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Suzuki-type fuzzy contractive inequalities in 1-$\mathfrak{Z}$-complete fuzzy metric-like spaces with an application*

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Abstract. In the piece of this note, we mention various Suzuki-type fuzzy contractive inequalities in 1-$\mathfrak{Z}$-complete fuzzy metric-like spaces for uniqueness and existence of a fixed point and prove a few fuzzy fixed point theorems, which are appropriate generalizations of some of the latest famed results in the literature. Mainly, we generalize fuzzy Θ-contraction in terms of Suzuki-type fuzzy Θ-contraction and also fuzzy Υ-contractive mapping in view of Suzuki-type. For this new group of Suzuki-type functions, acceptable conditions are formulated to ensure the existence of a unique fixed point. The attractive beauty of this fuzzy distance space lies in the symmetry of its variables, which play a crucial role in the construction of our contractive conditions to ensure the solution. Furthermore, a lot of considerable examples are presented to illustrate the significance of our results. In the end, we have discussed an application in an extensive way for the solution of a nonlinear fractional differential equation via Suzuki-type fuzzy contractive mapping.

Keywords: fuzzy Θ-contraction, Suzuki-type fuzzy Θ-contractive mapping, 1-$\mathfrak{Z}$-complete fuzzy metric-like space, nonlinear fractional differential equation.

1 Introduction

The theory of fuzzy sets was initiated by Zadeh [15] in 1965 in his influential note, which is based on the membership function. The membership value of any member from a nonempty set $\mathfrak{X}$ belongs to a continuous closed interval $[0,1]$, not on the Aristotelian set $\{0,1\}$. In 1975, Kramosil and Michalek [6] inaugurated the idea of a fuzzy metric space, which was further enhanced by George and Veeramani [2] in 1994 by defining Hausdorff topology in this fuzzy distance space and also proving "every metric induces a fuzzy metric".

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Furthermore, Harandi [1] introduced the concept of metric-like space by generalizing the concepts of metric space, partial metric space, and symmetric space, and giving some weak contraction conditions to put up new fixed point results. Later, Shukla and Abbas [10] in 2014 fuzzified the concepts of Harandi [1] and discovered the notion of a fuzzy metric-like space as a generalization of fuzzy metric space in the sense of George and Veeramani [2]. In fuzzy metric-like spaces, the membership degree of closeness of members x and y in space when $x = y$ is not identical, and this means the fuzzy-self distance may not be unity. Shukla and Abbas [10] gave many examples of this space and established several propositions. Also, provide satisfactory examples of the definitions of convergence of a sequence, the Cauchy sequence, and the completeness of this fuzzy distance space. Again, Shukla, Gopal, and Hierro [11] modified the concepts of convergence of a sequence, completeness, and Cauchy sequence in this fuzzy space in the sense of George and Veeramani [2] and gave the definitions of 1-$\mathcal{M}$-convergence of a sequence, 1-$\mathcal{M}$ Cauchy sequence, and 1-$\mathcal{M}$ completeness property in a more general way.

In this writing, Suzuki-type fuzzy Θ-contractive conditions in 1-$\mathfrak{Z}$-complete fuzzy metric-like spaces have been discussed, and some fuzzy results have been discovered to get a unique fixed point. Here the concept of Suzuki-type fuzzy Θ_1 and Θ_2-contractive conditions has been formulated, where the conclusion part of the fuzzy contractive condition is a Θ-contraction, and the hypothesis parts are different. We will prove at the end, with examples, that both concepts are independent. As well as the fuzzy Υ-contractive condition given by Mihet [8], it has been generalized in the Suzuki-type as a corollary. In the last section, we discuss an application to the solution of the nonlinear fractional differential equation.

2 Preliminaries

First, we remind some of the elementary concepts of fuzzy metric-like spaces.

Definition 1. (See [15].) A fuzzy set $\mathfrak{A}$ is a membership function with domain $\mathfrak{X}$ and codomain $[0, 1]$, where $\mathfrak{X}$ is a nonempty set.

Definition 2. (See [9].) A continuous triangular norm (t-norm in short) is a binary mapping $\circledast : [0, 1] \times [0, 1] \to [0, 1]$ if

(i) $\circledast$ satisfies commutative law, that is, $l \circledast m = m \circledast l$, and associative law, that is, $l \circledast (m \circledast n) = (l \circledast m) \circledast n$;

(ii) $\circledast$ is continuous;

(iii) $1 \circledast l = l$;

(iv) $l \circledast m \leqslant qn \circledast p$ whenever $l \leqslant n$ and $m \leqslant p$

for all $l, m, n, p \in [0, 1]$.

Shukla and Abbas [10] have declared the definition of a fuzzy metric-like space in the given manner.

Definition 3. (See [10].) The ordered triplet $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is called a fuzzy metric-like space, where $\mathfrak{X}$ is an arbitrary nonempty set, and t-norm $\circledast$ is continuous binary mapping, if the membership function $\mathfrak{Z} : \mathfrak{X}^2 \times (0, +\infty) \to [0, 1]$ satisfies the given assertions:

(i) $\mathfrak{Z}(\alpha, \beta, s) > 0$;
(ii) if $\mathfrak{Z}(\alpha, \beta, s) = 1$, then $\alpha = \beta$;
(iii) $\mathfrak{Z}(\alpha, \beta, s) = \mathfrak{Z}(\beta, \alpha, s)$ (symmetry);
(iv) $\mathfrak{Z}(\alpha, \eta, t + s) \geqslant \mathfrak{Z}(\alpha, \beta, t) \circledast \mathfrak{Z}(\beta, \eta, s)$ (triangle inequality);
(v) $\mathfrak{Z}(\alpha, \beta, \cdot) : (0, +\infty) \to [0, 1]$ is continuous

for all $\alpha, \beta, \eta \in \mathfrak{X}$ and $s, t > 0$.

