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Finite-time adaptive synchronization of fractional-order delayed quaternion-valued fuzzy neural networks*

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Abstract. Based on direct quaternion method, this paper explores the finite-time adaptive synchronization (FAS) of fractional-order delayed quaternion-valued fuzzy neural networks (FODQVFNNs). Firstly, a useful fractional differential inequality is created, which offers an effective way to investigate FAS. Then two novel quaternion-valued adaptive control strategies are designed. By means of our newly proposed inequality, the basic knowledge about fractional calculus, reduction to absurdity as well as several inequality techniques of quaternion and fuzzy logic, several sufficient FAS criteria are derived for FODQVFNNs. Moreover, the settling time of FAS is estimated, which is in connection with the order and initial values of considered systems as well as the controller parameters. Ultimately, the validity of obtained FAS criteria is corroborated by numerical simulations.

Keywords: quaternion-valued fuzzy neural networks, fractional order, finite-time synchronization, adaptive control, time delay.

1 Introduction

As we know, artificial neural networks (ANNs) consist of numerous processing units, which interconnect cheek by jowl. With the continuous development of neuroscience, the dynamics of ANNs are deemed to preferable simulate information processing in human

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brain. The research of ANNs has drawn considerable concern among scholars owing to their significant applications in extensive fields including pattern recognition, image encryption, optimal control, and so on [15, 16]. Time delay is frequently encountered in ANNs by reason of the limit on the speed from information transmission between neurons. And the existence of time delay not only increases complexity of dynamic analysis, but also degenerates the performance of considered systems involving chaos, oscillation, and instability. Hence, the dynamical behaviors of delayed neural networks (DNNs) have been a hot research topic [4, 5, 32]. In addition, it is meaningful to develop fuzzy DNNs by the combination of DNNs and fuzzy logic due to the ambiguity widely existing in real world. Since Yang et al. creatively proposed fuzzy cellular neural networks in [29], plenty of valuable results have been obtained for fuzzy neural networks (FNNs) such as dissipativity [20] and synchronization [19].

Note that the studies above are considered in real-valued neural networks (R-VNNs) or complex-valued neural networks (C-VNNs). However, the multidimensional data oftentimes encountering in color image processing, satellite attitude control, and wind forecasting is hard to be processed by R-VNNs and C-VNNs. Fortunately, this issue can be perfectly resolved through quaternion-valued neural networks (Q-VNNs) established via integrating quaternion with traditional ANNs. In addition, the approaches adopted in R-VNNs and C-VNNs cannot be directly used to deal with Q-VNNs owing to non-commutativity of quaternion multiplication, which increases the difficulty of Q-VNNs research. Three effective methods have been introduced to analyze the dynamic behavior of Q-VNNs up to now, namely, real decomposition method [2], plural decomposition method [8], and direct quaternion method [22, 31].

Nowadays, in contrast with integer-order calculus, fractional-order calculus has attached more attention for its excellent performances in depicting infinite memory and heritability characteristics of fractional systems in domains of chaotic maps [24], neural networks [1,6,7,10,12,13,18,21,25–28], etc. In consideration of these unique advantages, it is of practical significance to form fractional-order neural networks (F-ONNs) by the incorporation of ANNs and fractional derivative, and a wealth of interesting researches about F-ONNs have been published in stability [6], dissipativity [1], stabilization [7, 10], and so forth.

It is well known that the applications of ANNs are closely related with their dynamical behaviors. Synchronization, as an indispensable dynamical behavior, has stimulated research enthusiasm of many scholars. Until now a myriad of outstanding synchronization achievements in connection with fractional-order Q-VNNs (FOQ-VNNs) have been derived, which can be classified as finite-time synchronization and infinite-time synchronization on the basis of different convergence time. Compared with infinite-time synchronization, the finite-time synchronization of FOQ-VNNs has been received more focus because of its stronger robustness and faster convergence speed. In order to realize synchronization target, some effective control strategies have been put forward, including but not limited to linear feedback control [26], state feedback control [12], hybrid control [21, 28], sliding mode control [25], impulsive control [18], and adaptive control [27]. Thereinto the adaptive control strategy can realize self-regulation of control gains to reduce control costs in comparison to other ones. However, the finite-time adaptive synchronization of FODQVFNNs has not been investigated till now. Therefore it is spontaneous to propose the queries: whether FODQVFNNs can realize the finite-time adaptive synchronization via exploiting adaptive control method? If it can, how to design effective adaptive controllers? How to obtain finite-time adaptive synchronization criteria and estimate the relevant settling time? It is worth noting that these challenging and significative queries have not been considered yet, which inspires our research interests.

Sparked by the aforementioned analysis, we devote to addressing the issue of finite-time synchronization for FODQVFNNs with the help of adaptive control strategy. The novelties of this paper are summarized below:

(i) A novel Caputo fractional-order differential inequality is established, which extends the existing result derived in [12].
(ii) Different from the decomposition methods employed in [8, 19], the direct quaternion method is firstly used to investigate finite-time synchronization of FODQVFNNs.
(iii) Some sufficient finite-time synchronization criteria are obtained by designing two effective quaternion-valued adaptive controllers, and the relevant settling time is explicitly estimated.
(iv) The approaches adopted in this article can be further applied to fuzzy F-ONNs in real or complex fields.

The remainder of this work is organized as follows. Some indispensable preliminaries and the considered FODQVFNNs are provided in Section 2. In Section 3, some sufficient finite-time synchronization criteria are yielded under two different quaternion-valued adaptive controllers. A numerical example is given in Section 4 to illustrate validity of the obtained theoretical results. Section 5 offers the conclusion and future works.

Notations. $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$ and $\mathbb{Q}^n$ represent the set of real numbers, positive real numbers, quaternion numbers, and n-dimensional quaternion space, respectively. Let $\mathfrak{N} = \{1, 2, \dots, n\}$. For any $\delta_\iota = \delta_\iota^R + \mathfrak{i}\delta_\iota^I + \mathfrak{j}\delta_\iota^J + \mathfrak{k}\delta_\iota^K \in \mathbb{Q}$, where δ_ι^R, δ_ι^I, δ_ι^J, $\delta_\iota^K \in \mathbb{R}$, $\iota \in \mathfrak{N}$, $\mathfrak{i}$, $\mathfrak{j}$, and $\mathfrak{k}$ meet the Hamilton rules, i.e., $\mathfrak{k} = \mathfrak{i}\mathfrak{j} = -\mathfrak{j}\mathfrak{i}$, $\mathfrak{i} = \mathfrak{j}\mathfrak{k} = -\mathfrak{k}\mathfrak{j}$, $\mathfrak{j} = \mathfrak{k}\mathfrak{i} = -\mathfrak{i}\mathfrak{k}$, and $\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = -1$. The conjugate of δ_ι is defined as $\overline{\delta_\iota} = \delta_\iota^R - \mathfrak{i}\delta_\iota^I - \mathfrak{j}\delta_\iota^J - \mathfrak{k}\delta_\iota^K$, and $|\delta_\iota| = ((\delta_\iota^R)^2 + (\delta_\iota^I)^2 + (\delta_\iota^J)^2 + (\delta_\iota^K)^2)^{1/2}$ denotes the module of δ_ι. For $\delta = (\delta_1, \delta_2, \dots, \delta_n)^{\mathrm{T}} \in \mathbb{Q}^n$, the norm of δ is $\|\delta\| = (\sum_{\iota=1}^n |\delta_\iota|^2)^{1/2}$.

2 Preliminaries and model description

First several fundamental knowledge in relation to fractional calculus are retrospected in this section. A novel Caputo differential inequality is proposed to obtain the finite-time convergence after that, which is crucial to derive finite-time adaptive synchronization criteria. Furthermore, the model of FODQVFNN is formulated. Some of requisite assumptions and lemmas are rendered for subsequent study.

Definition 1. (See [17].) The fractional integral of $f(t)$ with $\varsigma > 0$ is defined by

$$ {}_{t_0}I_t^{\varsigma} f(t) = \frac{1}{\Gamma(\varsigma)} \int_{t_0}^{t} (t-\beta)^{\varsigma-1} f(\beta)\,\mathrm{d}\beta, $$

where $\Gamma(\varsigma) = \int_0^{+\infty} \mathrm{e}^{-\beta}\beta^{\varsigma-1}\,\mathrm{d}\beta$.

