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Nontrivial solutions for an asymptotically linear Δ_α-Laplace equation

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Abstract. In this paper, we study a class of degenerate unperturbed problems. We first investigate some properties of eigenvalues and eigenfunctions for the strongly degenerate elliptic operator and then obtain two existence theorems of nontrivial solutions when the nonlinearity is a function with an asymptotically condition.

Keywords: asymptotically linear, saddle point theorem, strongly degenerate elliptic operator.

1 Introduction

Our aim in this paper is to study the following nonlinear elliptic equation:

$$\begin{aligned} &-\Delta_\alpha u - \lambda u = f(x,u), \quad x \in \Omega,\\ &u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N > 2$), λ is a parameter, and Δ_α is a strongly degenerate elliptic operator of the form

$$\Delta_\alpha := \sum_{i=1}^{N} \partial_{x_i}\big(\alpha_i^2 \partial_{x_i}\big), \quad \alpha = (\alpha_1, \ldots, \alpha_N) : \mathbb{R}^N \to \mathbb{R}^N.$$

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The strong degenerate elliptic operator Δ_α was first introduced in [4], and the authors [5] remarked that Δ_α-Laplacian belong to the more general class of X-elliptic operators.

The Δ_α operator contains many degenerate elliptic operators such as the Grushin-type operator $G_a = \Delta_x + |x|^{2a}\Delta_y$, $a > 0$, where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $N_1 + N_2 = N$, and the operator of the form $P_{a,b,c} = \Delta_x + |x|^{2a}\Delta_y + |x|^{2b}|y|^{2c}\Delta_z$, $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, $N_1 + N_2 + N_3 = N$, where a, b and c are real positive constants. We refer readers to [1] for some important properties of this operator.

Many authors considered (1) with $\lambda = 0$, i.e.,

$$\begin{aligned} &-\Delta_\alpha u = f(x, u), \quad x \in \Omega, \\ &u = 0, \quad x \in \partial\Omega. \end{aligned} \tag{2}$$

In [1] the authors used the mountain pass theorem and the fountain theorem to study the existence and multiplicity of solutions for (2), where f satisfies a superlinear growth condition, and in [17] the authors examined the case where (2) has a nontrivial solution using sub-super solutions and variational methods. In [6] the authors adopted the three critical point theorem to consider the case where (2) has at least two solutions, and in [3, 15] the authors investigated the existence of infinitely many solutions when f satisfies a general superlinear growth condition. For more research to this kinds of equations, we also refer the readers to [7–14, 16, 19, 20] and the references therein.

Following [4], we denote by $W_\alpha^{1,2}(\Omega)$ the closure of $C_0^1(\Omega)$ with respect to the norm $\|u\|_{W_\alpha^{1,2}(\Omega)} = (\int_\Omega |\nabla_\alpha u|^2 \,\mathrm{d}x)^{1/2}$, which is a Hilbert space with the inner product $\langle u, v\rangle = \int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \,\mathrm{d}x$. Here $\nabla_\alpha = (\alpha_1\partial_{x_1}u, \dots, \alpha_N\partial_{x_N}u)$. For convenience, we abbreviate the norm $\|\cdot\|_{W_\alpha^{1,2}(\Omega)}$ as $\|\cdot\|$, and let $|\cdot|_q$ be the usual norm in the Lebesgue space $L^q(\Omega)$.

In order to study the asymptotically linear problem, we first present eigenvalues properties for Δ_α. We note that the author in [6] presented some properties for this operator, but the author did not provide proofs. For completeness, we first study the eigenvalue problem associated with (1),

$$\begin{aligned} &-\Delta_\alpha u = \lambda u, \quad x \in \Omega, \\ &u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{3}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue of the problem if there exists $u \in W_\alpha^{1,2} \setminus \{0\}$ such that (3) holds. Denote by $\sigma(-\Delta_\alpha)$ and $0 < \lambda_1 < \cdots < \lambda_k < \cdots$ the spectrum and the distinct eigenvalues of $-\Delta_\alpha$ in $W_\alpha^{1,2}(\Omega)$, respectively.

We note that problem (3) is equivalent to

$$\int_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \,\mathrm{d}x = \lambda \int_\Omega uv \,\mathrm{d}x, \quad u, v \in W_\alpha^{1,2}. \tag{4}$$

Theorem 1. *Let Ω be an open bounded set of $\mathbb{R}^N$. Then the eigenvalues and eigenfunctions of Δ_α have the following properties*:

(i) *Problem* (4) *has a positive eigenvalue* λ_1, *and its characteristic is*

$$\lambda_1 = \min_{\substack{|u|_2=1\\ u\in W_\alpha^{1,2}}} \int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x \tag{5}$$

or, equivalently,

$$\lambda_1 = \min_{u\in W_\alpha^{1,2}} \frac{\int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x}{\int_\Omega |u|^2\,\mathrm{d}x}.$$

(ii) *There exists a positive function* $e_1 \in W_\alpha^{1,2}$, *which is an eigenfunction corresponding to* λ_1, *attaining the minimum in* (4), *i.e.,* $|e_1|_2 = 1$ *and*

$$\lambda_1 = \int_\Omega |\nabla_\alpha e_1|^2\,\mathrm{d}x. \tag{6}$$

(iii) *The first eigenvalue* λ_1 *is simple, i.e., if* $u \in W_\alpha^{1,2}$ *is a solution of the following equation*

$$\int_\Omega \nabla_\alpha u\cdot\nabla_\alpha v\,\mathrm{d}x = \lambda_1\int_\Omega uv\,\mathrm{d}x \quad \forall v\in W_\alpha^{1,2},$$

then $u = \xi e_1$ *with* $\xi\in\mathbb{R}$.

(iv) *The set of eigenvalues of* (4) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with*

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \cdots \tag{7}$$

and

$$\lambda_k \to +\infty \quad \text{as } k\to+\infty. \tag{8}$$

Moreover, for any $k\in\mathbb{N}$, *the eigenvalues can be characterized as follows*:

$$\lambda_{k+1} = \min_{|u|_2=1 u\in\mathbb{P}_{k+1}} \int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x \tag{9}$$

or, equivalently,

$$\lambda_{k+1} = \min_{u\in\mathbb{P}_{k+1}\setminus\{0\}} \frac{\int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x}{\int_\Omega |u|^2\,\mathrm{d}x}, \tag{10}$$

where

$$\mathbb{P}_{k+1} := \bigl\{u\in W_\alpha^{1,2} \text{ s.t. } \langle u, e_j\rangle = 0\ \forall j = 1,\dots,k\bigr\}.$$

(v) *For any* $k\in\mathbb{N}$, *there exists a function* $e_{k+1}\in\mathbb{P}_{k+1}$, *which is an eigenfunction corresponding to* λ_{k+1}, *attaining the minimum in* (9), *i.e.,* $|e_{k+1}|_2 = 1$ *and*

$$\lambda_{k+1} = \int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x. \tag{11}$$

(vi) *The sequence* $\{e_k\}_k$ *of eigenfunctions corresponding to* λ_k *is an orthonormal basis of* $L^2(\Omega)$ *and orthogonal basis of* $W_\alpha^{1,2}$.

(vii) *Each eigenvalue* λ_k *has finite multiplicity. More precisely, if* λ_k *satisfies*

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1} \tag{12}$$

for some $h \in \mathbb{N}_0$*, then the set of all the eigenfunctions corresponding to* λ_k *agree with* $\operatorname{span}\{e_k \cdots e_{k+h}\}$.

Now, on the basis of the above theorem, we study the existence and multiplicity of solutions for (1). For the nonlinear term f and λ, we consider the following assumptions:

(f_1) f is a Carathéodory function, and $\sup_{|t|\leqslant r} |f(\cdot,t)| \in L^\infty(\Omega)$ for all $r > 0$.

