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Dynamic analysis and optimal control of a novel fractional-order 2I2SR rumor spreading model*

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Abstract. In this paper, a novel fractional-order 2I2SR rumor spreading model is investigated. Firstly, the boundedness and uniqueness of solutions are proved. Then the next-generation matrix method is used to calculate the threshold. Furthermore, the stability of rumor-free/spreading equilibrium is discussed based on fractional-order Routh–Hurwitz stability criterion, Lyapunov function method, and invariance principle. Next, the necessary conditions for fractional optimal control are obtained. Finally, some numerical simulations are given to verify the results.

Keywords: fractional order, rumor spreading, stability, optimal control.

1 Introduction

Rumor refers to the remarks that have no corresponding factual basis but are fabricated and promoted to spread by certain ways. With the development of science and technology, the rapid spread of rumors causes huge economic losses, disturb the normal order, and undermine social stability [6, 8]. Therefore, it is of great practical significance to study the dynamics of rumor propagation in social networks.

The mechanism of rumor transmission is similar to the process of epidemic transmission. In the 1960s, the classic DK model was proposed by Daley and Kendall, which divided the total population into three categories: ignorant, spreader, and removed, then numerical method is used to study the spread process of rumors, which is similar to those in infectious diseases [2, 3]. Based on the above research, the DK model was improved, and then MK model was obtained in 1973 [14]. With the efforts of many scholars, more and more modified rumor propagation models have been put forward in recent

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years [13, 22, 28, 31]. In 2019, Wang et al. considered the cross-propagation mechanism and established SIR rumor propagation model in a multilanguage environment. Then its stability also was deeply investigated [22]. In 2020, considering the network topology, Li et al. analyzed the dynamic behaviors of rumor propagation model with educational mechanism and carried out optimal control in the multilanguage environment [13]. In 2020, a time-delay SIR rumor propagation model considering the network topology and forcing silence function was proposed and the stability of the rumor propagation model was analyzed in [31]. Yu et al. established 2S2IR model based on multilanguage environment and studied Hopf bifurcation with time delay and the important parameter of the model, respectively [28].

Fractional calculus, called generalized calculus or arbitrary calculus, is a generalization of integral calculus and has short-term memory effect and genetic effect. With the continuous development and improvement of the fractional calculus theory, the fractional differential equation has been widely used in many fields [25, 29, 30]. Huang et al. considered fractional neural networks with double delays. Furthermore, the stability and bifurcation of the system were studied [9]. The dynamics of fractional SIR epidemic model with time delay and saturation function were studied by Wang et al. [23]. In 2019, Wang et al. proposed a fractional ecoepidemiological model with time delay. Then the Hopf bifurcation of the system was studied, and the control strategies were given [24]. In recent years, the problem of fractional optimal control has been studied extensively, and the conditions of fractional optimal control have been obtained [11, 15, 21]. Memory effect on information transmission process is studied in [17, 20, 26], which shows that multiple redundant contacts of the same rumor will change people's initial thoughts of it, and the cumulative feature will affect the behavior of individuals in social networks. Due to the memory effect of fractional calculus, rumor propagation process can be analyzed accurately by studying the rumor propagation process with fractional calculus. In 2019, Singh considered a SIR rumor propagation model with Atangana–Baleanu derivative, and the effect of fractional order on the population of each warehouse was studied [18]. A fractional-order SIR model, which is similar to the epidemic model, was established to examine the adoption and abandonment of online social networks by social network users. Then the properties of the solutions of the system were studied in [7]. Ren et al. established a fractional stochastic rumor propagation model in mobile social networks, and the stability conditions of the system were obtained [16]. Inspired by [7, 28], we consider a fractional-order 2I2SR rumor spreading model and study the properties of the solutions. The optimal control conditions of the system are given at last. The main contributions of this study are as follows:

- Compared with [28], a fractional 2I2SR rumor propagation model is generalized, which considers the unit dimension of the equation and memory effect in rumor propagation process.
- The properties and dynamics of the solutions to the given rumor propagation model are studied by means of fractional differential equation theory.
- The optimal control of the given fractional rumor propagation model is obtained by using the fractional optimal control theory.

The rest of this article is arranged as follows. In Section 2, some preparations related to fractional equations are introduced. In Section 3, a fraction-order 2S2IR rumor spreading model is proposed. In Section 4, the properties of the solutions are disscussed. Furthermore, the fractional-order optimal control strategies are presented. In Section 5, some numerical simulations are illustrated to verify the theoretical results. In Section 6, we have a brief summary for the whole paper.

2 Preliminaries

In this section, some preparations related to fractional differential equations are given, which will be used in the following discussion.

Definition 1. (See [5].) Let f be a function defined on $[a,b]$, and let $\kappa > 0$. The Riemann–Liouville fractional integral of order κ for the function f is defined by

$$ {}_aD_t^{-\kappa} f(t) = \frac{1}{\Gamma(\kappa)} \int_a^t (t-\tau)^{\kappa-1} f(\tau)\,\mathrm{d}\tau, \quad t \in [a,b], $$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2. (See [5].) The Caputo fractional derivative of order κ of a function $f(t)$ is defined as

$$ {}_{t_0}^{C}D_t^{\kappa} f(t) = \frac{1}{\Gamma(n-\kappa)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\kappa+1-n}}\,\mathrm{d}\tau, $$

where n is the positive integer, and $n-1 < \kappa < n$. $\Gamma(\cdot)$ is the gamma function, $\Gamma(s) = \int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$. When $0 < \kappa < 1$, one has

$$ {}_{t_0}^{C}D_t^{\kappa} f(t) = \frac{1}{\Gamma(1-\kappa)} \int_{t_0}^t \frac{f'(\tau)}{(t-\tau)^{\kappa}}\,\mathrm{d}\tau. $$

Definition 3. (See [15].) Let $f \in C[a,b]$, where $C[a,b]$ represents the space of absolutely continuous functions on $[a,b]$, the left and right Caputo fractional derivatives (CFDs) are as follows:

(i) left CFD

$$ {}_{a}^{C}D_t^{\kappa} f(t) = \frac{1}{\Gamma(n-\kappa)} \int_{a}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\kappa+1-n}}\,\mathrm{d}\tau; $$

(ii) right CFD

$$ {}_{t}^{C}D_b^{\kappa} f(t) = \frac{(-1)^n}{\Gamma(n-\kappa)} \int_{t}^b \frac{f^{(n)}(\tau)}{(t-\tau)^{\kappa+1-n}}\,\mathrm{d}\tau. $$

Lemma 1. (*See* [7].) *If f is continuous and $\kappa \geqslant 0$, then*

$$ {}_{a}^{C}D_t^{\kappa}\, {}_{a}D_t^{-\kappa} f(t) = f(t). $$

Lemma 2. (*See* [27].) *Let $0 < \alpha < 1$ and $t \geqslant 0$, then function $E_\alpha(\mu(t-t_0)^\alpha)$ is nonnegative. Furthermore, $0 \leqslant E_\alpha(\mu(t-t_0)^\alpha) \leqslant 1$ for $t \geqslant t_0$ when $\mu \leqslant 0$.*

Lemma 3. (*See* [12].) *Assume that $w(t)$ is continuous on $[t_0, +\infty)$ and satisfies*

$$ {}_{t_0}^{C}D_t^{\kappa} w(t) \leqslant -\lambda w(t) + \mu, \qquad w(t_0) = w_{t_0}, $$

where $0 < \kappa < 1$, $(\lambda, \mu) \in \mathbb{R}^2$, and $\lambda \neq 0$. Then

$$ w(t) \leqslant \left(w_{t_0} - \frac{\mu}{\lambda}\right) E_\kappa\left[-\lambda(t-t_0)^\kappa\right] + \frac{\mu}{\lambda}. $$

Lemma 4. (*See* [12].) *Consider the system*

$$ {}_{t_0}^{C}D_t^{\kappa} x(t) = f(t, x), \quad t > t_0, \tag{1} $$

with initial condition x_{t_0}, where $0 < \kappa \leqslant 1$, $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$, $\Omega \in \mathbb{R}^n$. Then there exists a unique solution of system (1) *on $[t_0, \infty) \times \Omega$ if $f(t, x)$ satisfies the locally Lipschitz condition with respect to x.*

Lemma 5. (*See* [4].) *Considering the following n-dimensional linear fractional differential system with multiple time delays*:

$$ \begin{aligned} {}_{0}^{C}D_t^{\kappa_1} x_1(t) &= b_{11}x_1(t-\tau_{11}) + b_{12}x_2(t-\tau_{12}) + \cdots + b_{1n}x_n(t-\tau_{1n}), \\ {}_{0}^{C}D_t^{\kappa_2} x_2(t) &= b_{21}x_1(t-\tau_{21}) + b_{22}x_2(t-\tau_{22}) + \cdots + b_{2n}x_n(t-\tau_{2n}), \\ &\cdots \\ {}_{0}^{C}D_t^{\kappa_n} x_n(t) &= b_{n1}x_1(t-\tau_{n1}) + b_{n2}x_2(t-\tau_{n2}) + \cdots + b_{nn}x_n(t-\tau_{nn}), \end{aligned} \tag{2} $$

where $0 < \kappa_i < 1$, and κ_i is real. The initial values $x_i = \phi_i(t)$ are given for $-\max_{i,j} \tau_{ij} = -\tau_{\max} \leqslant t \leqslant 0$ and $i = 1, 2, \ldots, n$. In this system, state variables $x_i(t)$, $x_i(t-\tau_{ij}) \in \mathbb{R}$, time-delay matrix $T = (\tau_{ij})_{n\times n} \in (\mathbb{R}^+)_{n\times n}$, coefficient matrix $B = (b_{ij})_{n\times n}$, and initial values $\phi_i(t) \in C^0[-\tau_{\max}, 0]$. Then the characteristic matrix of system (2) *can be labeled as*

$$ \Delta(s) = \begin{pmatrix} s^{\kappa_1} - b_{11}\mathrm{e}^{-s\tau_{11}} & -b_{12}\mathrm{e}^{-s\tau_{12}} & \cdots & -b_{1n}\mathrm{e}^{-s\tau_{1n}} \\ -b_{21}\mathrm{e}^{-s\tau_{21}} & s^{\kappa_2} - b_{22}\mathrm{e}^{-s\tau_{22}} & \cdots & -b_{2n}\mathrm{e}^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1}\mathrm{e}^{-s\tau_{n1}} & -b_{n2}\mathrm{e}^{-s\tau_{n2}} & \cdots & s^{\kappa_n} - b_{nn}\mathrm{e}^{-s\tau_{nn}} \end{pmatrix}. $$