Definition 2. (See [17].) Define the Caputo fractional derivative of $f(t)$ with $0 < \varsigma < 1$ as

$$ {}_{t_0}^{c}D_t^{\varsigma} f(t) = \frac{1}{\Gamma(1-\varsigma)} \int_{t_0}^{t} \frac{f'(\beta)}{(t-\beta)^{\varsigma}}\,\mathrm{d}\beta. $$

Consider a kind of FODQVFNNs described as follows:

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\varsigma} \nu_\iota(t) = {} & -a_\iota \nu_\iota(t) + \sum_{\kappa=1}^{n} b_{\iota\kappa} f_\kappa\big(\nu_\kappa(t)\big) + \sum_{\kappa=1}^{n} c_{\iota\kappa} \upsilon_\kappa \\ & + \bigwedge_{\kappa=1}^{n} \rho_{\iota\kappa} f_\kappa\big(\nu_\kappa(t-\chi)\big) + \bigvee_{\kappa=1}^{n} \varrho_{\iota\kappa} f_\kappa\big(\nu_\kappa(t-\chi)\big) \\ & + \bigwedge_{\kappa=1}^{n} \alpha_{\iota\kappa} \upsilon_\kappa + \bigvee_{\kappa=1}^{n} \beta_{\iota\kappa} \upsilon_\kappa + I_\iota(t), \end{aligned} \tag{1} $$

where $0 < \varsigma < 1$, $\nu_\iota(t)$ is the state variable of ιth neuron. $a_\iota \in \mathbb{R}^+$ stands for the neuron attenuation coefficient, $b_{\iota\kappa} \in \mathbb{Q}$ and $c_{\iota\kappa} \in \mathbb{R}$ are the synaptic connection weights of κth neuron, $f_\kappa(\cdot) : \mathbb{Q} \to \mathbb{Q}$ denotes the activation function. $\rho_{\iota\kappa}$ and $\varrho_{\iota\kappa}$ represent connection weights of the fuzzy feedback MIN template as well as MAX template, $\alpha_{\iota\kappa}$ and $\beta_{\iota\kappa}$ are the elements of fuzzy feed-forward MIN template as well as MAX template. $\chi \in \mathbb{R}^+$ is the constant delay, $\bigwedge$ and $\bigvee$ denote fuzzy AND as well as OR operations. υ_κ, $I_\iota(t)$ are the external input and bias of ιth neuron. The activation function $f_\kappa(\cdot)$ obeys the assumptions below.

Assumption 1. For fuzzy AND as well as OR operations, there exists a positive constant σ_κ such that $f_\kappa(\cdot)$ satisfies the following two inequalities:

$$ \begin{aligned} & \left(\bigwedge_{\kappa=1}^{n} \rho_{\iota\kappa} \big(\overline{f_\kappa\big(\omega_\kappa(t-\chi)\big)} - \overline{f_\kappa\big(\nu_\kappa(t-\chi)\big)} \big) \right) \\ & \quad \times \left(\bigwedge_{\kappa=1}^{n} \rho_{\iota\kappa} \big(f_\kappa\big(\omega_\kappa(t-\chi)\big) - f_\kappa\big(\nu_\kappa(t-\chi)\big) \big) \right) \\ & \leqslant \sum_{\kappa=1}^{n} \sigma_\kappa |\rho_{\iota\kappa}| \big(\overline{\omega_\kappa(t-\chi)} - \overline{\nu_\kappa(t-\chi)} \big) \big(\omega_\kappa(t-\chi) - \nu_\kappa(t-\chi) \big), \end{aligned} $$

$$\left(\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\big(\overline{f_\kappa\big(\omega_\kappa(t-\chi)\big)}-\overline{f_\kappa\big(\nu_\kappa(t-\chi)\big)}\big)\right)\\ \times\left(\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\big(f_\kappa\big(\omega_\kappa(t-\chi)\big)-f_\kappa\big(\nu_\kappa(t-\chi)\big)\big)\right)\\ \leqslant\sum_{\kappa=1}^{n}\sigma_\kappa|\varrho_{\iota\kappa}|\big(\overline{\omega_\kappa(t-\chi)}-\overline{\nu_\kappa(t-\chi)}\big)\big(\omega_\kappa(t-\chi)-\nu_\kappa(t-\chi)\big).$$

Assumption 2. For any $r, s \in \mathbb{Q}$, there exists positive constant o_κ such that

$$\big|f_\kappa(r)-f_\kappa(s)\big|\leqslant o_\kappa|r-s|.$$

FODQVFNN (1) is considered as derive system, then the matching response system is depicted by

$$\begin{aligned}{}_{t_0}^{c}D_t^{\varsigma}\omega_\iota(t) &= -a_\iota\omega_\iota(t)+\sum_{\kappa=1}^{n}b_{\iota\kappa}f_\kappa\big(\omega_\kappa(t)\big)+\sum_{\kappa=1}^{n}c_{\iota\kappa}\upsilon_\kappa\\ &\quad+\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\big(\omega_\kappa(t-\chi)\big)+\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\big(\omega_\kappa(t-\chi)\big)\\ &\quad+\bigwedge_{\kappa=1}^{n}\alpha_{\iota\kappa}\upsilon_\kappa+\bigvee_{\kappa=1}^{n}\beta_{\iota\kappa}\upsilon_\kappa+I_\iota(t)+u_\iota(t),\end{aligned}\tag{2}$$

where $u_\iota(t)\in\mathbb{Q}$ is the adaptive controller to be designed. Let $e_\iota(t)=\omega_\iota(t)-\nu_\iota(t)$, then the error system can be shown as

$$\begin{aligned}{}_{t_0}^{c}D_t^{\varsigma}e_\iota(t) &= -a_\iota e_\iota(t)+\sum_{\kappa=1}^{n}b_{\iota\kappa}f_\kappa\big(e_\kappa(t)\big)+\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\big(e_\kappa(t-\chi)\big)\\ &\quad+\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\big(e_\kappa(t-\chi)\big)+u_\iota(t),\end{aligned}\tag{3}$$

where

$$f_\kappa\big(e_\kappa(t)\big)=f_\kappa\big(\omega_\kappa(t)\big)-f_\kappa\big(\nu_\kappa(t)\big),$$
$$f_\kappa\big(e_\kappa(t-\chi)\big)=f_\kappa\big(\omega_\kappa(t-\chi)\big)-f_\kappa\big(\nu_\kappa(t-\chi)\big).$$

Remark 1. Since fuzzy logic, time delay, quaternion algebra, and fractional calculus are all taken into consideration, our model (1) is more practical and general than delayed Q-VNNs [31], FOQ-VNNs [1, 6, 7, 10, 12, 18, 21, 26, 28], fractional-order C-VNNs [27] as well as fuzzy F-ONNs [13].

The following definitions and lemmas are given throughout this work for the later study.

Definition 3. FODQVFNN (1) is said to realize the finite-time synchronization with FODQVFNN (2) if there exists constant T^* in dependence on the initial values of FODQVFNNs (1) and (2) such that

$$\lim_{t\to\mathsf{T}^*}\left|e_\iota(t)\right|=0,\quad \left|e_\iota(t)\right|\equiv 0\quad \forall t\geqslant \mathsf{T}^*,$$

where $\iota\in\mathfrak{N}$, and T^* is called the settling time of finite-time synchronization.

