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Existence and global asymptotic behavior of S-asymptotically periodic solutions for fractional evolution equation with delay*

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Abstract. This paper discusses the S-asymptotically periodic problem of fractional evolution equation with delay. By introducing a new noncompact measure theory involving infinite interval, we study the existence of S-asymptotically periodic mild solutions under the situation that the relevant semigroup is noncompact and the nonlinear term satisfies more general growth conditions instead of Lipschitz-type conditions. Moreover, by establishing a new Gronwall-type integral inequality corresponding to fractional differential equation with delay, we consider the global asymptotic behavior of S-asymptotically periodic mild solutions, which will make up for the blank of this field.

Keywords: fractional evolution equation with delay, S-asymptotically periodic solutions, existence and global asymptotic behavior, C_0-semigroup, measure of noncompactness, Gronwall-type inequality with delay.

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1 Introduction

In this paper, we investigate the existence and global asymptotic behavior of S-asymptotically periodic solutions for abstract fractional delayed evolution equation (FDEE)

$$\begin{aligned} &{}^{c}D_t^q u(t) + Au(t) = F\bigl(t, u(t), u_t\bigr), \quad t \geqslant 0, \\ &u(t) = \varphi(t), \quad t \in [-r, 0], \end{aligned} \tag{1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivation of order $q \in (0,1)$, X is a real Banach space, $A : D(A) \subset X \to X$ is a closed linear operator, and $-A$ generates a C_0-semigroup $T(t)$ $(t \geqslant 0)$ in X, $F : \mathbb{R}^+ \times X \times \mathcal{B} \to X$ is a given function, which will be specified later, $\varphi \in \mathcal{B}$; $r > 0$ is a constant, and $\mathcal{B} := C([-r, 0], X)$ denotes the space of continuous functions from $[-r, 0]$ into X provided with the uniform norm topology. For $t \geqslant 0$, u_t denotes the history function defined by $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, where u is a continuous function from $[-r, \infty)$ into X.

In the past decades, fractional calculus has attracted extensive attention from many scholars in different fields; see the monographs of Podlubny [22], Agrawal [1], Zhou [28]. Compared with integer-order calculus, the main advantage of fractional-order calculus is that it can accurately describe the memory or genetic characteristics of various new materials or better describe the process or behavior of real dynamic systems. In particular, many scholars have found that in many practical applications, fractional derivatives of time can more truthfully describe the process and phenomena of things' motion development than integer derivatives.

On the one hand, because the fractional evolution equations are abstract models in many practical applications, more and more mathematicians pay attention to the study of fractional evolution equations (see [8, 9, 17, 23]). On the other hand, the theory of delayed partial differential equations has a wide range of physical background and practical mathematical models. In the last decade, fractional evolution equations with delay have also been investigated extensively, and some interesting results have been obtained (see [6, 11, 18, 20]).

It is well known that the periodic law of the development or movement of things is a common phenomenon in nature and human activities. However, in real life, many phenomena do not have strict periodicity. Meanwhile, because fractional derivative has genetic or memory properties, the solutions of periodic boundary value problems of fractional differential equations cannot be extended periodically to time t in $\mathbb{R}^+$. Specially, Ren et al. [24] have proved the nonexistence of nonzero periodic solutions for Caputo-type linear fractional evolution equation.

The concept of S-asymptotically periodic function was first proposed and established by Henríquez et al. [14] in 2008. It is worth noting that S-asymptotically period function is a more general approximate period function. This opens up a new research direction for the periodic problems of fractional evolution equations (see [16, 24, 25]). Recently, in [17], we have also considered the existence of S-asymptotically periodic solutions for a time-space fractional evolution equation without delay; and in [19], we obtained the existence of maximal and minimal S-asymptotically periodic mild solutions for FDEE (1) in ordered Banach spaces.

It is not difficult to find that among the previous researches, the existence results of S-asymptotically periodic solutions for fractional evolution equations have been extensively studied in the case where the semigroups are compact or the nonlinear terms satisfy some compactness conditions or Lipschitz conditions. This is very convenient to the equations with compact resolvent (see, for example, [16, 17, 24]). But for the case that corresponding semigroups are noncompact or the nonlinear terms satisfy more general growth conditions, we have not seen the relevant papers to study S-asymptotically periodic mild solutions of FDEE (1). In addition, the global asymptotic behavior is one of the major problem encountered in applications and has attracted considerable attentions. For the mild solutions of fractional evolution equations without delay, the stability theories have been considered by using appropriate Gronwall-type integral inequalities (see [17, 20]). However, it is very difficult to establish Gronwall-type integral inequality corresponding to fractional evolution equations with delay. Moreover, as far as we know, the global asymptotic behavior of S-asymptotically mild periodic solution for FDEE (1) is an untreated topic in the literatures.

Motivated by the papers mentioned above, in this paper, we will investigate the existence and global asymptotic behavior of S-asymptotically periodic mild solutions of the abstract fractional delayed evolution equation (1). We point out that the work of this paper consists of the following two wedges: in Section 3, by introducing a new noncompact measure theory involving infinite interval, we are devoted to study the existence of S-asymptotically periodic mild solutions for FDEE (1) under the situation that the semigroup generated by $-A$ is noncompact and the nonlinear term F satisfies more general growth conditions instead of Lipschitz-type conditions; and in Section 4, by establishing a new Gronwall-type integral inequality corresponding to fractional differential equation with delay, we are concerned with the global asymptotic stability of S-asymptotically periodic mild solutions and global asymptotic periodicity for the FDEE (1), which will make up for the blank of this field. Some notions, definitions, and preliminary facts are introduced in Section 2, and an example is given to illustrate our main results in Section 5.

2 Preliminaries

Throughout this paper, let $(X, \|\cdot\|)$ be a Banach space. Assume that $A : D(A) \subset X \to X$ is a closed linear operator and that $-A$ generates a C_0-semigroup $T(t)(t \geqslant 0)$ in X. Generally, there exist constants $M \geqslant 1$ and $\nu \in \mathbb{R}$ such that (see [21])

$$\big\|T(t)\big\| \leqslant M\mathrm{e}^{\nu t}, \quad t \geqslant 0.$$

Specially, if $\|T(t)\| \leqslant M$ for any $t \geqslant 0$, then C_0-semigroup $T(t)$ $(t \geqslant 0)$ is called to be uniformly bounded. Let

$$\nu_0 = \inf\bigl\{\nu \in \mathbb{R} \bigm| \exists M \geqslant 1\colon \big\|T(t)\big\| \leqslant M\mathrm{e}^{\nu t}\ \forall t \geqslant 0\bigr\}.$$

Then ν_0 is said to be the growth exponent of the C_0-semigroup $T(t)$ $(t \geqslant 0)$. Moreover, the semigroup $T(t)$ $(t \geqslant 0)$ is said to be exponentially stable if $\nu_0 < 0$. Clearly, the exponentially stable C_0-semigroup $T(t)$ ($t \geqslant 0$) is uniformly bounded. If the semigroup $T(t)$ generated by $-A$ is continuous in the uniform operator topology for every $t > 0$ in X, then ν_0 can also be determined by $\sigma(A)$ (the spectrum of A),

$$\nu_0 = -\inf\bigl\{\operatorname{Re}\lambda \bigm| \lambda \in \sigma(A)\bigr\}.$$

As for the definition of Caputo fractional derivation, we can refer to many references (see [1, 7, 28] and so on), which will not be repeated here. In the following, we only give some operators needed in this paper and their related properties.

Definition 1. For any $q, p > 0$, the Mittag-Leffler function is defined by

$$E_{q,p}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(p+qn)}, \quad z \in \mathbb{C}.$$

According to [22], the Mittag-Leffler functions satisfy the following asymptotic representation as $z \to \infty$:

$$E_{q,p}(z) = \begin{cases} \frac{1}{q} z^{(1-p)/q} \mathrm{e}^{z^{1/q}} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(p-qn)} + O(|z|^{-N}), & |\arg z| \leqslant \frac{\pi q}{2}, \\ -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(p-qn)} + O(|z|^{-N}), & |\arg(-z)| \leqslant \pi - \frac{\pi q}{2}. \end{cases}$$

Denote $E_{q,1}$ by E_q. From the above formulae it follows that if $0 < q < 1$ and $z = \tau < 0$, then

$$E_q(\tau) \to 0 \quad \text{as } \tau \to -\infty. \tag{2}$$

From [27] one can find

$$E_q(-z) = \int_0^{\infty} \xi_q(s) \mathrm{e}^{-zs} \, \mathrm{d}s, \qquad E_{q,q}(-z) = q \int_0^{\infty} s \xi_q(s) \mathrm{e}^{-zs} \, \mathrm{d}s,$$

thus,

$$E_q(\tau) > 0, \quad E_{q,q}(\tau) > 0 \quad \text{for } \tau < 0,$$

where

$$\xi_q(s) = \frac{1}{\pi q} \sum_{n=1}^{\infty} (-s)^{n-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad s \in (0, \infty),$$

satisfies

$$\xi_q(s) \geqslant 0, \quad s \in (0, \infty), \qquad \int_0^{\infty} \xi_q(s) \, \mathrm{d}s = 1, \qquad \int_0^{\infty} s \xi_q(s) \, \mathrm{d}s = \frac{1}{\Gamma(1+q)}.$$

Note that $E_{q,q}(z) = q E_q'(z)$, thus for each $\nu > 0$,

$$(t-s)^{q-1} E_{q,q}\bigl(-\nu(t-s)^q\bigr) = \frac{\mathrm{d}}{\mathrm{d}s}\biggl(\frac{1}{\nu} E_q\bigl(-\nu(t-s)^q\bigr)\biggr),$$

which implies that

$$\int_{t_1}^{t_2} (t-s)^{q-1} E_{q,q}\big(-\nu(t-s)^q\big)\,\mathrm{d}s = \frac{1}{\nu}\big(E_q\big(-\nu(t-t_2)^q\big) - E_q\big(-\nu(t-t_1)^q\big)\big) \quad (3)$$

for all $t_1 \leqslant t_2$. For detailed definitions and more properties of the Mittag-Leffler functions, one can refer to [15, 22, 27] and references therein.