Lemma 6. (*See* [4].) *If all the roots of* $\det \Delta(s) = 0$ *satisfy* $|\arg s| > \pi/2$, *then the zero solution of system* (2) *is Lyapunov globally asymptotically stable.*

Remark 1. Assume that $\kappa_1 = \kappa_2 = \cdots = \kappa_n = \kappa \in (0,1)$, and $\det \Delta(\lambda) = 0$ is the characteristic equation of the following equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t,x).$$

Then we have the equivalent conditions:

$$|\arg s| > \frac{\pi}{2} \quad \Longleftrightarrow \quad |\arg \lambda| > \frac{\kappa\pi}{2}.$$

Lemma 7. (*See* [10].) *Assume that $u(t) \in \mathbb{R}^+$ is continuous and derivable. Then for any time instant $t \geqslant t_0$ and for all $\kappa \in (0,1)$,*

$${}_{t_0}^{C}D_t^{\kappa}\left[u(t) - u^* - u^* \ln \frac{u(t)}{u^*}\right] \leqslant \left(1 - \frac{u^*}{u(t)}\right){}_{t_0}^{C}D_t^{\kappa}u(t), \quad u^* \in \mathbb{R}^+.$$

Lemma 8. (*See* [10].) *Consider the following autonomous system*:

$$D^{\kappa}y(t) = g(y). \tag{3}$$

Suppose B is a bounded closed set, every solution of system (3) *starts from a point in B and remains in B for all time. There exists $V(y) : B \to \mathbb{R}$ with continuous first partial derivatives satisfying the following condition:*

$$D^{\kappa}V|_{(3)} \leqslant 0.$$

Let $F = \{y | D^{\kappa}V|_{(3)} = 0\}$, and let M be the largest invariant set of F. Then every solution $y(t)$ originating in B tends to M as $t \to +\infty$. Particularly, if $M = 0$, then $y \to 0$, $t \to +\infty$.

3 Model formulation

Many rumor models have been improved to better understand the rumor spreading process. A 2I2SR rumor propagation model in multilingual environment is introduced in [5]. The information may be in Chinese, English, or even other languages since these users come from different countries or regions. Assume that one of them is the official language of this social network, and others are unofficial languages, and all users understand official language. In this model, we consider five types of users: Ignorants-1 ($I_1(t)$), Ignorants-2 ($I_2(t)$), Spreaders-1 ($S_1(t)$), Spreaders-2 ($S_2(t)$) and Removers ($R(t)$). $I_1(t)$ represents users who can speak both official and other languages, but they prefer to publish information in unofficial languages, and they do not know the rumor information. $I_2(t)$ stands for users who only can use official language to exchange information, and they also do not know the rumor information. $S_1(t)$ refers to individuals who have received the rumor and can speak official language and other languages, but they prefer to spread rumor in unofficial languages. $S_2(t)$ describes the ones who know the rumor and propagate it in

Table 1. Descriptions of parameters for the model (4).

Symbols	Description	Units
Π_i	The immigration rates of $I_i(t)$, $i = 1, 2$	[Number] $\times$ [Unit of time]$^{-1}$
α_i	The probability of turning $I_i(t)$ into $S_i(t)$	[Number] $\times$ Unit of time]$^{-1}$
β_i	The probability of turning $S_i(t)$ into $R(t)$	[Unit of time]$^{-1}$
u	The probability of turning $I_1(t)$ into $S_2(t)$	[Number] $\times$ Unit of time]$^{-1}$
d	The removal rate for each compartment	[Unit of time]$^{-1}$

official language. $R(t)$ represents the rumor recovery individuals who know the rumor and no longer spread it. The population movement between the five warehouses is modeled as follows:

$$\begin{aligned}
\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} &= \Pi_1 - \alpha_1 I_1(t)S_1(t) - uI_1(t)S_2(t) - dI_1(t),\\
\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} &= \Pi_2 - \alpha_2 I_2(t)S_2(t) - dI_2(t),\\
\frac{\mathrm{d}S_1(t)}{\mathrm{d}t} &= \alpha_1 I_1(t)S_1(t) + uI_1(t)S_2(t) - dS_1(t) - \beta_1 S_1(t),\\
\frac{\mathrm{d}S_2(t)}{\mathrm{d}t} &= \alpha_2 I_2(t)S_2(t) - dS_2(t) - \beta_2 S_2(t),\\
\frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \beta_1 S_1(t) + \beta_2 S_2(t) - dR(t)
\end{aligned} \tag{4}$$

with the initial conditions

$$I_1(0) \geqslant 0, \qquad I_2(0) \geqslant 0, \qquad S_1(0) \geqslant 0, \qquad S_2(0) \geqslant 0, \qquad R(0) \geqslant 0.$$

People's acceptance of information and whether they choose to spread information are affected by individual's subjective will. For model (4), it was established with integer-order differential equations. A detailed description of the parameters can be seen in Table 1. However, the state of each moment does not depend on the historical status of the system. The memory effect of rumor transmission was not considered. It can be seen from [7, 9, 11, 15, 16, 18, 23, 24] that fractional calculus can better describe the dynamic processes with memory effect than integer calculus.

Fractional calculus is introduced to describe the memory effect. Through the application of fractional differential equations in dynamical systems in recent years [9, 11, 15, 23, 24], we can generalize system (4) into the following form in the sense of Caputo derivative:

$$\begin{aligned}
{}_0^C D_t^\kappa I_1(t) &= \Pi_1 - \alpha_1 I_1(t)S_1(t) - uI_1(t)S_2(t) - dI_1(t),\\
{}_0^C D_t^\kappa I_2(t) &= \Pi_2 - \alpha_2 I_2(t)S_2(t) - dI_2(t),\\
{}_0^C D_t^\kappa S_1(t) &= \alpha_1 I_1(t)S_1(t) + uI_1(t)S_2(t) - dS_1(t) - \beta_1 S_1(t),\\
{}_0^C D_t^\kappa S_2(t) &= \alpha_2 I_2(t)S_2(t) - dS_2(t) - \beta_2 S_2(t),\\
{}_0^C D_t^\kappa R(t) &= \beta_1 S_1(t) + \beta_2 S_2(t) - dR(t).
\end{aligned} \tag{5}$$

It is reasonable to generalize system (4) to system (5) because the memory effect of rumor propagation is considered. However, this approach does not take the time dimension into better account. The units on the left-hand side of system (5) are [Number] $\times$ [Unit of time]$^{-\kappa}$, while the units on the right-hand side of system (5) are [Number] $\times$ [Unit of time]$^{-1}$. In recent years, some scholars have considered the unity of fractional differential equations, which can be observed in [1, 7].

Inspired by the unit problem of considering parameters in [7], we generalized the 2I2SR rumor propagation model, which was studied in [29], into fractional ($0 < \kappa < 1$) differential equations. Firstly, system (4) is equivalent to the following integral equations:

$$
\begin{aligned}
I_1(t) &= I_1(0) + \int_0^t \big[\varPi_1 - \alpha_1 I_1(s)S_1(s) - uI_1(s)S_2(s) - dI_1(s)\big]\,\mathrm{d}s,\\
I_2(t) &= I_2(0) + \int_0^t \big[\varPi_2 - \alpha_2 I_2(s)S_2(s) - dI_2(s)\big]\,\mathrm{d}s,\\
S_1(t) &= S_1(0) + \int_0^t \big[\alpha_1 I_1(s)S_1(s) + uI_1(s)S_2(s) - dS_1(s) - \beta_1 S_1(s)\big]\,\mathrm{d}s,\\
S_2(t) &= S_2(0) + \int_0^t \big[\alpha_2 I_2(s)S_2(s) - dS_2(s) - \beta_2 S_2(s)\big]\,\mathrm{d}s,\\
R(t) &= R(0) + \int_0^t \big[\beta_1 S_1(s) + \beta_2 S_2(s) - dR(s)\big]\,\mathrm{d}s
\end{aligned}
\tag{6}
$$

with the initial conditions

$$
I_1(0) \geqslant 0, \qquad I_2(0) \geqslant 0, \qquad S_1(0) \geqslant 0, \qquad S_2(0) \geqslant 0, \qquad R(0) \geqslant 0.
$$

In order to consider the effect of memory effect on rumor spreading process, we rewrite system (6) into the following form with memory effect:

$$
\begin{aligned}
I_1(t) &= I_1(0) + \int_0^t k(t,s)\big[\varPi_1 - \alpha_1 I_1(s)S_1(s) - uI_1(s)S_2(s) - dI_1(s)\big]\,\mathrm{d}s,\\
I_2(t) &= I_2(0) + \int_0^t k(t,s)\big[\varPi_2 - \alpha_2 I_2(s)S_2(s) - dI_2(s)\big]\,\mathrm{d}s,\\
S_1(t) &= S_1(0) + \int_0^t k(t,s)\big[\alpha_1 I_1(s)S_1(s) + uI_1(s)S_2(s) - dS_1(s) - \beta_1 S_1(s)\big]\,\mathrm{d}s,
\end{aligned}
$$

$$S_2(t) = S_2(0) + \int_0^t k(t,s)\big[\alpha_2 I_2(s)S_2(s) - dS_2(s) - \beta_2 S_2(s)\big]\,\mathrm{d}s,$$

$$R(t) = R(0) + \int_0^t k(t,s)\big[\beta_1 S_1(s) + \beta_2 S_2(s) - dR(s)\big]\,\mathrm{d}s,$$

where $k(t,s)$ is the kernel function, and it has the following form:

$$k(t,s) = \frac{1}{\Gamma(\kappa)}(t-s)^{\kappa-1}, \quad \kappa \in (0,1).$$

Remark 2. Compared with integer calculus, the memory of fractional calculus is mainly reflected in the power law property of kernel function.