Lemma 1. (*See* [29].) *Let $\omega(t)$ as well as $\nu(t)$ be two states of FODQVFNN* (1)*, then*

$$\left|\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\big(\omega_\kappa(t)\big)-\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\big(\nu_\kappa(t)\big)\right|\leqslant\sum_{\kappa=1}^{n}|\rho_{\iota\kappa}|\left|f_\kappa\big(\omega_\kappa(t)\big)-f_\kappa\big(\nu_\kappa(t)\big)\right|,$$
$$\left|\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\big(\omega_\kappa(t)\big)-\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\big(\nu_\kappa(t)\big)\right|\leqslant\sum_{\kappa=1}^{n}|\varrho_{\iota\kappa}|\left|f_\kappa\big(\omega_\kappa(t)\big)-f_\kappa\big(\nu_\kappa(t)\big)\right|.$$

Lemma 2. (*See* [22].) *If α_1, $\alpha_2\in\mathbb{Q}$, and $\mathfrak{R}\in\mathbb{R}^+$, then*

$$\overline{\alpha_1}\alpha_2+\overline{\alpha_2}\alpha_1\leqslant\mathfrak{R}\overline{\alpha_1}\alpha_1+\frac{1}{\mathfrak{R}}\overline{\alpha_2}\alpha_2.$$

Lemma 3. (*See* [14].) *Assume that $\mathcal{V}(t)\in\mathbb{Q}$ is the differentiable function. Then*

$${}_{t_0}^{c}D_t^{\varsigma}\big(\overline{\mathcal{V}(t)}\mathcal{V}(t)\big)\leqslant\overline{\mathcal{V}(t)}{}_{t_0}^{c}D_t^{\varsigma}\mathcal{V}(t)+\big({}_{t_0}^{c}D_t^{\varsigma}\overline{\mathcal{V}(t)}\big)\mathcal{V}(t)\quad\forall\varsigma\in(0,1).$$

Lemma 4. (*See* [11].) *For the case of $0<\varsigma<1$, we have*

$${}_{t_0}I_t^{\varsigma}\,{}_{t_0}^{c}DD_t^{\varsigma}f(t)=f(t)-f(t_0).$$

Lemma 5. (*See* [9].) *Assume that $\xi_r>0$ for $r\in\mathfrak{N}$, $m\geqslant1$. Then*

$$\sum_{r=1}^{n}\xi_r^m\leqslant\left(\sum_{r=1}^{n}\xi_r\right)^m.$$

Lemma 6. (*See* [23].) *If $\varphi:\mathbb{R}\to\mathbb{R}$ and $\psi:[t_0,+\infty)\to\mathbb{R}$ are two continuously differentiable functions, moreover, ψ is convex, then*

$${}_{t_0}^{c}D_t^{\varsigma}\varphi\big(\psi(t)\big)\leqslant\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}{}_{t_0}^{c}D_t^{\varsigma}\psi(t)\quad\forall\varsigma\in(0,1).$$

Lemma 7. (*See* [30].) *If $\mathcal{V}(t)$ is the continuously differentiable function defined on $[t_0,p)$, then for any constant h and $t\in[t_0,p)$,*

$${}_{t_0}^{c}D_t^{\varsigma}\big(\mathcal{V}(t)-h\big)^2\leqslant2\big(\mathcal{V}(t)-h\big){}_{t_0}^{c}D_t^{\varsigma}\mathcal{V}(t)\quad\forall\varsigma\in(0,1).$$

Lemma 8. *Suppose $\varPhi(t)$ and $\varPsi(t)$ are two nonnegative continuously differentiable functions and satisfy*

$$ {}_{t_0}^{c}D_t^{\varsigma}\big(\varPhi(t)+\varPsi(t)\big) \leqslant -\theta\varPhi^{-m}(t)-\tau. $$

Then one has $\lim_{t\to\mathsf{T}^*}\varPhi(t)=0$, as well as $\varPhi(t)\equiv 0$ for all $t\geqslant \mathsf{T}^*$, where $0<\varsigma<1$, $\theta>0$, $m\geqslant 1$, $\tau>0$, and

$$ \mathsf{T}^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{\theta(1+m)}\left[\left(\varPhi(t_0)+\varPsi(t_0)+\left(\frac{\theta}{\tau}\right)^{1/m}\right)^{m+1} - \left(\frac{\theta}{\tau}\right)^{1+1/m}\right]\right\}^{1/\varsigma}. $$

Particularly, for the case of $\varPsi(t)=0$, we have

$$ \mathsf{T}^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{\theta(1+m)}\left[\left(\varPhi(t_0)+\left(\frac{\theta}{\tau}\right)^{1/m}\right)^{m+1} - \left(\frac{\theta}{\tau}\right)^{1+1/m}\right]\right\}^{1/\varsigma}. $$

Proof. Let $\hat{\varPhi}(t)=\varPhi(t)+\varPsi(t)+(\theta/\tau)^{1/m}$. Based on Lemma 5 and the nonnegativity of $\varPsi(t)$, we have

$$ \begin{aligned} \hat{\varPhi}^{-m}(t) &= \frac{1}{\big(\varPhi(t)+\varPsi(t)+(\frac{\theta}{\tau})^{1/m}\big)^m} \leqslant \frac{1}{\varPhi^m(t)+\varPsi^m(t)+\frac{\theta}{\tau}} \\ &\leqslant \frac{1}{\varPhi^m(t)+\frac{\theta}{\tau}} \leqslant \varPhi^{-m}(t)+\frac{\tau}{\theta}. \end{aligned} \tag{4} $$

Multiplying both sides of (4) by $-\theta$ gives

$$ -\theta\varPhi^{-m}(t)-\tau \leqslant -\theta\hat{\varPhi}^{-m}(t), \tag{5} $$

combining (4) and (5) as well as the definition of $\hat{\varPhi}(t)$ can derive that

$$ {}_{t_0}^{c}D_t^{\varsigma}\hat{\varPhi}(t) = {}_{t_0}^{c}D_t^{\varsigma}\big(\varPhi(t)+\varPsi(t)\big) \leqslant -\theta\varPhi^{-\theta}(t)-\tau \leqslant -\theta\hat{\varPhi}^{-m}(t), $$

which means

$$ \hat{\varPhi}^m(t)\,{}_{t_0}^{c}D_t^{\varsigma}\hat{\varPhi}(t) \leqslant -\theta. \tag{6} $$

Applying Lemma 6 to (6), we get

$$ {}_{t_0}^{c}D_t^{\varsigma}\hat{\varPhi}^{m+1}(t) \leqslant (m+1)\hat{\varPhi}^m(t)\,{}_{t_0}^{c}D_t^{\varsigma}\hat{\varPhi}(t) \leqslant -\theta(m+1). \tag{7} $$

Taking fractional integral for (7) with order ς from t_0 to t on the basis of Lemma 4 yields

$$ \hat{\varPhi}^{m+1}(t) \leqslant \hat{\varPhi}^{m+1}(t_0) - \frac{\theta(m+1)(t-t_0)^{\varsigma}}{\Gamma(1+\varsigma)}, $$

that is,

$$ \hat{\varPhi}(t) \leqslant \left[\hat{\varPhi}^{m+1}(t_0) - \frac{\theta(m+1)(t-t_0)^{\varsigma}}{\Gamma(1+\varsigma)}\right]^{1/(m+1)}. \tag{8} $$

It can follow from (8) and the definition of $\hat{\Phi}(t)$ as well as the nonnegativity of $\Psi(t)$ that

$$\Phi(t) + \left(\frac{\theta}{\tau}\right)^{1/m} \leqslant \hat{\Phi}(t) \leqslant \left[\left(\Phi(t_0) + \Psi(t_0) + \left(\frac{\theta}{\tau}\right)\right) - \frac{\theta(m+1)(t-t_0)^{\varsigma}}{\Gamma(1+\varsigma)}\right]^{1/(m+1)},$$

i.e.,

$$\Phi(t) \leqslant \left[\left(\Phi(t_0) + \Psi(t_0) + \left(\frac{\theta}{\tau}\right)\right) - \frac{\theta(m+1)(t-t_0)^{\varsigma}}{\Gamma(1+\varsigma)}\right]^{1/(m+1)} - \left(\frac{\theta}{\tau}\right)^{1/m}.$$

Due to the nonnegativity and continuity of $\Phi(t)$, we can easily obtain

$$\lim_{t\to\mathsf{T}^*} = \Phi(\mathsf{T}^*) = 0,$$

in which

$$\mathsf{T}^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{\theta(1+m)}\left[\left(\Phi(t_0) + \Psi(t_0) + \left(\frac{\theta}{\tau}\right)^{1/m}\right)^{m+1} - \left(\frac{\theta}{\tau}\right)^{1+1/m}\right]\right\}^{1/\varsigma}. \quad (9)$$