(f_2) $\lim_{|t|\to+\infty} f(x,t)/t = 0$ uniformly with respect to a.e. $x \in \Omega$.

(f_3) $\lim_{t\to 0} f(x,t)/t = \lambda_0 \in \mathbb{R}$ uniformly with respect to a.e. $x \in \Omega$.

(λ_1) $\lambda \notin \sigma(-\Delta_\alpha)$.

(λ_2) (Λ_2) There exist $h, k \in \mathbb{N}$ with $k \geqslant h$ such that $\lambda_0 + \lambda < \lambda_h \leqslant \lambda_k \leqslant \lambda$.

We use the saddle point theorem and the pseudoindex theory introduced in [2] to discuss the existence and multiplicity of solutions for (1). Next, we state the main results:

Theorem 2. *Assume that the nonlinearity* $f(x,u)$ *satisfies* (f_1), (f_2) *and* λ *satisfies* (λ_1). *Then equation* (1) *has at least a nontrivial weak solution.*

Theorem 3. *Assume that the nonlinearity* $f(x,u)$ *satisfies* (f_1), (f_2) *and* (f_3) *and* λ *satisfies* (λ_1), (λ_2). *Then equation* (1) *has at least* $k-h+1$ *distinct pairs of nontrivial weak solutions.*

Remark 1. Note that we were motivated partly by Theorem 3.1 in [2]. Here the nonlinearity is no longer superlinear, and we use the saddle point theorem to establish the existence of a solution. Also, we present eigenvalue properties of the operator Δ_α.

2 Preliminaries

We recall the functional setting in [3, 4]. Consider the operator of the form $\Delta_\alpha := \sum_{i=1}^N \partial_{x_i}(\alpha_i^2 \partial_{x_i})$, where $\partial_{x_i} = \partial/\partial x_i$, $i = 1,\dots,N$. Here the function $\alpha_i : \mathbb{R}^N \to \mathbb{R}$ is continuous, strictly positive and of C^1 outside the coordinate hyperplane, i.e., $\alpha_i \geqslant 0$, $i = 1\dots N$, in $\mathbb{R}^N \setminus \Pi$, where $\Pi = \{(x_1,\dots,x_N) \in \mathbb{R}^N\colon \Pi_{i=1}^N x_i = 0\}$. As in [4], we assume that α_i satisfy the following properties:

(i) $\alpha_1(x) \equiv 1$, $\alpha_i(x) = \alpha_i(x_1,\dots,x_{i-1})$, $i = 1,\dots,N$.

(ii) For every $x \in \mathbb{R}^N$, $\alpha_i(x) = \alpha_i(x^*)$, $i = 1,\dots,N$, where $x^* = (|x_1|,\dots,|x_N|)$ if $x = (x_1,\dots,x_N)$.

(iii) There exists a constant $\rho \geqslant 0$ such that $0 \leqslant x_k \partial_{x_k} \alpha_i(x) \leqslant \rho \alpha_i(x)$ for all $k \in \{1,\dots,i-1\}$, $i = 2,\dots,N$, and for every $x \in \mathbb{R}_+^N := \{(x_1,\dots,x_N) \in \mathbb{R}^N\colon x_i \geqslant 0\ \forall i = 1,\dots,N\}$.

(iv) There exists a group of dilations $\{\delta_t\}_{t>0}$, $\delta_t : \mathbb{R}^N \to \mathbb{R}^N$, $\delta_t(x) = \delta_t(x_1, \dots, x_N) = (t^{\epsilon_1} x_1, \dots, t^{\epsilon_N} x_N)$, where $1 \leqslant \epsilon_1 \leqslant \epsilon_2 \leqslant \dots \leqslant \epsilon_N$ such that α_i is δ_t-homogeneous of degree $\epsilon_i - 1$, i.e., $\alpha_i(\delta_t(x)) = t^{\epsilon_i - 1}\alpha(x)$ for all $x \in \mathbb{R}^N$, $t > 0$, $i = 1, \dots, N$. This implies that the operation Δ_α is δ_t-homogeneous of degree two, i.e., $\Delta_\alpha(u(\delta_t(x))) = t^2(\Delta_\alpha u)(\delta_t(x))$ for all $u \in C^\infty(\mathbb{R}^N)$.

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $\{\delta_t\}_{t>0}$, i.e., $Q := \epsilon_1 + \dots + \epsilon_N$. The homogeneous Q plays a crucial role, both in the geometry and in the functional associated with the operator Δ_α.

Proposition. (*See* [4].) *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N > 2)$. Then the embedding $W^{1,2}_\alpha(\Omega) \hookrightarrow L^p(\Omega)$ is compact for every $p \in [1, 2^*_\alpha)$, where $2^*_\alpha = 2Q/(Q-2)$.*

Remark 2. For all $s \in [1, 2^*_\alpha)$, there exists a positive constant C_s such that

$$|u|_s \leqslant C_s \|u\|. \tag{13}$$

Theorem 4. (*See* [18].) *Let $E = V \oplus X$, where E is a real Banach space, and $V \neq \{0\}$ is finite dimensional. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies the* (PS)*-condition, and let*

(i) *there is a constant α and a bounded neighborhood D of* 0 *in V such that $I|_{\partial D} \leqslant \alpha$;*
(ii) *there is a constant $\beta > \alpha$ such that $I|_X \geqslant \beta$.*

Then I possess a critical value $c \geqslant \beta$. Moreover, c can be characterized as $c = \inf_{h\in\Gamma} \max_{u\in\overline{D}} I(h(u))$, where $\Gamma = \{h \in C(\overline{D}, E)\colon\ h = id \text{ on } \partial D\}$.

Let X be a Banach space,

$$\Sigma = \Sigma(X) = \{A \subset X\colon\ A \text{ closed and symmetric w.r.t. the origin,} \\ \text{i.e., } -u \in A \text{ if } u \in A\}$$

and $\mathcal{H} = \{h \in C(X, X)\colon\ h \text{ odd}\}$. Taking $A \in \Sigma$, $A \neq \emptyset$, the genus of A is $\gamma(A) = \inf\{k \in \mathbb{N}^*\colon\ \exists \psi(-u) = -\psi(u)\ \forall u \in A\}$ if such an infimum exists, otherwise, $\gamma(A) = +\infty$. Assume that $\gamma(\emptyset) = 0$.

Theorem 5. (*See* [2].) *Let H be a real Hilbert space, $J \in C^1(H, \mathbb{R})$ an even functional, $(\Sigma, \mathcal{H}, \gamma)$ an index theory on H. Let $S \in \Sigma$ and consider the pseudoindex theory $(S, \mathcal{H}^*, \gamma^*)$, where $\mathcal{H}^* = \{h \in \mathcal{H}\colon\ h \text{ bounded homeomorphism s.t. } h(u) = u \text{ if } u \notin J^{-1}(]0, +\infty[)\}$, and $\gamma^* = \min_{h\in\mathcal{H}^*} \gamma(h(A) \cap S)$ for all $A \in \Sigma$. Taking $a, b, c_0, c_\infty \in \overline{R}$, $-\infty \leqslant a < c_0 < c_\infty < b \leqslant +\infty$, we assume that*:

(i) *the functional J satisfies $(PS) - condition$ in (a, b);*
(ii) *$S \subseteq J^{-1}([c_0, +\infty[)$;*
(iii) *there exist an integer $\overline{k} \geqslant 1$ and $\overline{A} \in \Sigma$ such that $\overline{A} \subset J^{c_\infty}$ and $\gamma^*(\overline{A}) \geqslant \overline{k}$.*

Then the numbers $c_i = \inf_{A\in\Sigma_i} \sup_{u\in A} J(u)$, $i \in \{1, \dots, \overline{k}\}$, with $\Sigma_i = \{A \in \Sigma\colon \gamma^* \geqslant i\}$ are critical values for J and $c_0 \leqslant c_1 \leqslant \dots \leqslant c_{\overline{k}} \leqslant c_\infty$. Furthermore, if $c = c_i = \dots = c_{i+r}$ with $i \geqslant 1$ and $i + r \leqslant \overline{k}$, then $\gamma(K_c) \geqslant r + 1$.

