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Optimal control for a two-sidedly degenerate aggregation equation

Mostafa Bendahmane[a,1], **Fahd Karami**[b], **Elmahdi Erraji**[b], **Abdelghafour Atlas**[c], **Lekbir Afraites**[d]

[a]Institut de mathématiques de Bordeaux,
Université de Bordeaux, France
mostafa.bendahmane@u-bordeaux.fr

[b]Ecole Supérieure de Technologie d'Essaouira,
Université Cadi Ayyad, Morocco
fa.karami@uca.ma; erraji0elmahdi@gmail.com

[c]Ecole Nationale des Sciences Appliquées de Marrakech,
Université Cadi Ayyad, Morocco
a.atlas@uca.ac.ma

[d]EMI FST Béni-Mellal,
Université Sultan Moulay Slimane, Morocco
l.afraites@usms.ma

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Abstract. In this paper, we are concerned with the study of the mathematical analysis for an optimal control of a nonlocal degenerate aggregation model. This model describes the aggregation of organisms such as pedestrian movements, chemotaxis, animal swarming. We establish the well-posedness (existence and uniqueness) for the weak solution of the direct problem by means of an auxiliary nondegenerate aggregation equation, the Faedo–Galerkin method (for the existence result) and duality method (for the uniqueness). Moreover, for the adjoint problem, we prove the existence result of minimizers and first-order necessary conditions. The main novelty of this work concerns the presence of a control to our nonlocal degenerate aggregation model. Our results are complemented with some numerical simulations.

Keywords: aggregation equation, nonlocal models, degenerate diffusion, finite volume, optimal control, adjoint problem.

1 Introduction

Nonlocal aggregation model has recently received great attention in biological applications. There exists a large variety of biological aggregation models such as the flocking of

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birds, aggregation of fish and the swarming insect. We refer the reader to a large number of works have focused on biological aggregation [6, 7, 10, 12, 16, 18–20, 22, 23].

In this paper, we consider the following nonlocal degenerate equation:

$$\begin{aligned} &\partial_t u - \operatorname{div}\big(a(u)\nabla u - u\nabla K * u\big) = F(u,w) \quad \text{in } \Omega_T, \\ &\big(a(u)\nabla u - u\nabla K * u\big)\cdot\eta = 0 \quad \text{on } \Sigma_T, \\ &u(x,0) = u_0(x) \quad \text{in } \Omega. \end{aligned} \tag{1}$$

Herein, $\Omega_T := \Omega\times(0,T)$, $\Sigma_T := \partial\Omega\times(0,T)$, $T>0$ is a fixed time, and Ω is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ and outer unit normal η. In the model above, the density of the population is represented by $u = u(x,t)$, $a(u)$ is a density-dependent diffusion coefficient. Furthermore, K is the sensing (interaction) kernel that models the long-range attraction. In the convolution term, u is extended by zero, outside of Ω. More precisely,

$$\nabla K * u(x) = \int\limits_{\Omega} \nabla K(x-y)u(y)\,\mathrm{d}y.$$

Note that system (1) arises in many models of biology and, in particular, in social organizations, which is one of the fundamental aspects of animal behaviors.

In this paper, we assume that the density-dependent diffusion coefficient $a(u)$ degenerates for $u=0$ and $u=\overline{u}$. This means that the diffusion vanishes when u approaches values close to the threshold $\overline{u}$ and also in the absence of the population. This interpretation was proposed in [2] and in the references therein for the chemotaxis model.

To summarize, the following main assumptions are made:

- (A1) $a\in C^1([0,1])$, $a(0)=a(\overline{u})=0$ and $a(s)>0$ for $0<s<\overline{u}$;
- (A2) $K\in C^2(\mathbb{R}^3)$ is a nonnegative radially nonincreasing function with the norm $\|K\|_{C^2(\mathbb{R}^3)}<\infty$ and $\int_{\mathbb{R}^3} K(x)\,\mathrm{d}x = 1$.

In addition, the reaction function F has the following form:

$$F(u,w) = \alpha u - wu^2,$$

where, $\alpha>0$ is the Malthusian growth coefficient, and $w(\cdot)$ (the control) is a nonnegative function of the intraspecific competition.

Regarding the degeneracy of the diffusion coefficient, a typical example of a is $a(u) = u(\overline{u}-u)$. Note that the degeneracy of the diffusion coefficient and the nonlocal term are major concerns for the mathematical and numerical treatment of equation (1).

To put this paper in the proper perspective, we mention that the nonlocal aggregation equation investigated analytically and numerically by many authors: [6, 22] for the study of the pure aggregation equation, i.e., $a(u)=0$ and $F=0$, [8] for the existence result, [3, 4] for the blowup of the solution and [13, 14] for the analysis of the numerical simulation.

Many studies have focused on the competition between the degenerate diffusion as a repellent force and nonlocal aggregation terms as attractive force (see, e.g., [15]). This competition is observed in many biological phenomenon from social pattern formation to microbiological dynamics under chemotaxis force [5]. From a mathematical perspective, we mention, for example, the work [17] where the author proposed and proved the existence results of local and global solutions to a class of aggregation equations depending on attraction kernel regularity. In passing, we want to mention that the authors in [5] considered and studied the model (1) with $F := 0$ and $a(u) = 0$ for $u = 0$.

In our study, we are concerned with the mathematical analysis and numerical simulations of an optimal control problem arising in the study of population dynamics. Our model is governed by a degenerate aggregation-diffusion equation. To this model, we introduce a notion of a weak solution for the direct problem and prove its well-posedness. Comparing to [5] (equation (1) with $F := 0$ and $a(u) = 0$ for $u = 0$), in this paper, we prove the existence of solutions by applying the Faedo–Galerkin method, deriving a priori estimates and then passing to the limit in the approximate solutions using monotonicity and compactness arguments. The uniqueness of these weak solutions is guaranteed by using the duality method. For the analysis of our optimal control problem, we use the Lagrangian framework in which the control problem is set as a constrained minimization problem. Note that, if there exists of a minimum to a suitable Lagrangian functional, it is a stationary point.

The numerical solution of our optimal control problem constrained by degenerate nonlocal aggregation model requires the proper discretization of the direct and the adjoint problems and the treatment of an optimization problem. From the standpoint of our specific application the main goal is to determine the control response to reduce the pattern formation generated by a nonlocal attraction term. More specifically, we are interested in determining the optimal intra species competition to insure a minimal pattern formation due the attraction force.

The structure of the paper is organized as follows. In Section 2, we present the main results, and we prove the well-posedness (existence and uniqueness) result to our degenerate aggregation model. Section 3 will be devoted to the optimal control problem. We present our optimal control approach, introduce a functional useful for minimize, prove the existence of the control, and we derive the adjoint-state problem. Finally, in Section 4, we introduce the numerical scheme for both direct and adjoint problem, present the optimal control procedure, and we demonstrate various realizations showing the effect of the optimal control solution on the overcrowding of the population.

2 Existence and uniqueness of weak solution

2.1 Weak solutions for the nonlocal degenerate equation

Before stating our main results, we give the definition of a weak solution for system (1).

Definition 1. A weak solution of (1) is a nonnegative function u satisfying the following conditions:

(C1) $u \in L^\infty(\Omega_T)$, $\mathcal{A}(u) := \int_0^u a(r)\,\mathrm{d}r \in L^2(0,T;H^1(\Omega))$, $u \in C_w(0,T;L^2(\Omega))$, $\partial_t u \in L^2(0;T;(H^1(\Omega))')$, $u(0) = u_0$;

(C2) For all $\varphi \in L^2(0,T;(H^1(\Omega))')$,

$$\int_0^T \langle \partial_t u, \varphi \rangle \,\mathrm{d}t + \iint_{\Omega_T} a(u)\nabla u \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} u(\nabla K * u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T} F(u,w)\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{2}$$

Here $C_w(0,T;L^2(\Omega))$ denotes the space of continuous functions with values in (a closed ball of) $L^2(\Omega)$ endowed with the weak topology, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$.

Our first result is the following well-posedness (existence and uniqueness) theorem for weak solutions.

Theorem 1. *Assume that conditions* (A1), (A2) *hold and* $u_0 \in L^\infty(\Omega)$ *with* $0 \leqslant u_0 \leqslant \overline{u}$, *where* $\overline{u}$ *is a positive constant in* $\mathbb{R}$. *Then there exists a unique weak solution to the nonlocal degenerate equation* (1) *in the sense of Definition* 1.