Remark 1. Symmetry defined in metric space is generalized by Kramosil and Michalek [6] in fuzzy setting, which is adopted by George and Veeramani [2], which means that the membership degree of closeness in the middle of α and β with regards to s is the same as the degree of closeness in the middle of β and α with regards to s. If we change assertion (ii) of Definition 3 by $\mathfrak{Z}(\alpha, \beta, s) = 1$ if and only if $\alpha = \beta$, then the ordered tripled $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ becomes a fuzzy metric [2].

Definition 4. (See [11].) Suppose $\{\zeta_n\}$ is any sequence of a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$. Then for all $s > 0$ and $n, m \in \mathbb{N}$,

(i) $\{\zeta_n\}$ is said to be a 1-$\mathfrak{Z}$-convergent to some $\zeta \in \mathfrak{X}$ if

$$\lim_{n \to +\infty} \mathfrak{Z}(\zeta_n, \zeta, s) = 1;$$

(ii) $\{\zeta_n\}$ is said to be a 1-$\mathfrak{Z}$ Cauchy sequence if

$$\lim_{n, m \to +\infty} \mathfrak{Z}(\zeta_n, \zeta_m, s) = 1;$$

(iii) the space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to have 1-$\mathfrak{Z}$-completeness property if every 1-$\mathfrak{Z}$ Cauchy sequence in $\mathfrak{X}$ converges to some $\zeta \in \mathfrak{X}$ such that $\mathfrak{Z}(\zeta, \zeta, s) = 1$.

Shukla, Gopal, and Hierro [11] introduced a new kind of control function known as family of $\varTheta$-function, which has been used to define the notion of fuzzy $\varTheta$-contraction. Such $\varTheta_{\mathfrak{B}}$-class of mappings will include continuous and discontinuous.

Definition 5. (See [10].) Suppose that $\varTheta_{\mathfrak{B}}$ indicates a group of mapping $\varTheta : [0, 1] \to [0, 1]$ holding the assertions:

(i) $\varTheta$ is increasing;
(ii) $\lim_{n \to +\infty} \varTheta^n(\omega) = 1$ for every $\omega \in (0, 1)$.

If $\varTheta \in \varTheta_{\mathfrak{B}}$, then $\omega < \varTheta(\omega)$ for all $\omega \in (0, 1)$ and $\varTheta(1) = \lim_{\omega \to 1^-} \varTheta^n(\omega) = 1$.

Example 1. (See [10].) The below written mappings $\varTheta : [0, 1] \to [0, 1]$ are few members of the family $\varTheta_{\mathfrak{B}}$:

(i) $\Theta(\omega) = 1$;
(ii) $\Theta(\omega) = 2\omega/(\omega + 1)$;
(iii) $\Theta(\omega) = \omega^k$;
(iv) $\Theta(\omega) = 1 - k(1 - \omega)$;
(v) $\Theta(\omega) = \omega/(k + k(1 - \omega))$, where $k \in (0, 1)$;
(vi) Let $\Theta : [0, 1] \to [0, 1]$ be any discontinuous function defined by

$$\Theta(\omega) = \begin{cases} a\omega & \text{if } \omega \in [0, b], \\ 1 & \text{if } \omega \in (b, 1], \end{cases}$$

where b and a are nonnegative real numbers such that $b < 1$, $a > 1$, and $ab < 1$.

Definition 6. (See [11].) Suppose that $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is a fuzzy metric-like space. A self-map $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ is said to be a Θ-contraction if there exists $\Theta \in \Theta_{\mathfrak{B}}$ such that

$$\Theta\big(\mathfrak{Z}(\zeta, \eta, s)\big) \leqslant \mathfrak{Z}(\mathcal{A}\zeta, \mathcal{A}\eta, s)$$

for all $\zeta, \eta \in \mathfrak{X}$ and all $s > 0$.

By above such class Shukla, Gopal, and Hierro [11] linkup the notations of various fuzzy inequalities.

In the literature of the crisp theory of fixed points, there are a lot of generalization of the Banach contraction result, but a remarkable and most interesting generalization was given by Suzuki [14] in 2007. Moreover, in discrete mathematics, we have learned the truth value of some conditional statements, like, if p and q are given statements, then the statement "if p, then q", denoted as p implies q (or $p \Longrightarrow q$), is called a conditional statement or implication, where p is a hypothesis, and q is a conclusion. Suzuki-type contractive conditions follow the truth value of statements that p implies q. Suzuki [14] adopted this concept and discovered a contraction as a tool to obtain a fixed point for a self-map.

Later, Khojasteh, Shukla, and Radenović [4] introduced the simulation function $\beta : [0, +\infty] \times [0, +\infty] \to \mathbb{R}$ and the notion of $\mathcal{Z}$-contraction. Later, Kumar, Gopal, and Bhudhiyi [7] adopted Suzuki idea to generalize $\mathcal{Z}$-contraction and introduced a new concept of Suzuki-type $\mathcal{Z}$-contraction in a complete metric space. The main motivational idea to introduce Suzuki-type contraction in the settings of fuzzy metric-like space has been inspired by the articles [3, 7, 12] and [13]. Now we are ready to present our main results.

3 Main results

First, we generalize fuzzy Θ-contraction in the view of Suzuki-type fuzzy Θ-contraction.

Definition 7. A self-map $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ defined on a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to be a Suzuki-type fuzzy Θ_1-contractive mapping if there exists $\Theta \in \Theta_{\mathfrak{B}}$ such that

$$\mathfrak{Z}(\zeta, \mathcal{A}\zeta, s) > q \cdot \mathfrak{Z}(\zeta, \eta, s) \quad \Longrightarrow \quad \mathfrak{Z}(\mathcal{A}\zeta, \mathcal{A}\eta, s) \geqslant \Theta\big(\mathfrak{Z}(\zeta, \eta, s)\big), \tag{1}$$

where $q \in (0, 1)$, for all $\zeta, \eta \in \mathfrak{X}$ and all $s > 0$.

The above definition of Suzuki-type fuzzy Θ_1-contractive condition motivates for the following definition to obtain constructed sequence is a 1-$\mathfrak{Z}$ Cauchy sequence.