3 Proof of the theorems

Before proving Theorem 1, we define the functional $I : W_\alpha^{1,2} \to \mathbb{R}$ as follows:

$$I(u) = \frac{1}{2} \int\limits_\Omega |\nabla_\alpha u|^2 \, \mathrm{d}x$$

and

$$\langle I'(u), v \rangle = \int\limits_\Omega \nabla_\alpha u \cdot \nabla_\alpha v \, \mathrm{d}x = \langle u, v \rangle.$$

In order to obtain Theorem 1, we prove the following lemmas.

Lemma 1. *If $\mathcal{A} \neq \emptyset$ is a weakly closed subspace of $W_\alpha^{1,2}$ and $\mathcal{M} := \{u \in \mathcal{A}\colon |u|_2 = 1\}$, then there exists $u_0 \in \mathcal{M}$ such that*

$$\min_{u \in \mathcal{M}} I(u) = I(u_0) \tag{14}$$

and

$$\langle u_0, v \rangle = \int\limits_\Omega \nabla_\alpha u_0 \cdot \nabla_\alpha v \, \mathrm{d}x = \lambda_0 \int\limits_\Omega u_0 v \, \mathrm{d}x \quad \forall v \in \mathcal{A}, \tag{15}$$

where $\lambda_0 := 2I(u_0) > 0$.

Proof. Let $\{u_j\}$ be the minimization sequence of I on $\mathcal{M}$, i.e., a sequence $u_j \in \mathcal{M}$ is such that

$$I(u_j) \to \inf_{u \in \mathcal{M}} I(u) \geqslant 0 > -\infty \tag{16}$$

as $j \to +\infty$. Then $I(u_j)$ is bounded in $\mathbb{R}$. From the definition of I we have that $\|u_j\|$ is bounded.

Note that $W_\alpha^{1,2}$ is a reflexive Banach space, and we have a subsequence still denoted as u_j and $u_j \rightharpoonup u_0$ in $W_\alpha^{1,2}$ for some $u_0 \in \mathcal{A}$. Thus,

$$\int\limits_\Omega \nabla_\alpha u_j \cdot \nabla_\alpha v \, \mathrm{d}x \to \int\limits_\Omega \nabla_\alpha u_0 \cdot \nabla_\alpha v \, \mathrm{d}x \quad \forall v \in W_\alpha^{1,2}, \quad j \to +\infty.$$

From $\|u_j\|$ bounded and the embedding theorem we have $u_j \to u_0$ in $L^2(\Omega)$ as $j \to \infty$. Then $|u_0|_2 = 1$, $u_0 \in \mathcal{M}$. By the weak lower semicontinuity we have

$$\begin{aligned} \lim_{j \to +\infty} I(u_j) &= \frac{1}{2} \lim_{j \to +\infty} \int\limits_\Omega |\nabla_\alpha u_j|^2 \, \mathrm{d}x \geqslant \frac{1}{2} \int\limits_\Omega |\nabla_\alpha u_0|^2 \, \mathrm{d}x \\ &= I(u_0) \geqslant \inf_{u \in \mathcal{M}} I(u). \end{aligned}$$

Therefore, from (16) $I(u_0) = \inf_{u \in \mathcal{M}} I(u)$. Hence, (14) is established.

Next, we let $\varepsilon \in (-1,1)$, $v \in \mathcal{M}$, $c_\varepsilon := |u_0 + \varepsilon v|_2$ and $u_\varepsilon := (u_0 + \varepsilon v)/c_\varepsilon$. We have that $u_\varepsilon \in \mathcal{M}$, and according to $|u_0|_2 = 1$, we get

$$\begin{aligned}
2J(u_\varepsilon) &= \int\limits_\Omega |\nabla_\alpha u_\varepsilon|^2 \,\mathrm{d}x = \int\limits_\Omega \left|\nabla_\alpha \frac{u_0 + \varepsilon v}{c_\varepsilon}\right|^2 \mathrm{d}x = \frac{\|u_0 + \varepsilon v\|^2}{c_\varepsilon^2} \\
&= \frac{\langle u_0 + \varepsilon v,\, u_0 + \varepsilon v\rangle}{(\int_\Omega |u_0 + \varepsilon v|^2 \,\mathrm{d}x)^2} = \frac{\|u_0\|^2 + 2\varepsilon\langle u_0, v\rangle + o(\varepsilon)}{|u_0|_2^2 + 2\varepsilon \int_\Omega u_0(x)v(x)\,\mathrm{d}x + o(\varepsilon)} \\
&= \big(2I(u_0) + 2\varepsilon\langle u_0, \varepsilon\rangle + o(\varepsilon)\big)\Bigg(1 - 2\int\limits_\Omega u_0(x)v(x)\,\mathrm{d}x + o(\varepsilon)\Bigg) \\
&= 2I(u_0) + 2\varepsilon\Bigg(\langle u_0, v\rangle - 2I(u_0)\int\limits_\Omega u_0(x)v(x)\,\mathrm{d}x\Bigg) + o(\varepsilon).
\end{aligned}$$

Note the minimum value of u_0, and we have (15). The proof is complete. □

Lemma 2. *Let $\lambda \neq \bar\lambda$ be different eigenvalues of problem* (4) *with eigenfunctions e and $\bar e \in W_\alpha^{1,2}$. Then*

$$\langle e, \bar e\rangle = 0 = \int\limits_\Omega e(x)\bar e(x)\,\mathrm{d}x.$$

Proof. If $e = 0$ or $\bar e = 0$, then the proof is complete. Now we consider the case when $e \neq 0$ and $\bar e \neq 0$. First, consider the characteristic function $f := e/|e|_2$ and $\bar f := \bar e/|\bar e|_2$. Substitute f, $\bar f$ into (4), and we have

$$\lambda\int\limits_\Omega f(x)\bar f(x)\,\mathrm{d}x = \int\limits_\Omega |\nabla_\alpha f|\cdot|\nabla_\alpha \bar f|\,\mathrm{d}x = \bar\lambda\int\limits_\Omega f(x)\bar f(x)\,\mathrm{d}x, \tag{17}$$

and then

$$(\lambda - \bar\lambda)\int_\Omega f(x)\bar f(x)\,\mathrm{d}x = 0.$$

Note that $\lambda \neq \bar\lambda$, we obtain

$$\int\limits_\Omega f(x)\bar f(x)\,\mathrm{d}x = 0. \tag{18}$$

Combine (17) and (18), and we get

$$\left\langle \frac{e}{|e|_2}, \frac{\bar e}{|\bar e|_2}\right\rangle = \langle f, \bar f\rangle = \int\limits_\Omega |\nabla_\alpha f|\cdot|\nabla_\alpha \bar f|\,\mathrm{d}x = 0.$$

Thus, $\langle e, \bar e\rangle = 0$ is established. The proof is complete. □

Lemma 3. *If e is an eigenfunction of* (4)*, the corresponding eigenvalue is λ, then*

$$\int\limits_\Omega |\nabla_\alpha e|^2\,\mathrm{d}x = \lambda|e|_2^2.$$

Proof. In (4), replacing v by e, we obtain

$$\int_{\Omega} |\nabla_\alpha e|^2 \,\mathrm{d}x = \lambda \int_{\Omega} \mathrm{e}^2 \,\mathrm{d}x.$$

The proof is complete. □

Proof of Theorem 1. (i) According to Lemma 1 (choosing $\mathcal{A} := W_\alpha^{1,2}$), we obtain that there is a λ_1, i.e.,

$$\lambda_1 = \min_{\substack{|u|_2=1\\ u\in W_\alpha^{1,2}}} \int_{\Omega} |\nabla_\alpha u|^2 \,\mathrm{d}x.$$

Moreover, it is an eigenvalue.