The proof by contradiction is adopted to verify $\Phi(t) \equiv 0$ for all $t \geqslant \mathsf{T}^*$ in the following part. Suppose that there exists $\check{\mathsf{T}}^* > \mathsf{T}^*$ satisfying $\Phi(\check{\mathsf{T}}^*) > 0$. Nevertheless, it follows from the monotonicity of $\theta(m+1)(t-t_0)^{\varsigma}/\Gamma(1+\varsigma)$ and (9) that

$$\begin{aligned}\Phi(\check{\mathsf{T}}^*) &\leqslant \left[\left(\Phi(t_0) + \Psi(t_0) + \left(\frac{\theta}{\tau}\right)\right) - \frac{\theta(m+1)(\check{\mathsf{T}}^*-t_0)^{\varsigma}}{\Gamma(1+\varsigma)}\right]^{1/(m+1)} - \left(\frac{\theta}{\tau}\right)^{1/m} \\ &< \left[\left(\Phi(t_0) + \Psi(t_0) + \left(\frac{\theta}{\tau}\right)\right) - \frac{\theta(m+1)(\mathsf{T}^*-t_0)^{\varsigma}}{\Gamma(1+\varsigma)}\right]^{1/(m+1)} - \left(\frac{\theta}{\tau}\right)^{1/m} \\ &= 0,\end{aligned}$$

which is in contradiction with the nonnegativity of $\Phi(t)$. Therefore $\Phi(t) \equiv 0$ for all $t \geqslant \mathsf{T}^*$. □

Remark 2. What is noteworthy is that Lemma 8 provides an effective and novel method to address the issue of FAS. Moreover, it can be reduced to Lemma 8 in [13] in the case of $\chi = 0$ and $\Psi(t) = 0$, it can also be reduced to Lemma 8 in [12] under circumstance of $\Psi(t) = 0$. Therefore our Lemma 8 extends and improves the results obtained in [12, 13] to some extent.

3 Main results

In this section, some sufficient finite-time synchronization criteria are derived via two different quaternion-valued adaptive control strategies. The novel quaternion-valued adaptive controller is designed as

$$u_\iota(t) = \begin{cases} -\hat{\mu}_\iota(t)e_\iota(t) - \frac{\gamma e_\iota(t)}{(\overline{e_\iota(t)}e_\iota(t))^{\vartheta}} - \frac{\lambda e_\iota(t)}{\overline{e_\iota(t)}e_\iota(t)}, & |e_\iota(t)| \neq 0, \\ 0, & |e_\iota(t)| = 0, \end{cases} \quad (10)$$

where $\vartheta \geqslant 2$, ${}_{t_0}^{c}D_t^{\varsigma}\hat{\mu}_\iota(t) = \eta_\iota\overline{e_\iota(t)}e_\iota(t)$.

Remark 3. Compared with linear feedback controller adopted in [26] and state feedback controller used in [12], the adaptive controller can vastly reduce control costs because of the fact that its control gain $\hat{\mu}_\iota(t)$ is updated by itself based on adaptive control law η_ι.

Theorem 1. *Under Assumption* 1 *as well as quaternion-valued adaptive controller* (10)*, FODQVFNNs* (1) *and* (2) *can achieve finite-time synchronization, and the settling time is estimated as*

$$\mathsf{T}_1^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{2\gamma\vartheta}\left[\left(\mathcal{V}_{11}(t_0) + \mathcal{V}_{12}(t_0) + \left(\frac{\gamma}{n\lambda}\right)^{1/(\vartheta-1)}\right)^{\vartheta} - \left(\frac{\gamma}{n\lambda}\right)^{\vartheta/(\vartheta-1)}\right]\right\}^{1/\varsigma}.$$

Proof. Consider the Lyapunov function as follows:

$$\mathcal{V}_1(t) = \underbrace{\sum_{\iota=1}^{n} \overline{e_\iota(t)} e_\iota(t)}_{\mathcal{V}_{11}(t)} + \underbrace{\sum_{\iota=1}^{n} \frac{1}{\eta_\iota}\left(\hat{\mu}_\iota(t) - \hat{\mu}_\iota^*\right)^2}_{\mathcal{V}_{12}(t)},$$

where

$$\hat{\mu}_\iota^* = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}n\pi_3 + \frac{1}{2}\sum_{\kappa=1}^{n}\pi_3^{-1} b_{\kappa\iota}^2 o_\iota^2 + \frac{1}{2}\varpi\zeta - a_\iota. \tag{11}$$

By means of Lemmas 3 and 7, calculating the Caputo derivative of $\mathcal{V}_1(t)$ with $0 < \varsigma < 1$ along the trajectory of (3) gives

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\varsigma}\mathcal{V}_1(t) \\
&\leqslant \sum_{\iota=1}^{n}\left[\overline{e_\iota(t)}\,{}_{t_0}^{c}D_t^{\varsigma} e_\iota(t) + \left({}_{t_0}^{c}D_t^{\varsigma}\overline{e_\iota(t)}\right) e_\iota(t)\right] + \sum_{\iota=1}^{n}\frac{2}{\eta_\iota}\left(\hat{\mu}_\iota(t) - \hat{\mu}_\iota^*\right){}_{t_0}^{c}D_t^{\varsigma}\hat{\mu}_\iota(t) \\
&= \sum_{\iota=1}^{n}\Bigg\{\overline{e_\iota(t)}\Bigg[-a_\iota e_\iota(t) + \sum_{\kappa=1}^{n} b_{\iota\kappa} f_\kappa\big(e_\kappa(t)\big) + \bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa} f_\kappa\big(e_\kappa(t-\chi)\big) \\
&\quad + \bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa} f_\kappa\big(e_\kappa(t-\chi)\big) - \hat{\mu}_\iota(t) e_\iota(t) - \frac{\gamma e_\iota(t)}{(\overline{e_\iota(t)} e_\iota(t))^{\vartheta}} - \frac{\lambda e_\iota(t)}{\overline{e_\iota(t)} e_\iota(t)}\Bigg] \\
&\quad + \Bigg[-a_\iota\overline{e_\iota(t)} + \sum_{\kappa=1}^{n}\overline{f_\kappa\big(e_\kappa(t)\big)}\,\overline{b_{\iota\kappa}} + \bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\overline{f_\kappa\big(e_\kappa(t-\chi)\big)} \\
&\quad + \bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\overline{f_\kappa\big(e_\kappa(t-\chi)\big)} - \hat{\mu}_\iota(t)\overline{e_\iota(t)} - \frac{\gamma\overline{e_\iota(t)}}{(\overline{e_\iota(t)} e_\iota(t))^{\vartheta}} - \frac{\lambda\overline{e_\iota(t)}}{\overline{e_\iota(t)} e_\iota(t)}\Bigg] e_\iota(t)\Bigg\} \\
&\quad + \sum_{\iota=1}^{n} 2(\hat{\mu}_\iota(t) - \hat{\mu}_\iota^*)\overline{e_\iota(t)} e_\iota(t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\iota=1}^{n}\left[\overline{e_\iota(t)}\left(\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\big(e_\kappa(t-\chi)\big)\right) + \left(\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\overline{f_\kappa\big(e_\kappa(t-\chi)\big)}\right)e_\iota(t)\right] \\
&\quad + \sum_{\iota=1}^{n}\left[\overline{e_\iota(t)}\left(\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\big(e_\kappa(t-\chi)\big)\right) + \left(\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\overline{f_\kappa\big(e_\kappa(t-\chi)\big)}\right)e_\iota(t)\right] \\
&\quad + \sum_{\iota,\kappa=1}^{n}\left[\overline{e_\iota(t)}b_{\iota\kappa}f_\kappa\big(e_\kappa(t)\big) + \overline{f_\kappa\big(e_\kappa(t)\big)}\overline{b_{\iota\kappa}}e_\iota(t)\right] - 2\gamma\sum_{\iota=1}^{n}\big(\overline{e_\iota(t)}e_\iota(t)\big)^{1-\vartheta} \\
&\quad - 2n\lambda - 2\sum_{\iota=1}^{n}(a_\iota + \hat{\mu}_\iota^*)\overline{e_\iota(t)}e_\iota(t).
\end{aligned} \tag{12}
$$