Definition 8. A self-map $\mathcal{A}$ defined on a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to have feature $\mathcal{M}_{k_1}$ if for any Picard sequence $\{\zeta_n = \mathcal{A}\zeta_{n-1},\ n \in \mathbb{N}\}$ with initial value $\zeta_0 \in \mathfrak{X}$, there exist subsequences $\{\zeta_{n_k}\}$ and $\{\zeta_{m_k}\}$ of $\{\zeta_n\}$ such that

$$\lim_{k\to+\infty} \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) = a(s) \in (0,1],$$

where $m_k > n_k > k$, $k \in \mathbb{N}$, and $q \in (0,1)$, then

$$\mathfrak{Z}(\zeta_{m_k}, \zeta_{m_k+1}, s) > q \cdot \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s)$$

for all $s > 0$ holds.

Now we present an explanatory example in the support of Definition 8.

Example 2. Consider $\mathfrak{X} = [0,1]$. Define a membership function $\mathfrak{Z} : \mathfrak{X}^2 \times (0,+\infty) \to [0,1]$ by

$$\mathfrak{Z}(\zeta, \eta, s) = \begin{cases} 1, & \zeta = \eta = 0, \\ \mathrm{e}^{-\max(\zeta,\eta)/s} & \text{otherwise.} \end{cases}$$

$(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is a fuzzy metric-like space (not a fuzzy metric space) with product t-norm. Define a map $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ such that

$$\mathcal{A}(\alpha) = \begin{cases} 1, & \alpha \in \{0,1\}, \\ \sqrt{\alpha} & \text{otherwise.} \end{cases}$$

Let $\zeta_n = 1 - 1/n$, $n \in \mathbb{N}$, be a sequence. Consider $3k > 2k > k$ such that $\zeta_{m_k} = \zeta_{3k} = 1 - 1/(3k)$ and $\zeta_{n_k} = \zeta_{2k} = 1 - 1/(2k)$. Since

$$\lim_{k\to+\infty} \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) = \mathfrak{Z}\left(1 - \frac{1}{3k},\ 1 - \frac{1}{2k},\ s\right) \in (0,1],$$

then $\mathfrak{Z}(\zeta_{3k}, \zeta_{3k+1}, s) > q \cdot \mathfrak{Z}(\zeta_{3k}, \zeta_{2k}, s)$ where $q = 1/2$, Definition 7 holds.

Next, we give another definition, which provides the facility to obtain fixed point.

Definition 9. A self-map $\mathcal{A}$ defined on a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to have feature $(\mathcal{M}_{k_2})$ if for any 1-$\mathfrak{Z}$-convergent sequence $\{\zeta_n\}$ converging to ζ^*,

$$\mathcal{A}(\zeta_n, \zeta_{n+1}, s) > q \cdot \mathcal{A}(\zeta_n, \zeta^*, s),$$

where $n \in \mathbb{N}$ and $q \in (0,1)$.

Now we are ready to put our major outcomes.

Theorem 1. *Suppose that $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ is a Suzuki-type fuzzy Θ_1-contractive mapping in a 1-$\mathfrak{Z}$-complete fuzzy metric-like space. Then $\mathcal{A}$ has a unique fixed point ζ^* in $\mathfrak{X}$, and for every $\zeta_0 \in \mathfrak{X}$, the Picard sequences $\{\zeta_n\}$, where $\zeta_n = \mathcal{A}\zeta_{n-1}$ for all $n \in \mathbb{N}$, converge to the fixed point of $\mathcal{A}$, provided that $\mathcal{A}$ has properties $\mathcal{M}_{k_1}, \mathcal{M}_{k_2}$ and $\mathfrak{Z}(\zeta^*, \zeta^*, s) = 1$ for all $s > 0$.*

Proof. Assume that $\mathcal{A}$ is a Suzuki-type Θ_1-contractive mapping with regards to $\Theta \in \Theta_{\mathfrak{B}}$. Let $\zeta_0 \in \mathfrak{X}$ and define a sequence by Picard iteration process $\{\zeta_n\}$ such as $\zeta_n = \mathcal{A}\zeta_{n-1}$ for all $n \in \mathbb{N}$.

If $\zeta_n = \zeta_{n-1}$, then $\mathcal{A}\zeta_{n-1} = \zeta_n = \zeta_{n-1}$ for all $n \in \mathbb{N}$, that is, ζ_{n-1} is a fixed point for self-map $\mathcal{A}$. So, no need to prove further. Thus, we consider $\zeta_n \neq \zeta_{n-1}$ for all $n \in \mathbb{N}$, that is, no successive terms of $\{\zeta_n\}$ are identical.

First, we must need to show that $\{\zeta_n\}$ is a Cauchy sequence.

Let

$$\mathfrak{I}_k = \inf\bigl\{\mathfrak{Z}(\zeta_m, \zeta_n, s)\colon\ m, n \geqslant k\bigr\},$$

where $k \in \mathbb{N}$, the sequence $\{\mathfrak{I}_k\}$ is monotonically increasing and bounded above sequences in $[0, 1]$. Thus the sequence $\{\mathfrak{I}_k\}$ is monotonically bounded, therefore it is convergent, that is, there exists $\mathfrak{I} \in [0, 1]$ such that

$$\lim_{k\to+\infty} \mathfrak{I}_k = \mathfrak{I}.$$

Next, we must present $\mathfrak{I} = 1$. We use contrapositive procedure. Let us suppose $\mathfrak{I} < 1$. By the definition of $\mathfrak{I}_k$ for every $k \in \mathbb{N}$, there exist two subsequences $\{\zeta_{m_k}\}$ and $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $m_k > n_k \geqslant k$ and

$$\mathfrak{I}_k \leqslant \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) < \mathfrak{I}_k + \frac{1}{k}. \tag{2}$$

Taking limit as k tends to $+\infty$, in (2), we get

$$\lim_{k\to+\infty} \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) = \mathfrak{I}.$$