(ii) For this, we get that e_1 is an eigenfunction corresponding to λ_1 by (15). Hence (with $\mathcal{A} := W_\alpha^{1,2}$ in Lemma 1), (6) is established. It can be seen from (14) that the minimum λ_1 is attained at some $e_1 \in W_\alpha^{1,2}$, where $|e_1|_2 = 1$. To complete the proof of (ii), we first show that if e is an eigenfunction corresponding to λ_1, with $|e|_2 = 1$, then both e and $|e|$ attain the minimum in (5), also, either $e \geqslant 0$ or $e \leqslant 0$ a.e. in Ω. From Lemma 3 and (6) we obtain

$$2I(e) = \int_{\Omega} |\nabla_\alpha e|^2 \,\mathrm{d}x = \lambda_1 = 2I(e_1).$$

Also, we get $I(|e|) = I(e) = I(e_1)$, where $|e| \in W_\alpha^{1,2}$ and $\|e\|^2 = \lambda_1$, and either $\{e > 0\}$ or $\{e < 0\}$ has zero measure. Hence, by replacing e with e_1, we obtain that $e_1 \geqslant 0$. Thus, there exists a function $e_1 \in W_\alpha^{1,2}$ with $e_1 \geqslant 0$ and is an eigenfunction relative to λ_1, attaining the minimum in (4).

(iii) Assume that λ_1 also corresponds to another eigenfunction u in $W_\alpha^{1,2}$ with $0 \neq u$ and $u \neq e_1$. It follows from the proof of (ii) that $u \geqslant 0$ or $u \leqslant 0$ a.e. in Ω. First, consider the case $u \geqslant 0$ a.e. in Ω. We set

$$g := \frac{u}{|u|_2}, \qquad g_1 := e_1 - g.$$

Next, we prove that

$$g_1(x) = 0 \quad \text{a.e. } x \in \Omega. \tag{19}$$

Suppose that $g_1(x) \neq 0$ a.e. $x \in \Omega$, and we can conclude that g_1 is an eigenfunction corresponding to λ_1. Using the proof of (ii) again, we get that $g_1 \geqslant 0$ or $g_1 \leqslant 0$ a.e. in Ω. Thus, either $e_1 \geqslant g$ or $e_1 \leqslant g$. From $e_1 \geqslant 0$ we have one of the following:

$$e_1^2 \geqslant g^2 \quad \text{or} \quad e_1^2 \leqslant g^2 \quad \text{a.e. in } \Omega.$$

Also,

$$\int_{\Omega} \big(e_1^2(x) - g^2(x)\big) \,\mathrm{d}x = |e_1|_2^2 - |g|_2^2 = 0.$$

According to the above, we get $e_1^2 - g^2 = 0$, so $e_1 = g$. Hence, $g_1 = 0$ a.e. in Ω. That is a contradiction, so (19) is established. Therefore, f_1 is proportional to e_1, i.e. $u = \xi e_1$, $\xi \in \mathbb{R}$. The situation when $u \leqslant 0$ a.e. in Ω is similar.

(iv) By Lemma 1 (choosing $\mathcal{A} := \mathbb{P}_{k+1}$) we see that there exists λ_{k+1} such that (9) holds, and it is attained at some $e_{k+1} \in \mathbb{P}_{k+1}$. Also, from $\mathbb{P}_{k+1} \subseteq \mathbb{P}_k \subseteq W_\alpha^{1,2}$ we have

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \lambda_{k+1}.$$

First, we prove (7). In fact, we show $\lambda_1 \neq \lambda_2$. Indeed, if $\lambda_1 = \lambda_2$ and $e_2 \in \mathbb{P}_2$ also is an eigenfunction relative to λ_1, from (iii) we get that $e_2 = \xi e_1$ with $\xi \in \mathbb{R}$ and $\xi \neq 0$ so $e_2 \neq 0$. From $e_2 \in \mathbb{P}_2$ we have

$$0 = \langle e_1, e_2 \rangle = \langle e_1, \xi e_1 \rangle = \xi \|e_1\|^2.$$

We conclude that $e_1 = 0$, which is a contradiction.

Now apply (15) with $\mathcal{A} = \mathbb{P}_{k+1}$, and we have

$$\int_\Omega \nabla_\alpha e_{k+1} \cdot \nabla_\alpha v \,\mathrm{d}x = \lambda_{k+1} \int_\Omega e_{k+1}(x) v(x) \,\mathrm{d}x \quad \forall v \in \mathbb{P}_{k+1}. \tag{20}$$

In order to show that λ_{k+1} is an eigenvalue with eigenfunction e_{k+1}, we need to show that the above formula holds for any $v \in W_\alpha^{1,2}$ not only in $\mathbb{P}_{k+1}$. We define

$$X_1 = \operatorname{span}\{e_1, \dots, e_k\}, \quad X_2 = X_1^\perp = \mathbb{P}_{k+1}, \qquad W_\alpha^{1,2} = X_1 \oplus X_2.$$

Hence, for any $v \in W_\alpha^{1,2}$, $v := v_1 + v_2$, where $v_2 \in \mathbb{P}_{k+1}$, and $v_1 = \sum_{j=1}^k c_j e_j$ for some $c_1, \dots, c_k \in \mathbb{R}$. Put $v_2 = v - v_1$ into (20), and with the definition of v_1 we deduce

$$\begin{aligned}
&\int_\Omega \nabla_\alpha e_{k+1} \cdot \nabla_\alpha v \,\mathrm{d}x - \lambda_{k+1} \int_\Omega e_{k+1}(x) v(x) \,\mathrm{d}x \\
&\quad = \int_\Omega \nabla_\alpha e_{k+1} \cdot \nabla_\alpha v_1 \,\mathrm{d}x - \lambda_{k+1} \int_\Omega e_{k+1}(x) v_1(x) \,\mathrm{d}x \\
&\quad = \sum_{j=1}^k c_j \biggl[\int_\Omega \nabla_\alpha e_{k+1} \cdot \nabla_\alpha e_j \,\mathrm{d}x - \lambda_{k+1} \int_\Omega e_{k+1}(x) e_j(x) \,\mathrm{d}x \biggr].
\end{aligned} \tag{21}$$

Test the eigenvalue equation (4) for e_j against e_{k+1} for $j = 1, \dots, k$, furthermore, $e_{k+1} \in \mathbb{P}_{k+1}$, and we have

$$0 = \int_\Omega \nabla_\alpha e_{k+1} \cdot \nabla_\alpha e_j \,\mathrm{d}x = \lambda_j \int_\Omega e_{k+1}(x) e_j(x) \,\mathrm{d}x,$$

so

$$0 = \int_\Omega \nabla_\alpha e_{k+1} \cdot \nabla_\alpha e_j \,\mathrm{d}x = \int_\Omega e_{k+1}(x) e_j(x) \,\mathrm{d}x \quad \forall j = 1, \dots, k.$$

Putting the above formula into (21), we deduce that (20) is established for any $v \in W_\alpha^{1,2}$. Hence λ_{k+1} is an eigenvalue with eigenfunction e_{k+1}.