2.2 Existence of weak solution

Note that a major difficulty for the analysis of equation (1) is the strong degeneracy of the diffusion term and the presence of the nonlocal term. To handle this difficulty, we replace the original diffusion term $a(u)$ by $a_\varepsilon(u) = a(u) + \varepsilon$ ($\varepsilon > 0$) and consider the following nonlocal nondegenerate equation:

$$\begin{aligned}&\partial_t u - \operatorname{div}\bigl(a_\varepsilon(u)\nabla u - u\nabla K * u\bigr) = F(u,w) \quad \text{in } \Omega_T,\\ &\bigl(a_\varepsilon(u)\nabla u - u\nabla K * u\bigr)\cdot\eta = 0 \quad \text{on } \Sigma_T,\\ &u(x,0) = u_0(x) \quad \text{in } \Omega.\end{aligned} \tag{3}$$

To prove Theorem 1, we first prove existence of solutions to the nondegenerate problem (3) by using the Faedo–Galerkin method (in an appropriate functional setting). Convergence is achieved by means of a priori estimates and compactness arguments.

In what follows, we use the abbreviation "a.e." to denote "almost everywhere", which means that a property or condition holds for all points in a set, except for a set of points that has measure zero.

2.2.1 The Faedo–Galerkin solution

To construct our Faedo–Galerkin approximation, we employ a classical Hilbert basis, which is orthonormal in L^2 and orthogonal in H^1. We look for solutions to the problem obtained from the projection of (3) onto the finite-dimensional subspace $S_n := \operatorname{span}\{e_1,\dots,e_n\}$. The approximate solution takes the following form:

$$u_n : [0,T] \to S_n, \quad u_n(t) = \sum_{l=1}^n c_{n,l}(t)e_l.$$

Herein, $\{e_l\}_{l=1}^{\infty}$ is an orthogonal basis in $H^1(\Omega)$ and an orthonormal basis in $L^2(\Omega)$. Our first goal is to determine the coefficients $c^n = \{c_{n,l}\}_{l=1}^{n}$ such that $(l = 1, \dots, n)$

$$\langle \partial_t u_n, e_l \rangle + \int_\Omega a_{\varepsilon_n}(u_n)\nabla u_n \cdot \nabla e_l \,\mathrm{d}x - \int_\Omega u_n(\nabla K * u_n) \cdot \nabla e_l \,\mathrm{d}x \\ = \int_\Omega (w u_n - u_n^2) e_l \,\mathrm{d}x \tag{4}$$

and with reference to the initial condition

$$u_n(0) = u_{0,n} = \sum_{l=1}^{n} c_l^n(0) e_l, \qquad c_l^n(0) := \int_\Omega u_{0,n}(x) e_l(x) \,\mathrm{d}x.$$

Herein, $\varepsilon_n := 1/n$, $n > 0$. More explicitly, we can write (4) as an equation of ordinary differential equation $(l = 1, \dots, n)$

$$c'_{n,l}(t) = -\int_\Omega a_{\varepsilon_n}(u_n)\nabla u_n \cdot \nabla e_l \,\mathrm{d}x + \int_\Omega u_n(\nabla K * u_n) \cdot \nabla e_l \,\mathrm{d}x \\ + \int_\Omega (w u_n - u_n^2) e_l \,\mathrm{d}x \\ := F_l\big(t, c_{n,1}(t), \dots, c_{n,n}(t)\big), \tag{5}$$

where we have used the orthonormality of the basis. Observe that F_l $(l = 1, \dots, n)$ is a Carathéoodory function. Therefore, using the standard ODE theory, there exists an absolutely continuous functions $\{c_{n,l}\}_{l=1}^{n}$ satisfying (5) for a.e. $t \in [0, t')$ for some $t' > 0$. The next is to show that the local solution constructed above can be extended to the whole time interval $[0, T)$ (independent of n), but this can be done as in [1], so we omit the details.

Observe that from (4) the Faedo–Galerkin solution satisfies the following weak formulation:

$$\int_0^T \langle \partial_t u_n, \varphi \rangle \,\mathrm{d}t + \iint_{\Omega_T} a_{\varepsilon_n}(u_n)\nabla u_n \cdot \nabla \varphi \ \mathrm{d}x \,\mathrm{d}t - \iint_{\Omega_T} u_n(\nabla K * u_n) \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \\ = \iint_{\Omega_T} F(u_n, w_n) \varphi \,\mathrm{d}x \,\mathrm{d}t \tag{6}$$

for all test functions $\varphi \in L^2(0, T; H^1(\Omega))$.

2.2.2 *Maximum principle*

In this section, we prove that the solution of the nonlocal degenerate equation (2) satisfies the following version of maximum principle.

Lemma 1. *Assume that* $0 \leqslant u_0 \leqslant \overline{u}$*, then the solution* u_n *to problem* (3) *satisfies*

$$0 \leqslant u_n(x,t) \leqslant \mathrm{e}^{\lambda t}\overline{u} \quad \textit{for a.e. } (x,t) \in \Omega_T,$$

where $\lambda \in \mathbb{R}$ *such that*

$$\lambda \geqslant -\|w\|_{L^\infty(\Omega_T)} - \big\|\mathrm{div}(\nabla K * u_n)\big\|_{L^\infty(\Omega_T)}. \tag{7}$$

Proof. For technical reasons, we need to extend the function $f(u) := \alpha u - wu^2$ so that it becomes measurable on Ω_T, continuous with respect to u. We do this by setting (recall that α and w are nonnegative)

$$\tilde{F}(u,w) = \begin{cases} F(u,w) & \text{if } u \geqslant 0, \\ 0 & \text{else.} \end{cases}$$

Next, we define the following new variable $\tilde{u}_n$ by setting $u_n = \mathrm{e}^{\lambda t}\tilde{u}_n$, where $\lambda > 0$ is defined in (7). It follows from (3) that $\tilde{u}_n$ satisfies

$$\begin{aligned} &\partial_t \tilde{u}_n - \mathrm{div}\big(a_m\big(\mathrm{e}^{\lambda t}\tilde{u}_n\big)\nabla\tilde{u}_n\big) + \mathrm{e}^{\lambda t}\,\mathrm{div}\big(\tilde{u}_n^+\nabla K * \tilde{u}_n\big) \\ &\quad = -\lambda\tilde{u}_n + \mathrm{e}^{-\lambda t}\tilde{F}\big(\mathrm{e}^{\lambda t}\tilde{u}_n^+, w\big), \end{aligned} \tag{8}$$

where $\tilde{u}_n^+ = \max\{\tilde{u}_n, 0\}$. Multiplying this equation by $u_n^- = \max\{-\tilde{u}_n, 0\}$ and integrating over Ω, the result is

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tilde{u}_n^-|^2\,\mathrm{d}x - \int_\Omega a_{\varepsilon_n}\big(\mathrm{e}^{\lambda t}\tilde{u}_n\big)\nabla\tilde{u}_n\cdot\nabla\tilde{u}_n^-\,\mathrm{d}x + \mathrm{e}^{\lambda t}\int_\Omega \tilde{u}_n^+\nabla K * \tilde{u}_n\cdot\nabla\tilde{u}_n^-\,\mathrm{d}x \\ &\quad = \int_\Omega \big(\lambda\tilde{u}_n - \mathrm{e}^{-\lambda t}\tilde{F}\big(\mathrm{e}^{-\lambda t}\tilde{u}_n^+, w\big)\big)\tilde{u}_n^-\,\mathrm{d}x. \end{aligned} \tag{9}$$

Observe that

$$\int_\Omega a_{\varepsilon_n}(\mathrm{e}^{\lambda t}\tilde{u}_n)\nabla\tilde{u}_n\cdot\nabla\tilde{u}_n^-\,\mathrm{d}x = -\int_\Omega a_{\varepsilon_n}(\mathrm{e}^{\lambda t}\tilde{u}_n)|\nabla\tilde{u}_n^-|^2\,\mathrm{d}x \leqslant 0,$$
$$\int_\Omega \tilde{u}_n^+\nabla K * \tilde{u}_n\cdot\nabla\tilde{u}_n^-\,\mathrm{d}x = 0$$

and

$$\int_\Omega \big(\lambda\tilde{u}_n - \mathrm{e}^{-\lambda t}F\big(\mathrm{e}^{-\lambda t}\tilde{u}_n^+, w\big)\big)\tilde{u}_n^-\,\mathrm{d}x = -\int_\Omega \lambda|\tilde{u}_n^-|^2\,\mathrm{d}x \leqslant 0.$$

Using this, we get easily from (9)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tilde{u}_n^-|^2\,\mathrm{d}x \leqslant 0.$$

Since the data u_0 is nonnegative, we deduce that $\tilde{u}_n^-(x,t) = 0$ for a.e. $(t,x) \in \Omega_T$.