Define two families of operators $U(t)$ $(t \geqslant 0)$ and $V(t)$ $(t \geqslant 0)$ in X as follows:

$$U(t) = \int_0^\infty \xi_q(s) T\big(t^q s\big)\,\mathrm{d}s, \qquad V(t) = q\int_0^\infty s\xi_q(s) T\big(t^q s\big)\,\mathrm{d}s. \quad (4)$$

Lemma 1. *The operators $U(t)$ $(t \geqslant 0)$ and $V(t)$ $(t \geqslant 0)$ defined by* (4) *have the following properties*:

(i) *$U(t)$ $(t \geqslant 0)$ and $V(t)$ $(t \geqslant 0)$ are strongly continuous operators, which means that for any $x \in X$ and $0 \leqslant t_1 \leqslant t_2$,*

$$\big\|U(t_2)x - U(t_1)x\big\| \to 0, \quad \big\|V(t_2)x - V(t_1)x\big\| \to 0 \quad \textit{as } t_2 - t_1 \to 0.$$

(ii) *$U(t)$ and $V(t)$ are linear bounded operators for any fixed $t \in \mathbb{R}^+$, i.e.,*

$$\big\|U(t)x\big\| \leqslant M\|x\|, \quad \big\|V(t)x\big\| \leqslant \frac{M}{\Gamma(q)}\|x\| \quad \forall\, x \in X.$$

(iii) *$U(t)$ and $V(t)$ are uniformly continuous for $t > 0$.*

(iv) *If $T(t)$ $(t \geqslant 0)$ is exponentially stable with the growth exponent $\nu_0 < 0$, then for every $t \geqslant 0$,*

$$\big\|U(t)\big\| \leqslant ME_q\big(\nu_0 t^q\big), \qquad \big\|V(t)\big\| \leqslant ME_{q,q}\big(\nu_0 t^q\big).$$

Proof. The proof of statements (i)–(iii) can be found in [7, 29], while the last one was proved in [4]. □

Next, let $C_b(\mathbb{R}^+, X)$ denote the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to X equipped with the norm $\|u\|_b = \sup_{t\in\mathbb{R}^+}\|u(t)\|$, and let $\mathcal{B} = C([-r, 0], X)$ denotes the space of continuous functions from $[-r, 0]$ into X endowed with the uniform norm $\|\phi\|_{\mathcal{B}} = \sup_{s\in[-r,0]}\|\phi(s)\|$, where $r > 0$ is a constant. Now, we introduce a standard definition of S-asymptotically ω-periodic function.

Definition 2. A function $u \in C_b(\mathbb{R}^+, X)$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}\|u(t+\omega) - u(t)\| = 0$. Thus, ω is called an asymptotic periodic of u.

Let $SAP_\omega(X)$ be the subspace of $C_b(\mathbb{R}^+, X)$ consisting of all the X-value S-asymptotically ω-periodic functions equipped with norm $\|\cdot\|_b$. Then $SAP_\omega(X)$ is a Banach space [14].

Definition 3. A function $u : [-r, \infty) \to E$ is said to be a mild solution of FDEE (1) if $u \in C([-r, \infty), E)$ and satisfies

$$u(t) = \begin{cases} U(t)\varphi(0) + \int_0^t (t-s)^{q-1} V(t-s) F(s, u(s), u_s) \, \mathrm{d}s, & t \geqslant 0, \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

Moreover, if $u|_{t \geqslant 0} \in SAP_\omega(X)$, then u is called S-asymptotically ω-periodic mild solution of Eq. (1).

Definition 4. Let u be an S-asymptotically ω-periodic mild solution of FDEE (1) with initial value $\varphi \in \mathcal{B}$, and let v be an arbitrary mild solution of FDEE (1) with initial value $\psi \in \mathcal{B}$. If there exist constants $\eta > 0$ and $\overline{M} > 0$ such that

$$\| u(t) - v(t) \| \leqslant \overline{M} \| \varphi - \psi \|_{\mathcal{B}} \cdot E_q(-\eta t^q), \quad t \geqslant 0,$$

then the S-asymptotically ω-periodic mild solution of FDEE (1) is said to be globally Mittag-Leffler stable.

Definition 5. FDEE (1) is said to be globally asymptotically ω-periodic if there exists a nonconstant ω-periodic function $u^*(t)$ such that all solutions of FDEE (1) converge to $u^*(t)$.

In the end of this section, we recall the definition and some properties about the Kuratowski measure of noncompactness.

Definition 6. (See [5].) The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded D of Banach space X is

$$\alpha(D) := \inf \left\{ \delta > 0 \;\middle|\; D = \bigcup_{n=1}^{m} D_n \text{ and } \operatorname{diam}(D_n) \leqslant \delta \text{ for } n = 1, 2, \dots, m \right\}.$$

Lemma 2. (*See* [5, 12].) *Let X be a Banach space and $D, D_1, D_2 \in X$ be bounded, and let the following properties are satisfied*:

(i) $\alpha(D) = 0$ *if and only if* $\overline{D}$ *is compact, where* $\overline{D}$ *is the closure hull of* D;
(ii) $\alpha(D) = \alpha(\overline{D}) = \alpha(\operatorname{conv} D)$, *where* $\operatorname{conv} D$ *is the convex hull of* D;
(iii) $\alpha(\kappa D) = |\kappa| \alpha(D)$ *for any* $\kappa \in \mathbb{R}$;
(iv) $D_1 \subset D_2$ *implies* $\alpha(D_1) \leqslant \alpha(D_2)$;
(v) $\alpha(D_1 \cup D_2) = \max\{\alpha(D_1), \alpha(D_2)\}$;
(vi) $\alpha(D_1 + D_2) \leqslant \alpha(D_1) + \alpha(D_2)$, *where* $D_1 + D_2 = \{x \mid x = x_1 + x_2, x_1 \in D_1, x_2 \in D_2\}$;
(vii) *If the map $Q : D \subset X \to Y$ is Lipschitz continuous with constant k, then $\alpha(Q(S)) \leqslant k\alpha(S)$ for any bounded subset $S \subset D$, where Y is another Banach space.*

We denote by $C([a,b],X)$ the Banach space of all continuous X-value functions on interval $[a,b]$. We use the notations $\alpha(\cdot)$, $\alpha_{C([a,b])}(\cdot)$, $\alpha_{\mathcal{B}}(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded sets of X, $C([a,b],X)$, and $\mathcal{B}$, respectively. For any $D \subset C([a,b],X)$ and $t \in [a,b]$, set $D(t) = \{u(t) \mid u \in D\}$, then $D(t) \subset X$. If $D \subset C([a,b],X)$ is bounded, then $D(t)$ is bounded in X, and $\alpha(D(t)) \leqslant \alpha_{C([a,b])}(D)$. For the detailed properties of the Kuratowski measure of noncompactness, see [5, 12].

Lemma 3. (*See* [5].) *Let X be a Banach space, and let $D \subset C([a,b],X)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $[a,b]$, and*

$$\alpha_{C([a,b])}(D) = \max_{t\in[a,b]} \alpha\bigl(D(t)\bigr).$$

Lemma 4. (*See* [10].) *Suppose that X is a metric space. Then for every bounded subset $B \subset X$, there is a countable subset B_0 of B such that $\alpha(B_0) = \alpha(B)$.*

Lemma 5. (*See* [13].) *Let X be a Banach space, and let $D = \{u_n\} \subset C([a,b],X)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $[a,b]$, and*

$$\alpha\left(\left\{\int_a^b u_n(s)\,\mathrm{d}s\right\}\right) \leqslant 2\int_a^b \alpha\bigl(D(t)\bigr)\,\mathrm{d}t.$$

3 Existence result

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and nondecreasing function such that $h(t) \geqslant 1$ for all $t \in \mathbb{R}^+$ and $\lim_{t\to\infty} h(t) = \infty$. Thus, we can define a Banach space

$$C_h(X) = \left\{u \in C\bigl(\mathbb{R}^+, X\bigr)\colon \lim_{t\to\infty} \frac{\|u(t)\|}{h(t)} = 0\right\}$$

with the norm $\|u\|_h = \sup_{t\geqslant 0} \|u(t)\|/h(t)$. In order to avoid confusion, we write $\alpha_h(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded sets of $C_h(X)$.

Lemma 6. *Let $D \subset C_h(X)$ be a bounded set, and let*

(i) *D is a locally equicontinuous family of function, i.e., for any constant $a > 0$, the functions in D are equicontinuous in $[0,a]$;*
(ii) $\lim_{t\to\infty} \|u(t)\|/h(t) = 0$ *uniformly for any $u \in D$.*

Then $\alpha_h(D) = \sup_{t\geqslant 0} \alpha(D(t)/h(t))$.