Since self-map $\mathcal{A}$ is a Suzuki-type fuzzy Θ_1-contractive with regards to $\Theta \in \Theta_{\mathfrak{B}}$ and by the property $\mathcal{M}_{k_1}$, for $m_k > n_k \geqslant k$, $k \in \mathbb{N}$, we have

$$\begin{aligned}&\mathfrak{Z}(\zeta_{m_k-1}, \zeta_{m_k}, s) > q \cdot \mathfrak{Z}(\zeta_{m_k-1}, \zeta_{n_k-1}, s)\\ &\quad\Longrightarrow\quad \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) \geqslant \Theta\bigl(\mathfrak{Z}(\zeta_{m_k-1}, \zeta_{n_k-1}, s)\bigr).\end{aligned}$$

Again,

$$\mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) \geqslant \Theta\bigl(\mathfrak{Z}(\zeta_{m_k-1}, \zeta_{n_k-1}, s)\bigr) \geqslant \Theta^2\bigl(\mathfrak{Z}(\zeta_{m_k-2}, \zeta_{n_k-2}, s)\bigr).$$

Inductively, we can have

$$\mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) \geqslant \Theta^{n_k}\bigl(\mathfrak{Z}(\zeta_0, \zeta_{m_k-n_k}, s)\bigr) \quad \text{for all } n_k \in \mathbb{N}.$$

Now taking that k tends to $+\infty$ and using definition 5, we obtain

$$\lim_{k\to+\infty} \mathfrak{I}_k = \mathfrak{I} = 1,$$

we get a contradiction. Thus

$$\lim_{n,m\to+\infty} \mathfrak{Z}(\zeta_n, \zeta_m, s) = 1.$$

Therefore $\{\zeta_n\}$ is a 1-$\mathfrak{Z}$ Cauchy sequence in $\mathfrak{X}$. By 1-$\mathfrak{Z}$-completeness property of the space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ there exists $\zeta^* \in \mathfrak{X}$ such that

$$\lim_{n\to+\infty} \mathfrak{Z}(\zeta_n, \zeta^*, s) = \lim_{n,m\to+\infty} \mathfrak{Z}(\zeta_n, \zeta_m, s) = \mathfrak{Z}(\zeta^*, \zeta^*, s) = 1.$$

Next, we prove that ζ^* is a fixed point of $\mathcal{A}$. Using property $\mathcal{M}_{k_2}$ and $\Theta(s) > s$ for all $s > 0$,

$$\begin{aligned}
&\mathfrak{Z}\left(\zeta_{n-1}, \zeta_n, \frac{s}{2}\right) > q \cdot \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right) \\
&\quad\Longrightarrow\quad \mathfrak{Z}\left(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta^*, \frac{s}{2}\right) \geqslant \Theta\left(\mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right)\right) \\
&\quad\Longrightarrow\quad \mathfrak{Z}\left(\zeta_n, \mathcal{A}\zeta^*, \frac{s}{2}\right) \geqslant \Theta\left(\mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right)\right) > \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right) \\
&\quad\Longrightarrow\quad \mathfrak{Z}\left(\zeta_n, \mathcal{A}\zeta^*, \frac{s}{2}\right) > \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right).
\end{aligned}$$

By the triangle inequality and using above inequality we have

$$\begin{aligned}
&\mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s) \\
&\quad\geqslant \mathfrak{Z}\left(\zeta^*, \zeta_n, \frac{s}{2}\right) \circledast \mathfrak{Z}\left(\zeta_n, \mathcal{A}\zeta^*, \frac{s}{2}\right) \geqslant \mathfrak{Z}\left(\zeta^*, \zeta_n, \frac{s}{2}\right) \circledast \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right).
\end{aligned}$$

Letting limit as n tends to $+\infty$, we can deduce that $\mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s) = 1$ implies $\mathcal{A}\zeta^* = \zeta^*$.

Next, we claim that fixed point ζ^* is unique. Let $v \in \mathfrak{X}$ is another fixed point of $\mathcal{A}$, that is, $\mathcal{A}v = v$ and for any $s_1 > 0$ such that $\mathfrak{Z}(\zeta^*, v, s_1) < 1$, we have $1 = \mathfrak{Z}(\zeta^*, \zeta^*, s_1) = \mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s_1) > q \cdot \mathfrak{Z}(\zeta^*, v, s_1)$. Since

$$\begin{aligned}
&\mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s_1) > q \cdot \mathfrak{Z}(\zeta^*, v, s_1) \\
&\quad\Longrightarrow\quad \mathfrak{Z}(\mathcal{A}\zeta^*, \mathcal{A}v, s_1) \geqslant \Theta\big(\mathfrak{Z}(\zeta^*, v, s_1)\big) \\
&\quad\Longrightarrow\quad \mathfrak{Z}(\zeta^*, v, s_1) \geqslant \Theta\big(\mathfrak{Z}(\zeta^*, v, s_1)\big) > \mathfrak{Z}(\zeta^*, v, s_1),
\end{aligned}$$

a contradiction. Thus ζ^* is a unique fixed point of $\mathcal{A}$, also, it verifies $\mathfrak{Z}(\zeta^*, \zeta^*, s) = 1$ for all $s > 0$. □

The significance of our findings and the accuracy of our generalization are demonstrated by the following example in which we will show that Suzuki-type fuzzy Θ_1-contractive mapping may also be discontinuous.

Example 3. Let $\mathfrak{X} = \mathbb{N} \cup \{0\}$. Define $\circledast$ by $l \circledast m = l \cdot m$ and a membership function $\mathfrak{Z} : \mathfrak{X}^2 \times (0, +\infty) \to [0, 1]$ by

$$\mathfrak{Z}(\alpha, \beta, s) = \begin{cases} \frac{\alpha}{\beta^3}, & \alpha \leqslant \beta, \\ \frac{\beta}{\alpha^3}, & \beta \leqslant \alpha, \\ \frac{1}{2}, & \alpha = \beta = 0, \end{cases}$$

where $\alpha, \beta \in \mathfrak{X}$. Then $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is a 1-$\mathfrak{Z}$-complete fuzzy metric-like space. Define a self-map $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ such that

$$\mathcal{A}(\gamma) = \begin{cases} 1, & \gamma \in \{0, 1\}, \\ 0, & \gamma \in \mathfrak{X} - \{0, 1\}. \end{cases}$$

Define $\Theta : [0, 1] \to [0, 1]$ be any discontinuous function

$$\Theta(\omega) = \begin{cases} 3\omega, & \omega \in [0, \frac{1}{4}], \\ 1, & \omega \in (\frac{1}{4}, 1], \end{cases}$$

where a and b nonnegative real numbers such that $3 = a > 1$, $1/4 = b < 1$, $ab = 3/4 < 1$.