Considering the unity of the units on both sides of this equations and applying Definition 1, we obtain

$$\begin{aligned}
I_1(t) - I_1(0) &= {}_0D_t^{-\kappa}\big[\Pi_1^\kappa - \alpha_1^\kappa I_1(t)S_1(t) - u^\kappa I_1(t)S_2(t) - d^\kappa I_1(t)\big],\\
I_2(t) - I_2(0) &= {}_0D_t^{-\kappa}\big[\Pi_2^\kappa - \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa I_2(t)\big],\\
S_1(t) - S_1(0) &= {}_0D_t^{-\kappa}\big[\alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) - d^\kappa S_1(t) - \beta_1^\kappa S_1(t)\big],\\
S_2(t) - S_2(0) &= {}_0D_t^{-\kappa}\big[\alpha_2^\kappa I_2(t)S_2(t) - d^\kappa S_2(t) - \beta_2^\kappa S_2(t)\big],\\
R(t) - R(0) &= {}_0D_t^{-\kappa}\big[\beta_1^\kappa S_1(t) + \beta_2^\kappa S_2(t) - d^\kappa R(t)\big].
\end{aligned} \tag{7}$$

Applying Lemma 1 and the Caputo derivative of order κ to both sides of Eq. (7), the following fractional-order 2S2IR rumor propagation model can be obtained:

$$\begin{aligned}
{}_0^CD_t^\kappa I_1(t) &= \Pi_1^\kappa - \alpha_1^\kappa I_1(t)S_1(t) - u^\kappa I_1(t)S_2(t) - d^\kappa I_1(t),\\
{}_0^CD_t^\kappa I_2(t) &= \Pi_2^\kappa - \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa I_2(t),\\
{}_0^CD_t^\kappa S_1(t) &= \alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) - d^\kappa S_1(t) - \beta_1^\kappa S_1(t),\\
{}_0^CD_t^\kappa S_2(t) &= \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa S_2(t) - \beta_2^\kappa S_2(t),\\
{}_0^CD_t^\kappa R(t) &= \beta_1^\kappa S_1(t) + \beta_2^\kappa S_2(t) - d^\kappa R(t)
\end{aligned} \tag{8}$$

with the initial conditions

$$I_1(0) \geqslant 0, \qquad I_2(0) \geqslant 0, \qquad S_1(0) \geqslant 0, \qquad S_2(0) \geqslant 0, \qquad R(0) \geqslant 0.$$

After a simple analysis of model (8), we can easily find that both sides of the equations have the same units. The specific parameters of this model are shown in Table 2. The following discussion and analysis are based on model (8).

Remark 3. When $\kappa \to 1^-$, system (8) is transformed into system (4) without memory effect. Furthermore, system (4) can be viewed as a special case of system (8), and it can be seen that the order κ is an intuitive embodiment of the memory effect of fractional-order system (8).

Table 2. Descriptions of parameters for the model (8).

Symbols	Description	Units
Π_i	The immigration rate of $I_i(t)$, $i = 1, 2$	[Number]$^{1/\kappa}$ $\times$ [Unit of time]$^{-1}$
α_i	The probability of turning $I_i(t)$ into $S_i(t)$	[Number]$^{-1/\kappa}$ $\times$ [Unit of time]$^{-1}$
β_i	The probability of turning $S_i(t)$ into $R(t)$	[Unit of time]$^{-1}$
u	The probability of turning $I_1(t)$ into $S_2(t)$	[Number]$^{-1/\kappa}$ $\times$ [Unit of time]$^{-1}$
d	The removal rate for each compartment	[Unit of time]$^{-1}$

4 Main results

In this section, we mainly prove the boundedness and uniqueness of the solutions of system (8). Next, the sufficient conditions for the stability of equilibriums are obtained. Furthermore, the necessary conditions for fractional optimal control are obtained.

4.1 Properties of the solutions

For convenience, let $\mathbb{D}^+ = \{(I_1(t), I_2(t), S_1(t), S_2(t), R(t))\colon I_1(t) \in \mathbb{R}^+,\ I_2(t) \in \mathbb{R}^+,\ S_1(t) \in \mathbb{R}^+,\ S_2(t) \in \mathbb{R}^+,\ R(t) \in \mathbb{R}^+\}$.

Theorem 1. *All the solutions of system* (8)*, which start in* $\mathbb{D}^+$*, are bounded.*

Proof. Define the function $W(t) = I_1(t) + I_2(t) + S_1(t) + S_2(t) + R(t)$. Then we can obtain

$$
\begin{aligned}
&{}_0^C D_t^\kappa W(t) + d^\kappa W(t) \\
&\quad = \Pi_1^\kappa - \alpha_1^\kappa I_1(t)S_1(t) - u^\kappa I_1(t)S_2(t) - d^\kappa I_1(t) \\
&\qquad + \Pi_2^\kappa - \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa I_2(t) + \alpha_1^\kappa I_1(t)S_1(t) \\
&\qquad + u^\kappa I_1(t)S_2(t) - d^\kappa S_1(t) - \beta_1^\kappa S_1(t) + \alpha_2^\kappa I_2(t)S_2(t) \\
&\qquad - d^\kappa S_2(t) - \beta_2^\kappa S_2(t) + \beta_1^\kappa S_1(t) + \beta_2^\kappa S_2(t) - d^\kappa R(t) \\
&\qquad + d^\kappa \big(I_1(t) + I_2(t) + S_1(t) + S_2(t) + R(t)\big) \\
&\quad = \Pi_1^\kappa + \Pi_2^\kappa.
\end{aligned}
$$

By Lemmas 2, 3 we can get

$$
W(t) \leqslant \left(W_{t_0} - \frac{\Pi_1^\kappa + \Pi_2^\kappa}{d^\kappa}\right) E_\kappa\big[-d^\kappa (t - t_0)\big] + \frac{\Pi_1^\kappa + \Pi_2^\kappa}{d^\kappa} \to \frac{\Pi_1^\kappa + \Pi_2^\kappa}{d^\kappa}
$$

as $t \to +\infty$. Therefore, all the solutions of system (8), which start in $\mathbb{D}^+$, are confined to the region

$$
\begin{aligned}
\Theta = \Bigg\{ &\big(I_1(t), I_2(t), S_1(t), S_2(t), R(t)\big) \in \mathbb{D}_+\colon \\
&\sum_{i=1}^{2} I_i(t) + \sum_{i=1}^{2} S_i(t) + R(t) \leqslant \frac{\Pi_1^\kappa + \Pi_2^\kappa}{d^\kappa} \Bigg\}.
\end{aligned}
$$

This completes the proof of theorem. $\square$

Theorem 2. *System* (8) *has a unique solution* $Y=(I_1(t),I_2(t),S_1(t),S_2(t),R(t))\in Z$ *for every* $(I_{10},I_{20},S_{10},S_{20},R_0)\in Z$*, where* $Z=\{(I_1(t),I_2(t),S_1(t),S_2(t),R(t))\in\mathbb{R}^5\colon \max\{|I_1(t)|,|I_2(t)|,|S_1(t)|,|S_2(t)|,|R(t)|\}\leqslant M\}$.

Proof. Consider a mapping $H(Y)=(H_1(Y),H_2(Y),H_3(Y),H_4(Y),H_5(Y))$, where

$$\begin{aligned}
H_1(Y)&=\Pi_1^\kappa-\alpha_1^\kappa I_1(t)S_1(t)-u^\kappa I_1(t)S_2(t)-d^\kappa I_1(t),\\
H_2(Y)&=\Pi_2^\kappa-\alpha_2^\kappa I_2(t)S_2(t)-d^\kappa I_2(t),\\
H_3(Y)&=\alpha_1^\kappa I_1(t)S_1(t)+u^\kappa I_1(t)S_2(t)-d^\kappa S_1(t)-\beta_1^\kappa S_1(t),\\
H_4(Y)&=\alpha_2^\kappa I_2(t)S_2(t)-d^\kappa S_2(t)-\beta_2^\kappa S_2(t),\\
H_5(Y)&=\beta_1^\kappa S_1(t)+\beta_2^\kappa S_2(t)-d^\kappa R(t).
\end{aligned}$$