Proof. First, we prove that $\alpha_h(D) \leqslant \sup_{t\geqslant 0} \alpha(D(t)/h(t))$. From condition (ii) we know that for any $\varepsilon > 0$, there exists $a > 0$ big enough such that

$$\frac{1}{h(t)}\bigl\|u(t)\bigr\| < \frac{\varepsilon}{2} \quad \text{for } t \geqslant a. \tag{5}$$

Denote by $D|_{[0,a]}$ the restriction of D on $[0,a]$. Clearly, $D|_{[0,a]} \subset C([0,a],X)$. Definition 6 and Lemma 3 ensure that

$$\alpha_h(D|_{[0,a]}) = \max_{t\in[0,a]} \alpha\left(\frac{D(t)}{h(t)}\right) \leqslant \sup_{t\geqslant 0} \alpha\left(\frac{D(t)}{h(t)}\right).$$

Thus there exists a partition of D such that $D = \bigcup_{i=1}^{n} D_i$, $D|_{[0,a]} = \bigcup_{i=1}^{n} D_i|_{[0,a]}$, and

$$\operatorname{diam}_h(D_i|_{[0,a]}) < \sup_{t\geqslant 0} \alpha\left(\frac{D(t)}{h(t)}\right) + \varepsilon, \quad i = 1,2,\dots,n, \tag{6}$$

where $\operatorname{diam}_h(\cdot)$ denotes the diameter of bounded subset of $C_h(X)$. Furthermore, for any $u_1, u_2 \in D_i$ ($i = 1,2,\dots,n$) and $t \geqslant a$, it follows from (5) and (6) that

$$\begin{aligned}\frac{1}{h(t)}\left\|u_2(t) - u_1(t)\right\| &\leqslant \left\|\frac{u_2(t)}{h(t)} - \frac{u_2(a)}{h(a)}\right\| + \left\|\frac{u_2(a)}{h(a)} - \frac{u_1(a)}{h(a)}\right\| + \left\|\frac{u_1(a)}{h(a)} - \frac{u_1(t)}{h(t)}\right\| \\ &< 3\varepsilon + \sup_{t\geqslant 0} \alpha\left(\frac{D(t)}{h(t)}\right).\end{aligned}$$

Combining the above inequality with (6), we can have $\operatorname{diam}_h(D_i) \leqslant \sup_{t\geqslant 0} \alpha(D(t)/h(t)) + 3\varepsilon$. Hence, noticing that $D = \bigcup_{i=1}^{n} D_i$ and ε is arbitrary, we obtain that $\alpha_h(D) \leqslant \sup_{t\geqslant 0} \alpha(D(t)/h(t))$.

Conversely, we prove that $\sup_{t\geqslant 0} \alpha(D(t)/h(t)) \leqslant \alpha_h(D)$. Given $\varepsilon > 0$, there exists a partition of $D = \bigcup_{i=1}^{n} D_i$ such that $\operatorname{diam}_h(D_i) < \alpha_h(D) + \varepsilon$, thus, for any $u_1, u_2 \in D_i$ ($i = 1,2,\dots,n$) and $t \geqslant 0$, one can see

$$\frac{1}{h(t)}\left\|u_2(t) - u_1(t)\right\| \leqslant \|u_2 - u_1\|_h < \alpha_h(D) + \varepsilon.$$

According to $D(t) = \bigcup_{i=1}^{n} D_i(t)$, we have $\alpha(D(t)/h(t)) < \alpha_h(D) + \varepsilon$ for $t \geqslant 0$. From the arbitrariness of ε it follows that $\sup_{t\geqslant 0} \alpha(D(t)/h(t)) \leqslant \alpha_h(D)$.

This completes the proof of Lemma 6. □

Next, we state and prove the existence of S-asymptotically periodic mild solutions for FDEE (1).

Theorem 1. *Assume that $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates an exponentially stable equicontinuous C_0-semigroup $T(t)$ ($t \geqslant 0$) in a Banach space X, whose growth exponent $\nu_0 < 0$. Let the nonlinear functions $F(t,\cdot,\cdot) : X \times \mathcal{B} \to X$ be continuous for every $t \geqslant 0$. Let the following conditions hold*:

(H1) *There exists $\omega > 0$ such that*

$$\lim_{t\to\infty} \left\|F(t+\omega,\, x,\, \phi) - F(t,x,\phi)\right\| = 0$$

uniformly for $x \in X$, $\phi \in \mathcal{B}$;

(H2) *There exist positive functions* $h_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ *such that for any* $t \geqslant 0$ *and bounded countable sets* $D \subset X$, $\mathcal{D} \in \mathcal{B}$,

$$\alpha\bigl(F(t, D, \mathcal{D})\bigr) \leqslant h_1(t)\alpha(D) + h_2(t) \sup_{\tau\in[-r,0]} \alpha(\mathcal{D}(\tau)),$$

the functions $s \mapsto (t-s)^{q-1}h_i(s)$ *belong to* $L(\mathbb{R}^+, \mathbb{R}^+)$ *for all* $t > s$, *which means that there exist constants* $0 < \gamma_1, \gamma_2 < \infty$ *such that*

$$\sup_{t\geqslant 0} \int_0^t (t-s)^{q-1} h_i(s)\,\mathrm{d}s = \gamma_i, \quad i = 1, 2,$$

and $2M(\gamma_1 + \gamma_2) < \Gamma(q)$;

(H3) *There are nondecreasing functions* $\Phi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\liminf_{R\to\infty} \Phi_i(R)/R = \rho_i < \infty$ $(i = 1, 2)$ *such that*

$$\bigl\|F\bigl(t, h(t)x, h(t)\phi\bigr)\bigr\| \leqslant \Phi_1\bigl(\|x\|\bigr) + \Phi_2\bigl(\|\phi\|_{\mathcal{B}}\bigr), \quad t \geqslant 0,\ x \in X,\ \phi \in \mathcal{B}.$$

Then for a given $\varphi \in \mathcal{B}$, *FDEE* (1) *has at least one* S*-asymptotically* ω*-periodic mild solution, provided that*

$$M(\rho_1 + \rho_2) < |\nu_0|. \tag{7}$$

Proof. For a given $\varphi \in \mathcal{B}$ and each $u \in C_h$, we define $u[\varphi](t) : [-r, +\infty) \to X$ by

$$u[\varphi](t) = \begin{cases} u(t), & t \in [0, +\infty), \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

We denote a closed subspace of C_h by

$$C_{\varphi,h} := \bigl\{u \in C_h \bigm| u(0) = \varphi(0)\bigr\}. \tag{8}$$

It is not difficult to verify that $C_{\varphi,h}(X)$ is a Banach space with the norm $\|\cdot\|_h$.

Now, we consider an operator $\mathcal{Q}$ on $C_{\varphi,h}(X)$ defined by

$$\mathcal{Q}u(t) = U(t)\varphi(0) + \int_0^t (t-s)^{q-1} V(t-s) F\bigl(s, u(s), u[\varphi]_s\bigr)\,\mathrm{d}s, \quad t \geqslant 0. \tag{9}$$

For every $u \in C_{\varphi,h}(X)$ and $t \geqslant 0$, we have $\|u(t)\| \leqslant h(t)\|u\|_h$ and

$$\begin{aligned}\bigl\|u[\varphi]_t\bigr\|_{\mathcal{B}} &= \sup_{\tau\in[-r,0]} \bigl\|u[\varphi](t+\tau)\bigr\| \leqslant \sup_{\tau\in[-r,0]} \bigl\|\varphi(\tau)\bigr\| + \sup_{t\in[0,\infty)} \bigl\|u(t)\bigr\| \\ &\leqslant \|\varphi\|_{\mathcal{B}} + h(t)\|u\|_h\end{aligned}$$

By condition (H3),

$$\begin{aligned}
&\frac{1}{h(t)}\left\| \int_0^t (t-s)^{q-1} V(t-s) F\big(s, u(s), u[\varphi]_s\big)\,\mathrm{d}s \right\| \\
&\quad \leqslant \frac{1}{h(t)} \int_0^t (t-s)^{q-1} \big\|V(t-s)\big\| \left(\Phi_1\left(\frac{\|u(s)\|}{h(s)}\right) + \Phi_2\left(\frac{\|u[\varphi]_s\|_{\mathcal{B}}}{h(s)}\right)\right) \mathrm{d}s \\
&\quad \leqslant \frac{M(\Phi_1(\|u\|_h) + \Phi_2(\|\varphi\|_{\mathcal{B}} + \|u\|_h))}{|\nu_0| h(t)}.
\end{aligned}$$

Thus, from (8) and (9) one can easily find that $\mathcal{Q} : C_{\varphi,h}(X) \to C_{\varphi,h}(X)$ is well defined. Therefore, by Definition 3, we can assert that the fixed point u of the operator $\mathcal{Q}$ in $C_{\varphi,h}(X)$ implies that $u[\varphi]$ is the mild solution of FDEE (1). Moreover, if $u \in SAP_\omega(X)$, then $u[\varphi]$ is the S-asymptotically ω-periodic mild of the problem (1).

Now, we complete the proof by five steps.

Step 1. We show that $\mathcal{Q}$ is continuous on $C_{\varphi,h}(X)$.

