A. Alqudah, On fractional differential inclusion problems involving fractional order derivative with respect to another function, *Fractals*, **28**(8):2040002, 2020, https://doi.org/https://doi.org/10.1142/S0218348X20400022.
7. S. Belmor, C. Ravichandran, F. Jarad, Nonlinear generalized fractional differential equations with generalized fractional integral conditions, *J. Taibah Univ. Sci.*, **14**(1):114–123, 2020, https://doi.org/10.1080/16583655.2019.1709265.
8. M. Benchohra, J.R. Graef, S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.*, **87**:851–863, 2008, https://doi.org/10.1080/00036810802307579.
9. J.V. da C. Sousa, E.C. de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60**:72–91, 2018, https://doi.org/10.1016/j.cnsns.2018.01.005.
10. J.V. da C. Sousa, E.C. de Oliveira, Leibniz type rule: ψ-Hilfer fractional operator, *Commun. Nonlinear Sci. Numer. Simul.*, **77**:305–311, 2019, https://doi.org/https://doi.org/10.1016/j.cnsns.2019.05.003.
11. J.V. da C. Sousa, K.D. Kucche, E.C. de Oliveira, Stability of ψ-Hilfer impulsive fractional differential equations, *Appl. Math. Lett.*, **88**:73–80, 2019, https://doi.org/https://doi.org/10.1016/j.aml.2018.08.013.
12. F. Echenique, A short and constructive proof of Tarski's fixed-point theorem, *Int. J. Game Theory*, **33**(2):215–218, 2005, https://doi.org/10.1007/s001820400192.
13. S. Hamani, M. Benchohra, J.R. Graef, Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions, *Electron. J. Differ. Equ.*, **2010**:20, 2010.
14. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Contin. Dyn. Syst., Ser. S*, **13**(3):709–722, 2020, https://doi.org/https://doi.org/10.3934/dcdss.2020039.
15. F. Jarad, S. Harikrishnan, K. Shah, K. Kanagarajan, Existence and stability results to a class of fractional random implicit differential equations involving a generalized Hilfer fractional derivative, *Discrete Contin. Dyn. Syst., Ser. S*, **13**(3):723–739, 2020, https://doi.org/https://doi.org/10.3934/dcdss.2020040.
16. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006.

17. B. Mohammadi, V. Parvaneh, H. Aydi, H. Işik, Extended Mizoguchi-Takahashi type fixed point theorems and an application, *Mathematics*, **7**:575, 2019, https://doi.org/10.3390/math7070575.

18. S.B. Nadler, Multivalued contraction mappings, *Pacific J. Math.*, **30**(2):475–88, 1969.

19. D.S. Oliveira, E.C. de Oliveira, On a Caputo-type fractional derivative, *Adv. Pure Appl. Math.*, **10**(2):81–91, 2019, https://doi.org/https://doi.org/10.1515/apam-2017-0068.

20. M. Parvaneh, A.P. Farajzadeh, Existence on weak-Wardowski-Prešić fixed point theorems via measure of noncompactness with applications to a system of fractional integral equations, *J. Nonlinear Convex Anal.*, **24**(1):1–15, 2023.

21. I. Podlubny, *Fractional Differential Equations*, Math. Sci. Eng., Vol. 198, Academic Press, San Diego, CA, 1999.

22. M.E. Samei, M.M. Matar, S. Etemad, S. Rezapour, On the generalized fractional snap boundary problems via G-Caputo operators: Existence and stability analysis, *Adv. Difference Equ.*, **2021**:498, 2021, https://doi.org/https://doi.org/10.1186/s13662-021-03654-9.

23. B. Samet, H. Aydi, Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative, *J. Inequal. Appl.*, **2018**:286, 2018, https://doi.org/https://doi.org/10.1186/s13660-018-1850-4.