For any $Y,\widehat{Y}\in Z$,

$$\begin{aligned}
&\big\|H(Y)-H(\widehat{Y})\big\|\\
&\quad=\big\|H_1(Y)-H_1(\widehat{Y})\big\|+\big\|H_2(Y)-H_2(\widehat{Y})\big\|+\big\|H_3(Y)-H_3(\widehat{Y})\big\|\\
&\qquad+\big\|H_4(Y)-H_4(\widehat{Y})\big\|+\big\|H_5(Y)-H_5(\widehat{Y})\big\|\\
&\quad=\big|\Pi_1^\kappa-\alpha_1^\kappa I_1(t)S_1(t)-u^\kappa I_1(t)S_2(t)-d^\kappa I_1(t)\\
&\qquad-(\Pi_1^\kappa-\alpha_1^\kappa\widehat{I}_1(t)\widehat{S}_1(t)-u^\kappa\widehat{I}_1(t)\widehat{S}_2(t)-d^\kappa\widehat{I}_1(t))\big|\\
&\qquad+\big|\Pi_2^\kappa-\alpha_2^\kappa I_2(t)S_2(t)-d^\kappa I_2(t)-\big(\Pi_2^\kappa-\alpha_2^\kappa\widehat{I}_2(t)\widehat{S}_2(t)-d^\kappa\widehat{I}_2(t)\big)\big|\\
&\qquad+\big|\alpha_1^\kappa I_1(t)S_1(t)+u^\kappa I_1(t)S_2(t)-d^\kappa S_1(t)-\beta_1^\kappa S_1(t)\\
&\qquad-\big(\alpha_1^\kappa\widehat{I}_1(t)\widehat{S}_1(t)+u^\kappa\widehat{I}_1(t)\widehat{S}_2(t)-d^\kappa\widehat{S}_1(t)-\beta_1^\kappa\widehat{S}_1(t)\big)\big|\\
&\qquad+\big|\alpha_2^\kappa I_2(t)S_2(t)-d^\kappa S_2(t)-\beta_2^\kappa S_2(t)\\
&\qquad-\big(\alpha_2^\kappa\widehat{I}_2(t)\widehat{S}_2(t)-d^\kappa\widehat{S}_2(t)-\beta_2^\kappa\widehat{S}_2(t)\big)\big|\\
&\qquad+\big|\beta_1^\kappa S_1(t)+\beta_2^\kappa S_2(t)-d^\kappa R(t)-\big(\beta_1^\kappa\widehat{S}_1(t)+\beta_2^\kappa\widehat{S}_2(t)-d^\kappa\widehat{R}(t)\big)\big|\\
&\quad\leqslant 2\big|\alpha_1^\kappa I_1(t)S_1(t)-\alpha_1^\kappa I_1(t)\widehat{S}_1(t)+\alpha_1^\kappa I_1(t)\widehat{S}_1(t)-\alpha_1^\kappa\widehat{I}_1(t)\widehat{S}_1(t)\big|\\
&\qquad+2\big|u^\kappa I_1(t)S_2(t)-u^\kappa I_1(t)\widehat{S}_2(t)+u^\kappa I_1(t)\widehat{S}_2(t)-u^\kappa\widehat{I}_1(t)\widehat{S}_2(t)\big|\\
&\qquad+2\big|\alpha_2^\kappa I_2(t)S_2(t)-\alpha_2^\kappa I_2(t)\widehat{S}_2(t)+\alpha_2^\kappa I_2(t)\widehat{S}_2(t)-\alpha_2^\kappa\widehat{I}_2(t)\widehat{S}_2(t)\big|\\
&\qquad+\big(d^\kappa+2\beta_1^\kappa\big)\big|S_1(t)-\widehat{S}_1(t)|+(d^\kappa+2\beta_2^\kappa)|S_2(t)-\widehat{S}_2(t)\big|\\
&\qquad+d^\kappa\big|I_2(t)-\widehat{I}_2(t)\big|+d^\kappa|R(t)-\widehat{R}(t)\big|\\
&\leqslant\big(2\alpha_1^\kappa\big|I_1(t)\big|+2\beta_1^\kappa+d^\kappa\big)\big|S_1(t)-\widehat{S}_1(t)\big|\\
&\qquad+\big(2\alpha_1^\kappa\big|\widehat{S}_1(t)\big|+2u^\kappa\big|\widehat{S}_2(t)\big|+d^\kappa\big)\big|I_1(t)-\widehat{I}_1(t)\big|\\
&\qquad+\big(2u^\kappa\big|I_1(t)\big|+2\alpha_2^\kappa\big|I_2(t)\big|+2\beta_2^\kappa+d^\kappa\big)\big|S_2(t)-\widehat{S}_2(t)\big|\\
&\qquad+\big(2\alpha_2^\kappa\big|\widehat{S}_2(t)\big|+\beta_2^\kappa+d^\kappa\big)\big|I_2(t)-\widehat{I}_2(t)\big|+d^\kappa|R(t)-\widehat{R}(t)\big|.
\end{aligned}$$

Let $M_1 = \max\{|I_1(t)|, |I_2(t)|, |\widehat{S}_1(t)|, |\widehat{S}_2(t)|\}$, then we have

$$\begin{aligned}\|H(Y) - H(\widehat{Y})\| = {}& \bigl(2\alpha_1^\kappa M + 2\beta_1^\kappa + d^\kappa\bigr)\bigl|S_1(t) - \widehat{S}_1(t)\bigr| \\ &+ \bigl(2u^\kappa M + 2\alpha_2^\kappa M + 2\beta_2^\kappa + d^\kappa\bigr)\bigl|S_2(t) - \widehat{S}_2(t)\bigr| \\ &+ \bigl(2\alpha_1^\kappa M + 2u^\kappa M + d^\kappa\bigr)\bigl|I_1(t) - \widehat{I}_1(t)\bigr| \\ &+ \bigl(2\alpha_2^\kappa M + d^\kappa\bigr)\bigl|I_2(t) - \widehat{I}_2(t)\bigr| + d^\kappa\bigl|R(t) - \widehat{R}(t)\bigr|.\end{aligned}$$

Let

$$\begin{aligned}L = \max\bigl\{&2\alpha_1^\kappa M_1 + 2\beta_1^\kappa + d^\kappa,\ 2u^\kappa M_1 + 2\alpha_2^\kappa M_1 + 2\beta_2^\kappa + d^\kappa, \\ &2\alpha_1^\kappa M_1 + 2u^\kappa M_1 + d^\kappa\bigr\}.\end{aligned}$$

Furthermore, we can obtain $\|H(Y) - H(\widehat{Y})\| \leqslant L\|Y - \widehat{Y}\|$. This completes the proof by applying Lemma 4. $\square$

4.2 Stability analysis

In this section, we will give the threshold R_0^α and discuss the stability of the equilibrium of system (8). The main results and proofs are as follows.

Firstly, the threshold of system (8) is calculated by using the next-generation matrix method. Now, we only need to investigate the following subsystem:

$$\begin{aligned}{}_0^C D_t^\kappa I_1(t) &= \Pi_1^\kappa - \alpha_1^\kappa I_1(t)S_1(t) - u^\kappa I_1(t)S_2(t) - d^\kappa I_1(t), \\ {}_0^C D_t^\kappa I_2(t) &= \Pi_2^\kappa - \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa I_2(t), \\ {}_0^C D_t^\kappa S_1(t) &= \alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) - d^\kappa S_1(t) - \beta_1^\kappa S_1(t), \\ {}_0^C D_t^\kappa S_2(t) &= \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa S_2(t) - \beta_2^\kappa S_2(t).\end{aligned} \tag{9}$$

By calculation it is clear that the rumor-free equilibrium $E_0 = (\Pi_1^\kappa/d^\kappa, \Pi_2^\kappa/d^\kappa, 0, 0)$. Let $\chi(t) = (I_1(t), I_2(t), S_1(t), S_2(t))^{\mathrm{T}}$, then system (9) can be rewritten as

$${}_0^C D_t^\kappa \chi(t) = \mathcal{F}(\chi) - \mathcal{V}(\chi),$$

where

$$\mathcal{F}(\chi) = \begin{pmatrix} \alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) \\ \alpha_2^\kappa I_2(t)S_2(t) \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}(\chi) = \begin{pmatrix} (d^\kappa + \beta_1^\kappa)S_1(t) \\ (d^\kappa + \beta_2^\kappa)S_2(t) \\ \alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) + d^\kappa I_1(t) - \Pi_1^\kappa \\ \alpha_2^\kappa I_2(t)S_2(t) + d^\kappa I_2(t) - \Pi_2^\kappa \end{pmatrix}.$$

We can obtain the Jacobian matrices $D\mathcal{F}(\chi)$ and $D\mathcal{V}(\chi)$ at E_0 as follows:

$$D\mathcal{F}(\chi) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \qquad D\mathcal{V}(\chi) = \begin{pmatrix} V & 0 \\ A_1 & A_2 \end{pmatrix},$$

where

$$F_1 = \begin{pmatrix} \frac{\alpha_1^\kappa \varPi_1^\kappa}{d^\kappa} & \frac{u^\kappa \varPi_1^\kappa}{d^\kappa} \\ 0 & \frac{\alpha_2^\kappa \varPi_2^\kappa}{d^\kappa} \end{pmatrix}, \qquad V_1 = \begin{pmatrix} d^\kappa + \beta_1^\kappa & 0 \\ 0 & d^\kappa + \beta_2^\kappa \end{pmatrix},$$

$$A_1 = \begin{pmatrix} \frac{\alpha_1^\kappa \varPi_1^\kappa}{d^\kappa} & \frac{u^\kappa \varPi_1^\kappa}{d^\kappa} \\ 0 & \frac{\alpha_2^\kappa \varPi_2^\kappa}{d^\kappa} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} d^\kappa & 0 \\ 0 & d^\kappa \end{pmatrix}.$$

Thus,

$$F_1 V_1^{-1} = \begin{pmatrix} \frac{\alpha_1^\kappa \varPi_1^\kappa}{d^\kappa (d^\kappa + \beta_1^\kappa)} & 0 \\ 0 & \frac{\alpha_2^\kappa \varPi_2^\kappa}{d^\kappa (d^\kappa + \beta_2^\kappa)} \end{pmatrix}.$$

The threshold of system (8) is given by

$$R_0^\kappa = \sum_{i=1}^{2} \frac{\alpha_i^\kappa \varPi_i^\kappa}{d^\kappa (d^\kappa + \beta_i^\kappa)} = \sum_{i=1}^{2} R_{0i}^\kappa.$$

Remark 4. R_0^κ is a threshold quantity. When $\kappa = 1$, R_0^κ is the basic reproduction number of model (8). It refers to the number of people that a ignorant can turn into a spreader during the average period of transmission when everyone is ignorant at the initial stage of rumor transmission.

Remark 5. In a multilingual environment, the system threshold R_0^κ is equal to the sum of the thresholds R_{0i}^κ of each group i. In this paper, R_{01}^κ and R_{02}^κ are thresholds in two language environments, respectively. A detailed description can be found in [19].