Let $\{u^{(n)}\} \subset C_{\varphi,h}(X)$ and $u^{(n)} \to u$ in $C_{\varphi,h}(X)$ as $n \to \infty$, thus, $u^{(n)}[\varphi]_t \to u[\varphi]_t$ in $\mathcal{B}$ as $n \to \infty$ for all $t \geqslant 0$. Combining this with the continuity of function F, we known that for arbitrary $\varepsilon > 0$ and sufficiently large n

$$\big\|F\big(t, u^{(n)}(t), u^{(n)}[\varphi]_t\big) - F\big(t, u(t), u[\varphi]_t\big)\big\| \leqslant \frac{|\nu_0|\varepsilon}{M}, \quad t \geqslant 0. \tag{10}$$

Hence, according to the dominated convergence theorem, (9), and (10), we obtain

$$\begin{aligned}
&\big\|\mathcal{Q}u^{(n)}(t) - \mathcal{Q}u(t)\big\| \\
&\quad \leqslant \int_0^t (t-s)^{q-1} \big\|V(t-s)\big\| \big\|F\big(s, u^{(n)}(s), u^{(n)}[\varphi]_s\big) - F\big(s, u(s), u[\varphi]_s\big)\big\|\,\mathrm{d}s \\
&\quad \leqslant |\nu_0|\varepsilon \int_0^t \int_0^\infty q\sigma\xi_q(\sigma)(t-s)^{q-1} \mathrm{e}^{\nu_0 (t-s)^q \sigma}\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant \varepsilon,
\end{aligned}$$

thus

$$\big\|\mathcal{Q}u^{(n)} - \mathcal{Q}u\big\|_h = \left(\sup_{t \geqslant 0} \frac{1}{h(t)} \big\|\mathcal{Q}u_n(t) - \mathcal{Q}u(t)\big\|\right) \to 0 \quad \text{as } n \to \infty,$$

which implies that $\mathcal{Q} : C_{\varphi,h}(X) \to C_{\varphi,h}(X)$ is continuous.

For any $R > \|\varphi\|_{\mathcal{B}}$, set

$$\overline{\Omega}_R = \big\{u \in C_{\varphi,h}(X) \;\big|\; \|u\|_h \leqslant R - \|\varphi\|_{\mathcal{B}}\big\}.$$

Clearly, $\overline{\Omega}_R$ is a closed ball in $C_{\varphi,h}(X)$ with centre θ.

Step 2. We prove that there exists a positive constant R_0 such that $\mathcal{Q}(\overline{\Omega}_{R_0}) \subset \overline{\Omega}_{R_0}$ and $\lim_{t\to\infty} \|\mathcal{Q}u(t)\|/h(t) = 0$ for any $u \in \overline{\Omega}_{R_0}$.

Suppose on the contrary that for any $R > \|\varphi\|_{\mathcal{B}}$, there exist $u \in \overline{\Omega}_R$ such that $\|\mathcal{Q}u\|_h > R - \|\varphi\|_{\mathcal{B}}$, which implies that $\sup_{t\geqslant 0} \|(\mathcal{Q}u)(t)\|/h(t) > R - \|\varphi\|_{\mathcal{B}}$. Meanwhile, by condition (H3), one can see for $t \geqslant 0$,

$$\begin{aligned}
&\frac{1}{h(t)}\big\|(\mathcal{Q}u)(t)\big\| \\
&\quad\leqslant \big\|(\mathcal{Q}u)(t)\big\| \\
&\quad\leqslant M\|\varphi\|_{\mathcal{B}} + \int\limits_0^t (t-s)^{q-1}\big\|V(t-s)\big\|\left(\varPhi_1\left(\frac{\|u(s)\|}{h(s)}\right) + \varPhi_2\left(\frac{\|u[\varphi]_s\|_{\mathcal{B}}}{h(s)}\right)\right)\mathrm{d}s \\
&\quad\leqslant M\|\varphi\|_{\mathcal{B}} + \frac{M(\varPhi_1(R)+\varPhi_2(R))}{|\nu_0|},
\end{aligned}$$

thus,

$$R - \|\varphi\|_{\mathcal{B}} \leqslant M\|\varphi\|_{\mathcal{B}} + \frac{M(\varPhi_1(R)+\varPhi_2(R))}{|\nu_0|}.$$

Dividing on both sides by R and taking the lower limit as $R \to \infty$, one can obtain that $1 \leqslant M(\rho_1+\rho_2)/|\nu_0|$, which contradicts (7). Hence, there is a positive constant R_0 such that $\mathcal{Q}(\overline{\Omega}_{R_0}) \subset \overline{\Omega}_{R_0}$.

Furthermore, combining the above proof with the property of the function $h(t)$, we can easily obtain that for any $u \in \overline{\Omega}_{R_0}$,

$$\lim_{t\to\infty} \frac{1}{h(t)}\big\|\mathcal{Q}u(t)\big\| = 0. \tag{11}$$

Step 3. We demonstrate that $\mathcal{Q}(\overline{\Omega}_{R_0})$ is a locally equicontinuous family of functions in $C_{\varphi,h}(X)$.

Suppose that $0 < a < \infty$ is an arbitrary constant. Without loss of generality, let $0 \leqslant t_1 < t_2 \leqslant a$. For any $u \in \overline{\Omega}_{R_0}$, by (9), one can find

$$\begin{aligned}
&\big\|\mathcal{Q}u(t_2) - \mathcal{Q}u(t_1)\big\| \\
&\quad\leqslant \big\|U(t_2)\varphi(0) - U(t_1)\varphi(0)\big\| \\
&\qquad + \int\limits_0^{t_1} \big((t_1-s)^{q-1} - (t_2-s)^{q-1}\big)\cdot\big\|V(t_2-s)\big\|\big\|F\big(s,u(s),u[\varphi]_s\big)\big\|\,\mathrm{d}s \\
&\qquad + \int\limits_0^{t_1} (t_1-s)^{q-1}\big\|V(t_2-s) - V(t_1-s)\big\|\big\|F\big(s,u(s),u[\varphi]_s\big)\big\|\,\mathrm{d}s \\
&\qquad + \int\limits_{t_1}^{t_2} (t_2-s)^{q-1}\big\|V(t_2-s)\big\|\big\|F\big(s,u(s),u[\varphi]_s\big)\big\|\,\mathrm{d}s \\
&\quad:= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Next, we check if J_i tend to 0 independently of $u \in \overline{\Omega}_{R_0}$ as $t_2 - t_1 \to 0$ $(i = 1, 2, 3, 4)$. By Lemma 1(i), it is easy to test that $J_1 \to 0$ as $t_2 - t_1 \to 0$. From the continuity of function F and condition (H3) there exist positive constant M_f such that for all $u \in \overline{\Omega}_{R_0}$,

$$\sup_{t \geqslant 0} \big\| F\big(t, u(t), u[\varphi]_s\big) \big\| \leqslant M_f. \tag{12}$$

According to Lemma 1(iv), let

$$M_q = M \max\Big\{ \sup_{t \geqslant 0} E_q\big(\nu_0 t^q\big)(1+t)^q, \sup_{t \geqslant 0} E_{q,q}\big(\nu_0 t^q\big)(1+t)^{2q} \Big\},$$

then

$$\big\|U(t)\big\| \leqslant \frac{M_q}{(1+t)^q}, \quad \big\|V(t)\big\| \leqslant \frac{M_q}{(1+t)^{2q}}, \quad t \in \mathbb{R}^+. \tag{13}$$

Thus, by (12) and (13), one can obtain

$$\begin{aligned} J_2 &\leqslant \int_0^{t_1} \big((t_1 - s)^{q-1} - (t_2 - s)^{q-1}\big) \big\|V(t_2 - s)\big\| \big\|F\big(s, u(s), u[\varphi]_s\big)\big\| \,\mathrm{d}s \\ &\leqslant M_f M_q \int_0^{t_1} \frac{(t_1 - s)^{q-1} - (t_2 - s)^{q-1}}{(1 + t_2 - s)^{2q}} \,\mathrm{d}s \\ &\leqslant \frac{2 M_f M_q}{q} (t_2 - t_1)^q \to 0 \quad \text{as } t_2 - t_1 \to 0 \end{aligned}$$

and

$$\begin{aligned} J_4 &\leqslant \int_{t_1}^{t_2} (t_2 - s)^{q-1} \big\|V(t_2 - s)\big\| \big\|F\big(s, u(s), u[\varphi]_s\big)\big\| \,\mathrm{d}s \\ &\leqslant \frac{M_f M_q}{q} (t_2 - t_1)^q \to 0 \quad \text{as } t_2 - t_1 \to 0. \end{aligned}$$

For $t_1 = 0$ and $t_2 > 0$, it easy to see that $J_3 = 0$. For $t_1 > 0$ and $\varepsilon > 0$ small enough, by (12), (13), and Lemma 1(iii), we get that

$$\begin{aligned} J_3 &= \int_0^{t_1} (t_1 - s)^{q-1} \big\|V(t_2 - s) - V(t_1 - s)\big\| \big\|F\big(s, u(s), u[\varphi]_s\big)\big\| \,\mathrm{d}s \\ &\leqslant M_f \int_0^{t_1 - \varepsilon} (t_1 - s)^{q-1} \big\|V(t_2 - s) - V(t_1 - s)\big\| \,\mathrm{d}s \\ &\quad + M_f \int_{t_1 - \varepsilon}^{t_1} (t_1 - s)^{q-1} \big\|V(t_2 - s) - V(t_1 - s)\big\| \,\mathrm{d}s \end{aligned}$$

$$\leqslant M_f\biggl(\sup_{s\in[0,t_1-\varepsilon]}\bigl\|V(t_2-s)-V(t_1-s)\bigr\|\frac{t_1^q-\varepsilon^q}{q}+\frac{2M_q\varepsilon^q}{q}\biggr)$$
$$\to 0\quad \text{as } \varepsilon\to 0,\ t_2-t_1\to 0.$$

As a result, we have proved that $\|\mathcal{Q}u(t_2)-\mathcal{Q}u(t_1)\|$ tends to 0 independently of $u\in\overline{\Omega}_{R_0}$ as $t_2-t_1\to 0$, which means that $\mathcal{Q}:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ is equicontinuous in $[0,a]$ for arbitrary constant $0<a<\infty$. Hence, the operator $\mathcal{Q}:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ is locally equicontinuous.

Step 4. We verify that $\mathcal{Q}:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ is a condensing operator.