It follows from Assumption 1 and Lemma 2 that

$$
\begin{aligned}
&\overline{e_\iota(t)}\left(\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\big(e_\kappa(t-\chi)\big)\right) + \left(\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\overline{f_\kappa\big(e_\kappa(t-\chi)\big)}\right)e_\iota(t) \\
&\quad = \overline{e_\iota(t)}\left[\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\big(f_\kappa\big(\omega_\kappa(t-\chi)\big) - f_\kappa\big(\nu_\kappa(t-\chi)\big)\big)\right] \\
&\qquad + \left[\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\big(\overline{f_\kappa\big(\omega_\kappa(t-\chi)\big)} - \overline{f_\kappa\big(\nu_\kappa(t-\chi)\big)}\big)\right]e_\iota(t) \\
&\quad \leqslant \pi_1\overline{e_\iota(t)}e_\iota(t) + \pi_1^{-1}\left[\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\big(\overline{f_\kappa\big(\omega_\kappa(t-\chi)\big)} - \overline{f_\kappa\big(\nu_\kappa(t-\chi)\big)}\big)\right] \\
&\qquad \times \left[\bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\big(f_\kappa\big(\omega_\kappa(t-\chi)\big) - f_\kappa\big(\nu_\kappa(t-\chi)\big)\big)\right] \\
&\quad \leqslant \pi_1\overline{e_\iota(t)}e_\iota(t) + \pi_1^{-1}\sum_{\kappa=1}^{n}\sigma_\kappa|\rho_{\iota\kappa}|\overline{e_\iota(t-\chi)}e_\iota(t-\chi)
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\overline{e_\iota(t)}\left(\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\big(e_\kappa(t-\chi)\big)\right) + \left(\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\overline{f_\kappa\big(e_\kappa(t-\chi)\big)}\right)e_\iota(t) \\
&\quad = \overline{e_\iota(t)}\left[\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\big(f_\kappa\big(\omega_\kappa(t-\chi)\big) - f_\kappa\big(\nu_\kappa(t-\chi)\big)\big)\right] \\
&\qquad + \left[\bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}\big(\overline{f_\kappa\big(\omega_\kappa(t-\chi)\big)} - \overline{f_\kappa\big(\nu_\kappa(t-\chi)\big)}\big)\right]e_\iota(t) \\
&\quad \leqslant \pi_2\overline{e_\iota(t)}e_\iota(t) + \pi_2^{-1}\sum_{\kappa=1}^{n}\sigma_\kappa|\varrho_{\iota\kappa}|\overline{e_\iota(t-\chi)}e_\iota(t-\chi).
\end{aligned} \tag{14}
$$

It follows from Assumption 2 and Lemma 2 that

$$\begin{aligned}
&\sum_{\iota,\kappa=1}^{n} \big[\overline{e_\iota(t)} b_{\iota\kappa} f_\kappa\big(e_\kappa(t)\big) + \overline{f_\kappa\big(e_\kappa(t)\big)}\,\overline{b_{\iota\kappa}} e_\iota(t)\big] \\
&\quad \leqslant \sum_{\iota,\kappa=1}^{n} \big[\pi_3 \overline{e_\iota(t)} e_\iota(t) + \pi_3^{-1} \overline{f_\kappa\big(e_\kappa(t)\big)}\,\overline{b_{\iota\kappa}} b_{\iota\kappa} f_\kappa\big(e_\kappa(t)\big)\big] \\
&\quad \leqslant \sum_{\iota,\kappa=1}^{n} \big[\pi_3 \overline{e_\iota(t)} e_\iota(t) + \pi_3^{-1} b_{\iota\kappa}^2 o_\kappa^2 \overline{e_\kappa(t)} e_\kappa(t)\big] \\
&\quad = \sum_{\iota=1}^{n} \left(n\pi_3 + \sum_{\kappa=1}^{n} \pi_3^{-1} b_{\kappa\iota}^2 o_\iota^2\right) \overline{e_\iota(t)} e_\iota(t).
\end{aligned} \tag{15}$$

In the light of fractional-order Razumikhin theorem [3], one has

$$\begin{aligned}
&\sum_{\iota=1}^{n} \left[\pi_1^{-1} \sum_{\kappa=1}^{n} \sigma_\kappa |\rho_{\iota\kappa}| \overline{e_\iota(t-\chi)} e_\iota(t-\chi) + \pi_2^{-1} \sum_{\kappa=1}^{n} \sigma_\kappa |\varrho_{\iota\kappa}| \overline{e_\iota(t-\chi)} e_\iota(t-\chi)\right] \\
&\quad = \sum_{\iota=1}^{n} \left[\sum_{\kappa=1}^{n} \sigma_\iota \big(\pi_1^{-1} |\rho_{\kappa\iota}| + \pi_2^{-1} |\varrho_{\kappa\iota}|\big)\right] \overline{e_\iota(t-\chi)} e_\iota(t-\chi) \\
&\quad \leqslant \varpi \sum_{\iota=1}^{n} \overline{e_\iota(t-\chi)} e_\iota(t-\chi) = \varpi \mathcal{V}_1(t-\chi) \\
&\quad \leqslant \varpi\zeta \mathcal{V}_1(t) = \varpi\zeta \sum_{\iota=1}^{n} \overline{e_\iota(t)} e_\iota(t),
\end{aligned} \tag{16}$$

where $\varpi = \max_{1\leqslant\iota\leqslant n}\{\sum_{\kappa=1}^{n} \sigma_\iota(\pi_1^{-1}|\rho_{\kappa\iota}| + \pi_2^{-1}|\varrho_{\kappa\iota}|)\}$, $\zeta > 1$. Combining (12)–(16) with (11) derives

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\varsigma} \mathcal{V}_1(t) &\leqslant -2\gamma \sum_{\iota=1}^{n} \big(\overline{e_\iota(t)} e_\iota(t)\big)^{1-\vartheta} - 2n\lambda \leqslant -2\gamma \left(\sum_{\iota=1}^{n} \overline{e_\iota(t)} e_\iota(t)\right)^{1-\vartheta} - 2n\lambda \\
&= -2\gamma \mathcal{V}_{11}^{1-\vartheta}(t) - 2n\lambda.
\end{aligned}$$

Based on Lemma 8, we yield that $\lim_{t\to \mathsf{T}_1^*} \mathcal{V}_1(t) = 0$ and $\mathcal{V}_1(t) \equiv 0$ for all $t \geqslant \mathsf{T}_1^*$, where

$$\begin{aligned}
\mathsf{T}_1^* = t_0 + \Bigg\{ \frac{\Gamma(1+\varsigma)}{2\gamma\vartheta} \Bigg[\left(\mathcal{V}_{11}(t_0) + \mathcal{V}_{12}(t_0) + \left(\frac{\gamma}{n\lambda}\right)^{1/(\vartheta-1)}\right)^{\vartheta} \\
- \left(\frac{\gamma}{n\lambda}\right)^{\vartheta/(\vartheta-1)} \Bigg] \Bigg\}^{1/\varsigma},
\end{aligned}$$

which means $\lim_{t\to \mathsf{T}_1^*} e_\iota(t) = 0$ and $e_\iota(t) \equiv 0$ for all $t \geqslant \mathsf{T}_1^*$. According to Definition 3, FODQVFNNs (1) and (2) can realize finite-time synchronization within settling time T_1^*.

The other novel quaternion-valued adaptive controller is designed by

$$u_\iota(t) = \begin{cases} -\check{\mu}_\iota(t)e_\iota(t) - \frac{\gamma e_\iota(t)}{(\overline{e_\iota(t)}e_\iota(t))^\vartheta} - \frac{\lambda e_\iota(t)}{\overline{e_\iota(t)}e_\iota(t)}, & |e_\iota(t)| \neq 0, \\ 0, & |e_\iota(t)| = 0, \end{cases} \tag{17}$$

where $\vartheta \geqslant 2$, ${}^{c}_{t_0}D^{\varsigma}_{t}\check{\mu}_\iota(t) = \overline{e_\iota(t)}e_\iota(t) - \delta_\iota \operatorname{sign}(\check{\mu}_\iota(t) - \check{\mu}^*_\iota)|\check{\mu}_\iota(t) - \check{\mu}^*_\iota|^{1-2\vartheta}$. □

Remark 4. The adaptive controller (17) offers an other effective means to investigate finite-time synchronization of FODQVFNNs (1) and (2), which owns a different adaptive control law in contrast to (10). The conclusion "${}^{c}_{t_0}D^{\varsigma}_{t}\mathcal{V}_2(t) \leqslant -q_1\mathcal{V}_2^{1-\vartheta}(t) - q_2$" can be gained by employing controller (17), this is different with "${}^{c}_{t_0}D^{\varsigma}_{t}(\mathcal{V}_{11}(t) + \mathcal{V}_{12}(t)) \leqslant -2\gamma\mathcal{V}_{11}^{1-\vartheta}(t) - 2n\gamma$" derived in Theorem 1.