Let $\bar{I}_1(t) = I_1(t) - I_1^*$, $\bar{I}_2(t) = I_2(t) - I_2^*$, $\bar{S}_1(t) = S_1(t) - S_1^*$, $\bar{S}_2(t) = S_2(t) - S_2^*$, $\bar{R}(t) = R(t) - R_*$. The linear system of system (8) is

$$\begin{aligned}
{}_0^C D_t^\kappa \bar{I}_1(t) &= (-\alpha_1^\kappa S_1^* - u^\kappa S_2^* - d^\kappa)\bar{I}_1(t) - \alpha_1^\kappa I_1^* \bar{S}_1(t) - u^\kappa I_1^* \bar{S}_2(t), \\
{}_0^C D_t^\kappa \bar{I}_2(t) &= (-\alpha_2^\kappa S_2^* - d^\kappa)\bar{I}_2(t) - \alpha_2^\kappa I_2^* \bar{S}_2(t), \\
{}_0^C D_t^\kappa \bar{S}_1(t) &= (\alpha_1^\kappa S_1^* + u^\kappa S_2^*)\bar{I}_1(t) + (\alpha_1^\kappa I_1^* - \beta_1^\kappa - d^\kappa)\bar{S}_1(t) + u^\kappa I_1^* \bar{S}_2(t), \\
{}_0^C D_t^\kappa \bar{S}_2(t) &= \alpha_2^\kappa S_2^* \bar{I}_2(t) + (\alpha_2^\kappa I_2^* - \beta_2^\kappa - d^\kappa)\bar{S}_2(t), \\
{}_0^C D_t^\kappa \bar{R}(t) &= \beta_1^\kappa \bar{S}_1(t) + \beta_2^\kappa \bar{S}_2(t) - d^\kappa \bar{R}(t).
\end{aligned}$$

Obviously, $R(t)$ is independent of the first four equations, so we can just consider the first four equations in the following study.

Theorem 3. *The rumor-free equilibrium E_0 of system* (8) *is locally asymptotically stable when $R_{01}^\kappa < 1$ and $R_{02}^\kappa < 1$.*

Proof. The characteristic matrix of system (8) at $E_0 = (\varPi_1^\kappa/d^\kappa, \varPi_2^\kappa/d^\kappa, 0, 0)$ is

$$\Delta(s) = \begin{pmatrix} s^\kappa + d^\kappa & 0 & \frac{\alpha_1^\kappa \varPi_1^\kappa}{d^\kappa} & \frac{u^\kappa \varPi_1^\kappa}{d^\kappa} \\ 0 & s^\kappa + d^\kappa & 0 & \frac{\alpha_2^\kappa \varPi_2^\kappa}{d^\kappa} \\ 0 & 0 & s^\kappa - \frac{\alpha_1^\kappa \varPi_1^\kappa}{d^\kappa} + \beta_1^\kappa + d^\kappa & \frac{u^\kappa \varPi_1^\kappa}{d^\kappa} \\ 0 & 0 & 0 & s^\kappa - \frac{\alpha_2^\kappa \varPi_2^\kappa}{d^\kappa} + \beta_2^\kappa + d^\kappa \end{pmatrix}.$$

Hence, the characteristic equation that corresponds to system (8) is

$$\left(s^\kappa + d^\kappa\right)\left(s^\kappa + d^\kappa\right)\left(s^\kappa - \frac{\alpha_1^\kappa \Pi_1^\kappa}{d^\kappa} + \beta_1^\kappa + d^\kappa\right)\left(s^\kappa - \frac{\alpha_2^\kappa \Pi_2^\kappa}{d^\kappa} + \beta_2^\kappa + d^\kappa\right) = 0. \quad (10)$$

Let $\lambda = s^\kappa$, the roots of Eq. 10 are given by

$$\lambda_1 = \lambda_2 = -d^\kappa, \qquad \lambda_3 = \left(\beta_1^\kappa + d^\kappa\right)\left(R_{01}^\kappa - 1\right), \qquad \lambda_4 = \left(\beta_2^\kappa + d^\kappa\right)\left(R_{02}^\kappa - 1\right).$$

Obviously, $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$ and $\lambda_4 < 0$ when $R_{01}^\kappa < 1$ and $R_{02}^\kappa < 1$. Furthermore, $\arg(\lambda_i) > \pi/2 > \kappa\pi/2$, which means $\arg(s_i) > \pi/2$, $i = 1, 2, 3, 4$. By applying Lemma 6 E_0 is locally asymptotically stable when $R_{01}^\kappa < 1$ and $R_{02}^\kappa < 1$. □

Next, we discuss the stability of rumor-spreading equilibrium, system (8) satisfies the following equations at $E^* = (I_1^*, I_2^*, S_1^*, S_2^*, R^*)$:

$$\begin{aligned}
&\Pi_1^\kappa - \alpha_1^\kappa I_1^* S_1^* - u^\kappa I_1^* S_2^* - d^\kappa I_1^* = 0,\\
&\Pi_2^\kappa - \alpha_2^\kappa I_2^* S_2^* - d^\kappa I_2^* = 0,\\
&\alpha_1^\kappa I_1^* S_1^* + u^\kappa I_1^* S_2^* - d^\kappa S_1^* - \beta_1^\kappa S_1^* = 0,\\
&\alpha_2^\kappa I_2^* S_2^* - d^\kappa S_2^* - \beta_2^\kappa S_2^* = 0.
\end{aligned} \qquad (11)$$

When $\alpha_1^\kappa \Pi_1^\kappa - d^\kappa(d^\kappa + \beta_1^\kappa) > 0$, we can get $E_1^* = (I_1^*, I_2^*, S_1^*, S_2^*, R^*)$, where

$$I_1^* = \frac{d^\kappa + \beta_1^\kappa}{\alpha_1^\kappa}, \qquad I_2^* = \frac{\Pi_2^\kappa}{d^\kappa}, \qquad S_1^* = \frac{d^\kappa(R_{01}^\kappa - 1)}{\alpha_1^\kappa}, \qquad S_2^* = 0.$$

When $\alpha_2^\kappa \Pi_2^\kappa - d^\kappa(d^\kappa + \beta_2^\kappa) \geqslant 0$, we can obtain that $E_2^* = (I_1^*, I_2^*, S_1^*, S_2^*, R^*)$, where

$$I_1^* = \frac{\alpha_2^\kappa \Pi_1^\kappa}{\alpha_1^\kappa \alpha_2^\kappa S_1^* + \alpha_2^\kappa d^\kappa + u^\kappa d^\kappa (R_{02}^\kappa - 1)}, \qquad I_2^* = \frac{d^\kappa + \beta_2^\kappa}{\alpha_2^\kappa}, \qquad S_2^* = \frac{d^\kappa(R_{02}^\kappa - 1)}{\alpha_2^\kappa}.$$

Combining the expressions of I_1^*, S_2^* and the third equation of system (11), it can be seen that

$$a_0(S_1^*)^2 + b_0(S_1^*) + c_0 = 0, \qquad (12)$$

where

$$\begin{aligned}
a_0 &= -\alpha_1^\kappa \alpha_2^\kappa\left(d^\kappa + \beta_1^\kappa\right),\\
b_0 &= \alpha_2^\kappa\left(\alpha_1^\kappa \Pi_1^\kappa - d^\kappa\left(d^\kappa + \beta_1^\kappa\right)\right) - u^\kappa\left(d^\kappa + \beta_1^\kappa\right)\frac{\alpha_2^\kappa \Pi_2^\kappa - d^\kappa(d^\kappa + \beta_2^\kappa)}{d^\kappa + \beta_2^\kappa},\\
c_0 &= \Pi_1^\kappa u^\kappa \frac{\alpha_2^\kappa \Pi_2^\kappa - d^\kappa(d^\kappa + \beta_2^\kappa)}{d^\kappa + \beta_2^\kappa}.
\end{aligned}$$

The following results can be proved by simple calculation:

$$\begin{aligned}
c_0 > 0 \quad &\Longleftrightarrow \quad \alpha_2^\kappa \Pi_2^\kappa - d^\kappa\left(d^\kappa + \beta_2^\kappa\right) > 0, \quad c_0 = 0\\
&\Longleftrightarrow \quad \alpha_2^\kappa \Pi_2^\kappa - d^\kappa\left(d^\kappa + \beta_2^\kappa\right) = 0.
\end{aligned}$$

So we discuss the roots of Eq. (12) in two different cases in the following study.

(i) For $c_0 > 0$, Eq. (12) has a unique positive root S_1^*.
(ii) For $c_0 = 0$, Eq. (12) is equivalent to $a_0(S_1^*)^2 + v_0 S_1^* = 0$ in which $v_0 = \alpha_2^\kappa(\alpha_1^\kappa \Pi_1^\kappa - d^\kappa(d^\kappa + \beta_1^\kappa))$.

Theorem 4. *The following results can be derived*:

(i) *If* $R_{01}^\kappa > 1$, *system* (8) *has a unique rumor-spreading equilibrium* E_1^*.
(ii) *If* $R_{02}^\kappa > 1$, *system* (8) *has a unique rumor-spreading equilibrium* E_2^*.

Theorem 5. *If* $R_{01}^\kappa > 1$ *and* $R_{02}^\kappa < 1$ *are satisfied, the rumor-spreading equilibrium* E_1^* *of system* (8) *is locally asymptotically stable.*

Proof. By a simple calculation it can be obtained that the rumor-spreading equilibrium $E_1^* = ((d^\kappa + \beta_1^\kappa)/\alpha_1^\kappa,\ \Pi_2^\kappa/d^\kappa,\ d^\kappa(R_{01}^\kappa - 1)/\alpha_1^\kappa,\ 0)$. The characteristic matrix of system (8) at E_1^* is

$$\Delta(s) = \begin{pmatrix} s^\kappa + d^\kappa + \alpha_1^\kappa S_1^* & 0 & \alpha_1^\kappa I_1^* & u^\kappa I_1^* \\ 0 & s^\kappa + d^\kappa & 0 & \alpha_2^\kappa I_2^* \\ -\alpha_1^\kappa S_1^* & 0 & s^\kappa - \alpha_1^\kappa I_1^* + (d^\kappa + \beta_1^\kappa) & u^\kappa I_1^* \\ 0 & 0 & 0 & s^\kappa - \alpha_2^\kappa I_2^* + (d^\kappa + \beta_2^\kappa) \end{pmatrix}.$$

Hence, the characteristic equation that corresponds to system (8) is

$$\begin{aligned} &\left[s^\kappa - \alpha_2^\kappa I_2^* + \left(d^\kappa + \beta_2^\kappa\right)\right] \\ &\quad \times \left(s^\kappa + d^\kappa\right)\left[s^{2\kappa} + \left(d^\kappa + \alpha_1^\kappa S_1^*\right)s^\kappa + \alpha_1^\kappa I_1^* \alpha_1^\kappa S_1^*\right] = 0. \end{aligned} \tag{13}$$

Let $\lambda = s^\kappa$, then substituting S_1^*, I_2^* into Eq. (13), we can get

$$\begin{aligned} &\left[\lambda - \left(d^\kappa + \beta_2^\kappa\right)\left(R_{02}^\kappa - 1\right)\right]\left(\lambda + d^\kappa\right) \\ &\quad \times \left\{\lambda^2 + \frac{\alpha_1^\kappa \Pi_1^\kappa}{d^\kappa + \beta_1^\kappa}\lambda + \left[d^\kappa\left(d^\kappa + \beta_1^\kappa\right)\left(R_{01}^\kappa - 1\right)\right]\right\} = 0. \end{aligned}$$

Obviously, $\lambda_i < 0$ ($i = 1, 2, 3, 4$) when $R_{02}^\kappa < 1$ and $R_{01}^\kappa > 1$. At the same time, $\arg(\lambda_i) > \pi/2 > \kappa\pi/2$, which means $\arg(s_i) > \pi/2$ ($i = 1, 2, 3, 4$). By applying Lemma 6 E_0 is locally asymptotically stable when $R_{02}^\kappa < 1$ and $R_{01}^\kappa > 1$ are satisfied. □

We make the following hypotheses to obtain our result.