Consider $q = 1/2$ and take the following cases to verify the Suzuki-type Θ_1-contractive mapping.

Case 1. If $\gamma_1 = 1$ and $\gamma_2 = 1$, $\mathfrak{Z}(1, 1, s) > q \cdot \mathfrak{Z}(1, 1, s)$, that is, $1 > 1/2$ implies $\mathfrak{Z}(\mathcal{A}1, \mathcal{A}1, s) \geqslant \Theta(\mathfrak{Z}(1, 1, s))$ implies $1 = \mathfrak{Z}(1, 1, s) \geqslant \Theta(\mathfrak{Z}(1, 1, s)) = \Theta(1) = 1$. Inequality holds.

Case 2. If $\gamma_1 = 0$ and $\gamma_2 = 0$, $\mathfrak{Z}(0, 1, s) > q \cdot \mathfrak{Z}(0, 0, s)$, that is, $0 > 1/4$, hypothesis of inequality (1) does not hold, so that no need to further investigation, but even then $1 = \mathfrak{Z}(1, 1, s) \geqslant \Theta(\mathfrak{Z}(1, 1, s)) = \Theta(1) = 1$, conclusion part of inequality (1) holds. So, inequality (1) holds for this case.

Case 3. For all γ_1 and $\gamma_2 \in \mathfrak{X} - \{0, 1\}$ such that $\gamma_1 \leqslant \gamma_2$ or $\gamma_2 \leqslant \gamma_1$, $\mathfrak{Z}(\gamma_1, 0, s) > \mathfrak{Z}(\gamma_1, \gamma_2, s)/2$, that is, $0 > (\gamma_1/\gamma_2^3)/2$ does not hold. Now $\mathfrak{Z}(\mathcal{A}\gamma_1, \mathcal{A}\gamma_2, s) \geqslant \Theta(\mathfrak{Z}(\gamma_1, \gamma_2, s))$, that is, $\mathfrak{Z}(0, 0, s) \geqslant \Theta(\gamma_1/\gamma_2^3)$.

Let us consider $\gamma_1 = 2$ and $\gamma_2 = 2$. So, $\mathfrak{Z}(0, 0, s) \geqslant \Theta(\gamma_1/\gamma_2^3)$ implies $1/2 \geqslant 3/4$, which is not possible.

Thus inequality (1) holds for this case also.

To understand the beauty of Suzuki-type fuzzy contractive conditions, we need to verify contraction conditions for those points for which the hypothesis of the contractive condition is satisfied, as well as the conclusion part. For those points where the hypothesis does not hold, there is no need to verify the conclusion part. But the interesting part of Example 3 is that for those points, the hypothesis does not hold, and even the conclusion part does not hold. So, by this, we can prove with the help of the above Example 3 that the self-map $\mathcal{A}$ is Suzuki-type fuzzy Θ_1-contractive but not a fuzzy Θ-contraction in the sense of [11]. Also, $(\mathfrak{X}, \mathfrak{Z}, \mathcal{A}, \Theta)$ has features $\mathcal{M}_{k_1}, \mathcal{M}_{k_2}$, trivially. Thus, every assertion of Theorem 1 holds. So, $\gamma = 1$ is the unique fixed point.

Mihet [8] introduced fuzzy Υ-contractive mappings in fuzzy metric space. Let Φ be the category of all functions $\Upsilon : [0,1] \to [0,1]$ such that Υ is continuous, nondecreasing, and $\Upsilon(t) > t$ for all $t \in (0,1)$. The category Φ is known to be contained in the family $\Theta_{\mathfrak{B}}$ (see [11]). In the line of this, we can obtain a few corollaries.

Corollary 1. *Suppose a self-map $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ defined on a 1-$\mathfrak{Z}$-complete fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ satisfies*

$$\mathfrak{Z}(\zeta, \mathcal{A}\zeta, s) > q \cdot \mathfrak{Z}(\zeta, \eta, s) \quad \Longrightarrow \quad \mathfrak{Z}(\mathcal{A}\zeta, \mathcal{A}\eta, s) \geqslant \Upsilon\big(\mathcal{A}(\zeta, \eta, s)\big),$$

where $\Upsilon \in \Phi$, $q \in (0,1)$, for all $\zeta, \eta \in \mathfrak{X}$ with properties $\mathcal{M}_{k_1}, \mathcal{M}_{k_2}$. Then $\mathcal{A}$ has unique fixed point $u \in \mathfrak{X}$, and $\mathfrak{Z}(u, u, s) = 1$.

Proof. The proof is similar to that of Theorem 1. □

Corollary 2. *Suppose a self-map $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ defined on a 1-$\mathfrak{Z}$-complete fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ satisfies*

$$\mathfrak{Z}(\zeta, \mathcal{A}\zeta, s) > q \cdot \mathfrak{Z}(\zeta, \eta, s) \quad \Longrightarrow \quad \mathfrak{Z}(\mathcal{A}^n\zeta, \mathcal{A}^n\eta, s) \geqslant \Theta\big(\mathfrak{Z}(\zeta, \eta, s)\big),$$

where $q \in (0,1)$, for all $\zeta, \eta \in \mathfrak{X}$, $s > 0$ and with properties $\mathcal{M}_{k_1}, \mathcal{M}_{k_2}$. Then $\mathcal{A}$ has a unique fixed point $u \in \mathfrak{X}$, and $\mathfrak{Z}(u, u, s) = 1$.