Theorem 2. *Under Assumption* 1 *as well as quaternion-valued adaptive controller* (17)*, FODQVFNNs* (1) *and* (2) *can realize finite-time synchronization, and the settling time is reckoned as*

$$\mathsf{T}_2^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{q_1\vartheta}\left[\left(\mathcal{V}_2(t_0) + \left(\frac{q_1}{q_2}\right)^{1/(\vartheta-1)}\right)^\vartheta - \left(\frac{q_1}{q_2}\right)^{\vartheta/(\vartheta-1)}\right]\right\}^{1/\varsigma},$$

where $q_1 = 2\min_{1\leqslant\iota\leqslant n}\{\gamma, \delta_\iota\}$, $q_2 = 2n\lambda$.

Proof. Construct a Lyapunov function as follows:

$$\mathcal{V}_2(t) = \sum_{\iota=1}^{n}\overline{e_\iota(t)}e_\iota(t) + \sum_{\iota=1}^{n}\left(\check{\mu}_\iota(t) - \check{\mu}^*_\iota\right)^2,$$

where

$$\check{\mu}^*_\iota = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}n\pi_3 + \frac{1}{2}\sum_{\kappa=1}^{n}\pi_3^{-1}b^2_{\kappa\iota}o^2_\iota + \frac{1}{2}\varpi\zeta - a_\iota. \tag{18}$$

By utilizing Lemmas 3 and 7, similarly calculating Caputo derivative of $\mathcal{V}_2(t)$ with the $0 < \varsigma < 1$ along the trajectory of (3) gives

$$\begin{aligned}
&{}^{c}_{t_0}D^{\varsigma}_{t}\mathcal{V}_2(t) \\
&\quad\leqslant \sum_{\iota=1}^{n}\left[\overline{e_\iota(t)}\,{}^{c}_{t_0}D^{\varsigma}_{t}e_\iota(t) + \left({}^{c}_{t_0}D^{\varsigma}_{t}\overline{e_\iota(t)}\right)e_\iota(t)\right] + \sum_{\iota=1}^{n}2\left(\check{\mu}_\iota(t) - \check{\mu}^*_\iota\right){}^{c}_{t_0}D^{\varsigma}_{t}\mu_\iota(t) \\
&\quad= \sum_{\iota=1}^{n}\left\{\overline{e_\iota(t)}\left[-a_\iota e_\iota(t) + \sum_{\kappa=1}^{n}b_{\iota\kappa}f_\kappa\left(e_\kappa(t)\right) + \bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}f_\kappa\left(e_\kappa(t-\chi)\right)\right.\right. \\
&\qquad\left. + \bigvee_{\kappa=1}^{n}\varrho_{\iota\kappa}f_\kappa\left(e_\kappa(t-\chi)\right) - \check{\mu}_\iota(t)e_\iota(t) - \frac{\gamma e_\iota(t)}{(\overline{e_\iota(t)}e_\iota(t))^\vartheta} - \frac{\lambda e_\iota(t)}{\overline{e_\iota(t)}e_\iota(t)}\right] \\
&\qquad + \left[-a_\iota\overline{e_\iota(t)} + \sum_{\kappa=1}^{n}\overline{f_\kappa\left(e_\kappa(t)\right)}\,\overline{b_{\iota\kappa}} + \bigwedge_{\kappa=1}^{n}\rho_{\iota\kappa}\overline{f_\kappa\left(e_\kappa(t-\chi)\right)}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \bigvee_{\kappa=1}^{n} \varrho_{\iota\kappa} \overline{f_\kappa\big(e_\kappa(t-\chi)\big)} - \check{\mu}_\iota(t)\overline{e_\iota(t)} - \frac{\gamma \overline{e_\iota(t)}}{(\overline{e_\iota(t)}e_\iota(t))^{\vartheta}} - \frac{\lambda \overline{e_\iota(t)}}{\overline{e_\iota(t)}e_\iota(t)}\Bigg] e_\iota(t)\Bigg\} \\
&+ \sum_{\iota=1}^{n} 2\big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)\big[\overline{e_\iota(t)}e_\iota(t) - \delta_\iota \operatorname{sign}\big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)\big|\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big|^{1-2\vartheta}\big] \\
&- 2\gamma \sum_{\iota=1}^{n} \big(\overline{e_\iota(t)}e_\iota(t)\big)^{1-\vartheta} - 2\sum_{\iota=1}^{n} \delta_\iota \big(\big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)^2\big)^{1-\vartheta} - 2n\lambda, \\
= &\sum_{\iota=1}^{n} \left[\overline{e_\iota(t)}\left(\bigwedge_{\kappa=1}^{n} \rho_{\iota\kappa} f_\kappa\big(e_\kappa(t-\chi)\big)\right) + \left(\bigwedge_{\kappa=1}^{n} \rho_{\iota\kappa} \overline{f_\kappa\big(e_\kappa(t-\chi)\big)}\right) e_\iota(t)\right] \\
&+ \sum_{\iota=1}^{n} \left[\overline{e_\iota(t)}\left(\bigvee_{\kappa=1}^{n} \varrho_{\iota\kappa} f_\kappa\big(e_\kappa(t-\chi)\big)\right) + \left(\bigvee_{m=1}^{n} \varrho_{\iota\kappa} \overline{f_\kappa\big(e_\kappa(t-\chi)\big)}\right) e_\iota(t)\right] \\
&+ \sum_{\iota,\kappa=1}^{n} \big[\overline{e_\iota(t)} b_{\iota\kappa} f_\kappa\big(e_\kappa(t)\big) + \overline{f_\kappa\big(e_\kappa(t)\big)}\,\overline{b_{\iota\kappa}} e_\iota(t)\big] \\
&- 2\gamma \sum_{\iota=1}^{n} \big(\overline{e_\iota(t)}e_\iota(t)\big)^{1-\vartheta} - 2\sum_{\iota=1}^{n} \delta_\iota \big(\big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)^2\big)^{1-\vartheta} \\
&- 2n\lambda - 2\sum_{\iota=1}^{n} (a_\iota + \check{\mu}_\iota^*)\overline{e_\iota(t)}e_\iota(t) \\
\leqslant &\sum_{\iota=1}^{n} \left[-2a_\iota - 2\check{\mu}_\iota^* + \pi_1 + \pi_2 + n\pi_3 + \sum_{\kappa=1}^{n} \pi_3^{-1} b_{\kappa\iota}^2 o_\iota^2 + \varpi\zeta\right] \overline{e_\iota(t)}e_\iota(t) \\
&- 2\gamma \sum_{\iota=1}^{n} \big(\overline{e_\iota(t)}e_\iota(t)\big)^{1-\vartheta} - 2\sum_{\iota=1}^{n} \delta_\iota \big(\big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)^2\big)^{1-\vartheta} - 2n\lambda. \qquad (19)
\end{aligned}
$$

Combining (18) with (19) derives

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\varsigma}\mathcal{V}_2(t) &\leqslant -2\gamma \sum_{\iota=1}^{n} \big(\overline{e_\iota(t)}e_\iota(t)\big)^{1-\vartheta} - 2\sum_{\iota=1}^{n} \delta_\iota \big(\big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)^2\big)^{1-\vartheta} - 2n\lambda \\
&\leqslant -q_1 \left[\sum_{\iota=1}^{n} \big(\overline{e_\iota(t)}e_\iota(t)\big) + \sum_{\iota=1}^{n} \big(\check{\mu}_\iota(t) - \check{\mu}_\iota^*\big)^2\right]^{1-\vartheta} - q_2 \\
&= -q_1 \mathcal{V}_2^{1-\vartheta}(t) - q_2. \qquad (20)
\end{aligned}
$$

It can follow from Lemma 8 and (20) that $\lim_{t\to \mathsf{T}_2^*} \mathcal{V}_2(t) = 0$ and $\mathcal{V}_2(t) \equiv 0$ for all $t \geqslant \mathsf{T}_2^*$, where

$$
\mathsf{T}_2^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{q_1\vartheta}\left[\left(\mathcal{V}_2(t_0) + \left(\frac{q_1}{q_2}\right)^{1/(\vartheta-1)}\right)^{\vartheta} - \left(\frac{q_1}{q_2}\right)^{\vartheta/(\vartheta-1)}\right]\right\}^{1/\varsigma}, \qquad (21)
$$

which means $\lim_{t\to\mathsf{T}_2^*} e_\iota(t) = 0$ and $e_\iota(t) \equiv 0$ for all $t \geqslant \mathsf{T}_2^*$. According to Definition 3, FODQVFNNs (1) and (2) can realize finite-time synchronization within settling time T_2^*. □

Remark 5. In [2, 19], the dynamics of Q-VNNs were studied by utilizing decomposition method. The direct quaternion method is adopted in this paper, which cannot only reduce greatly the computational complexity, but keep system properties.