In the following step, we show that $\tilde{u}_n(x,t) \leqslant \overline{u}$, for a.e. $(x,t) \in \Omega_T$. To do this, it suffices to prove that $(\tilde{u}_n - \overline{u})^+ = 0$. We multiply equation (8) by $(\tilde{u}_n - \overline{u})^+$, and we integrate over Ω to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|(\tilde{u}_n - \overline{u})^+\right|^2 \mathrm{d}x + \int_\Omega a_{\varepsilon_n}(\mathrm{e}^{\lambda t}\tilde{u}_n)\nabla \tilde{u}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \\
&\qquad - \mathrm{e}^{\lambda t}\int_\Omega \tilde{u}_n \mathcal{K}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \\
&\quad = \int_\Omega \bigl(-\lambda\tilde{u}_n + \mathrm{e}^{-\lambda t}\tilde{F}\bigl(\mathrm{e}^{\lambda t}\tilde{u}_n, w\bigr)\bigr)(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \leqslant \int_\Omega (-\lambda\tilde{u}_n + \alpha\tilde{u}_n)(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \\
&\quad = \int_\Omega (-\lambda + \alpha)\left|(\tilde{u}_n - \overline{u})^+\right|^2 \mathrm{d}x + \int_\Omega (-\lambda + \alpha)\overline{u}(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x, \qquad (10)
\end{aligned}$$

where $\mathcal{K}_n := \nabla K * u_n$. Regarding the degenerate diffusion term, we have

$$\int_\Omega a_{\varepsilon_n}\bigl(\mathrm{e}^{\lambda t}\tilde{u}_n\bigr)\nabla\tilde{u}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x = \int_\Omega a_{\varepsilon_n}(\mathrm{e}^{\lambda t}\tilde{u}_n)\left|\nabla(\tilde{u}_n - \overline{u})^+\right|^2 \mathrm{d}x \geqslant 0. \qquad (11)$$

For the nonlocal term, we use an integration by part to deduce

$$\begin{aligned}
&\int_\Omega \tilde{u}_n \mathcal{K}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \\
&\quad = \int_\Omega (\tilde{u}_n - \overline{u})^+ \mathcal{K}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x + \int_\Omega \overline{u}\mathcal{K}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_\Omega \mathcal{K}_n \cdot \nabla\left|(\tilde{u}_n - \overline{u})^+\right|^2 \mathrm{d}x + \int_\Omega \overline{u}\mathcal{K}_n \cdot \nabla(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x \\
&\quad = -\frac{1}{2}\int_\Omega \operatorname{div}(\mathcal{K}_n)\left|(\tilde{u}_n - \overline{u})^+\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} \left|(\tilde{u}_n - \overline{u})^+\right|^2 \mathcal{K}_n \cdot \eta d\sigma(x) \\
&\qquad - \int_\Omega \overline{u}\operatorname{div}(\mathcal{K}_n)(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x + \int_{\partial\Omega} \overline{u}(\tilde{u}_n - \overline{u})^+ \mathcal{K}_n \cdot \eta d\sigma(x) \\
&\quad \leqslant -\frac{1}{2}\int_\Omega \operatorname{div}(\mathcal{K}_n)\left|(\tilde{u}_n - \overline{u})^+\right|^2 \mathrm{d}x - \int_\Omega \overline{u}\operatorname{div}(\mathcal{K}_n)(\tilde{u}_n - \overline{u})^+ \,\mathrm{d}x, \qquad (12)
\end{aligned}$$

where we have used (recall that in (A2), K is radially nonincreasing)

$$\mathcal{K}_n \cdot \eta \leqslant 0 \quad \text{on } \Sigma_T.$$

Collecting the previous estimates (11) and (12), we readily conclude from (10)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega}\left|(\tilde{u}_n-\overline{u})^+\right|^2\mathrm{d}x+\int\limits_{\Omega}\left(\lambda-w+\frac{1}{2}\operatorname{div}(\mathcal{K}_n)\right)\left|(\tilde{u}_n-\overline{u})^+\right|^2\mathrm{d}x$$
$$+\int\limits_{\Omega}\overline{u}\big(\lambda-w+\operatorname{div}(\mathcal{K}_n)\big)(\tilde{u}_n-\overline{u})^+\,\mathrm{d}x\leqslant 0. \tag{13}$$

Now, by the choice of λ in (7) we deduce from (13)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega}\left|(\tilde{u}_n-\overline{u})^+\right|^2\mathrm{d}x\leqslant 0.$$

Using that $u_0\leqslant\overline{u}$, we conclude from this $u_n(t,\cdot)\leqslant \mathrm{e}^{\lambda t}\overline{u}$ in Ω for $t\in(0,T)$. This concludes the proof of the lemma. □

2.2.3 A priori estimates

First, we denote

$$A(r)=\int\limits_0^r a(s)\,\mathrm{d}s\quad\text{and}\quad \mathcal{A}(r)=\int\limits_0^r A(s)\,\mathrm{d}s.$$

To pass to the limit in (6) and prove the existence of the solution u, we need the following a priori estimates lemma.

Lemma 2. *The solution u_n to problem* (3) *satisfies*

$$\left\|\mathcal{A}\big(u_n(x,t)\big)\right\|_{L^\infty(0,T;L^1(\Omega))}+\left\|\nabla A(u_n)\right\|_{L^2(\Omega_T)}$$
$$+\varepsilon_n\|u_n\|_{L^\infty(0,T;L^2(\Omega))}+\varepsilon_n\|\nabla u_n\|_{L^2(\Omega_T)}\leqslant C, \tag{14}$$

$$\|\partial_t u_n\|_{L^2(0,T;(H^1(\Omega))')}\leqslant C \tag{15}$$

for some constant $C>0$ not depending of n.

Proof. We substitute $\varphi=A_{\varepsilon_n}(u_n):=A(u_n)+\varepsilon_n u_n$ in (6), and we integrate over $(0,\tau)$ with $\tau<T$ to obtain

$$\int\limits_0^\tau\big\langle\partial_t u_n,A_{\varepsilon_n}(u_n)\big\rangle\,\mathrm{d}t+\iint\limits_{\Omega_\tau}\left|\nabla A(u_n)\right|^2\mathrm{d}x\,\mathrm{d}t+\varepsilon_n\iint\limits_{\Omega_\tau}|\nabla u_n|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-\iint\limits_{\Omega_\tau}u_n(\nabla K*u_n)\cdot\nabla A_{\varepsilon_n}(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\iint\limits_{\Omega_\tau}F(u_n,w)A_{\varepsilon_n}(u_n)\,\mathrm{d}x\,\mathrm{d}t. \tag{16}$$

Next, using Young inequality and Lemma 1, we obtain

$$
\begin{aligned}
&\biggl|\iint\limits_{\Omega_\tau} u_n(\nabla K * u_n)\cdot\nabla A_{\varepsilon_n}(u_n)\,\mathrm{d}x\,\mathrm{d}t\biggr|\\
&\quad\leqslant \iint\limits_{\Omega_\tau} |\nabla K * u_n|^2|u_n|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint\limits_{\Omega_\tau}\bigl|\nabla A(u_n)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon_n}{2}\iint\limits_{\Omega_\tau}|\nabla u_n|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant C_1 + \frac{1}{2}\iint\limits_{\Omega_\tau}\bigl|\nabla A(u_n)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon_n}{2}\iint\limits_{\Omega_\tau}|\nabla u_n|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{17}
$$

and

$$
\iint\limits_{\Omega_\tau}\bigl|F(w,u_n)A_{\varepsilon_n}(u_n)\bigr|\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2 \tag{18}
$$

for some constants $C_1, C_2 > 0$. Now, exploiting (17) and (18), we deduce from (16)

$$
\begin{aligned}
&\sup_{0<\tau\leqslant T}\int\limits_\Omega \mathcal{A}\bigl(u_n(\tau)\bigr)\,\mathrm{d}x + \varepsilon_n\sup_{0<\tau\leqslant T}\int\limits_\Omega\bigl|u_n(\tau)\bigr|^2\,\mathrm{d}x + \frac{1}{2}\iint\limits_{\Omega_T}\bigl|\nabla A(u_n)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \frac{\varepsilon_n}{2}\iint\limits_{\Omega_\tau}|\nabla u_n|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant C_3
\end{aligned}\tag{19}
$$

for some constant $C_3 > 0$. This implies the desired estimate (14).