For any $D\subset\overline{\Omega}_{R_0}$ with $\alpha_h(D)>0$, by Lemma 4, there exists a countable set $D_0=\{u^{(n)}\}\subset D$ such that

$$\alpha_h(\mathcal{Q}(D))=\alpha_h\bigl(\mathcal{Q}(D_0)\bigr). \tag{14}$$

Since $\mathcal{Q}(D_0)\subset\mathcal{Q}(\overline{\Omega}_{R_0})$ is a locally equicontinuous family of functions in $C_{\varphi,h}(X)$, we obtain from Lemma 6 and (11) that

$$\alpha_h\bigl(\mathcal{Q}(D_0)\bigr)=\sup_{t\in[0,\infty)}\alpha\biggl(\frac{\mathcal{Q}(D_0)(t)}{h(t)}\biggr). \tag{15}$$

In addition, for $t\geqslant 0$,

$$\begin{aligned}\sup_{\tau\in[-r,0]}\alpha\bigl(\bigl\{u^{(n)}[\varphi]_t(\tau)\bigr\}\bigr)&=\sup_{\tau\in[-r,0]}\alpha\bigl(\bigl\{u^{(n)}[\varphi](t+\tau)\bigr\}\bigr)\\&\leqslant\alpha\bigl(\bigl\{u^{(n)}(t)\bigr\}\bigr).\end{aligned} \tag{16}$$

Hence, by Lemmas 5, 2, condition (H2), properties of function h, and (16), we have

$$\begin{aligned}&\alpha\biggl(\frac{\mathcal{Q}(D_0)(t)}{h(t)}\biggr)\\&\quad\leqslant\frac{2}{h(t)}\int_0^t(t-s)^{q-1}\bigl\|V(t-s)\bigr\|\alpha\bigl(\bigl\{F\bigl(s,u^{(n)}(s),u^{(n)}[\varphi]_s\bigr)\bigr\}\bigr)\,\mathrm{d}s\\&\quad\leqslant 2\int_0^t(t-s)^{q-1}\bigl\|V(t-s)\bigr\|\bigl(h_1(s)+h_2(s)\bigr)\alpha\biggl(\frac{D_0(s)}{h(s)}\biggr)\,\mathrm{d}s\\&\quad\leqslant\frac{2M(\gamma_1+\gamma_2)}{\Gamma(q)}\alpha_h(D_0).\end{aligned}$$

Combining above inequality with (7), (14), (15), and condition (H2), we obtain

$$\alpha_h\bigl(\mathcal{Q}(D)\bigr)\leqslant\frac{2M(\gamma_1+\gamma_2)}{\Gamma(q)}\alpha_h(D_0)<\alpha_h(D_0)\leqslant\alpha_h(D).$$

Therefore, $\alpha_h(\mathcal{Q}(D))<\alpha_h(D)$ for any bounded set $D\in C_{\varphi,h}(X)$, which implies that $\mathcal{Q}:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ is a condensing operator.

Step 5. We prove that $\mathcal{Q}(SAP_{\omega,\varphi}(X)) \subset SAP_{\omega,\varphi}(X)$, where $SAP_{\omega,\varphi}(X) = \{u \in SAP_\omega(X) \mid u(0) = \varphi(0)\}$.

Obviously, $SAP_{\omega,\varphi}(X)$ is a closed subspace of $C_{\varphi,h}$, and $u \in SAP_{\omega,\varphi}(X)$ implies that the function $t \mapsto u[\varphi]_t$ belongs to $SAP_\omega(\mathcal{B})$. Thus, for every $u \in SAP_{\omega,\varphi}(X)$, it suffices to show that the function

$$f : t \to U(t)\varphi(0) + \int\limits_0^t (t-s)^{q-1} V(t-s) F\bigl(s, u(s), u[\varphi]_s\bigr)\,\mathrm{d}s \in SAP_\omega(X).$$

Since $u \in SAP_{\omega,\varphi}(X)$ and $u[\varphi]_t \in SAP_\omega(\mathcal{B})$ for all $t \geqslant 0$, for any $\varepsilon > 0$, there exists a constant $t_{\varepsilon,1} > 0$ such that $\|u(t+\omega) - u(t)\| \leqslant \varepsilon$ and $\|u[\varphi]_{t+\omega} - u[\varphi]_t\|_{\mathcal{B}} \leqslant \varepsilon$ for every $t \geqslant t_{\varepsilon,1}$. Thus, by continuity of F and condition (H1), for $t \geqslant t_{\varepsilon,1}$, we have

$$\bigl\|F\bigl(t, u(t+\omega), u[\varphi]_{t+\omega}\bigr) - F\bigl(t, u(t), u[\varphi]_t\bigr)\bigr\| \leqslant \frac{|\nu_0|}{2M}\varepsilon, \tag{17}$$

and we can find a positive constant $t_{\varepsilon,2}$ sufficiently large such that for $t \geqslant t_{\varepsilon,2}$,

$$\bigl\|F\bigl(t+\omega,\, u(t+\omega),\, u[\varphi]_{t+\omega}\bigr) - F\bigl(t,\, u(t+\omega),\, u[\varphi]_{t+\omega}\bigr)\| \leqslant \frac{|\nu_0|}{2M}\varepsilon. \tag{18}$$

Then for $t > t_\varepsilon := \max\{t_{\varepsilon,1}, t_{\varepsilon,2}\}$, we have

$$\begin{aligned}
f(t+\omega) - f(t) &= U(t+\omega)\varphi(0) - U(t)\varphi(0) \\
&\quad + \int\limits_0^\omega (t+\omega-s)^{q-1} V(t+\omega-s) F\bigl(s, u(s), u[\varphi]_s\bigr)\,\mathrm{d}s \\
&\quad + \int\limits_0^t (t-s)^{q-1} V(t-s)\bigl(F\bigl(s+\omega,\, u(s+\omega),\, u[\varphi]_{s+\omega}\bigr) \\
&\qquad - F\bigl(s, u(s), u[\varphi]_s\bigr)\bigr)\,\mathrm{d}s \\
&:= I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

Then

$$\bigl\|f(t+\omega) - f(t)\bigr\| \leqslant \bigl\|I_1(t)\bigr\| + \bigl\|I_2(t)\bigr\| + \bigl\|I_3(t)\bigr\|.$$

By (12) and (13), we have

$$\begin{aligned}
\bigl\|I_1(t)\bigr\| &\leqslant \bigl\|U(t+\omega)\varphi(0)\bigr\| + \bigl\|U(t)\varphi(0)\bigr\| \leqslant \bigl(\bigl\|U(t+\omega)\bigr\| + \bigl\|U(t)\bigr\|\bigr)\|\varphi\|_{\mathcal{B}} \\
&\leqslant \frac{2M_q\|\varphi\|_{\mathcal{B}}}{(1+t)^q}, \\
\bigl\|I_2(t)\bigr\| &\leqslant \int\limits_0^\omega (t+\omega-s)^{q-1}\bigl\|V(t+\omega-s)\bigr\|\bigl\|F\bigl(s, u(s), u[\varphi]_s\bigr)\bigr\|\,\mathrm{d}s \\
&\leqslant \frac{M_q M_f((t+\omega)^q - t^q)}{q(1+t)^{2q}} \leqslant \frac{M_q M_f \omega^q}{q(1+t)^{2q}},
\end{aligned}$$

hence, we deduce that $\|I_1\|$, $\|I_2\|$ tend to 0 as $t \to \infty$. By (12), (13), (17), and (18), we obtain

$$\begin{aligned}\|I_3(t)\| &\leqslant \int_0^t (t-s)^{q-1}\|V(t-s)\|\|F(s+\omega, u(s+\omega), u[\varphi]_{s+\omega}) \\ &\quad - F(s, u(s+\omega), u[\varphi]_{s+\omega})\| \,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\|V(t-s)\|\|F(s, u(s+\omega), u[\varphi]_{s+\omega}) \\ &\quad - F(s, u(s), u[\varphi]_s)\| \,\mathrm{d}s \\ &\leqslant 4M_f M_q \int_0^{t_\varepsilon} \frac{(t-s)^{q-1}}{(1+t-s)^{2q}} \,\mathrm{d}s + \int_{t_\varepsilon}^t (t-s)^{q-1}\|V(t-s)\| \,\mathrm{d}s \frac{|\nu_0|\varepsilon}{M} \\ &\leqslant 4M_f M_q \frac{(t-t_\varepsilon)^{-q} - t^{-q}}{q} + \varepsilon,\end{aligned}$$

which implies that $\|I_3(t)\|$ tends to 0 as $t \to \infty$.

Thus, from the above results we can deduce that for $t \geqslant 0$,

$$t \to U(t)\varphi(0) + \int_0^t (t-s)^{q-1} V(t-s) F(s, u(s), u[\varphi]_s) \,\mathrm{d}s \in SAP_\omega(X).$$

Combining this with the definition of $\mathcal{Q}$, we can conclude that $\mathcal{Q}u \in SAP_{\omega,\varphi}(X)$ for any $u \in SAP_{\omega,\varphi}(X)$, which implies that $\mathcal{Q}(SAP_{\omega,\varphi}(X)) \subset SAP_{\omega,\varphi}(X)$.