Remark 6. The asymptotic synchronization issues of FOQ-VNNs were investigated in [18, 21, 26], the finite-time synchronization is considered in this paper, which has faster convergence speed and stronger robustness. In [12, 13, 27], the finite-time synchronization of F-ONNs has been explored via adaptive control strategy in real, complex, and quaternion fields, respectively. However, the above researches do not take into account of quaternion or fuzzy factor. To our knowledge, the direct quaternion method-based finite-time synchronization of FODQVFNNs is studied by designing two effective quaternion-valued adaptive controllers for the first time.

If the adaptive control gain items in (10) and (17) are changed into feedback one, then quaternion-valued adaptive controller can degrade into quaternion-valued feedback controller as follows:

$$u_\iota(t) = \begin{cases} -\mu_\iota e_\iota(t) - \frac{\gamma e_\iota(t)}{(\overline{e_\iota(t)} e_\iota(t))^\vartheta} - \frac{\lambda e_\iota(t)}{\overline{e_\iota(t)} e_\iota(t)}, & |e_\iota(t)| \neq 0, \\ 0, & |e_\iota(t)| = 0. \end{cases}$$

Combining quaternion-valued feedback controller (21) with Theorems 1 and 2 can easily obtain the following corollary.

Corollary 1. *Let Assumption* 1 *holds, FODQVFNNs* (1) *and* (2) *can achieve finite-time synchronization under quaternion-valued feedback controller* (21) *if control gain* μ_ι *satisfies*

$$\mu_\iota \geqslant \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}n\pi_3 + \frac{1}{2}\sum_{\kappa=1}^{n} \pi_3^{-1} b_{\kappa\iota}^2 o_\iota^2 + \frac{1}{2}\varpi\zeta - a_\iota. \tag{22}$$

Furthermore, the settling time is estimated as

$$\mathsf{T}_3^* = t_0 + \left\{ \frac{\Gamma(1+\varsigma)}{2\vartheta\gamma} \left[\left(\mathcal{V}_3(t_0) + \left(\frac{\gamma}{n\lambda}\right)^{1/(\vartheta-1)} \right)^{\vartheta} - \left(\frac{\gamma}{n\lambda}\right)^{\vartheta/(\vartheta-1)} \right] \right\}^{1/\varsigma}.$$

Proof. Consider the Lyapunov function as follows: $\mathcal{V}_3(t) = \sum_{\iota=1}^{n} \overline{e_\iota(t)} e_\iota(t)$.

Analogously to the proof of Theorems 1 and 2, one has

$$\begin{aligned} {}_{t_0}^{c}D_t^{\varsigma} \mathcal{V}_3(t) \leqslant{}& \sum_{\iota=1}^{n} \left[-2a_\iota - 2\mu_\iota + \pi_1 + \pi_2 + n\pi_3 + \sum_{\kappa=1}^{n} \pi_3^{-1} b_{\kappa\iota}^2 o_\iota^2 + \varpi\zeta \right] \overline{e_\iota(t)} e_\iota(t) \\ &- 2\gamma \left(\sum_{\iota=1}^{n} \overline{e_\iota(t)} e_\iota(t) \right)^{1-\vartheta} - 2n\lambda. \end{aligned} \tag{23}$$

Combining (23) with condition (22) gives

$$ {}_{t_0}^{c}D_t^{\varsigma}\mathcal{V}_3(t) \leqslant -2\gamma\left(\sum_{\iota=1}^{n}\overline{e_\iota(t)}e_\iota(t)\right)^{1-\vartheta} - 2n\lambda = -2\gamma V_3^{1-\vartheta}(t) - 2n\lambda. \tag{24} $$

It follows from Lemma 8 and (24) that $\lim_{t\to\mathsf{T}_3^*}\mathcal{V}_3(t) = 0$ and $\mathcal{V}_3(t) \equiv 0$ for all $t \geqslant \mathsf{T}_3^*$, where

$$ \mathsf{T}_3^* = t_0 + \left\{\frac{\Gamma(1+\varsigma)}{2\vartheta\gamma}\left[\left(\mathcal{V}_3(t_0) + \left(\frac{\gamma}{n\lambda}\right)^{1/(\vartheta-1)}\right)^{\vartheta} - \left(\frac{\gamma}{n\lambda}\right)^{\vartheta/(\vartheta-1)}\right]\right\}^{1/\varsigma}, $$

which implies $\lim_{t\to\mathsf{T}_3^*} e_\iota(t) = 0$ and $e_\iota(t) \equiv 0$ for all $t \geqslant \mathsf{T}_3^*$. In the light of Definition 3, FODQVFNNs (1) and (2) can realize finite-time synchronization within settling time T_3^*. □

Remark 7. The criteria obtained in Theorems 1 and 2 as well as Corollary 1 are also true when $\varsigma = 1$, that is, our conclusions can be ulteriorly applied to integer-order systems.

4 Numerical example

Some numerical simulations are provided to verify the feasibility of our novel quaternion-valued adaptive control strategies and obtained FAS criteria in this section. In this section, we give a numerical example to demonstrate the effectiveness of the theoretical results.

Example 1. Consider FODQVFNN in the case of $\iota \in \{1, 2\}$ as below:

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\varsigma}\nu_\iota(t) = {} & -a_\iota\nu_\iota(t) + \sum_{\kappa=1}^{2} b_{\iota\kappa}f_\kappa\big(\nu_\kappa(t)\big) + \bigwedge_{\kappa=1}^{2}\rho_{\iota\kappa}f_\kappa\big(\nu_\kappa(t-\chi)\big) \\ & + \bigvee_{\kappa=1}^{2}\varrho_{\iota\kappa}f_\kappa\big(\nu_\kappa(t-\chi)\big) + I_\iota(t), \quad \iota = 1, 2, \end{aligned} \tag{25} $$

where $\varsigma = 0.9$, $(\nu_1(t), \nu_2(t))^{\mathrm{T}} \in \mathbb{Q}^2$,

$$ f_\kappa\big(\nu_\kappa(t)\big) = \tanh\big(\nu_\kappa^R(t)\big) + \tanh\big(\nu_\kappa^I(t)\big)\mathfrak{i} + \tanh\big(\nu_\kappa^J(t)\big)\mathfrak{j} + \tanh\big(\nu_\kappa^K(t)\big)\mathfrak{k} $$

with $\nu_\kappa^R(t), \nu_\kappa^I(t), \nu_\kappa^J(t), \nu_\kappa^K(t) \in \mathbb{R}$, $I_\iota(t) = 0$, $a_1 = a_2 = 1$,

$$ \begin{aligned} b_{11} &= 1 + 1\mathfrak{i} + 1\mathfrak{j} - 1\mathfrak{k}, & b_{12} &= -1 + 0.8\mathfrak{i} - 0.6\mathfrak{j} - 0.6\mathfrak{k}, \\ b_{21} &= -2 - 1.5\mathfrak{i} + 2\mathfrak{j} - 2\mathfrak{k}, & b_{22} &= -1 - \mathfrak{i} - \mathfrak{j} - \mathfrak{k}, \end{aligned} $$

$\chi = 2$, $\zeta = 2$, $\sigma_1 = \sigma_2 = \pi_1 = \pi_2 = \pi_3 = o_1 = o_2 = 1$, $\rho_{11} = 0.2$, $\rho_{12} = -0.1$, $\rho_{21} = -0.1$, $\rho_{22} = 0.2$, $\varrho_{11} = -0.1$, $\varrho_{12} = 0.2$, $\varrho_{21} = 0.1$, $\varrho_{22} = -0.2$, the initial value of FODQVFNN (25) is chosen as

$$ \nu(0) = (0.5 + 0.4\mathfrak{i} + 0.8\mathfrak{j} + 0.6\mathfrak{k},\ -0.5 + 0.6\mathfrak{i} - 0.1\mathfrak{j} + 0.1\mathfrak{k})^{\mathrm{T}}. $$