(H1) $\alpha_1^\kappa I_1^* - (d^\kappa + \beta_1^\kappa) \leqslant 0$.
(H2) $A u^\kappa I_1^* S_1^* + B\alpha_2^\kappa I_2^* S_2^* - B(d^\kappa + \beta_2^\kappa)S_2^* \leqslant 0$.
(H3) $u^\kappa I_1^* S_2^* - \alpha_1^\kappa I_1^* S_1^* - (d^\kappa + \beta_1^\kappa)S_1^* \leqslant 0$.

Theorem 6. *If* $R_{02}^\kappa > 1$ *and* (H1)–(H3) *are satisfied, the rumor-spreading equilibrium* E_2^* *of system* (8) *is globally asymptotically stable.*

Proof. Lyapunov function is constructed as follows:

$$V(t) = A\left(I_1^* g\left(\frac{I_1(t)}{I_1^*}\right)\right) + B\left(I_2^* g\left(\frac{I_2(t)}{I_2^*}\right)\right) + A\left(S_1^* g\left(\frac{S_1(t)}{S_1^*}\right)\right) + B\left(S_2^* g\left(\frac{S_2(t)}{S_2^*}\right)\right),$$

where A, B are positive constants, and $g(y) = y - 1 - \ln y \geqslant 0$ for $y > 0$. Let

$$x_1 = \frac{I_1(t)}{I_1^*}, \quad x_2 = \frac{I_2(t)}{I_2^*}, \qquad y_1 = \frac{S_1(t)}{S_1^*}, \quad y_2 = \frac{S_2(t)}{S_2^*}.$$

Next, differentiating $V(t)$ along system (8), one has

$$\begin{aligned} {}_0^C D_t^\kappa V(t) \leqslant{}& A\left(1 - \frac{1}{x_1}\right){}_0^C D_t^\kappa I_1(t) + B\left(1 - \frac{1}{x_2}\right){}_0^C D_t^\kappa I_2(t) \\ &+ A\left(1 - \frac{1}{y_1}\right){}_0^C D_t^\kappa S_1(t) + B\left(1 - \frac{1}{y_2}\right){}_0^C D_t^\kappa S_2(t). \end{aligned}$$

Combining Eq. (8) with Eq. (11), we have

$$\begin{aligned} {}_0^C D_t^\kappa I_1(t) &= -\alpha_1^\kappa\big(I_1(t)S_1(t) - I_1^* S_1^*\big) - u^\kappa\big(I_1(t)S_2(t) - I_1^* S_2^*\big) - d^\kappa\big(I_1(t) - I_1^*\big) \\ &= \alpha_1^\kappa I_1^* S_1^*(1 - x_1 y_1) + u^\kappa I_1^* S_2^*(1 - x_1 y_2) - d^\kappa I_1^*(x_1 - 1), \\ {}_0^C D_t^\kappa I_2(t) &= -\alpha_2^\kappa\big(I_2(t)S_2(t) - I_2^* S_2^*\big) - d^\kappa\big(I_2(t) - I_2^*\big) \\ &= \alpha_2^\kappa I_2^* S_2^*(1 - x_2 y_2) - d^\kappa I_2^*(x_2 - 1), \\ {}_0^C D_t^\kappa S_1(t) &= \alpha_1^\kappa\big(I_1(t)S_1(t) - I_1^* S_1^*\big) + u^\kappa\big(I_1(t)S_2(t) - I_1^* S_2^*\big) \\ &\quad - \big(d^\kappa + \beta_1^\kappa\big)\big(S_1(t) - S_1^*\big) \\ &= \alpha_1^\kappa I_1^* S_1^*(1 - x_1 y_1) + u^\kappa I_1^* S_2^*(1 - x_1 y_2) - d^\kappa I_1^*(x_1 - 1), \\ {}_0^C D_t^\kappa S_2(t) &= \alpha_2^\kappa\big(I_2(t)S_2(t) - I_2^* S_2^*\big) - \big(d^\kappa + \beta_2^\kappa\big)\big(S_2(t) - S_2^*\big) \\ &= \alpha_1^\kappa I_1^* S_1^*(1 - x_1 y_1) + u^\kappa I_1^* S_2^*(1 - x_1 y_2) - d^\kappa I_1^*(x_1 - 1). \end{aligned}$$

Furthermore, we can obtain

$$\begin{aligned} {}_0^C D_t^\kappa V(t) \leqslant{}& A\bigg[\alpha_1^\kappa I_1^* S_1^* g(y_1) - \alpha_1^\kappa I_1^* S_1^* g\left(\frac{1}{x_1}\right) - \alpha_1^\kappa I_1^* S_1^* g(x_1 y_1) + u^\kappa I_1^* S_2^* g(y_2) \\ &- u^\kappa I_1^* S_2^* g\left(\frac{1}{x_1}\right) - u^\kappa I_1^* S_2^* g(x_1 y_2) - d^\kappa I_1^* g(x_1) - d^\kappa I_1^* g\left(\frac{1}{x_1}\right)\bigg] \\ &+ B\bigg[\alpha_2^\kappa I_2^* S_2^* g(y_2) - \alpha_2^\kappa I_2^* S_2^* g\left(\frac{1}{x_2}\right) - \alpha_2^\kappa I_2^* S_2^* g(x_2 y_2) \\ &- d^\kappa I_2^* g(x_2) - d^\kappa I_2^* g\left(\frac{1}{x_2}\right)\bigg] \\ &+ A\bigg[-\alpha_1^\kappa I_1^* S_1^* g(x_1) - \alpha_1^\kappa I_1^* S_1^* g\left(\frac{1}{y_1}\right) + \alpha_1^\kappa I_1^* S_1^* g(x_1 y_1) \end{aligned}$$

$$
\begin{aligned}
&+ u^\kappa I_1^* S_2^* g(x_1 y_2) - u^\kappa I_1^* S_2^* g\left(\frac{x_1 y_2}{y_1}\right) + u^\kappa I_1^* S_2^* g\left(\frac{1}{y_1}\right) \\
&- \left(d^\kappa + \beta_1^\kappa\right) S_1^* g(y_1) - \left(d^\kappa + \beta_1^\kappa\right) S_1^* g\left(\frac{1}{y_1}\right)\bigg] \\
&+ B\bigg[\alpha_2^\kappa I_2^* S_2^* g(x_2 y_2) - \alpha_2^\kappa I_2^* S_2^* g(x_2) - \alpha_2^\kappa I_2^* S_2^* g\left(\frac{1}{y_1}\right) \\
&- \left(d^\kappa + \beta_2^\kappa\right) S_2^* g(y_2) - \left(d^\kappa + \beta_2^\kappa\right) S_2^* g\left(\frac{1}{y_2}\right)\bigg] \\
= {}& A\left(-d^\kappa I_1^* - \alpha_1^\kappa I_1^* S_1^*\right) g(x_1) + B\left(-d^\kappa I_2^* - \alpha_2^\kappa I_2^* S_2^*\right) g(x_2) \\
&+ A\left[\alpha_1^\kappa I_1^* S_1^* - \left(d^\kappa + \beta_1^\kappa\right) S_1^*\right] g(y_1) \\
&+ \left[A u^\kappa I_1^* S_1^* + B \alpha_2^\kappa I_2^* S_2^* - B\left(d^\kappa + \beta_2^\kappa\right) S_2^*\right] g(y_2) \\
&+ A\left(-\alpha_1^\kappa I_1^* S_1^* - d^\kappa I_1^* - u^\kappa I_1^* S_2^*\right) g\left(\frac{1}{x_1}\right) \\
&+ A\left[u^\kappa I_1^* S_2^* - \alpha_1^\kappa I_1^* S_1^* - \left(d^\kappa + \beta_1^\kappa\right) S_1^*\right] g\left(\frac{1}{y_1}\right) \\
&+ B\left[-\alpha_2^\kappa I_2^* S_2^* - \left(d^\kappa + \beta_2^\kappa\right) S_2^*\right] g\left(\frac{1}{y_2}\right) - A u^\kappa I_1^* S_2^* g\left(\frac{x_1 y_2}{y_1}\right).
\end{aligned}
$$

Choose appropriate A and B to make sure that

$$
\begin{aligned}
&\alpha_1^\kappa I_1^* - \left(d^\kappa + \beta_1^\kappa\right) \leqslant 0, \\
&A u^\kappa I_1^* S_1^* + B \alpha_2^\kappa I_2^* S_2^* - B\left(d^\kappa + \beta_2^\kappa\right) S_2^* \leqslant 0, \\
&u^\kappa I_1^* S_2^* - \alpha_1^\kappa I_1^* S_1^* - \left(d^\kappa + \beta_1^\kappa\right) S_1^* \leqslant 0.
\end{aligned}
$$

It can be seen that ${}_0^C D_t^\kappa V(t) \leqslant 0$, and ${}_0^C D_t^\kappa V(t) = 0$ if and only if

$$\big(I_1(t), I_2(t), S_1(t), S_2(t)\big) = (I_1^*, I_2^*, S_1^*, S_2^*).$$

By Lemma 7 the rumor-spreading equilibrium E_2^* of system (8) is globally asymptotically stable when $R_{02}^\kappa > 1$ is satisfied. □