Therefore, from the above results we can deduce that $\mathcal{Q} : \overline{\Omega_{R_0} \cap SAP_{\omega,\varphi}(X)} \to \overline{\Omega_{R_0} \cap SAP_{\omega,\varphi}(X)}$ is a condensing operator. It follows from Sadovskii fixed point theorem that $\mathcal{Q}$ has a fixed point $u \in \overline{\Omega_{R_0} \cap SAP_{\omega,\varphi}(X)}$. Let $\{u^{(n)}\} \subset \overline{\Omega}_{R_0} \cap SAP_{\omega,\varphi}(X)$ converges to $u \in SAP_{\omega,\varphi}(X)$, then $\{\mathcal{Q}u^{(n)}\}$ converges to $Qu = u$ uniformly in $[0, \infty)$. Hence, $u[\varphi]$ is an S-asymptotically ω-periodic mild solution of FDEE (1). This completes the proof of Theorem 1. □

4 Global asymptotic behavior

The following Gronwall-type inequality plays an important role in the proof of the globally Mittag-Leffler stability.

Lemma 7. *Suppose that there exists a continuous function $w : [-r, \infty) \to \mathbb{R}^+$ and positive constants c_1, c_2, η with $c_1 + c_2 < \eta$ such that*

$$\begin{aligned}w(t) &\leqslant E_q(-\eta t^q)\phi(0) \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}(-\eta(t-s)^q)\Big(c_1 w(s) + c_2 \sup_{\tau\in[-r,0]} w(s+\tau)\Big) \,\mathrm{d}s \qquad (19)\end{aligned}$$

for $t \geqslant 0$*, and* $w(t) = \phi(t)$ *for* $t \in [-r, 0]$*. Then for any* $t \geqslant 0$*,*

$$w(t) \leqslant \max_{\tau \in [-r,0]} \phi(\tau) E_q\bigl(-\delta t^q\bigr),$$

where $\phi \in C([-r, 0], \mathbb{R}^+)$*,* $E_q(\cdot)$*,* $E_{q,q}(\cdot)$ *are the Mittag-Leffler functions,* $r > 0$*,* $q \in (0, 1)$ *are constants, and* δ *satisfies the following equation:*

$$c_1 + c_2 \frac{\sup_{\tau \in [-r,0]} E_q(-\delta(t+\tau)^q)}{E_q(-\delta t^q)} + \delta = \eta. \tag{20}$$

Proof. Denote the right-hand side of inequality (19) by $\rho(t)$ for $t \geqslant 0$, and let $\rho(t) = \phi(t)$ for $t \in [-r, 0]$, then $w(t) \leqslant \rho(t)$ for $t \in [-r, \infty)$. From the positiveness of $E_{q,q}(\mu)$ for $\mu < 0$ it follows that

$$\begin{aligned} \rho(t) &\leqslant E_q\bigl(-\eta t^q\bigr)\phi(0) \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}\bigl(-\eta(t-s)^q\bigr)\Bigl(c_1\rho(s) + c_2 \sup_{\tau \in [-r,0]} \rho(s+\tau)\Bigr)\,\mathrm{d}s \\ &:= \varrho(t) \end{aligned} \tag{21}$$

for $t \geqslant 0$ and $\rho(t) = \phi(t)$ for $t \in [-r, 0]$. Notice that the function ρ is continuous, and hence, by [15, Ex. 4.9], the function $\varrho(t)$ is the solution of the problem

$$\begin{aligned} &{}^cD_t^\alpha \varrho(t) = -\eta\varrho(t) + c_1\rho(t) + c_2 \sup_{t \in [-r,0]} \rho(t+\tau), \quad t \geqslant 0, \\ &\varrho(t) = \phi(t), \quad t \in [-r, 0]. \end{aligned}$$

Thus,

$$\begin{aligned} &{}^cD_t^\alpha \varrho(t) \leqslant -\eta\varrho(t) + c_1\varrho(t) + c_2 \sup_{t \in [-r,0]} \varrho(t+\tau), \quad t \geqslant 0, \\ &\varrho(t) = \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{22}$$

Let $v(t)$ be a solution of the following differential equation:

$${}^cD_t^\alpha v(t) = -\eta v(t) + c_1 v(t) + c_2 \sup_{\tau \in [-r,0]} v(t+\tau), \quad t \geqslant 0, \tag{23}$$

and let $v(t) \geqslant \varrho(t)$ for $t \in [-r, 0]$. Based on the comparison principle of fractional differential systems, we obtain that $\varrho(t) \leqslant v(t)$ for $t \in [-r, \infty)$. Now, we will show that $v(t) = \phi(0)E_q(-\delta t^q)$ is a solution of the differential equation (23). Similar to the discussion of [15, Lemma2.23], one can derive that for $t \geqslant 0$,

$${}^cD_t^\alpha v(t) = -\delta\phi(0)E_q\bigl(-\delta t^q\bigr). \tag{24}$$

By means of (20) and (24), we know that

$$
\begin{aligned}
{}^{c}D_t^{\alpha} v(t) &= \phi(0)\bigl(-\eta E_q\bigl(-\delta t^q\bigr) + c_1 E_q\bigl(-\delta t^q\bigr) + c_2 \sup_{\tau\in[-r,0]} E_q\bigl(-\delta(t+\tau)^q\bigr)\bigr) \\
&= -\eta v(t) + c_1 v(t) + c_2 \sup_{\tau\in[-r,0]} v(t+\tau).
\end{aligned}
$$

Denote

$$
g(\delta) = c_1 + c_2 \frac{\sup_{\tau\in[-r,0]} E_q(-\delta(t+\tau)^q)}{E_q(-\delta t^q)} - \eta + \delta.
$$

From $c_1 + c_2 < \eta$ it follows that $g(0) = c_1 + c_2 - \eta < 0$ and

$$
g(\eta) = c_1 + c_2 \frac{\sup_{\tau\in[-r,0]} E_q(-\eta(t+\tau)^q)}{E_q(-\eta t^q)} > 0.
$$

Consequently, from the continuity of g it follows that there is a constant $\delta > 0$ such that $g(\delta) = 0$, which implies that there exist a positive constant such that $v(t) = \phi(0)E_q(-\delta t^q)$ is a solution of the differential equation (23).

Finally, by (19), (21), (22), and (24), we can easy find that

$$
w(t) \leqslant \max_{\tau\in[-r,0]} \phi(\tau) E_q\bigl(-\delta t^q\bigr) \quad \text{for } t \geqslant 0.
$$

This completes the proof of Lemma 7. □

Theorem 2. *Assume that $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates an exponentially stable C_0-semigroup $T(t)$ ($t \geqslant 0$) in X, whose growth exponent $\nu_0 < 0$. Let the nonlinear functions $F : \mathbb{R}^+ \times X \times \mathcal{B} \to X$ be continuous, and let* (H1) *and the following condition hold*:

(H4) *There exist positive constants L_1, L_2 such that*

$$
\bigl\|F(t,x,\phi) - F(t,y,\psi)\bigr\| \leqslant L_1\|x-y\| + L_2\|\phi-\psi\|_{\mathcal{B}}
$$

for all $t \geqslant 0$ and $x, y \in X$, $\phi, \psi \in \mathcal{B}$.

Then for a given $\varphi_0 \in \mathcal{B}$, FDEE (1) *has a unique S-asymptotically periodic mild solution, provided that*

$$
M(L_1 + L_2) < |\nu_0|. \tag{25}
$$

Moreover, if there exists constant $\delta > 0$ such that

$$
M\left(L_1 + L_2 \frac{\sup_{\tau\in[-r,0]} E_q(-\delta(t+\tau)^q)}{E_q(-\delta t^q)}\right) + \delta = |\nu_0|, \tag{26}
$$

then the unique S-asymptotically periodic mild solution is globally Mittag-Leffler stable.

Proof. Given $\varphi \in \mathcal{B}$ and $u \in C_b([0,\infty),X)$, we define the function $u[\varphi] : [-r,\infty) \to X$ as follows:

$$u[\varphi](t) = \begin{cases} u(t) & \text{for } t \geqslant 0, \\ \varphi(t) & \text{for } t \in [-r,0]. \end{cases}$$

We denote

$$C_{\varphi,b} = \bigl\{u \in C_b\bigl([0,\infty),X\bigr)\colon\ u(0) = \varphi(0)\bigr\}.$$

Then $C_{\varphi,b}$ is a closed subspace of $C_b([0,\infty),X)$, and $SAP_{\omega,\varphi}(X) \subset C_{\varphi,b}$.

For $u \in C_{\varphi,b}$ and $t \geqslant 0$, let $\mathcal{Q}u(t)$ defined by (9). It is easy to find that $\mathcal{Q} : C_{\varphi,b}(X) \to C_{\varphi,b}(X)$ is well defined and continuous. Moreover, by the proof of Theorem 1, we know that $\mathcal{Q}u \in SAP_{\omega,\varphi}(X)$ for $u \in SAP_{\omega,\varphi}(X)$ and the fixed point u of the operator $\mathcal{Q}$ in $SAP_{\omega,\varphi}(X)$ can ensure that $u[\varphi]$ is the S-asymptotic ω-periodic mild solution of problem (1).

For all $t \geqslant 0$ and $u, v \in SAP_{\omega,\varphi}(X)$, by (9), we get

$$\begin{aligned} &\bigl\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\bigr\| \\ &\quad\leqslant \int_0^t (t-s)^{q-1}\bigl\|V(t-s)\bigr\|\bigl\|F\bigl(s,u(s),u[\varphi]_s\bigr) - F\bigl(s,v(s),v[\varphi]_s\bigr)\bigr\|\,\mathrm{d}s \\ &\quad\leqslant M(L_1+L_2)\int_0^\infty \xi_q(\sigma)\,\mathrm{d}\sigma \int_0^\infty \mathrm{e}^{-|\nu_0|s}\,\mathrm{d}s\cdot\|u-v\|_b, \end{aligned}$$

thus

$$\bigl\|\mathcal{Q}u - \mathcal{Q}v\bigr\|_b \leqslant \frac{M(L_1+L_2)}{|\nu_0|}\,\|u-v\|_b.$$

By (25), we can conclude that $\mathcal{Q}$ is a contraction mapping. Thus, the famous Banach fixed point theorem ensure that $\mathcal{Q}$ has a unique fixed point $u^* \in SAP_{\omega,\varphi}(X)$, which implies that $u^*[\varphi]$ is the unique S-asymptotically ω-periodic mild solution for FDEE (1).