Now prove our other theorem in the view of Suzuki-type contraction, which is a generalization of Theorem 2 given in [11].

Theorem 2. *Suppose that a self-map $\mathcal{A}$ defined on a 1-$\mathfrak{Z}$-complete fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$, and a function $g : (0, +\infty) \to (0, +\infty)$ is such that $\mathfrak{Z}(\zeta, \eta, g(s)) \geqslant \mathfrak{Z}(\zeta, \eta, s)$ and $\lim_{n\to+\infty} \mathfrak{Z}(\zeta, \eta, g^n(s)) = 1$ for all $\zeta, \eta \in \mathfrak{X}$, $s > 0$.*

(i) *For all $\zeta, \eta \in \mathfrak{X}$, $s > 0$, and $q \in (0,1)$, mapping $\mathcal{A}$ satisfies*

$$\mathfrak{Z}(\zeta, \mathcal{A}\zeta, s) > q \cdot \mathfrak{Z}(\zeta, \eta, s) \quad \Longrightarrow \quad \mathfrak{Z}(\mathcal{A}\zeta, \mathcal{A}\eta, s) \geqslant \mathfrak{Z}\big(\zeta, \eta, g(s)\big); \tag{3}$$

(ii) *$\mathcal{A}$ has the properties $\mathcal{M}_{k_1}$ and $\mathcal{M}_{k_2}$.*

Then $\mathcal{A}$ has a unique fixed point $\zeta^* \in \mathfrak{X}$ and $\mathfrak{Z}(\zeta^*, \zeta^*, s) = 1$ for all $s > 0$.

Proof. Formulation of Picard sequence $\{\zeta_n\}$ is similar to Theorem 1 such that no consecutive terms of sequence $\{\zeta_n\}$ are same. We must show that $\{\zeta_n\}$ is a Cauchy sequence.

For this, let

$$\mathfrak{I}_k = \inf\big\{\mathfrak{Z}(\zeta_m, \zeta_n, s)\colon\ m, n \geqslant k\big\},$$

where $k \in \mathbb{N}$, the sequence $\{\mathfrak{I}_k\}$ is monotonically increasing and bounded above sequences in $[0,1]$. Thus the sequence $\{\mathfrak{I}_k\}$ is monotonic bounded, therefore it is convergent, that is, there exists $\mathfrak{I} \in [0,1]$ such that

$$\lim_{k\to+\infty} \mathfrak{I}_k = \mathfrak{I}.$$

We must show $\mathfrak{I} = 1$. Taking a contrary, let us suppose $\mathfrak{I} < 1$. By the definition of $\mathfrak{I}_k$ for every $k \in \mathbb{N}$ there exist subsequences $\{\zeta_{m_k}\}$ and $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $m_k > n_k \geqslant k$ and

$$\mathfrak{I}_k \leqslant \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) < \mathfrak{I}_k + \frac{1}{k}.$$

Taking k tends to $+\infty$, we get

$$\lim_{k\to+\infty} \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) = \mathfrak{I}.$$

Since $\mathcal{A}$ satisfies contraction (3) and by the property $\mathcal{M}_{k_1}$, for $m_k > n_k \geqslant k$, $k \in \mathbb{N}$, we have

$$\begin{aligned} &\mathfrak{Z}(\zeta_{m_k-1}, \zeta_{m_k}, s) > q \cdot \mathfrak{Z}(\zeta_{m_k-1}, \zeta_{n_k-1}, s) \\ &\quad\Longrightarrow\quad \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) \geqslant \mathfrak{Z}\big(\zeta_{m_k-1}, \zeta_{n_k-1}, g(s)\big). \end{aligned}$$

Again,

$$\mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) \geqslant \mathfrak{Z}\big(\zeta_{m_k-1}, \zeta_{n_k-1}, g(s)\big) \geqslant \mathfrak{Z}\big(\zeta_{m_k-2}, \zeta_{n_k-2}, g^2(s)\big).$$

By induction

$$\mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) \geqslant \mathfrak{Z}\big(\zeta_0, \zeta_{m_k-n_k}, g^{n_k}(s)\big) \quad \text{for all } n_k \in \mathbb{N}.$$

Letting limit k tends to $+\infty$,

$$\lim_{k\to\infty} \mathfrak{I}_k = \mathfrak{I} = 1,$$

we get a contradiction. Thus

$$\lim_{n,m\to+\infty} \mathfrak{Z}(\zeta_n, \zeta_m, s) = 1.$$

Thus $\{\zeta_n\}$ is a 1-$\mathfrak{Z}$ Cauchy sequence in $\mathfrak{X}$. By 1-$\mathfrak{Z}$-completeness of the space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ there exist $\zeta^* \in \mathfrak{X}$ such that

$$\lim_{n\to+\infty} \mathfrak{Z}(\zeta_n, \zeta^*, s) = \lim_{n,m\to+\infty} \mathfrak{Z}(\zeta_n, \zeta_m, s) = \mathfrak{Z}(\zeta^*, \zeta^*, s) = 1. \tag{4}$$

Next, we must show that ζ^* is a fixed point of $\mathcal{A}$. Using the feature $\mathcal{M}_{k_2}$, we can have

$$\begin{aligned} &\mathfrak{Z}\left(\zeta_{n-1}, \zeta_n, \frac{s}{2}\right) > q \cdot \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right) \\ &\quad\Longrightarrow\quad \mathfrak{Z}\left(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta^*, \frac{s}{2}\right) \geqslant \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, g\left(\frac{s}{2}\right)\right) \\ &\quad\Longrightarrow\quad \mathfrak{Z}\left(\zeta_n, \mathcal{A}\zeta^*, \frac{s}{2}\right) \geqslant \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, g\left(\frac{s}{2}\right)\right) > \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right) \\ &\quad\Longrightarrow\quad \mathfrak{Z}\left(\zeta_n, \mathcal{A}\zeta^*, \frac{s}{2}\right) > \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right). \end{aligned}$$

By the triangle inequality we have

$$\begin{aligned}&\mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s)\\&\quad\geqslant \mathfrak{Z}\left(\zeta^*, \zeta_n, \frac{s}{2}\right) \circledast \mathfrak{Z}\left(\zeta_n, \mathcal{A}\zeta^*, \frac{s}{2}\right) \geqslant \mathfrak{Z}\left(\zeta^*, \zeta_n, \frac{s}{2}\right) \circledast \mathfrak{Z}\left(\zeta_{n-1}, \zeta^*, \frac{s}{2}\right).\end{aligned}$$

Using (4), we can deduce that $\mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s) = 1$ implies $\mathcal{A}\zeta^* = \zeta^*$.