To prove estimate (15), we take $\varphi \in L^2(0,T;H^1(\Omega))$, and we use the weak formulation (6) to obtain

$$
\begin{aligned}
\biggl|\int\limits_0^T\langle\partial_t u_n,\varphi\rangle\,\mathrm{d}t\biggr| &\leqslant \iint\limits_{\Omega_T}\bigl|\nabla A_{\varepsilon_n}(u_n)\cdot\nabla\varphi\bigr|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \iint\limits_{\Omega_T}\bigl|u_n(\nabla K * u_n)\cdot\nabla\varphi\bigr|\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_{\Omega_T}\bigl|F(u_n,w)\varphi\bigr|\,\mathrm{d}x\,\mathrm{d}t,\\
&\leqslant \bigl\|\nabla\mathcal{A}_{\varepsilon_n}(u_n)\bigr\|_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}\\
&\quad + \|\nabla K * u_n\|_{L^\infty(\Omega_T)}\|u_n\|_{L^2(\Omega_T)}\|\nabla\varphi\|_{L^2(\Omega_T)}\\
&\quad + \bigl\|F(u_n,w)\bigr\|_{L^2(\Omega_T)}\|\varphi\|_{L^2(\Omega_T)}\\
&\leqslant C_4\|\varphi\|_{L^2(0,T;H^1(\Omega))}
\end{aligned}
$$

for some constant $C_4 > 0$, where we have used (19). This implies

$$
\|\partial_t u_m\|_{L^2(0,T;(H^1(\Omega))')} \leqslant C_4. \qquad \square
$$

2.2.4 Passing to the limit

Thanks to Lemma 2 and Aubin–Simon compactness theorem (see, e.g., [21]), we can extract subsequences, which we do not relabel, such that, as $n \to \infty$,

$$\begin{aligned} u_n &\to u && \text{weakly-}{*} \text{ in } L^\infty(\Omega_T), \\ A(u_n) &\to \overline{A} && \text{weakly in } L^2\bigl(0,T;H^1(\Omega)\bigr), \\ \sqrt{\varepsilon_n} u_\varepsilon &\to 0 && \text{weakly in } L^2\bigl(0,T;H^1(\Omega)\bigr), \\ \partial_t u_\varepsilon &\to \partial_t u && \text{weakly in } L^2\bigl(0,T;\bigl(H^1(\Omega)\bigr)'\bigr). \end{aligned} \tag{20}$$

Next, we use the compact embedding $L^\infty(\Omega) \subset (H^1(\Omega))'$ and Corollary 4 of [21] to deduce that u_n is a Cauchy sequence in $C(0,T;(H^1(\Omega))')$.

Observe that $A(u_\varepsilon)$ is uniformly bounded in $\mathcal{S}$, where

$$\mathcal{S} = \bigl\{u \in L^2\bigl(0,T,H^1(\Omega)\bigr)\colon \partial_t u \in L^2\bigl(0,T;\bigl(H^1(\Omega)\bigr)'\bigr)\bigr\}.$$

From the compact imbedding $\mathcal{S} \subset L^2(\Omega_T)$ we deduce that there exists a subsequence of u_n such that

$$A(u_n) \to \overline{A} \quad \text{strongly in } L^2(\Omega_T).$$

Since A is monotone, we get $A(u) = \overline{A}$. Therefore,

$$A(u_n) \to A(u) \quad \text{strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Moreover, as A^{-1} is well defined and continuous, we apply the dominated convergence theorem to $u_n = A^{-1}(A(u_n))$ to obtain

$$u_n \to u \quad \text{strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Using this and the weak-$*$ convergence of u_n to u in $L^\infty(\Omega_T)$, we obtain

$$u_n \to u \quad \text{strongly in } L^q(\Omega_T) \text{ for } 1 \leqslant q < \infty.$$

With the above convergences, we are ready to identify the limit u as a weak solution of (1). Let $\varphi \in L^2(0,T;H^1(\Omega))$ be a test function in (6). By (20) it is clear that, as $\varepsilon \to 0$,

$$\int_0^T \langle \partial_t u_n, \varphi \rangle \,\mathrm{d}t \to \int_0^T \langle \partial_t u, \varphi \rangle \,\mathrm{d}t$$

and

$$\iint_{\Omega_T} a_{\varepsilon_n}(u_n) \nabla u_n \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \to \iint_{\Omega_T} a(u) \nabla u \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

Since $u_n(\nabla K * u_n)$ is bounded in $L^\infty(\Omega_T)$, we also have that, as $n \to \infty$,

$$\iint_{\Omega_T} u_n(\nabla K * u_n) \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \to \iint_{\Omega_T} u(\nabla K * u) \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

Similarly, we have, as $n \to \infty$,

$$\iint_{\Omega_T} F(u_n, w)\varphi \,\mathrm{d}x\,\mathrm{d}t \to \iint_{\Omega_T} F(u, w)\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

We have finally identified u as a solution of (1).

2.3 Uniqueness of the weak solution

In this section, we prove uniqueness of weak solutions to our nonlocal degenerate aggregation model, thereby completing the well-posedness analysis. The uniqueness proof of weak solutions is proved by using duality technique.

First, we consider u_1 and u_2 two solutions of system (1). We set $U = u_1 - u_2$, then U satisfies (for $i = 1, 2$)

$$\begin{aligned} &\partial_t U - \Delta\bigl(A(u_1) - A(u_2)\bigr) + \operatorname{div}(u_1 \nabla K * u_1 - u_2 \nabla K * u_2) \\ &\quad = F(u_1, w) - F(u_2, w) \quad \text{in } \Omega_T, \\ &\bigl(\nabla A(u_i) - u_i \nabla K * u_i\bigr) \cdot \eta = 0 \quad \text{on } \Sigma_T, \\ &u_i(x, 0) = u_0(x) \quad \text{in } \Omega. \end{aligned} \tag{21}$$

Now, we define the function φ solution of the problem

$$-\Delta\varphi(t, \cdot) = U(t, \cdot) \quad \text{in } \Omega \quad \text{and} \quad \nabla\varphi(t, \cdot) \cdot \eta = 0 \quad \text{on } \partial\Omega \tag{22}$$

for a.e. $t \in (0, T)$. Since u_1 and u_2 are bounded in L^∞, then we get from the theory of linear elliptic equations the existence, uniqueness and regularity of solution φ satisfying

$$\varphi \in C\bigl([0, T]; H^2(\Omega)\bigr) \quad \text{with} \int_{\Omega} \varphi(t, \cdot)\,\mathrm{d}x = 0.$$

Note that from the boundary condition of φ in (22) and $U(0, \cdot) = 0$ we deduce that

$$\nabla\varphi(0, \cdot) = 0 \quad \text{in } L^2(\Omega). \tag{23}$$

Multiplying the second equation in (21) by $\psi \in L^2(0, T; H^1(\Omega))$ and integrating over $\Omega_t := (0, t) \times \Omega$, we get

$$\begin{aligned} &\int_0^t \langle \partial_s U, \psi \rangle \,\mathrm{d}s + \iint_{\Omega_t} \nabla\bigl(A(u_1) - A(u_2)\bigr) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \iint_{\Omega_t} (u_1 \nabla K * u_1 - u_2 \nabla K * u_2) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \iint_{\Omega_t} \bigl(F(u_1, w) - F(u_2, w)\bigr)\psi \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{24}$$

Since $\varphi \in L^2(0,T;H^1(\Omega))$, we can take $\psi = \varphi$ in (24), and we obtain from (22), (23)

$$2\int_0^t \langle \partial_s U, \varphi\rangle \,\mathrm{d}s = -2\int_0^t \langle \partial_s \Delta\varphi, \varphi\rangle \,\mathrm{d}s = \int_\Omega \bigl|\nabla\varphi(t,x)\bigr|^2 \,\mathrm{d}x - \int_\Omega \bigl|\nabla\varphi(0,x)\bigr|^2 \,\mathrm{d}x$$
$$= \int_\Omega \bigl|\nabla\varphi(t,x)\bigr|^2 \,\mathrm{d}x \tag{25}$$

and

$$\int_0^t \langle \partial_s U, \varphi\rangle \,\mathrm{d}s - \iint_{\Omega_t} \bigl(A(u_1) - A(u_2)\bigr)\Delta\varphi \,\mathrm{d}x\,\mathrm{d}s$$
$$= \iint_{\Omega_t} (u_1\nabla K * u_1 - u_2\nabla K * u_2)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \iint_{\Omega_t} \bigl(F(u_1,w) - F(u_2,w)\bigr)\varphi \,\mathrm{d}x\,\mathrm{d}s. \tag{26}$$

Since u_1 and u_2 are bounded in L^∞, then there exist constants $C_5, C_6 > 0$ such that

$$\bigl|F(u_1,w) - F(u_2,w)\bigr| \leqslant C_5|u_1 - u_2|, \tag{27}$$
$$\bigl|u_1\nabla K * u_1 - u_2\nabla K * u_2)\bigr| \leqslant C_6|u_1 - u_2|. \tag{28}$$

Using (22), (27), (28), Hölder's, Young's, Sobolev–Poincaré's inequalities, (26) yields

$$\int_0^t \langle \partial_s U, \varphi\rangle \,\mathrm{d}s \leqslant -C_a \iint_{\Omega_t} |U|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{C_a}{4}\iint_{\Omega_t} |U|^2 \,\mathrm{d}x\,\mathrm{d}s + C_7\int_0^t \|\nabla\varphi\|^2_{L^2(\Omega)} \,\mathrm{d}s$$
$$+ \frac{C_a}{4}\iint_{\Omega_t} |U|^2 \,\mathrm{d}x\,\mathrm{d}s + C_8\int_0^t \|\varphi\|^2_{L^2(\Omega)} \,\mathrm{d}s$$
$$\leqslant C_9\int_0^t \|\nabla\varphi\|^2_{L^2(\Omega)} \,\mathrm{d}s$$

for some constants $C_7, C_8, C_9 > 0$. Using this and (25), we deduce

$$\int_\Omega \bigl|\nabla\varphi(t,x)\bigr|^2 \,\mathrm{d}x = 2\int_0^t \langle \partial_s U, \varphi_j\rangle \,\mathrm{d}s \leqslant 2C_9\int_0^t \|\nabla\varphi\|^2_{L^2(\Omega)} \,\mathrm{d}s. \tag{29}$$

Finally, we use Gronwall's lemma to conclude from (29)

$$\nabla\varphi = 0 \quad \text{a.e. in } \Omega_T,$$

ensuring the uniqueness of weak solutions.