The controlled FODQVFNN is depicted as

$$\begin{aligned} {}_{t_0}^{c}D_t^{\varsigma}\omega_\iota(t) = &-a_\iota\omega_\iota(t) + \sum_{\kappa=1}^{2} b_{\iota\kappa} f_\kappa\big(\omega_\kappa(t)\big) + \bigwedge_{\kappa=1}^{2} \rho_{\iota\kappa} f_\kappa\big(\omega_\kappa(t-\chi)\big) \\ &+ \bigvee_{\kappa=1}^{2} \varrho_{\iota\kappa} f_\kappa\big(\omega_\kappa(t-\chi)\big) + I_\iota(t) + u_\iota(t), \quad \iota = 1,2, \end{aligned} \tag{26}$$

where $\varsigma = 0.9$,

$$f_\kappa\big(\omega_\kappa(t)\big) = \tanh\big(\omega_\kappa^R(t)\big) + \tanh\big(\omega_\kappa^I(t)\big)\mathfrak{i} + \tanh\big(\omega_\kappa^J(t)\big)\mathfrak{j} + \tanh\big(\omega_\kappa^K(t)\big)\mathfrak{k}$$

with $\omega_\kappa^R(t), \omega_\kappa^I(t), \omega_\kappa^J(t), \omega_\kappa^K(t) \in \mathbb{R}$, $I_\iota(t) = 0$, the initial value of FOFQDNN (26) is chosen as

$$\omega(0) = (0.6 - 0.1\mathfrak{i} - 0.3\mathfrak{j} + 0.5\mathfrak{k},\ -0.6 - 0.3\mathfrak{i} + 0.5\mathfrak{j} + 0.4\mathfrak{k})^{\mathrm{T}}.$$

The other parameters of FODQVFNN (26) are the same as ones of FODQVFNN (25).

Remark 8. When $u_\iota(t) = 0$ in FODQVFNN (26), the time evolution curves of $e_\iota(t)$ ($\iota = 1,2$) are exhibited in Fig. 1, which displays that FODQVFNNs (25) and (26) cannot realize FAS in defect of control. When adaptive controller $u_\iota(t)$ in FODQVFNN (26) is designed as (10), set $\gamma = 0.8$, $\lambda = 0.2$, $\vartheta = 2.1$, $\eta_1 = \eta_2 = 0.4$, $\hat{\mu}_1(0) = 0.1$, $\hat{\mu}_2(0) = 0.2$. By simple calculation, one has $\hat{\mu}_1^* = 10.825$, $\hat{\mu}_2^* = 4.88$. On the basis of Theorem 1, FODQVFNNs (25) and (26) can achieve FAS, which is depicted in Fig. 2. The state trajectories of $\hat{\mu}_1(t)$ and $\hat{\mu}_2(t)$ are displayed in Fig. 3. Obviously, it is necessary to design effective control strategy in order to achieve the goal of synchronization. By the comparison between Figs. 1 and 2, it is not difficult to see that the theoretical result of Theorem 1 is feasible, and the new adaptive controller (10) is valid.

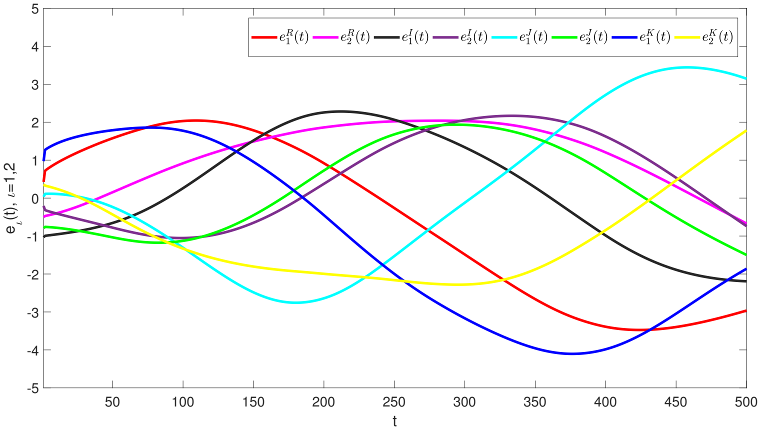

Figure 1. The error curves between FODQVFNNs (25) and (26) without controller.

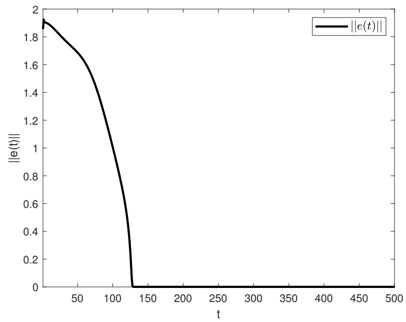

Figure 2. The time response curve of error norm $\|e(t)\|$ under controller (10).

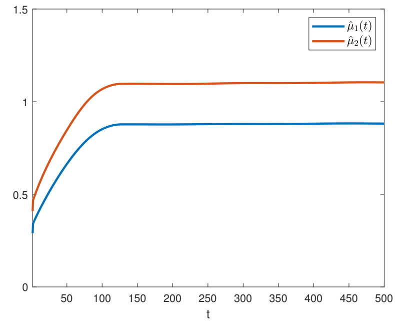

Figure 3. Time evolutions of adaptive gains $\hat{\mu}_1(t)$ and $\hat{\mu}_2(t)$.

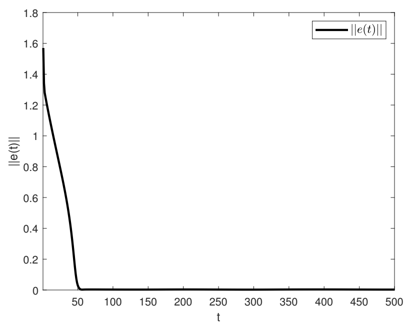

Figure 4. The time response curve of error norm $\|e(t)\|$ under controller (17).

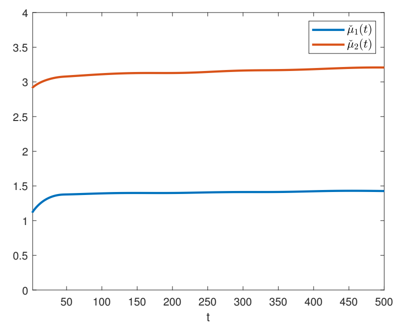

Figure 5. Time evolutions of adaptive gains $\check{\mu}_1(t)$ and $\check{\mu}_2(t)$.

Remark 9. When the adaptive controller $u_\iota(t)$ in FODQVFNN (26) is designed as (17), let $\gamma = 0.6$, $\lambda = 0.3$, $\vartheta = 2.1$, $\delta_1 = \delta_2 = 0.5$, $\check{\mu}_1(0) = 0.545$, $\check{\mu}_2(0) = 2.36$. After simple calculation, we get $\check{\mu}_1^* = 8.545$, $\check{\mu}_2^* = 6.36$. On the basis of Theorem 2, FODQVFNNs (25) and (26) can realize FAS, which is depicted in Fig. 4. The time evolutions of $\check{\mu}_1(t)$ and $\check{\mu}_2(t)$ are displayed in Fig. 5. From Figs. 4, 5 it is evident that the FAS criterion derived in Theorem 2 is correct, and the novel adaptive controller (17) is effective.

5 Conclusion

In this paper, the FAS issue of FODQVFNNs has been addressed via direct quaternion method. At first, a valuable fractional differential inequality was developed to present a novel thinking for the study of FAS. Moreover, some sufficient criteria have been yielded to guarantee FAS of FODQVFNNs by designing two different quaternion-valued adaptive controllers. Finally, the numerical simulations were given to illustrate the validity of derived FAS criteria. As we know, the parameter uncertainties, stochastic effects, external disturbance, and other factors are inevitable in nature, therefore the dynamic analysis of FODQVFNNs with above one or more factors may be a part of our future research.

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