Next, in order to obtain (8), we prove that if $k, h \in \mathbb{N}$, $k \neq h$, then $\langle e_k, e_h \rangle = 0 = \int_\Omega e_k(x) e_h(x)\,\mathrm{d}x$. In fact, let $k > h$, then $k - 1 \geqslant h$, and

$$e_k \in \mathbb{P}_k = \bigl(\operatorname{span}\{e_1, \dots, e_{k-1}\}\bigr)^\perp \subseteq \bigl(\operatorname{span}\{e_h\}\bigr)^\perp.$$

Then $\langle e_k(x), e_h(x) \rangle = 0$. However, e_k is an eigenfunction. We substitute e_k into (4) and replace v with e_h, and we have

$$\int\limits_\Omega \nabla_\alpha e_k \cdot \nabla_\alpha e_h \,\mathrm{d}x = \lambda_k \int\limits_\Omega e_k e_h \,\mathrm{d}x,$$

so $\langle e_k, e_h \rangle = 0 = \int_\Omega e_k(x) e_h(x)\,\mathrm{d}x$. To prove $\lambda_k \to +\infty$, we assume $\lambda_k \to c$, $k \to +\infty$ for some $c \in \mathbb{R}$, so λ_k is bounded in $\mathbb{R}$. By Lemma 3 we get $\|e_k\|^2 = \lambda_k$, and there exists a subsequence $\{e_{k_j}\}$ and some $e_\infty \in L^2(\Omega)$ with

$$e_{k_j} \to e_\infty \quad \text{in } L^2(\Omega) \text{ as } k_j \to \infty.$$

According to the previous analysis, we see that e_{k_j} and e_{k_i} are orthogonal in $L^2(\Omega)$, and we get

$$|e_{k_j} - e_{k_i}|^2 = |e_{k_j}|^2 + |e_{k_i}|^2 = 2.$$

We have a contradiction since e_{k_j} is a Cauchy sequence in $L^2(\Omega)$. Thus, (8) is established.

Finally, we show (9). Suppose that there exists an eigenvalue $\lambda \notin \{\lambda_k\}_{k \in \mathbb{N}}$, and let $e \in W_\alpha^{1,2}$ be an eigenfunction corresponding to λ, so $|e|_2 = 1$ is obtained by normalization. According to Lemma 3, we get

$$2I(e) = \int\limits_\Omega |\nabla_\alpha e|^2 \,\mathrm{d}x = \lambda. \tag{22}$$

Also, by (5) and (6) we have

$$\lambda = 2I(e) \geqslant 2I(e_1) = \lambda_1.$$

From $\lambda \notin \{\lambda_k\}_{k \in \mathbb{N}}$ and (8) we see that there exists $k \in \mathbb{N}$ such that

$$\lambda_k < \lambda < \lambda_{k+1}.$$

Assume that $e \in \mathbb{P}_{k+1}$ and (22) and (9) imply that $\lambda = 2I(e) \geqslant \lambda_{k+1}$, which is a contradiction. Thus, we have $e \notin \mathbb{P}_{k+1}$, and there exists $j \in \{1, \dots, k\}$ such that $\langle e, e_j \rangle \neq 0$, so this contradicts Lemma 2. This completes the proof of (iv).

(v) Apply Lemma 1, let $\mathcal{A}$ be replaced by $\mathbb{P}_{k+1}$, and the minimum defining λ_{k+1} is attained for some $e_{k+1} \in \mathbb{P}_{k+1}$. By Lemma 1 we have (11). According to the proof of (iv), we see that (21) holds for any $v \in W_\alpha^{1,2}$, so we can conclude that e_{k+1} is an eigenfunction relative to λ_{k+1}. This completes the proof of (v).

(vi) From the proof of (iv) we see that the sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to λ_k is an orthonormal basis. Next, to complete the proof of (vi), we claim that the $\{e_k\}_{k\in\mathbb{N}}$ is a basis for both $W_\alpha^{1,2}$ and $L^2(\Omega)$.

First, we prove that it is a basis of $W_\alpha^{1,2}$. We prove that if $v \in W_\alpha^{1,2}$ is such that for all $k \in \mathbb{N}$, $\langle v, e_k\rangle = 0$, then $v \equiv 0$. Assume that $v \neq 0$ and there exists a nontrivial $v \in W_\alpha^{1,2}$ such that for all $k \in \mathbb{N}$, $\langle v, e_k\rangle = 0$, and by normalization we assume that $|v|_2 = 1$. Therefore, from (8) there exists a $k \in \mathbb{N}$ such that

$$2I(v) < \lambda_{k+1} = \min_{\substack{|u|_2=1\\ u\in\mathbb{P}_{k+1}}} \int_\Omega |\nabla_\alpha u|^2 \,\mathrm{d}x.$$

We see $v \notin \mathbb{P}_{k+1}$ and there exists $j \in \mathbb{N}$ such that $\langle v, e_j\rangle \neq 0$. This contradicts the assumption. Now we define $E_i := e_i/\|e_i\|$, and let $g \in W_\alpha^{1,2}$, $g_j := \sum_{i=1}^{j}\langle g, E_i\rangle E_i$. Also, $g_j \in \operatorname{span}\{e_1, \dots, e_j\}$ for all $j \in \mathbb{N}$. Define $G_j := g - g_j$, and by the orthogonality of $\{e_k\}_{k\in\mathbb{N}}$ in $W_\alpha^{1,2}$

$$\begin{aligned}
0 \leqslant \|G_j\|^2 &= \langle G_j, G_j\rangle = \langle g - g_j, -g_j\rangle \\
&= \|g\|^2 + \|g_j\|^2 - 2\langle g, g_j\rangle = \|g\|^2 + \langle g_j, g_j\rangle - 2\left\langle g, \sum_{i=1}^{j}\langle g, E_i\rangle E_i\right\rangle \\
&= \|g\|^2 - 2\sum_{i=1}^{j}\langle g, E_i\rangle^2.
\end{aligned}$$

Then $2\sum_{i=1}^{j}\langle g, E_i\rangle^2 \leqslant \|g\|^2$ for all $j \in \mathbb{N}$. We deduce that $\sum_{i=1}^{+\infty}\langle g, E_i\rangle^2$ is a convergent series. Now we assume that $\omega_j := \sum_{i=1}^{j}\langle g, E_i\rangle^2$, and since ω_j is a convergent series, it is a Cauchy sequence in $\mathbb{R}$. Also, from the orthogonality of $\{e_k\}_{k\in\mathbb{N}}$ in $W_\alpha^{1,2}$ we get

$$\begin{aligned}
\|G_j - G_J\|^2 = \|g_J - g_j\|^2 &= \left\|\sum_{i=j+1}^{J}\langle g, E_i\rangle E_i\right\|^2 \\
&= \sum_{i=j+1}^{J}\langle g, E_i\rangle^2 = \omega_J - \omega_j \quad \text{if } J > j.
\end{aligned}$$

We obtain that G_j is a Cauchy sequence in $W_\alpha^{1,2}$, and by the completeness of $W_\alpha^{1,2}$ there exists a $G \in W_\alpha^{1,2}$ such that

$$G_j \to G \quad \text{in } W_\alpha^{1,2},\ j \to \infty. \tag{23}$$

Moreover,

$$\langle G_j, E_k\rangle = \langle g, E_k\rangle - \langle g_j, E_k\rangle = \langle g, E_k\rangle - \langle g, E_k\rangle = 0.$$

From (23), for any $k \in \mathbb{N}$, we have $\langle G, E_k\rangle = 0$, that is, $G = 0$. Hence,

$$g_j \to g \quad \text{as } j \to \infty \text{ in } W_\alpha^{1,2}.$$

Finally, we prove that $\{e_k\}_{k\in\mathbb{N}}$ is a basis for $L^2(\Omega)$. Let $v \in L^2(\Omega)$ and $v_j \in C_0^1(\Omega)$ such that $|v_j - v|_2 \leqslant 1/j$. From $C_0^1(\Omega) \subseteq W_\alpha^{1,2}$ it follows that $v_j \in W_\alpha^{1,2}$. Since $\{e_k\}_{k\in\mathbb{N}}$ is a basis for $W_\alpha^{1,2}$, hence there exists $k_j \in \mathbb{N}$ and function μ_j, and $\mu_j \in \operatorname{span}\{e_1, \dots, e_{k_j}\}$ such that $\|v_j - \mu_j\| \leqslant 1/j$. Thus,

$$|v_j - \mu_j|_2 \leqslant C\|v_j - \mu_j\| \leqslant \frac{C}{j}$$

and

$$|v - \mu_j|_2 \leqslant |v - v_j|_2 + |v_j - \mu_j|_2 \leqslant \frac{C+1}{j}.$$

Therefore, the sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions of (4) is a basis in $L^2(\Omega)$.