3 The optimal control problem

In this subsection, we provide the existence of the solution for the optimal control problem of the nonlocal degenerate equation (1). We considered the following cost functional for the optimization of the population density and aggregation term:

$$J(w,u) = \frac{\varepsilon_1}{2} \int_0^T \int_{\Omega_c} |w|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon_2}{2} \iint_{\Omega_T} \left|u(\nabla K * u)\right|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{30}$$

where ε_1 and ε_2 denote regularization parameters. Herein, $\Omega_c \subset \Omega$ is the control subdomain. We define the set of admissible controls $\mathcal{U}$ by

$$\mathcal{U} = \left\{w \in L^\infty(\Omega_T)\colon\ \underline{w} \leqslant w(t,x) \leqslant \overline{w}\right\},$$

where $\underline{w} \in \mathbb{R}_+^*$ and $\overline{w} \in \mathbb{R}_+^*$ are the minimal and the maximal intraspecific competition rates, respectively. We consider the following minimization problem:

$$\min_w J(w,u) \quad \text{subject to (1)}. \tag{31}$$

3.1 Existence of the control

In this subsection, we show the existence of the optimal solution $w^* \in \mathcal{U}$ for the problem (31).

Lemma 3. *Assume that $u_0 \in L^\infty(\Omega)$. Then there exists a solution $w^* \in \mathcal{U}$ of the optimal control problem* (30).

Proof. Let w_n be a minimizing sequence of J such that

$$\inf_{w\in\mathcal{U}}\{J\} \leqslant J(w_n) \leqslant \inf_{w\in\mathcal{U}}\{J\} + \frac{1}{n}.$$

Thanks to the definition of J, the sequence $(w_n)_n$ is bounded in $L^2(\Omega_T)$. This implies that w_n converge weakly to an w^*. Let u_n be a solution to problem (1) with respect to the control w_n. Working exactly as in Section 2, we deduce the following convergence (upon a subsequence):

$$u_n \to u^* \quad \text{strongly in } L^q(\Omega_T) \text{ and a.e. in } \Omega_T \text{ for } 1 \leqslant q < \infty.$$

Note that, since the cost functional $J(\cdot,\cdot)$ is continuous and convex on $L^2(\Omega_T) \times \mathcal{U}$, it follows that $J(\cdot,\cdot)$ is weakly lower semicontinuous. Hence, by exploiting the strong convergence of u_n combined with the weak lower semicontinuity of J we arrive to

$$J(u^*,w^*) \leqslant \liminf_{n\to\infty} J(w_n) \leqslant \inf_{w\in\mathcal{U}}\left\{J(w)\right\} = J(u^*,w^*).$$

This implies the existence result of our optimal control solution (31). $\square$

3.2 Optimal conditions and dual problem

In this subsection, we derive the optimality conditions based on the Lagrangian formulation. We introduce the Lagrange functional L defined by (recall that $A(u) = \int_0^u a(s)\,\mathrm{d}s$)

$$
\begin{aligned}
L(u,w,p) &= J(w,u) + \iint_{\Omega_T} \bigl(u_t - \operatorname{div}\bigl(a(u)\nabla u - u(\nabla K * u)\bigr) - F(u,w)\bigr)p\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{\varepsilon_1}{2}\int_0^T\int_{\Omega_c} |w|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon_2}{2}\iint_{\Omega_T} \bigl|u(\nabla K * u)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{\Omega_T} \bigl(u_t - \operatorname{div}\bigl(a(u)\nabla u - u(\nabla K * u)\bigr) - F(u,w)\bigr)p\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{\varepsilon_1}{2}\int_0^T\int_{\Omega_c} |w|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon_2}{2}\iint_{\Omega_T} \bigl|u(\nabla K * u)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} \partial_t u\,p\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{\Omega_T} \bigl(\nabla A(u) - u(\nabla K * u)\bigr)\nabla p\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} F(u,w)p\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{\varepsilon_1}{2}\int_0^T\int_{\Omega_c} |w|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon_2}{2}\iint_{\Omega_T} \bigl|u(\nabla K * u)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} \partial_t u\,p\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \iint_{\Omega_T} A(u)\Delta p\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Sigma_T} A(u)\nabla p\cdot\eta\,\mathrm{d}\sigma(x)\,\mathrm{d}t \\
&\quad - \iint_{\Omega_T} u(\nabla K * u)\cdot\nabla p\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} F(u,w)p\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

The first-order optimality system characterizing the adjoint variables is given by the Lagrange multipliers, which result from equating the partial derivative of L with respect to u equal to zero:

$$
\begin{aligned}
&\left(\frac{\partial L(u,w,p)}{\partial u}, \delta u\right) \\
&\quad = \varepsilon_2 \iint_{\Omega_T} u(\nabla K * u)\bigl((\nabla K * u)\delta u + u(\nabla K * \delta u)\bigr)\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} \partial_t\,p\,\delta u \\
&\qquad - \iint_{\Omega_T} a(u)\Delta p\,\delta u\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T} \bigl(\delta u(\nabla K * u) + u(\nabla K * \delta u)\bigr)\nabla p\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \iint_{\Omega_T} \partial_u F(u,w)p\,\delta u\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon_2 \iint\limits_{\Omega_T} u|\nabla K * u|^2 \delta u \,\mathrm{d}x\,\mathrm{d}t + \varepsilon_2 \iint\limits_{\Omega_T} |u|^2 (\nabla K * u)(\nabla K * \delta u)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \iint\limits_{\Omega_T} \partial_t\, p\,\delta u - \iint\limits_{\Omega_T} a(u)\Delta p\,\delta u\,\mathrm{d}x\,\mathrm{d}t - \iint\limits_{\Omega_T} (\nabla K * u)\nabla p\,\delta u\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \iint\limits_{\Omega_T} u(\nabla K * \delta u)\nabla p\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_{\Omega_T} \partial_u F(u,w) p\,\delta u\,\mathrm{d}x\,\mathrm{d}t \\
&= \iint\limits_{\Omega_T} \bigl[-\partial_t p - a(u)\Delta p - (\nabla K * u)\nabla p + \varepsilon_2 u|\nabla K * u|^2 - \partial_u F(u,w)p\bigr]\delta u\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \iint\limits_{\Omega_T} \bigl[u\nabla p - \varepsilon_2|u|^2(\nabla K * u)\bigr]\cdot(\nabla K * \delta u)\,\mathrm{d}x\,\mathrm{d}t. \qquad (32)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\mathcal{B} &:= \iint\limits_{\Omega_T} \bigl(u\nabla p - \varepsilon_2|u|^2(\nabla K * u)\bigr)(x)\cdot(\nabla K * \delta u)(x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int\limits_{\Omega\times\Omega} \bigl(u\nabla p - \varepsilon_2|u|^2(\nabla K * u)\bigr)(x)\cdot\nabla K(x-y)\delta u(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int\limits_{\Omega\times\Omega} \bigl(u\nabla p - \varepsilon_2|u|^2(\nabla K * u)\bigr)(x)\cdot\nabla K(y-x)\delta u(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&= -\iint\limits_{\Omega_T} \bigl(u\nabla p - \varepsilon_2|u|^2(\nabla K * u)\bigr) * \nabla K\,\delta u\,\mathrm{d}x\,\mathrm{d}t. \qquad (33)
\end{aligned}
$$