Now, we verify the global Mittag-Leffler stability of the unique S-asymptotically periodic mild solution. Clearly, (26) implies that (25) holds, in fact,

$$|\nu_0| > M\biggl(L_1 + L_2\frac{\sup_{\tau\in[-r,0]} E_q(-\delta(t+\tau)^q)}{E_q(-\delta t^q)}\biggr) \geqslant M(L_1+L_2). \tag{27}$$

Let $u^*[\varphi]$ be the unique S-asymptotically periodic mild solution of FDEE (1). In addition, for any $\psi \in \mathcal{B}$, one can deduce that FDEE (1) has a unique mild solution $v[\psi] : [-r,\infty) \to X$ with $v \in C_b([0,\infty),X)$ and the new initial value $v[\psi](t) = \psi(t)$ for $t \in [-r,0]$.

From Lemma 1, (9), and condition (H4) it is easy to find for every $t \geqslant 0$,

$$\begin{aligned}\left\|u^*[\varphi](t) - v[\psi](t)\right\| &= \left\|u^*(t) - v(t)\right\| \\ &\leqslant ME_q(\nu_0 t^q)\|\varphi - \psi\|_{\mathcal{B}} \\ &\quad + M\int_0^t (t-s)^{q-1} E_{q,q}\big(\nu_0(t-s)^q\big)\Big(L_1\left\|u^*(s) - v(s)\right\| \\ &\quad + L_2 \sup_{\tau\in[-r,0]} \left\|u^*[\varphi](s+\tau) - v[\psi](s+\tau)\right\|\Big)\,\mathrm{d}s.\end{aligned}$$

Now, from Lemma 7 it follows that for all $t \geqslant 0$,

$$\left\|u^*[\varphi](t) - v[\psi](t)\right\| \leqslant M\|\varphi - \psi\|_{\mathcal{B}} E_q\big(-\delta t^q\big),$$

where $\delta > 0$ is a solution of the following equation:

$$M\left(L_1 + L_2 \frac{\sup_{\tau\in[-r,0]} E_q(-\delta(t+\tau)^q)}{E_q(-\delta t^q)}\right) + \delta = |\nu_0|. \tag{28}$$

Therefore, according to Definition 4, the unique S-asymptotic ω-periodic mild solution of FDEE (1) is globally Mittag-Leffler stable. □

Theorem 3. *Let $A : D(A) \subset X \to X$ be a closed linear operator, and let $-A$ generate an exponentially stable C_0-semigroup $T(t)$ ($t \geqslant 0$) in X. Assume that $F : \mathbb{R}^+ \times X \times \mathcal{B} \to X$ is a continuous function, $\sup_{t\geqslant 0} \|F(t,\theta,\theta)\| < \infty$, and there exists a constant $\omega > 0$ such that $F(t+\omega, x, \phi) = F(t, x, \phi)$ for every $t \in \mathbb{R}^+$ and $x \in X$, $\phi \in \mathcal{B}$. If condition* (H4) *holds, then FDEE* (1) *is globally asymptotically ω-periodic for any initial value $\varphi \in \mathcal{B}$ provided with* (26).

Proof. We prove the theorem by the following three propositions.

Proposition 1. *All mild solutions of the initial value problem for FDEE* (1) *are bounded.*

Denote $L_0 := \sup_{t\geqslant 0} \|F(t,\theta,\theta)\| < \infty$. From (H4) it follows that

$$\begin{aligned}\left\|F(t,x,\phi)\right\| &< \left\|F(t,x,\phi) - F(t,\theta,\theta)\right\| + \left\|F(t,\theta,\theta)\right\| \\ &\leqslant L_1\|x\| + L_2\|\phi\|_{\mathcal{B}} + L_0\end{aligned} \tag{29}$$

for any $t \in \mathbb{R}^+$, $x \in X$, $\phi \in \mathcal{B}$.

Let $u[\varphi] : [-r,\infty) \to X$ be a mild solution of the initial value problem for FDEE (1) satisfying $u[\varphi](t) = \varphi(t)$ for $t \in [-r, 0]$. From Definition 3 and (29) it follows that for any $t \geqslant 0$,

$$\left\|u(t)\right\| \leqslant M\|\varphi\|_{\mathcal{B}} + \frac{M}{|\nu_0|}\Big((L_1 + L_2)\sup_{t\in[0,\infty)} \|u(t)\| + L_2\|\varphi\|_{\mathcal{B}} + L_0\Big).$$

Hence,

$$\left(1-\frac{M(L_1+L_2)}{|\nu_0|}\right)\sup_{t\in[0,\infty)}\left\|u(t)\right\|\leqslant M\|\varphi\|_{\mathcal{B}}+\frac{L_2M}{|\nu_0|}\|\varphi\|_{\mathcal{B}}+\frac{ML_0}{|\nu_0|}.$$

By (26) and (27), $1-M(L_1+L_2)/|\nu_0|>0$, which implies that $u\in C_{\varphi,b}(X)$, namely, the mild solution $u[\varphi](t)$ of FDEE (1) is bounded.

Proposition 2. *All mild solutions of FDEE* (1) *are S-asymptotically ω-periodic.*

Let $u[\varphi]$ be a mild solution of FDEE (1) with $u\in C_{\varphi,b}(X)$. We only need to verify $\lim_{t\to\infty}\|u(t+\omega)-u(t)\|=0$. By Definition 3 and the periodicity of F, we have

$$\begin{aligned}
&\left\|u(t+\omega)-u(t)\right\|\\
&\quad\leqslant\left\|U(t+\omega)\varphi(0)-U(t)\varphi(0)\right\|\\
&\qquad+\left\|\int_0^{\omega}(t+\omega-s)^{q-1}V(t+\omega-s)F\big(s,u(s),u[\varphi]_s\big)\,\mathrm{d}s\right\|\\
&\qquad+\left\|\int_0^{t}(t-s)^{q-1}V(t-s)\big(F\big(s,u(s+\omega),u[\varphi]_{s+\omega}\big)-F\big(s,u(s),u[\varphi]_s\big)\big)\,\mathrm{d}s\right\|\\
&\quad:=\left\|K_1(t)\right\|+\left\|K_2(t)\right\|+\left\|K_3(t)\right\|.
\end{aligned}$$

First of all, from Lemma 1(iv) and the properties of Mittag-Leffler function $E_q(\cdot)$ we have

$$\begin{aligned}
\left\|K_1(t)\right\|&\leqslant\left\|U(t+\omega)\varphi(0)\right\|+\left\|U(t)\varphi(0)\right\|\\
&\leqslant M\big(E_q\big(\nu_0(t+\omega)^q\big)+E_q\big(\nu_0t^q\big)\big)\|\varphi\|_{\mathcal{B}}\\
&\leqslant 2M\|\varphi\|_{\mathcal{B}}E_q\big(\nu_0t^q\big).
\end{aligned}$$

Secondly, since the mild solution $u\in C_{\varphi,b}(X)$, hence, there is a constant $R>\|\varphi\|_{\mathcal{B}}$ such that $\|\varphi\|_{\mathcal{B}}+\|u\|_b\leqslant R$. Combining this with (H4) and (29), one can find

$$\begin{aligned}
\left\|F\big(t,u(t),u[\varphi]_t\big)\right\|&\leqslant L_1\left\|u(t)\right\|+L_2\left\|u[\varphi]_t\right\|_{\mathcal{B}}+L_0\\
&\leqslant(L_1+L_2)R+L_0:=L
\end{aligned}$$

for any $t\geqslant 0$. Therefore, from (3) and Lemma 1(iv) it follows that

$$\begin{aligned}
\left\|K_2(t)\right\|&\leqslant\int_0^{\omega}(t+\omega-s)^{q-1}\left\|V(t+\omega-s)\right\|\left\|F\big(s,u(s),u[\varphi]_s\big)\right\|\,\mathrm{d}s\\
&\leqslant\frac{ML}{|\nu_0|}\big(E_q\big(\nu_0t^q\big)-E_q\big(\nu_0(t+\omega)^q\big)\big)\leqslant\frac{ML}{|\nu_0|}E_q\big(\nu_0t^q\big).
\end{aligned}$$

Finally, by Lemma 1(iv) and (H4) we have

$$\begin{aligned}\|K_3(t)\| &\leqslant \int_0^t (t-s)^{q-1}\|V(t-s)\| \\ &\quad \times \|F(s,u(s+\omega),u[\varphi]_{s+\omega}) - F(s,u(s),u[\varphi]_s)\|\,\mathrm{d}s \\ &\leqslant M\int_0^t (t-s)^{q-1}E_{q,q}(\nu_0(t-s)^q) \\ &\quad \times (L_1\|u(s+\omega)-u(s)\| + L_2\|u[\varphi]_{s+\omega}-u[\varphi]_s\|_{\mathcal{B}})\,\mathrm{d}s.\end{aligned}$$

In conclusion, we have

$$\begin{aligned}&\|u(t+\omega)-u(t)\| \\ &\quad \leqslant \left(2M\|\varphi\|_{\mathcal{B}} + \frac{ML}{|\nu_0|}\right)E_q(\nu_0 t^q) \\ &\qquad + M\int_0^t (t-s)^{q-1}E_{q,q}(\nu_0(t-s)^q)\Big(L_1\|u(s+\omega)-u(s)\| \\ &\qquad + L_2 \sup_{\tau\in[-r,0]}\|u[\varphi](s+\omega+\tau)-u[\varphi](s+\tau)\|\Big)\,\mathrm{d}s.\end{aligned}$$

Thus, from Lemma 7, (H4), and (2) we can obtain that there is a constant $\delta > 0$ such that

$$\begin{aligned}\|u(t+\omega)-u(t)\| &\leqslant \left(2M\|\varphi\|_{\mathcal{B}} + \frac{M(L_1+L_2)}{|\nu_0|}\right)E_q(-\delta t^q) \\ &\to 0 \quad \text{as } t\to\infty,\end{aligned} \tag{30}$$

which implies that $u \in SAP_{\omega,\varphi}(X)$, where $\delta > 0$ is a solution of equation (28).