(vii) Consider some $h \in \mathbb{N}_0$ such that $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1}$. We let

$$W_\alpha^{1,2} = \operatorname{span}\{e_k, \dots, e_{k+h}\} \oplus \big(\operatorname{span}\{e_k, \dots, e_{k+h}\}\big)^\perp,$$

and $\phi = \phi_1 + \phi_2$, where $\phi_1 \in \operatorname{span}\{e_k, \dots, e_{k+h}\}$, and $\phi_2 \in (\operatorname{span}\{e_k, \dots, e_{k+h}\})^\perp$. We have

$$\langle \phi_1, \phi_2 \rangle = 0. \tag{24}$$

Since ϕ is an eigenfunction relative to λ_k, substituting ϕ into (4), from (24) we have

$$\|\phi_1\|^2 + \|\phi_2\|^2 = \|\phi\|^2 = \int\limits_\Omega |\nabla_\alpha \phi|^2 \,\mathrm{d}x = \lambda_k \int\limits_\Omega \phi^2 \,\mathrm{d}x. \tag{25}$$

According to (v), we have that $e_k, \dots, e_{k+h}$ are eigenfunctions relative to $\lambda_k = \cdots = \lambda_{k+h}$ and ϕ is also an eigenfunction corresponding to λ_k. Hence, substituting both ϕ_1 and ϕ_2 into (4), we have

$$\lambda_k \int\limits_\Omega \phi_1 \phi_2 \,\mathrm{d}x = \int\limits_\Omega |\nabla_\alpha \phi_1| \cdot |\nabla_\alpha \phi_2| \,\mathrm{d}x = \langle \phi_1, \phi_2 \rangle = 0 \quad \Longrightarrow \quad \int\limits_\Omega \phi_1 \phi_2 \,\mathrm{d}x = 0$$

and

$$|\phi_1|_2^2 = |\phi_1 + \phi_2|_2^2 = |\phi_1|_2^2 + |\phi_2|_2^2. \tag{26}$$

Let

$$\phi_1 = \sum_{i=k}^{k+h} c_j e_j, \quad c_j \in \mathbb{R}.$$

By (v) and the orthogonality in (vi) we have

$$\begin{aligned} \|\phi_1\|^2 = \langle \phi_1, \phi_1 \rangle = \sum_{j=k}^{k+h} \langle c_j e_j, c_j e_j \rangle = \sum_{j=k}^{k+h} c_j^2 \|e_j\|^2 \\ = \sum_{j=k}^{k+h} c_j^2 \lambda_j = \lambda_k \sum_{j=k}^{k+h} c_j^2 = \lambda_k |\phi_1|_2^2. \end{aligned} \tag{27}$$

Since ϕ_1 and ϕ are eigenfunctions corresponding to λ_k, hence we deduce that ϕ_2 is also an eigenfunction corresponding to λ_k. From Lemma 2 and (12) we get

$$\langle\phi_2, e_1\rangle = \langle\phi_2, e_2\rangle = \cdots = \langle\phi_2, e_{k-1}\rangle = 0$$

and

$$\phi_2 \in \bigl(\operatorname{span}\{e_k, \ldots, e_{k+h}\}\bigr)^\perp = \mathbb{P}_{k+h+1}.$$

Now we prove that $\phi_2 = 0$ via contradiction. If $\phi_2 \neq 0$, from (10)

$$\lambda_k < \lambda_{k+h+1} = \min_{u\in\mathbb{P}_{k+h+1}\setminus\{0\}} \frac{\int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x}{\int_\Omega |u|^2\,\mathrm{d}x} \leqslant \frac{\int_\Omega |\nabla_\alpha \phi_2|^2\,\mathrm{d}x}{\int_\Omega |\phi_2|^2\,\mathrm{d}x} = \frac{\|\phi_2\|^2}{|\phi_2|_2^2}. \tag{28}$$

Also, from (25)–(28) we have

$$\lambda_k|\phi|_2^2 = \|\phi_1\|^2 + \|\phi_2\|^2 > \lambda_k|\phi_1|_2^2 + \lambda_k|\phi_2|_2^2 = \lambda_k|\phi|_2^2.$$

This is a contradiction. Therefore, we deduce that $\phi = \phi_1 \in \operatorname{span}\{e_k, \ldots, e_{k+h}\}$. The proof is complete. □

Now we define the following energy functional:

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x,$$

where $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$. From the hypotheses on f we observe that J_λ is well defined on $W_\alpha^{1,2}(\Omega)$ and $J_\lambda \in C^1(W_\alpha^{1,2}(\Omega), \mathbb{R})$ with

$$\bigl\langle J_\lambda'(u), v\bigr\rangle = \int_\Omega \nabla_\alpha u \nabla_\alpha v\,\mathrm{d}x - \lambda\int_\Omega uv\,\mathrm{d}x - \int_\Omega f(x,u)v\,\mathrm{d}x \quad \forall v \in W_\alpha^{1,2}. \tag{29}$$

To establish Theorems 2 and 3, we first provide the following lemmas, which shows that the (PS)-condition is satisfied.

Lemma 4. *Let* (f_1), (f_2) *be satisfied. Then any* (PS)*-sequence* $\{u_j\}$ *of* J_λ *is bounded in* $W_\alpha^{1,2}$.

Proof. Let $\{u_j\} \subset W_\alpha^{1,2}$ be a (PS)-sequence such that

$$J_\lambda(u_j) \leqslant c, \qquad J_\lambda'(u_j) \to 0, \tag{30}$$

and then we have

$$\langle u_j, \varphi\rangle - \lambda\int_\Omega u_j\varphi\,\mathrm{d}x - \int_\Omega f(x,u_j)\varphi\,\mathrm{d}x = o(1) \quad \forall\varphi \in W_\alpha^{1,2}. \tag{31}$$

From (f_1) and (f_2), for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\bigl|f(x,t)\bigr| \leqslant \varepsilon|t| + C_\varepsilon \quad \text{a.e. } x \in \Omega\ \forall t \in \mathbb{R}. \tag{32}$$

Assume that $\|u_j\| \to \infty$ as $j \to \infty$. Set $v_j := u_j/\|u_j\|$, and then $\|v_j\|$=1. Thus, we have $v_j \rightharpoonup v$ in $W_\alpha^{1,2}$, $v_j \to v$ in L^p for $1 \leqslant p < 2_\alpha^*$. In (31), φ is replaced by $v_j - v$, and dividing by $\|u_j\|$, one has

$$\langle v_j,\, v_j - v\rangle = \lambda \int\limits_\Omega v_j(v_j - v)\,\mathrm{d}x + \int\limits_\Omega \frac{f(x,u_j)}{\|u_j\|}(v_j - v)\,\mathrm{d}x + o(1).$$

From $v_j \to v$ in L^p for $1 \leqslant p < 2_\alpha^*$ and Hölder's inequality we have

$$\biggl|\int\limits_\Omega v_j(v_j - v)\,\mathrm{d}x\biggr| \leqslant |v_j|_2|v_j - v|_2 = o(1). \tag{33}$$

Moreover, from (32) we have that

$$\begin{aligned}\biggl|\int\limits_\Omega \frac{f(x,u_j)}{\|u_j\|}(v_j - v)\,\mathrm{d}x\biggr| &\leqslant \biggl|\int\limits_\Omega \biggl(\varepsilon v_j + \frac{C_\varepsilon}{\|u_j\|}\biggr)(v_j - v)\,\mathrm{d}x\biggr| \\ &\leqslant \varepsilon|v_j|_2|v_j - v|_2 + C_\varepsilon\frac{|v_j - v|_1}{\|u_j\|} \\ &= o(1). \end{aligned}\tag{34}$$