Next, we exploit (32) and (33) to deduce the adjoint equation of the nonlocal degenerate aggregation model (1)

$$
\begin{aligned}
&-\partial_t p - a(u)\Delta p - (\nabla K * u)\nabla p + (u\nabla p) * \nabla K \\
&\quad = \partial_u F(u,w)p + \mathcal{F}_K(u) \quad \text{in } \Omega_T, \\
&\nabla p\cdot\eta = 0 \quad \text{on } \Sigma_T, \\
&p(x,T) = p_T(x) = 0 \quad \text{in } \Omega,
\end{aligned} \qquad (34)
$$

where

$$
\partial_u F(u,w) = \alpha - 2wu \quad \text{and} \quad \mathcal{F}_K(u) := \varepsilon_2\bigl(|u|^2\nabla K * u\bigr) * \nabla K - \varepsilon_2 u|\nabla K * u|^2.
$$

To find the optimal conditions, we calculate the gradient of the functional $J(w,u)$:

$$
\left(\frac{\partial L}{\partial w}, \delta w\right) = \varepsilon_1 \int_0^T\int\limits_{\Omega_c} w\,\delta w\,\mathrm{d}x\,\mathrm{d}t - \iint\limits_{\Omega_c} u^2 p\,\delta w\,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \nabla J(w,u) = \frac{\partial L}{\partial w}.
$$

Therefore, the optimality condition can be written as follows:

$$\int_0^T\int_{\Omega_c} \left(\varepsilon_1 w + u^2 p\right) \mathrm{d}x \, \mathrm{d}t = 0.$$

Remark 1. Note that in aggregation equations, it is common to use an even kernel ($K(-x) = K(x)$). We mention that the gradient of an even function became odd (i.e., $\nabla K(-x) = -\nabla K(x)$.)

4 Numerical discretization

In this section, we present numerical methods to solve the nonlocal aggregation problem (1). We propose a numerical scheme to approximate the solution of the associated adjoint problem (34), and we implement the optimal control solver of the minimization problem (30). To approximate the solution of the direct problem (1) and adjoint problem (34), we will use the numerical scheme introduced in [9]. First, let us consider a Cartesian mesh with the step h_i in the direction $i \in \{1, \dots, d\}$ and $h = \max_i h_i$. Denote by $(C_J)_{J\in\mathbb{Z}^d}$ the space cells, where each cell C_J has a center $x_J := (x_1 \dots, x_d)$ with $x_i = J_i h_i$ for $i \in \{1, \dots, d\}$. Next, we let e_i the canonical basis of $\mathbb{Z}^d$, and we denote $(u_J^n)_{J\in\mathbb{Z}^d}$ the approximation of cell average of $u(t, \cdot)$ at a given time $t_n = n\tau$.

We propose the following numerical approximation for the direct problem (1) (recall that $(s)^+ = \max\{0, s\}$ and $(s)^- = \max\{0, -s\}$ for a real number s):

$$\begin{aligned} u_J^{n+1} = u_J^n &+ \sum_{i=1}^d \frac{\tau}{h_i}\Big(\big(\mathcal{B}_{i,J}^n\big)^+ a\big(u_J^n\big) - \big(\mathcal{B}_{i,J+e_i}^n\big)^- a\big(u_{J+e_i}^n\big) \\ &\qquad - \big(\mathcal{B}_{i,J-e_i}^n\big)^+ a\big(u_{J-e_i}^n\big) + \big(\mathcal{B}_{i,J}^n\big)^- a\big(u_J^n\big)\Big) \\ &- \sum_{i=1}^d \frac{\tau}{h_i}\Big(\big(\mathcal{A}_{i,J}^n\big)^+ u_J^n - \big(\mathcal{A}_{i,J+e_i}^n\big)^- u_{J+e_i}^n \\ &\qquad - \big(\mathcal{A}_{i,J-e_i}^n\big)^+ u_{J-e_i}^n + \big(\mathcal{A}_{i,J}^n\big)^- u_J^n\Big) \\ &+ \tau F\big(u_J^n, w_J^n\big). \end{aligned} \tag{35}$$

The numerical discrete aggregation and diffusion velocities are defined respectively by

$$\mathcal{A}_{i,J}^n := -\sum_{L\in\mathbb{Z}^d} u_L^n D_i K_J^L \quad \text{and} \quad \mathcal{B}_{i,J}^n := \frac{u_{J+e_i}^n - u_J^n}{h_i},$$

where $D_i K_J^L = \partial_{x_i} K(x_J - x_L)$ for a pointy potential K.

Now, for the solution of adjoint problem (34), we consider $(p_J^n)_{J\in\mathbb{Z}}$ the finite-volume approximation of cell average of $p(t, \cdot)$ at a given time $t = n\tau$. We use the following numerical approximation of the adjoint problem (34):

$$\begin{aligned}p_J^{n-1} = p_J^n &+ \sum_{i=1}^{d} \frac{\tau}{h_i}\Big(\big(\overline{\mathcal{B}}_{i,J}^n\big)^+ a\big(u_J^n\big) - \big(\overline{\mathcal{B}}_{i,J+e_i}^n\big)^- a\big(u_{J+e_i}^n\big)\\&\qquad - \big(\overline{\mathcal{B}}_{i,J-e_i}^n\big)^+ a\big(u_{J-e_i}^n\big) + \big(\overline{\mathcal{B}}_{i,J}^n\big)^- a\big(u_J^n\big)\Big)\\&- \sum_{i=1}^{d} \frac{\tau}{h_i}\Big(\big(\overline{\mathcal{A}}_{i,J}^n\big)^+ u_J^n - \big(\overline{\mathcal{A}}_{i,J+e_i}^n\big)^- u_{J+e_i}^n\\&\qquad - \big(\overline{\mathcal{A}}_{i,J-e_i}^n\big)^+ u_{J-e_i}^n + \big(\overline{\mathcal{A}}_{i,J}^n\big)^- u_J^n\Big)\\&- \sum_{i=1}^{d} \frac{\tau}{h_i}\big(\mathcal{F}_{i,J}^n - \mathcal{F}_{i,J+e_i}^n\big) + \tau\big(\mathcal{F}_K\big(u_J^n\big) + \partial_u F\big(u_J^n, w_J^n\big)p_J^n\big),\end{aligned} \tag{36}$$

where

$$\overline{\mathcal{B}}_{i,J}^n := \frac{p_{J+e_i}^n - p_J^n}{h_i}, \quad \text{and} \quad \overline{\mathcal{A}}_{i,J}^n := \sum_{i=1}^{d} \sum_{L\in\mathbb{Z}^d} u_L^n \frac{p_{L+e_i}^n - p_{L-e_i}^n}{2h_i} D_i K_J^L.$$

The term $\mathcal{F}_{i,J}^n$ can be computed as

$$\mathcal{F}_{i,J}^n := \psi_{i,J}^n u_{J-e_i}^n + \phi^n\big(r_{i,J}^n\big)\psi_{i,J}^n\left(1 - \frac{\tau}{h_i}\psi_{i,J}^n\right)\big[u_J^n - u_{J-e_i}^n\big],$$

where the convection velocity is given by

$$\psi_{i,J}^n := a'\big(u_J^n\big)\frac{u_J^n - u_{J-e_i}^n}{h_i} - \sum_{L\in\mathbb{Z}} u_L^n D_i K_J^L \quad \text{and} \quad r_{i,J}^n := \frac{u_{J-e_i}^n - u_{J-2e_i}^n}{u_J^n - u_{J-e_i}^n}.$$

Following the establishment of the essential discretization of the direct and adjoint problems, we must develop a numerical approach to minimize the specified cost function (30). It is common knowledge that the basic gradient descent approach does not ensure global convergence relative to the initial guest. Therefore, we implemented the nonlinear conjugate gradient technique [11] to achieve global convergence performance (see Algorithm 1). However, this class of methods has many limitations in terms of convergence to a global minimum.

5 Numerical simulations

In this section, we present an efficient implementation of the proposed numerical schemes (35), (36) with the optimal control Algorithm 1 to simulate the population dynamics under attractive forces. We focus our simulations on the effect of the optimal control (30) on the pattern formation induced by attractive forces under several initial conditions.

To compute u_J^n for direct (35) and p_J^n for adjoint (36) problems, we choose the computational domain $\Omega := (-4, 4) \times (-4, 4)$, the diffusion function $a(u) = u(\overline{u} - u)$, the time step $\tau = 0.001$ and $h_1 = h_2 = 0.1$. For the aggregation kernel, we choose the

Algorithm 1. The optimal control solver.