Next, we claim that fixed point ζ^* is unique. Taking contrary, suppose $v \in \mathfrak{X}$ is another fixed point of $\mathcal{A}$, that is, $\mathcal{A}v = v$, and for any $s_1 > 0$ such that $\mathfrak{Z}(\zeta^*, v, s_1) < 1$, $1 = \mathfrak{Z}(\zeta^*, \zeta^*, s_1) = \mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s_1) > q \cdot \mathfrak{Z}(\zeta^*, v, s_1)$. Since

$$\begin{aligned}&\mathfrak{Z}(\zeta^*, \mathcal{A}\zeta^*, s_1) > q \cdot \mathfrak{Z}(\zeta^*, v, s_1)\\&\quad\Longrightarrow\quad \mathfrak{Z}(\mathcal{A}\zeta^*, \mathcal{A}v, s_1) \geqslant \mathfrak{Z}\big(\zeta^*, v, g(s_1)\big)\\&\quad\Longrightarrow\quad \mathfrak{Z}(\zeta^*, v, s_1) \geqslant \mathfrak{Z}\big(\zeta^*, v, g(s_1)\big) > \mathfrak{Z}(\zeta^*, v, s_1)\end{aligned}$$

is a contradiction, thus ζ^* is a unique fixed point of $\mathcal{A}$, and it verifies $\mathfrak{Z}(\zeta^*, \zeta^*, s) = 1$ for all $s > 0$. □

Now we give another contraction condition by changing our hypothesis in the Suzuki-type view.

Definition 10. A mapping $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ defined on a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to be a Suzuki-type fuzzy $\varTheta_2$-contraction if there exists $\varTheta \in \varTheta_{\mathfrak{B}}$ such that

$$\mathfrak{Z}(\zeta, \mathcal{A}\zeta, s) > \mathfrak{Z}\left(\zeta, \eta, \frac{s}{2}\right) \quad\Longrightarrow\quad \mathfrak{Z}(\mathcal{A}\zeta, \mathcal{A}\eta, s) \geqslant \varTheta\big(\mathfrak{Z}(\zeta, \eta, s)\big) \tag{5}$$

for all $\zeta, \eta \in \mathfrak{X}$ and for all $s > 0$.

In the view of above Definition 10, it is essential to define when the sequence $\{\zeta_n\}$ is a 1-$\mathfrak{Z}$ Cauchy sequence.

Definition 11. A self-map $\mathcal{A}$ defined on a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to have feature $\mathcal{M}_{l_1}$ if for any Picard sequence $\{\zeta_n = \mathcal{A}\zeta_{n-1},\ n \in \mathbb{N}\}$ with initial value $\zeta_0 \in \mathfrak{X}$, there exist two subsequences $\{\zeta_{n_k}\}$ and $\{\zeta_{m_k}\}$ of $\{\zeta_n\}$ such that if

$$\lim_{k\to+\infty} \mathfrak{Z}(\zeta_{m_k}, \zeta_{n_k}, s) = a(s) \in (0, 1],$$

where $m_k > n_k > k$, $k \in \mathbb{N}$, and $q \in (0, 1)$, then

$$\mathfrak{Z}(\zeta_{m_k}, \zeta_{m_k+1}, s) > \mathfrak{Z}\left(\zeta_{m_k}, \zeta_{n_k}, \frac{s}{2}\right)$$

holds for all $s > 0$.

Now we need the following definition to ensure that the self-map $\mathcal{A}$ has a fixed point.

Definition 12. A self-map $\mathcal{A}$ defined on a fuzzy metric-like space $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is said to have feature $(\mathcal{M}_{l_2})$ if for any 1-$\mathcal{A}$-convergent sequence $\{\zeta_n\}$ converging to ζ^*,

$$\mathfrak{Z}(\zeta_n, \zeta_{n+1}, s) > \mathfrak{Z}\left(\zeta_n, \zeta^*, \frac{s}{2}\right),$$

where $n \in \mathbb{N}$ and $s > 0$.

Theorem 3. *Suppose that $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ be a Suzuki-type fuzzy Θ_2-contractive mapping in a 1-$\mathfrak{Z}$-complete fuzzy metric-like space. Then $\mathcal{A}$ has a unique fixed point ζ^* in $\mathfrak{X}$, and for every $\zeta_0 \in \mathfrak{X}$, the Picard sequences $\{\zeta_n\}$, where $\zeta_n = \mathcal{A}\zeta_{n-1}$, $n \in \mathbb{N}$, converge to the fixed point of $\mathcal{A}$, provided that $\mathcal{A}$ has properties $\mathcal{M}_{l_1}, \mathcal{M}_{l_2}$ and $\mathfrak{Z}(\zeta^*, \zeta^*, s) = 1$ for all $s > 0$.*

Proof. The proof is similar to that of Theorem 1. □

Example 4. Let $\mathfrak{X} = \{1, 2, 4\}$. Define a membership function $\mathfrak{Z}$ in $\mathfrak{X}^2 \times (0, +\infty)$ with product t-norm by

$$\mathfrak{Z}(\gamma_1, \gamma_2, s) = \begin{cases} \frac{\gamma_1}{\gamma_2^3}, & \gamma_1 \leqslant \gamma_2, \\ \frac{\gamma_2}{\gamma_1^3}, & \gamma_2 \leqslant \gamma_1. \end{cases}$$

is a 1-$\mathfrak{Z}$-complete fuzzy metric-like space but not fuzzy metric space. Consider the mapping $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ such that

$$\mathcal{A}(\gamma_1) = \begin{cases} 1, & \gamma_1 = 1, 2, \\ 2, & x = 4. \end{cases}$$

Consider $\Theta(\omega) > \omega$, where $\omega > 0$. For $\gamma_1 = 4$ and $\gamma_2 = 4$, the hypothesis is not satisfied, the rest of the points satisfy condition (5). Thus $\mathcal{A}$ is Suzuki-type Θ_2-contractive mapping, and $\gamma = 1$ is the unique fixed point of $\mathcal{A}$.