Hence, from (33), (34), we get $\langle v_j,\, v_j - v\rangle = o(1)$. Thus, $v_j \to v$ strongly in $W_\alpha^{1,2}$. If $v = 0$, we obtain $\|v_j\| \to 0$, a contradiction. Hence, $v \neq 0$. Now, dividing (31) by $\|u_j\|$, we obtain

$$\langle v_j, \varphi\rangle - \lambda\int\limits_\Omega v_j\varphi\,\mathrm{d}x - \int\limits_\Omega \frac{f(x,u_j)\varphi}{\|u_j\|}\,\mathrm{d}x = o(1). \tag{35}$$

From (f_2) we have

$$\lim_{j\to+\infty}\int\limits_\Omega \frac{f(x,u_j)\varphi}{\|u_j\|}\,\mathrm{d}x = \lim_{j\to+\infty}\int\limits_\Omega \frac{f(x,u_j)}{u_j}v_j\varphi\,\mathrm{d}x = 0.$$

Passing to the limit in (35), from $v_j \to v$ strongly in $W_\alpha^{1,2}$ we have that $\langle v, \varphi\rangle = \lambda\int_\Omega v\varphi\,\mathrm{d}x$ for all $\varphi \in W_\alpha^{1,2}$. This implies that $\lambda \in \sigma(-\Delta_\alpha)$, which contradicts (λ_1). Therefore, $\{u_j\}$ is a bounded sequence. □

Lemma 5. *Let* (f_1)*,* (f_2) *be satisfied. Then any* (PS)*-sequence* $\{u_j\}$ *has a convergent subsequence.*

Proof. We see that $\{u_j\}$ is bounded in $W_\alpha^{1,2}$, and therefore, we can assume that there is a subsequence, still denoted by $\{u_j\}$ and there exists $u_1 \in W_\alpha^{1,2}$ such that $u_j \rightharpoonup u_1$ in $W_\alpha^{1,2}$ and $u_j \to u_1$ in L^p for all $p \in [1, 2_\alpha^*)$. From (29) and (30) we get

$$\langle J'_\lambda(u_j),\, u_j - u_1\rangle \to 0, \quad j \to +\infty. \tag{36}$$

Moreover, from (32) and the Hölder inequality we have

$$\int_\Omega \left|f(x,u_j)\right||u_j-u_1|\,\mathrm{d}x \leqslant \int_\Omega \left(\varepsilon|u_j|+C_\varepsilon\right)|u_j-u_1|\,\mathrm{d}x$$
$$\leqslant \varepsilon|u_j|_2|u_j-u_1|_2+C_\varepsilon|u_j-u_1|_1 \to 0, \quad j\to+\infty.$$

Therefore, from (29) and (36) it follows that

$$\langle J'_\lambda(u_j-u_1),\, u_j-u_1\rangle = \int_\Omega \left|\nabla_\alpha(u_j-u_1)\right|^2\mathrm{d}x - \lambda\int_\Omega |u_j-u_1|^2\,\mathrm{d}x$$
$$-\int_\Omega f(x,\, u_j-u_1)|u_j-u_1|\,\mathrm{d}x \to 0.$$

Thus, $\|u_j-u_1\|^2\to 0$. Hence $u_j\to u_1$ in $W^{1,2}_\alpha$. The proof is complete. □

Proof of Theorem 2. Let $\{e_k\}_k$ the eigenfunctions corresponding to λ_k be the orthonormal basis of $W^{1,2}_\alpha$. According to the proof of Theorem 1, we have

$$W^{1,2}_\alpha = X_1\oplus X_2.$$

Consider $\lambda>\lambda_1$, and $\lambda\notin\sigma(\Delta_\alpha)$. By the definition of the eigenvalues, we get

$$\|u\|^2\leqslant\lambda_k|u|_2^2 \quad \forall u\in X_1 \quad\text{and}\quad \|u\|^2\geqslant\lambda_{k+1}|u|_2^2 \quad \forall u\in X_2. \tag{37}$$

According to (32), we have $|F(x,t)|\leqslant C_1(1+t^2)$ for a.e. $x\in\Omega$, for all $t\in\mathbb{R}$. For every $u\in X_1$, we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x\\
&\leqslant \frac{\lambda_k}{2}|u|_2^2 - \frac{\lambda}{2}|u|_2^2 + \frac{\varepsilon}{2}\int_\Omega u^2\,\mathrm{d}x + C_\varepsilon\int_\Omega u\,\mathrm{d}x\\
&\leqslant \frac{1}{2}(\lambda_k-\lambda+\varepsilon)|u|_2^2 + C_2|u|_2.
\end{aligned}\tag{38}$$

Let $\lambda_k<\lambda$ be such that $\lambda_k+\varepsilon<\lambda$. Then, since X_1 is a finite dimensional subspace and $J_\lambda\to-\infty$ as $\|u\|$ diverges in X_1, there exists a positive constant C_3 such that $J_\lambda(u)\leqslant -C_3$ for all $u\in X_1$. On the other hand, from (37), for every $u\in X_2$, we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x\\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{2\lambda_{k+1}}\|u\|^2 - \int_\Omega C_1\left(1+u^2\right)\mathrm{d}x\\
&\geqslant \frac{1}{2}\left(1-\frac{\lambda+C_4}{\lambda_{k+1}}\right)\|u\|^2 - C_5.
\end{aligned}$$

We obtain that $J_\lambda(u) \geqslant C_6$ for all $u \in X_2$. Therefore, let $E = W_\alpha^{1,2}$, $V = X_1$ and $X = X_2$, and it follows from Lemmas 4 and 5 that all the conditions of Theorem 4 are satisfied.

If $\lambda < \lambda_1$, the functional J_λ is coercive and can be shown to a global minimum using the method of the Weierstrass Theorem. The proof is complete. □

Lemma 6. *Let* (f_1)–(f_3)*,* (λ_1)*,* (λ_2) *be satisfied. Then there exist* $\rho > 0$ *and* $c_0 > 0$ *such that* $J_\lambda(u) \geqslant c_0$ *for all* $u \in S_\rho \cap X_2$*,* $S_\rho := \{u \in W_\alpha^{1,2}\colon \|u\| = \rho\}$.

Proof. According to (Λ_2), we have that $\lambda_0 < 0$. Therefore, it follows from (f_2) that for every $\varepsilon > 0$, there exist $\delta_1 \geqslant 1$ and $\delta_2 \geqslant 0$ such that

$$\bigl|F(x,t)\bigr| \leqslant \frac{\varepsilon}{2}t^2 \quad \text{if } |t| > \delta_1 \text{ for a.e. } x \in \Omega, \tag{39}$$

and

$$\left|F(x,t) - \frac{\lambda_0}{2}t^2\right| \leqslant \frac{\varepsilon}{2}t^2 \quad \text{if } |t| < \delta_2 \text{ for a.e. } x \in \Omega. \tag{40}$$

From (f_1), choosing any constant $p \in [0, 4Q/(Q-2))$, there exists $\epsilon > 0$ such that

$$\bigl|F(x,t)\bigr| \leqslant \epsilon|t|^{p+2} \quad \text{if } \delta_2 \leqslant |t| \leqslant \delta_1 \text{ for a.e. } x \in \Omega. \tag{41}$$

Hence, it follows from (39)–(41) that there exists ϵ_1 such that $|F(x,t)| \leqslant (\lambda_0 + \varepsilon)t^2/2 + \epsilon_1|t|^{p+2}$, $t \in \mathbb{R}$, for a.e. $x \in \Omega$. Integrate both sides of the above formula, and we have that

$$\int_\Omega \bigl|F(x,u)\bigr| \,\mathrm{d}x \leqslant \frac{\lambda_0 + \varepsilon}{2}|u|_2^2 + \epsilon_1|u|_{p+2}^{p+2} \quad \forall u \in W_\alpha^{1,2}. \tag{42}$$