Input: $u_0, err \leftarrow 1$
Initialize: $w^0, \alpha, tol, k \leftarrow 0$

while $\|\nabla J(w^k)\| > tol$ **do**
  for $t = t^1, \ldots, t^{\text{final}}$ **do**
    Giving τ_2^k Compute u^h from the direct problem;
  end for
  Compute the cost function $J(w^k, u^h)$
  for $t = t^{\text{final}}, \ldots, t^0$ **do**
    Giving w^k, u^h, compute p^h by solving the adjoint problem;
  end for
  Compute the gradient $g_{k+1} = \nabla J(w^k, p^h)$;
  Compute $y_k = g_{k+1} - g_k$
  Compute step length α_k
  Update the values of w: $w^{k+1} = w^k + \alpha_k d_k$;
  Compute $\beta^k = (y_k - 2d_k\|y_k\|^2/(d_k^T y_k))^T g_{k+1}/(d_k^T y_k)$
  $d_k = -g_k + \beta_k d_{k-1}$;
  Update the direction $d_k = g_k + \beta^k d_{k-1}$
  $k \leftarrow k + 1$
end while

Gaussian distribution

$$K(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{\sigma^2}\right) \quad \text{for all } x \in \mathbb{R}^2,$$

where $\sigma > 0$ is a given parameter.

In the next subsections, we will present various tests and simulations. The first test is devoted to examine the effect of the aggregation and degenerate diffusion terms in the absence of the reaction term. Afterward, we will investigate the effect of the reaction term under various parameters. The final subsection focuses on evaluating the efficiency of the optimal control algorithm by considering two different initial conditions, and examining the impact of the resulting optimal control on the aggregation dynamics in each case.

5.1 Aggregation diffusion equation

In order to show the effect of the attractive force under a degenerate diffusion coefficient, we run different numerical simulations under different initial conditions.

Figure 1 presents the dynamic of a three group under attractive force. In Fig. 2, we depict the evolution of a randomly distributed initial population density.

It is well known that a diffusion process drives individuals of a given population towards lower densities according to gradient direction. In the other hand, the attraction force forms a velocity field that drives individuals from lower density groups to higher density groups. These two effects drive the solution to various forms of steady states. For example, in Fig. 1, we notice that a single group is formed from three different population densities. Starting from a randomly distributed initial condition with a sharper attraction

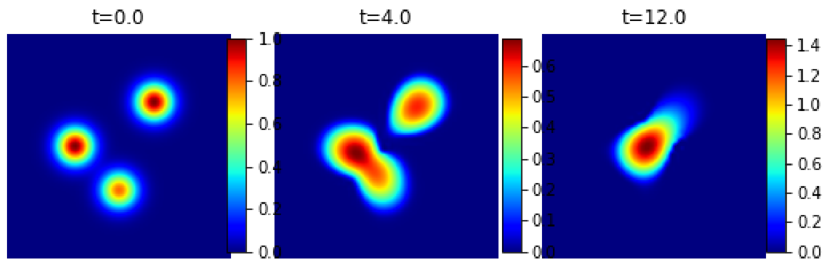

Figure 1. Evolution of the population dynamics using Gaussian attraction (with $\sigma = 0.8$), the initial condition $u_0 = \exp(c((X+1)^2 + Y^2)) + 0.8\exp(c(X^2 + (Y-1)^2)) + \exp(c((X-0.8)^2 + (Y+1)^2))$, where $c = -1$.

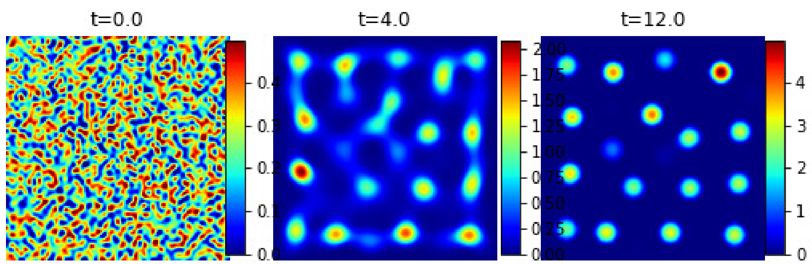

Figure 2. Evolution of the population dynamics using Gaussian attraction (with $\sigma = 0.2$) and a random initial population density u_0.

kernel (i.e., $\sigma = 0.2$), Fig. 2 shows the formation of several groups due to attraction forces.

5.2 Aggregation equation with nonlinear interaction term

To lessen the effect of over-crowding phenomenon, a degenerate diffusion plays a counter effect role. In more realistic phenomenon inspired from nature, the overcrowding effect comes with costs on the population. The mortality rate of the population rises due to some limited resources. This can be modeled by using logistic reaction term $F(u,w) := \alpha u - wu^2$.

Note that the logistic reaction term F (with $\alpha = 0.25$ and $w = 0.25$) eliminates the aggregation phenomenon in Fig. 3. Moreover, the solution achieves in short time range a constant steady-state solution $u \equiv 1$. When $\alpha = 0.1$ and $w = 0.1$, we observe that the evolution of the population is closer to the aggregation dynamics (see Fig. 4). In the last experiment of the direct problem (consult Fig. 5), we notice that the reaction term drives a random distributed initial condition to a more regular steady-state solution comparing to Fig. 2. We conclude that, according to the value of α, we can control the dominance of the logistic reaction term.

In the next subsection, we study and control the aggregation effect acting on the intraspecific competition rate w.

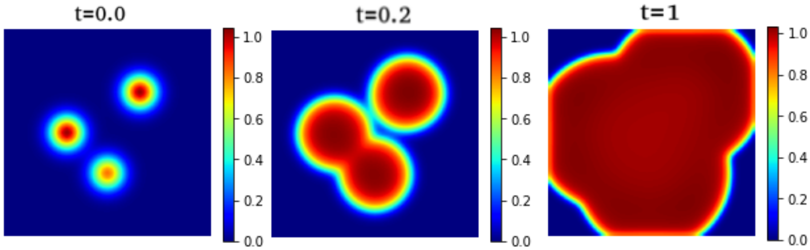

Figure 3. Evolution of the population dynamics using the logistic source term F (with $\alpha = 0.25$ and $w = 0.25$), a nonlocal pointy Gaussian potential (with $\sigma = 0.8$) and the initial population $u_0 = \exp(c((X+1)^2+Y^2)) + 0.8\exp(c(X^2 + (Y-1)^2)) + \exp(c((X-0.8)^2 + (Y+1)^2))$, where $c = -1$.

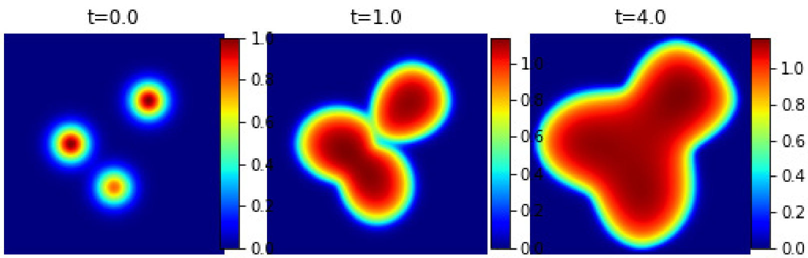

Figure 4. Evolution of the population dynamics using the logistic source term F (with $\alpha = 0.1$ and $w = 0.1$), the nonlocal pointy Gaussian potential (with $\sigma = 0.8$) and the initial population $u_0 = \exp(c((X+1)^2 + Y^2)) + 0.8\exp(c(X^2 + (Y-1)^2)) + \exp(c((X-0.8)^2 + (Y+1)^2))$, where $c = -1$.

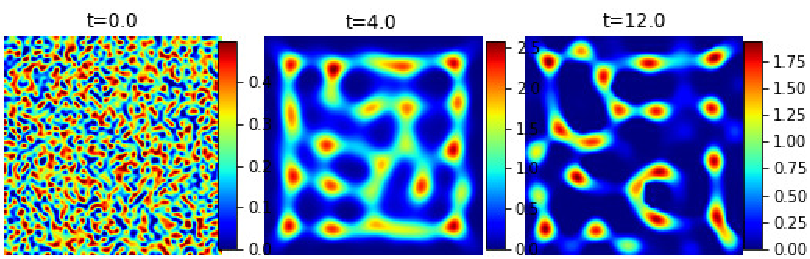

Figure 5. Evolution of the population dynamics using the logistic source term F (with $\alpha = 0.01$ and $w = 0.01$), the nonlocal pointy Gaussian potential (with $\sigma = 0.2$) and a random initial population density.

5.3 Optimal control simulation of the degenerate aggregation model

In this subsection, we implement several tests showing the efficiency of the proposed optimal control procedure to eliminate the hoarding effect. In each test, we plot the comparison between the controlled and the uncontrolled dynamics. In Figs. 6 and 8, we present a comparison between the controlled and uncontrolled dynamics of a given initial population density and a given attraction kernel. In the first and second rows of each figure, we illustrate the effects of an uncontrolled and controlled dynamic, respectively, and the third raw illustrates the development of the optimal solution w.