Proposition 3. *The solution of FDEE* (1) *asymptotically converges to a nonconstant ω-periodic function.*

Let $u[\varphi]$ be an S-asymptotically ω-periodic mild solution of problem (1) with $u \in SAP_{\omega,\varphi}(X)$. Obviously, the sequence $\{u(t+n\omega)\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded. From Arzelà–Ascoli theorem it follows that there is a subsequence of $\{n\omega\}$ (for convenience, we still denote the subsequence as $\{n\omega\}$), such that $\{u(t+n\omega)\}$ uniformly converges to a continuous function $u^*(t)$ on any compact set of $[0,\infty)$. Clearly, $u^* \in C_{\varphi,b}(X)$ is a periodic function satisfying $u^*(t+\omega) = u^*(t)$ for $t \in [0,\infty)$.

Now, for $t \in [0,\infty)$ and $n \in \mathbb{N}$, we consider

$$\begin{aligned}\|u[\varphi](t) - u^*[\varphi](t)\| &\leqslant \|u(t)-u(t+\omega)\| + \|u(t+\omega)-u(t+n\omega)\| \\ &\quad + \|u(t+n\omega)-u^*(t)\|.\end{aligned} \tag{31}$$

From S-asymptotic ω-periodicity, the Mittag-Leffler stability, and the definition of u^* we have

$$\lim_{t\to\infty} \|u(t) - u(t+\omega)\| = 0, \tag{32}$$

$$\lim_{t\to\infty} \|u(t+\omega) - u(t+n\omega)\| = 0 \quad \text{for each } n \in \mathbb{N}, \tag{33}$$

$$\lim_{n\to\infty} \|u(t+n\omega) - u^*(t)\| = 0 \quad \text{for any } t \in [0,\infty). \tag{34}$$

By the definition of $u[\varphi]$, (31)–(34), one can easily find

$$\lim_{t\to\infty} \|u[\varphi](t) - u^*[\varphi](t)\| = 0.$$

Therefore, FDEE (1) is globally asymptotically periodic. □

5 Example

Consider the initial-boundary value problem for the time fractional-order delayed partial differential equation

$$\begin{aligned} &\frac{\partial^{1/4} u(x,t)}{\partial t^{1/4}} + A(x,D)u(x,t) \\ &\quad = \frac{l_1 \sin t}{\mathrm{e}^{bt}} \sqrt{u(x,t)+1} + \frac{l_2 \sin t}{\mathrm{e}^{bt}} \sin u(x,\, t+\tau), \quad x \in \Theta,\ t \in \mathbb{R}^+, \\ &Bu(x,t) = 0, \quad x \in \partial\Theta,\ t \in \mathbb{R}^+, \\ &u(x,\tau) = \varphi(x,\tau), \quad x \in \Theta, \end{aligned} \tag{35}$$

where $\tau \in [-r, 0]$, $\partial^{1/4}/\partial t^{1/4}$ is the Caputa fractional partial derivative of order $1/4$, b, r, l_1, l_2 are nonnegative constants, $\overline{\Theta} \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Theta$.

We assume that $A(x,D)$ in $\overline{\Theta}$ is a uniformly elliptic differential operator defined by

$$A(x,D)u := -\sum_{i,j=1}^{N} a_{ij}(x) D_i D_j u + \sum_{j=1}^{N} a_j(x) D_j u + a_0(x) u,$$

whose coefficients $a_{ij}(x)$, $a_j(x)$ $(i,j = 1,\dots,n)$, and $a_0(x)$ are Höder-continuous on $\overline{\Theta}$, and $a_0(x) \geqslant 0$; B is a boundary operator on $\partial\Theta$ defined by

$$Bu := b_0(x)u + \delta \frac{\partial u}{\partial \beta},$$

where either $\delta = 0$ and $b_0(x) \equiv 1$ (Dirichlet boundary operator) or $\delta = 1$ and $b_0(x) \geqslant 0$ (regular oblique derivative boundary operator; in this case, we assume that $a_0(x) \not\equiv 0$ or $b_0(x) \not\equiv 0$), β is an outward pointing, nowhere tangent vector field on $\partial\Theta$. Let λ_1 be the first eigenvalue of operator $A(x,D)$ under the boundary condition $Bu = 0$. From [2, Thm. 1.16] it follows that $\lambda_1 > 0$.

In order to write the value-boundary problem (35) into the abstract form FDEE (1), we denote $X = L^2(\Theta)$ with the norm $\|\cdot\|_2$. Then X is a Banach space. Define an operator $A : D(A) \subset X \to X$ by

$$D(A) = \left\{u \in W^{2,2}(\Theta) \;\middle|\; Bu = 0,\ x \in \partial\Theta\right\}, \quad Au = A(x,D)u.$$

From [3] it follows that $-A$ generates an exponentially stable analytic semigroup $T(t)$ $(t \geqslant 0)$ in X satisfying $\|T(t)\| \leqslant M = 1$ for every $t \geqslant 0$. Denote λ_1 be the first eigenvalue of operator A, then $\lambda_1 > 0$ from [2, Thm. 1.16]. On the other hand, analytic semigroup is continuous in the uniform operator topology for every $t > 0$ (see [21, 26]), hence the growth exponent $\nu_0 = -\lambda_1$, which implies that $\|T(t)\| \leqslant \mathrm{e}^{-\lambda_1 t}$ for $t \geqslant 0$. Hence, by Theorems 1 and 2, we have the following results.

Theorem 4. *Assume that $b > 0$ and $l_1 + l_2 < b^{1/4}/4$. Then the initial-boundary value problem for the time fractional-order delayed partial differential equation* (35) *has at least one S-asymptotically periodic mild solution.*

Proof. For every $t \geqslant 0$, $x \in \Theta$, and $\tau \in [-r, 0]$, let $u(t)(x) = u(x,t)$, $u_t(\tau)(x) = u(x, t+\tau)$, and

$$F\bigl(t, u(t), u_t\bigr)(x) = \frac{l_1 \sin t}{\mathrm{e}^{bt}}\sqrt{u(x,t)+1} + \frac{l_2 \sin t}{\mathrm{e}^{bt}} \sin u(x,\ t+\tau).$$

Then the initial-boundary value problem for the time fractional-order delayed partial differential equation (35) can be rewritten into the abstract form of FDEE (1).

For any $\omega > 0$, we have

$$\lim_{t\to\infty}\left|\frac{\sin(t+\omega)}{\mathrm{e}^{b(t+\omega)}} - \frac{\sin t}{\mathrm{e}^{bt}}\right| = 0.$$

Thus, we easily see that F is continuous and satisfies condition (H1). Moreover, we can test and verify condition (H3) and (7) with $\Phi_1(R) = l_1\sqrt{R+1}$, $\Phi_2(R) = l_2$, $\rho_1 = \rho_2 = 0$. On the other hand, from the definition of nonlinear function F, Lemma 2, the assumption $l_1 + l_2 < b^{1/4}/4$, and the following inequality

$$\int_0^t (t-s)^{-3/4}\mathrm{e}^{-bs}\,\mathrm{d}s \leqslant 2b^{-1/4}\Gamma\left(\frac{1}{4}\right)$$

we deduce that condition (H2) is satisfied with $h_1(t) = l_1\mathrm{e}^{-bt}$, $h_2(t) = l_2\mathrm{e}^{-bt}$, $\gamma_1 = 2l_1 b^{-1/4}\Gamma(1/4)$, $\gamma_2 = 2l_2 b^{-1/4}\Gamma(1/4)$.

Now, all the conditions of Theorem 1 are satisfied, which means that the initial-boundary value problem for the time fractional-order delayed partial differential equation (35) has at least one S-asymptotically periodic mild solution. This completes the proof of Theorem 4. □

Theorem 5. *Assume that $l_1 + l_2 < \lambda_1$. Then the initial-boundary value problem for the time fractional-order delayed partial differential equation* (35) *has unique S-asymptotically periodic mild solution. Specially, if there exists constant $\delta > 0$ such that*

$$\left(l_1 + l_2 \frac{\sup_{\tau\in[-r,0]} E_{1/4}(-\delta(t+\tau)^{1/4})}{E_{1/4}(-\delta t^{1/4})}\right) + \delta = \lambda_1, \tag{36}$$

then the unique S-asymptotically periodic mild solution is globally Mittag-Leffler stable. Furthermore, if $b = 0$, then problem (35) *is globally asymptotically 2π-periodic.*

Proof. From the definition of nonlinear function F we know that $F(t, u(t), u_t)$ is Lipschitz continuous about variable $u(t)$ and u_t with Lipschitz constants l_1 and l_2, respectively. If $b = 0$, one can find that F is 2π-periodic in t.

Now, by the assumption $l_1 + l_2 < \lambda_1$ and (36), Theorems 2 and 3 can guarantee Theorem 5. This completes the proof of Theorem 5. □

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