Then from (13), (37) and (42) we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla_\alpha u|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2 \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}|u|_2^2 - \frac{\lambda_0 + \varepsilon}{2}|u|_2^2 - \epsilon_1|u|_{p+2}^{p+2} \\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{2\lambda_{k+1}}\|u\|^2 - \frac{\lambda_0 + \varepsilon}{2\lambda_{k+1}}\|u\|^2 - \epsilon_1\|u\|^{p+2} \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda + \lambda_0 + \varepsilon}{\lambda_h}\right)\|u\|^2 - \epsilon_1\|u\|^{p+2}.
\end{aligned}$$

Based on (λ_2), note that ε can be small enough, there exists a constant α such that $J_\lambda(u) \geqslant \alpha\|u\|^2 - \epsilon'\|u\|^{p+2}$. If ρ is small enough, there exist $c_0 > 0$ such that $J_\lambda(u) \geqslant c_0$. □

Lemma 7. *Let* (f_1)*,* (f_3) *and* (λ_2) *be satisfied. Then there exist* $c_\infty > c_0$ *such that* $J_\lambda(u) \leqslant c_\infty$ *for all* $u \in X_1$.

Proof. From (38), for every $u \in X_1$, we have that $J_\lambda(u) \leqslant (1/2)(1-(\lambda+C_2)/\lambda_k)\|u\|^2$ as X_1 is a finite dimensional subspace, and $J_\lambda \to -\infty$ as $\|u\|$ diverges in X_1. We see that there exists $c_\infty = c_\infty(\varepsilon)(c_\infty > c_0)$ such that $J_\lambda(u) \leqslant c_\infty$. □

Proof of Theorem 3. From Lemmas 4 and 5 we see that J_λ satisfies the (PS)-condition. Also, by Lemmas 6 and 7 we consider the pseudoindex theory $(S_\rho \cap X_2, \mathcal{H}^*, \gamma^*)$ related to the genus, $S_\rho \cap X_2$ and J_λ. By Theorem A.2 in [18], with $V = X_1$, $\partial B = S_\rho$ and $W = X_2$, we get $\gamma(X_1 \cap h(S_\rho \cap X_2)) \geqslant \dim X_1 - \operatorname{codim} X_2$ for all $h \in X_1$, which implies that $\gamma^*(X_1) \geqslant k-h-1$. Hence, with $\overline{A} = X_1$, $S = S_\rho \cap X_2$, all the conditions of Theorem 5 are satisfied. Thus, J_λ has at least $k-h-1$ distinct pairs of critical points corresponding to at most $k-h-1$ distinct critical values c_i. The proof is complete. □

References

1. C.T. Anh, B.K. My, Existence of solutions to Δ_λ-Laplace equations without the Ambrosetti–Rabinowitz condition, *Complex Var. Elliptic Equ.*, **61**(1):137–150, 2016, `https://doi.org/10.1080/17476933.2015.1068762`.

2. R. Bartolo, A.M. Candela, A. Salvatore, Perturbed asymptotically linear problems, *Ann. Mat. Pura Appl. (4)*, **193**(1):89–101, 2014, `https://doi.org/10.1007/s10231-012-0267-9`.

3. J. Chen, X. Tang, Z. Gao, Infinitely many solutions for semilinear Δ_λ-Laplace equations with sign-changing potential and nonlinearity, *Stud. Sci. Math. Hung.*, **54**(4):536–549, 2017, `https://doi.org/10.1556/012.2017.54.4.1382`.

4. A.E. Kogoj, E. Lanconelli, On semilinear Δ_λ-Laplace equation, *Nonlinear Anal., Theory Methods Appl.*, **75**(12):4637–4649, 2012, `https://doi.org/10.1016/j.na.2011.10.007`.

5. A.E. Kogoj, S. Sonner, Hardy type inequalities for Δ_λ-Laplacians, *Complex Var. Elliptic Equ.*, **61**(3):422–442, 2016, `https://doi.org/10.1080/17476933.2015.1088530`.

6. D.T. Luyen, Two nontrivial solutions of boundary-value problems for semilinear Δ_γ-differential equations, *Math. Notes*, **101**(5–6):815–823, 2017, `https://doi.org/10.1134/S0001434617050078`.

7. D.T. Luyen, Existence of nontrivial solution for fourth-order semilinear Δ_γ-Laplace equation in $\mathbb{R}^N$, *Electron. J. Qual. Theory Differ. Equ.*, **2019**:78, 2019, `https://doi.org/10.14232/ejqtde.2019.1.78`.

8. D.T. Luyen, Sign-changing solutions of boundary value problems for semilinear Δ_γ-Laplace equations, *Rend. Semin. Mat. Univ. Padova*, **143**:113–134, 2020, `https://doi.org/10.4171/rsmup/42`.

9. D.T. Luyen, Nontrivial solutions to boundary value problems for semilinear Δ_γ-differential equations, *Appl. Math.*, **66**(4):461–478, 2021, `https://doi.org/10.21136/AM.2021.0363-19`.

10. D.T. Luyen, Picone's identity for Δ_γ-Laplace operator and its applications, *Ukr. Math. J.*, **73**(4):601–609, 2021, `https://doi.org/10.1007/s11253-021-01946-7`.

11. D.T. Luyen, Multiple solutions for Δ_γ-Laplace problems without the Ambrosetti–Rabinowitz condition, *Asian-Eur. J. Math.*, **15**(3):2250051, 2022, https://doi.org/10.1142/S1793557122500516.
12. D.T. Luyen, L.T.H. Hanh, Infinitely many solutions for semilinear Δ_γ-differential equations in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition, *Minimax Theory Appl.*, **5**(1):7–18, 2020.
13. D.T. Luyen, L.T.H. Hanh, Infinitely many solutions for perturbed $\Delta\gamma$-Laplace equations, *Georgian Math. J.*, **29**(6):863–882, 2022, https://doi.org/10.1515/gmj-2022-2179.
14. D.T. Luyen, L.T.H. Hanh, Three nontrivial solutions of boundary value problems for semilinear Δ_γ-Laplace equation, *Bol. Soc. Parana. Mat. (3)*, **40**:1–10, 2022, https://doi.org/10.5269/bspm.45841.
15. D.T. Luyen, D.T. Huong, L.T.H. Hanh, Existence of infinitely many solutions for Δ_γ-Laplace problems, *Math. Notes*, **103**(5–6):724–736, 2018, https://doi.org/10.1134/S000143461805005X.
16. D.T. Luyen, N.T. Nam, Infinitely many solutions for fourth-order semilinear Δ_γ-Laplace equation in $\mathbb{R}^N$, *J. Elliptic Parabol. Equ.*, **7**(2):977–988, 2021, https://doi.org/10.1007/s41808-021-00129-6.
17. D.T. Luyen, N.M. Tri, Existence of solutions to boundary-value problems for similinear Δ_γ differential equations, *Math. Notes*, **97**(1–2):73–84, 2015, https://doi.org/10.1134/S0001434615010101.
18. P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math., Vol. 65, AMS, Providence, RI, 1986, https://doi.org/10.1090/cbms/065.
19. B. Rahal, Liouville-type theorems with finite Morse index for semilinear Δ_λ-Laplace operators, *NoDEA, Nonlinear Differ. Equ. Appl.*, **25**(3):21, 2018, https://doi.org/10.1007/s00030-018-0512-z.
20. B. Rahal, M.K. Hamdani, Infinitely many solutions for Δ_α-Laplace equations with sign-changing potential, *J. Fixed Point Theory Appl.*, **20**(4):137, 2018, https://doi.org/10.1007/s11784-018-0617-3.