4.3 Fractional-order optimal control for 2I2SR rumor spreading model

Optimal control theory has a wide range of applications in biological systems, infectious diseases, and rumor control. Therefore, considering the memory effect of the system, it is necessary to study the fractional-order optimal control problem [11]. In this part, the optimal control method can effectively suppress rumors in the expected time period at the lowest cost. We formulate an optimal problem for model (8) to find a suitable compromise between minimizing the number of the spreaders and the cost of the control. We introduce Lebesgue square-integrable control function $u_i(t) \in \Delta$, where $\Delta = \{u_i(t)$ is Lebesgue measurable on $(0, T_m]$, $0 \leqslant u_i(t) \leqslant 1$, $t \in (0, T_m]$, $i = 1, 2\}$ denotes the set of admissible controls, and T_m is the ending time. The improved controlled model is as follows:

$$\begin{aligned}
{}_0^C D_t^\kappa I_1(t) &= \Pi_1^\kappa - \alpha_1^\kappa I_1(t)S_1(t) - u^\kappa I_1(t)S_2(t) - d^\kappa I_1(t),\\
{}_0^C D_t^\kappa I_2(t) &= \Pi_2^\kappa - \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa I_2(t),\\
{}_0^C D_t^\kappa S_1(t) &= \alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) - d^\kappa S_1(t) - \beta_1^\kappa S_1(t) - u_1(t)S_1(t),\\
{}_0^C D_t^\kappa S_2(t) &= \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa S_2(t) - \beta_2^\kappa S_2(t) - u_2(t)S_2(t),\\
{}_0^C D_t^\kappa R(t) &= \beta_1^\kappa S_1(t) + \beta_2^\kappa S_2(t) - d^\kappa R(t) + u_1(t)S_1(t) + u_2(t)S_2(t)
\end{aligned} \tag{14}$$

with the initial condition

$$I_1(0) \geqslant 0, \qquad I_2(0) \geqslant 0, \qquad S_1(0) \geqslant 0, \qquad S_2(0) \geqslant 0, \qquad R(0) \geqslant 0.$$

In order to study the optimal level of the number of spreaders and cost under the control function $u_1(t)$ and $u_2(t)$, the objective function $J(u_1(t), u_2(t))$ is constructed as follows:

$$J\bigl(u_1(t), u_2(t)\bigr) = \int_0^{T_m} \bigl[\psi_1 S_1(t) + \psi_2 S_2(t) + \phi_1 u_1^2(t) + \phi_2 u_2^2(t)\bigr]\,\mathrm{d}t, \tag{15}$$

where ψ_1, ψ_2, ϕ_1, ϕ_2 are positive weights, $\phi_i u_i^2(t)$ $(i = 1, 2)$ represents the average cost of applying control $u_i(t)$ $(i = 1, 2)$ to control and educate $S_i(t)$ $(i = 1, 2)$.

The Lagrangian function is given by

$$L\bigl(S_1(t), S_2(t), u_1(t), u_2(t)\bigr) = \psi_1 S_1(t) + \psi_2 S_2(t) + \phi_1 u_1^2(t) + \phi_2 u_2^2(t).$$

To solve the optimal control problem, the Hamiltonian function of Eq. (15) is defined as

$$\begin{aligned}
&H\bigl(I_i(t), S_i(t), R(t), u_i(t), \lambda_j(t)\bigr)\\
&\quad = L\bigl(S_1(t), S_2(t), u_1(t), u_2(t)\bigr)\\
&\qquad + \lambda_1(t)\bigl[\Pi_1^\kappa - \alpha_1^\kappa I_1(t)S_1(t) - u^\kappa I_1(t)S_2(t) - d^\kappa I_1(t)\bigr]\\
&\qquad + \lambda_2(t)\bigl[\Pi_2^\kappa - \alpha_2^\kappa I_2(t)S_2(t) - d^\kappa I_2(t)\bigr]\\
&\qquad + \lambda_3(t)\bigl[\alpha_1^\kappa I_1(t)S_1(t) + u^\kappa I_1(t)S_2(t) - d^\kappa S_1(t) - \beta_1^\kappa S_1(t) - u_1(t)S_1(t)\bigr]\\
&\qquad + \lambda_4(t)\bigl[\alpha_2^\kappa I_2(t)S_2(t) - d^\kappa S_2(t) - \beta_2^\kappa S_2(t) - u_2(t)S_2(t)\bigr]\\
&\qquad + \lambda_5(t)\bigl[\beta_1^\kappa S_1(t) + \beta_2^\kappa S_2(t) - d^\kappa R(t) + u_1(t)S_1(t) + u_2(t)S_2(t)\bigr].
\end{aligned}$$

Let $(I_1^*(t), I_2^*(t), S_1^*(t), S_2^*(t), R^*(t))$ be the optimal solution with $(u_1^*(t), u_2^*(t))$ for the controlled system (14). From [11] we can get the necessary conditions for optimal control that $\lambda_i(t)$ $(i = 1, \dots, 5)$ satisfy

$$\begin{aligned}
{}_0^C D_t^\kappa \lambda_1(T_m - t) &= \lambda_1(T_m - t)\bigl(-\alpha_1^\kappa S_1^*(T_m - t) - u^\kappa S_2^*(T_m - t) - d^\kappa\bigr)\\
&\qquad + \lambda_3(T_m - t)\bigl(\alpha_1^\kappa S_1^*(T_m - t) + u^\kappa S_2^*(T_m - t)\bigr),\\
{}_0^C D_t^\kappa \lambda_2(T_m - t) &= \lambda_2(T_m - t)\bigl(-\alpha_2^\kappa S_2^*(T_m - t) - d^\kappa\bigr) + \lambda_4(T_m - t)\alpha_2^\kappa S_2^*(T_m - t),
\end{aligned}$$

$$
\begin{aligned}
{}_0^C D_t^\kappa \lambda_3(T_m - t) &= \psi_1 - \lambda_1(T_m - t)\alpha_1^\kappa I_1^*(T_m - t) \\
&\quad + \lambda_3(T_m - t)\big(\alpha_1^\kappa I_1^*(T_m - t) - d^\kappa - \beta_1^\kappa - u_1(T_m - t)\big) \\
&\quad + \lambda_5(T_m - t)\big(\beta_1^\kappa + u_1(T_m - t)\big), \\
{}_0^C D_t^\kappa \lambda_4(T_m - t) &= \psi_2 - \lambda_1(T_m - t)u^\kappa I_1^*(T_m - t) - \lambda_2(T_m - t)\alpha_2^\kappa I_2^*(T_m - t) \\
&\quad + \lambda_5(T_m - t)\big(\beta_2^\kappa u_2(T_m - t)\big) \\
&\quad + \lambda_4(T_m - t)\big(\alpha_2^\kappa I_2^*(T_m - t) - d^\kappa - \beta_2^\kappa - u_2(T_m - t)\big) \\
&\quad + \lambda_3(T_m - t)u^\kappa I_1^*(T_m - t), \\
{}_0^C D_t^\kappa \lambda_5(T_m - t) &= \lambda_5(T_m - t)d^\kappa
\end{aligned}
$$

with the terminal condition

$$\lambda_1(T_m) = \lambda_2(T_m) = \lambda_3(T_m) = \lambda_4(T_m) = \lambda_5(T_m) = 0.$$

From [11] we have

$$\left.\frac{\partial H(t)}{\partial u_1(t)}\right|_{u_1(t)} = u_1^*(t) = 2\phi_1 u_1^*(t) - \lambda_3(t)S_1^*(t) + \lambda_5(t)S_1^*(t) = 0, \tag{16}$$

$$\left.\frac{\partial H(t)}{\partial u_2(t)}\right|_{u_2(t)} = u_2^*(t) = 2\phi_2 u_2^*(t) - \lambda_4(t)S_2^*(t) + \lambda_5(t)S_2^*(t) = 0. \tag{17}$$

Solving Eqs. (16), (17), we can obtain

$$u_1^*(t) = \frac{(\lambda_3(t) - \lambda_5(t))S_1^*(t)}{2\phi_1}, \qquad u_2^*(t) = \frac{(\lambda_4(t) - \lambda_5(t))S_2^*(t)}{2\phi_2}.$$

Furthermore, $u_1^*(t)$ and $u_2^*(t)$ can be given as follows:

$$
\begin{aligned}
u_1^*(t) &= \min\left\{\max\left\{\frac{(\lambda_3(t) - \lambda_5(t))S_1^*(t)}{2\phi_1}, 0\right\}, 1\right\}, \\
u_2^*(t) &= \min\left\{\max\left\{\frac{(\lambda_4(t) - \lambda_5(t))S_2^*(t)}{2\phi_2}, 0\right\}, 1\right\}.
\end{aligned}
$$

Remark 6. In [13, 22, 28], rumor propagation models in multilingual environment are established, and the stability of these systems is studied. Different from their work, a fractional-order 2I2SR rumor model is proposed in multilingual environment in which the memory effect in rumor propagation is considered by Caputo fractional derivative. More importantly, the necessary conditions for fractional optimal control of rumor propagation are obtained.

Remark 7. Compared with [11,15,23,24], not only a fractional 2I2SR rumor propagation model is established in a multilingual environment and the stability conditions of the fractional-order system are obtained, but also the unity of units on the left and right sides of the generalized equation is taken into account, which is worth being considered in future research.

5 Numerical examples

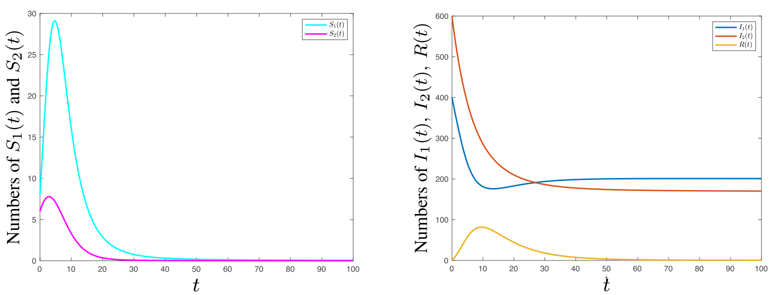
To solve fractional differential equations, we mainly use predictor-corrector method, which is described in [15].