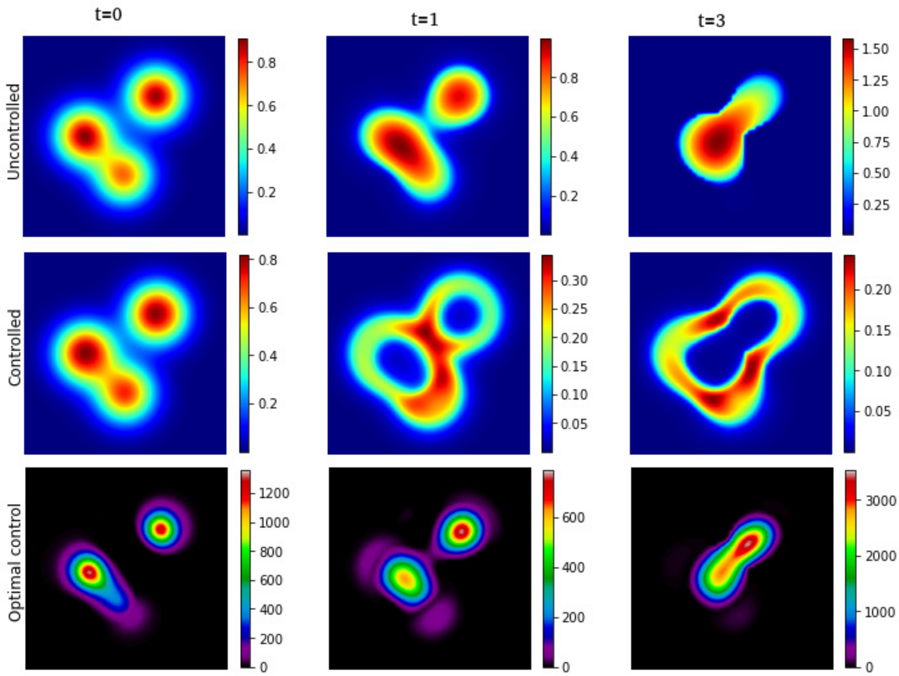

Figure 6. A comparison between controlled and uncontrolled dynamics acting on competition coefficient w, where $a(u) = u(\overline{u} - u)$, $\sigma = 0.5$, $T = 3$, $c = -1$, $u_0(x, y) = 0.8\exp(c((X+1)^2 + Y^2)) + 0.64\exp(c(X^2 + (Y-1)^2)) + 0.8\exp((X-0.8)^2 + (Y+1)^2)$.

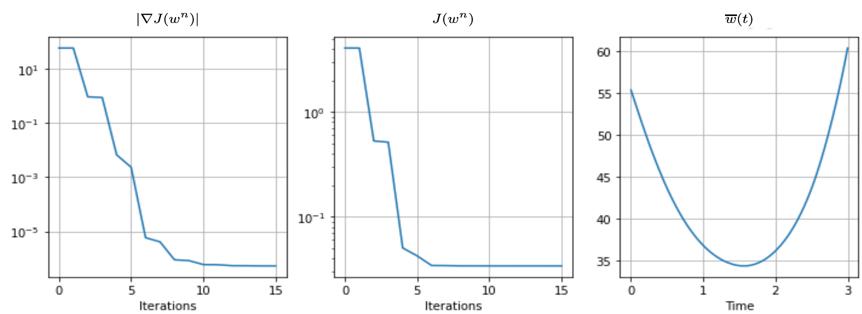

Figure 7. The outputs of the algorithm 1 with respect to Fig. 6 with $\varepsilon_1 = 1$ and $\varepsilon_2 = 10^{-8}$. The L^2-norm of the gradient of the cost functional at the left, the minimization values of the cost functional $J(w, u)$ at the middle and the average of the optimal control $\overline{w}(t) = \int_\Omega w(x, t)\, \mathrm{d}x$ at the right.

The gradient of the functional, as depicted in Figs. 7 and 9, is an important metric for evaluating the performance of the optimization process. A decrease in the gradient to a value less than 10^{-5} indicates a significant improvement in the minimization process. This decrease in gradient is accompanied by a significant reduction in the cost functional, which approaches values close to 0. This leads to a reduced crowding effect compared

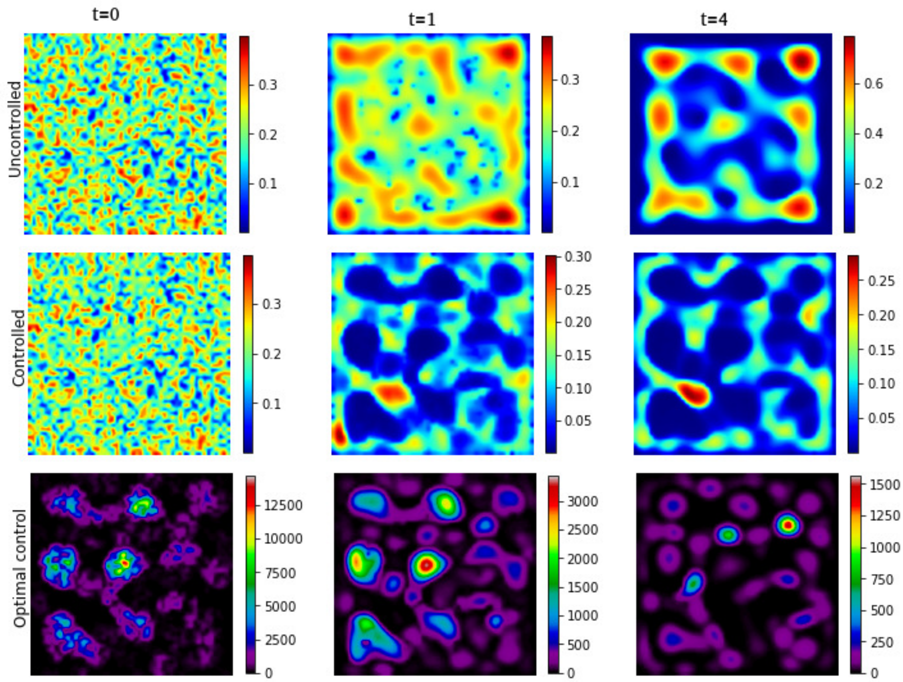

Figure 8. A comparison between controlled and uncontrolled dynamics acting on competition coefficient w where $\sigma = 0.25$, $T = 4$, $\alpha = 0.005$. The initial condition $u_0(x, y)$ is a uniform random distribution.

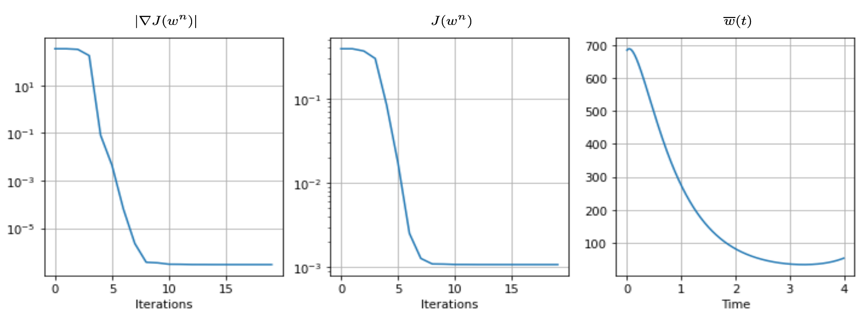

Figure 9. The outputs of the algorithm 1 with respect to Fig. 8 with $\varepsilon_1 = 1$ and $\varepsilon_2 = 10^{-10}$. The L^2-norm of the gradient of the cost functional at the left, the minimization values of the cost functional $J(w, u)$ at the middle and the average of the optimal control $\overline{w}(t) = \int_\Omega w(x, t)\,\mathrm{d}x$ at the right.

to the outcome of the direct problem, where the population dynamics is not guided by a control mechanism. Observations of the controlled solution in comparison to the uncontrolled solution in Fig. 6 reveal that the optimal control, represented by the control variable w, is effective in minimizing the attraction force between groups of individuals. By targeting the centers of the groups the optimal control minimizes the attraction force and prevents the formation of a single crowd at $t = 3.0$. This same effect is observed in

Fig. 8 with a random initial distribution, where the optimal control decreases the gradient and the resulting attractive force preventing the formation of multiple crowds. These observations demonstrate the effectiveness of the optimal control approach in reducing crowding effects.

6 Conclusion

In this paper, we dealt with an optimal control to a two-sidedly degenerate aggregation equation with logistic source term. We provided a rigorous analysis of the mathematical model. We have proposed an optimal control procedure to reduce the over-crowding and pattern formation. We derived the adjoint state equation with the corresponding explicit formulation of the gradient of the cost functional. We showed a numerical simulation of the natural dynamics of different initial population under different attraction forces. Moreover, we have computed the optimal carrying capacity that reduces the pattern formation and the over-crowding effect.

Finally, we want to mention that the well-posedness of the adjoint problem (34) will be the subject of a forthcoming paper.

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