5.1 Stability of rumor-free equilibrium E_0

In this part, we select $\varPi_1 = 30$, $\varPi_2 = 25$, $\alpha_1 = 0.002$, $\alpha_2 = 0.001$, $u = 0.001$, $d = 0.12$, $\beta_1 = 0.55$, $\beta_2 = 0.45$, $\kappa = 0.96$. By calculation it follows that $R^{\kappa}_{01} = 0.7408$, $R^{\kappa}_{02} = 0.3727$, and $E_0 = (200, 168, 0, 0, 0)$. Based on Theorem 3, E_0 is locally asymptotically stable when $R^{\kappa}_{01} < 1$ and $R^{\kappa}_{02} < 1$. The results are shown in Fig. 1.

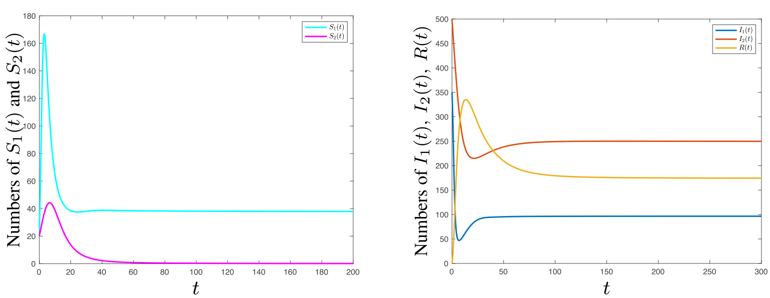
5.2 Stability of rumor equilibrium E_1^* and E_2^*

In order to discuss the stability of E_1^*, we select $\varPi_1 = 25$, $\varPi_2 = 20$, $\alpha_1 = 0.003$, $\alpha_2 = 0.001$, $u = 0.001$, $d = 0.06$, $\beta_1 = 0.3$, $\beta_2 = 0.35$, $\kappa = 0.95$. By calculation it follows that $R^{\kappa}_{01} = 3.1886$, $R^{\kappa}_{02} = 0.8041$, and $E_1^* = (97, 249, 38, 0, 174)$. Based on Theorem 5, E_1^* is locally asymptotically stable when $R^{\kappa}_{01} > 1$ and $R^{\kappa}_{02} < 1$. The results are shown in Fig. 2.



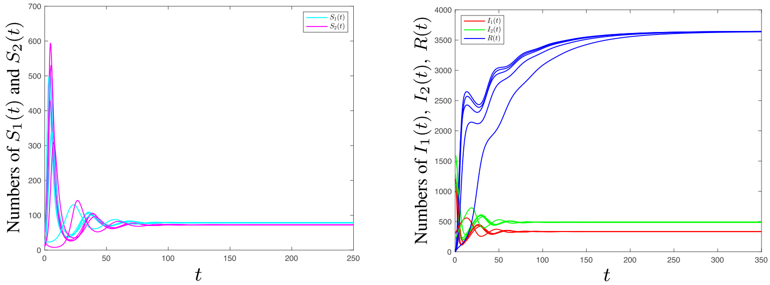
(a) (b)

Figure 1. Local asymptotic stability of E_0, $R^{\kappa}_{01} = 0.7408 < 1$, $R^{\kappa}_{02} = 0.3727 < 1$.



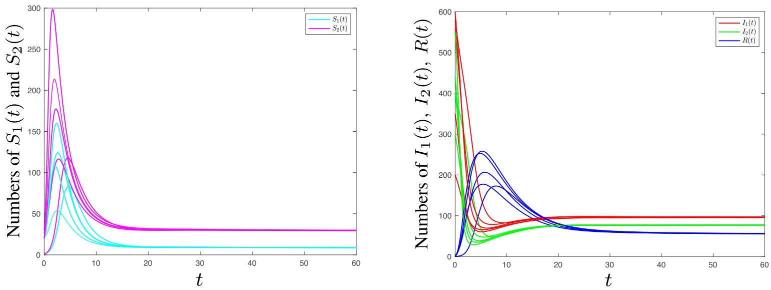
(a) (b)

Figure 2. Local asymptotic stability of E_1^*, $R^{\kappa}_{01} = 3.1886 > 1$, $R^{\kappa}_{02} = 0.8041 < 1$.



(a) (b)

Figure 3. Global asymptotic stability of E_2^*, $R_{01}^\kappa = 4.725 > 1$, $R_{02}^\kappa = 4.725 > 1$.



(a) (b)

Figure 4. Global asymptotic stability of E_2^*, $R_{01}^\kappa = 0.4644 < 1$, $R_{02}^\kappa = 1.8421 > 1$.

We will discuss the stability of E_2^* in the following two cases: (i) $R_{02}^\kappa > 1$, $R_{01}^\kappa > 1$; (ii) $R_{02}^\kappa > 1$, $R_{01}^\kappa < 1$.

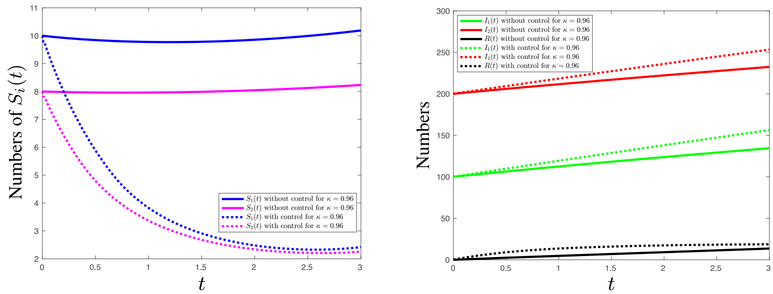
For case (i), we choose $\mathit{\Pi}_1 = 50$, $\mathit{\Pi}_2 = 50$, $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $u = 0.0005$, $d = 0.02$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $\kappa = 0.99$. By calculation it follows that $R_{01}^\kappa = 4.725$, $R_{02}^\kappa = 4.725$, and $E_2^* = (334, 489, 78, 72, 3649)$. Based on Theorem 6, E_2^* is globally asymptotically stable when $R_{01}^\kappa > 1$ and $R_{02}^\kappa > 1$. The results are shown in Fig. 3.

For case (ii), we choose $\mathit{\Pi}_1 = 30$, $\mathit{\Pi}_2 = 35$, $\alpha_1 = 0.002$, $\alpha_2 = 0.005$, $u = 0.001$, $d = 0.2$, $\beta_1 = 0.45$, $\beta_2 = 0.25$, $\kappa = 0.96$. By calculation it follows that $R_{01}^\kappa = 0.4644$, $R_{02}^\kappa = 1.8421$, and $E_2^* = (96, 77, 8, 29, 54)$. Based on Theorem 6, E_2^* is globally asymptotically stable when $R_{01}^\kappa < 1$ and $R_{02}^\kappa > 1$. The results are shown in Fig. 4.

5.3 The effect of control $u_i(t)$ $(i = 1, 2)$ on system (8)

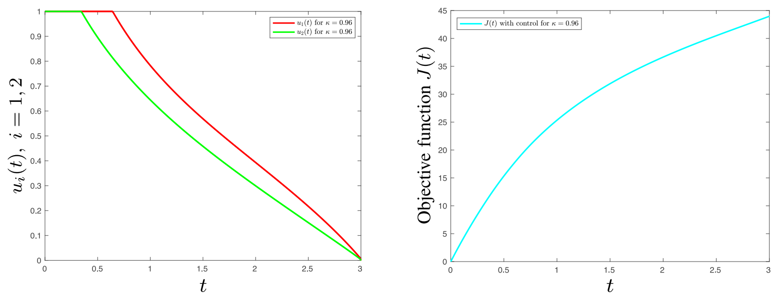
To test our theoretical results, we discuss the influence of different orders on system (14), and the influence of control $u_i(t)$ $(i = 1, 2)$ on the controlled system is simulated. Choose $\mathit{\Pi}_1 = 25$, $\mathit{\Pi}_2 = 25$, $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $u = 0.001$, $d = 0.01$, $\beta_1 = 0.25$, $\beta_2 = 0.25$, $\psi_1 = 1.5$, $\psi_2 = 1.5$, $\phi_1 = 4$, $\phi_2 = 6$.

Firstly, each state of uncontrolled (8) and controlled (14) system is compared, respectively. The solutions are plotted in Fig. 5. It is clear that control variables $u_i(t)$ $(i = 1, 2)$



(a) (b)

Figure 5. Comparison of the number of individuals in controlled and uncontrolled systems.



(a) (b)

Figure 6. The trajectories of optimal control $u_i(t)$ $(i = 1, 2)$ and consumption $J(t)$ with $\kappa = 0.96$.

have a great influence on rumor spreader $S_i(t)$ $(i = 1, 2)$, which can effectively control the rumor propagation. Next, the optimal control curve and the cost of official control curve are given in Fig. 6.

6 Conclusions

A fractional-order 2I2SR rumor spreading model is investigated in this paper. Firstly, the boundedness and uniqueness of the solutions of the fractional-order system are proved. Then the next-generation matrix method is used to calculate the threshold. Based on generalized fractional-order Routh–Hurwitz judgment, the local asymptotic stability of the rumor-free equilibrium E_0 and the rumor-spreading equilibrium E_1^* is studied. The global asymptotic stability of rumor-spreading equilibrium E_2^* is discussed by means of Lyapunov function method and invariance principle. It can be obtained by detailed proof that if $R_{01}^{\kappa} < 1$ and $R_{02}^{\kappa} < 1$, E_0 is locally asymptotically stable, if $R_{01}^{\kappa} > 1$ and $R_{02}^{\kappa} < 1$ are satisfied, E_1^* is locally asymptotically stable, and if $R_{02}^{\kappa} > 1$, E_2^* is globally asymptotically stable. Finally, the necessary conditions for fractional optimal control of the rumor spreading model are obtained.

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