Example 4 shows that $\mathcal{A}$ is Suzuki-type Θ_2-contractive mapping but Θ-contraction given in [11] also. Now we present an example in support of only Suzuki-type fuzzy Θ_2-contractive conditions, which are not fuzzy Θ-contractions. In this way, we can ensure that our generalization of fuzzy Θ-contraction in terms of Suzuki-type Θ-contraction is genuine.

Example 5. Let $\mathfrak{X} = [0, 1]$. Define a membership function $\mathfrak{Z} : \mathfrak{X}^2 \times (0, +\infty) \to [0, 1]$ with product t-norm by

$$\mathfrak{Z}(\gamma_1, \gamma_2, s) = \begin{cases} 1, & \gamma_1 = \gamma_2 = 0, \\ \frac{\gamma_1 + \gamma_2}{2} & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is a 1-$\mathfrak{Z}$-complete fuzzy metric-like space. Define $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ such that

$$\mathcal{A}(\gamma) = \begin{cases} 0, & \gamma \in \{0, \frac{1}{2}\}, \\ 1 - \gamma, & \gamma \in \mathfrak{X} - \{0, \frac{1}{2}\}. \end{cases}$$

Take $\Theta \in \Theta_{\mathfrak{B}}$ such that $\Theta(s) > s$ for all $s \in (0,1)$. Consider the sequence $\gamma_n = 1/n$, $n \in \mathbb{N}$, and consider $3k > 2k > k$ such that $\gamma_{m_k} = \gamma_{3k} = 1/(3k)$ and $\gamma_{n_k} = \gamma_{2k} = 1/(2k)$. We can see that Definitions 11 and 12 hold, trivially.

Next, we have to show that $\mathcal{A}$ satisfies the contraction condition (5). For this, we use the hypothesis

$$\mathfrak{Z}(\gamma_1, \mathcal{A}\gamma_1, s) > \mathfrak{Z}\left(\gamma_1, \gamma_2, \frac{s}{2}\right) \quad \Longrightarrow \quad 1 > \gamma_1 + \gamma_2$$

for some $\gamma_1, \gamma_2 \in \mathfrak{X}$, the inequality

$$\begin{aligned} &\mathfrak{Z}(\mathcal{A}\gamma_1, \mathcal{A}\gamma_2, s) \geqslant \Theta\big(\mathfrak{Z}(\gamma_1, \gamma_2, s)\big) \\ &\quad \Longrightarrow \quad \mathfrak{Z}(1-\gamma_1,\, 1-\gamma_2, s) \geqslant \Theta\left(\frac{\gamma_1+\gamma_2}{2}\right) > \frac{\gamma_1+\gamma_2}{2} \\ &\quad \Longrightarrow \quad 1 > \gamma_1 + \gamma_2 \end{aligned}$$

for those points for which hypothesis is satisfied. For some $\gamma_1, \gamma_2 \in \mathfrak{X}$ such that $\gamma_1 + \gamma_2 \geqslant 1$, the hypothesis is not satisfied as well as conclusion $(1 > \gamma_1 + \gamma_2)$ still not holds. So, this is one of a suitable example, which shows that a self $\mathcal{A}$ is Suzuki-type fuzzy Θ_2-contraction but not a fuzzy Θ-contraction given by [11].

One more observation in the sense of Definition 7 is that self-map $\mathcal{A}$ is not a Suzuki-type Θ_1-contraction. Also, $(\mathfrak{X}, \mathfrak{Z}, \mathcal{A}, \Theta)$ possess features $\mathcal{M}_{l_1}, \mathcal{M}_{l_2}$, thus all the conditions of the above Theorem 3 satisfied by the self-map $\mathcal{A}$. Hence $\gamma = 0$ is a unique fixed point of $\mathcal{A}$.

4 Application to fractional calculus

In the application part, we have mentioned a theorem for ensuring the solution of a nonlinear fractional differential equation with few boundary conditions via Suzuki-type fuzzy Θ-contraction.

The Caputo derivative of fractional order α for a continuous function $g : [0, +\infty) \to \mathbb{R}$ is defined as

$$^{C}D^{\alpha} g(t_1) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t_1 - s_1)^{n-\alpha-1} g^n(s_1)\, \mathrm{d}s_1$$

$(n-1 < \alpha < n,\ n = [\alpha]+1)$, where $[\alpha]$ indicates integer part of real number α, and gamma function is denoted by Γ (see [5]).

Consider the nonlinear fractional differential equation with boundary condition

$$^{C}D^{\alpha}(\zeta(t_1)) + \xi\big(t_1, \zeta(t_1)\big) = 0 \quad 0 \leqslant t_1 < 1,\ \alpha < 1, \tag{6}$$

$\zeta(0) = 0 = \zeta(1)$, where $\zeta \in \mathfrak{C}([0,1], \mathbb{R})$, the set of all continuous function from $[0,1]$ into $\mathbb{R}$ is indicated by $\mathfrak{C}([0,1], \mathbb{R})$, and $\xi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is also continuous function.

Remind that analogous Green function with problem (6) is written by

$$\mathcal{G}(t_1, s_1) = \begin{cases} (t_1(1-s_1))^{\alpha-1} - (t_1 - s_1)^{\alpha-1} & \text{if } 0 \leqslant s_1 \leqslant t_1 \leqslant 1, \\ \frac{(t_1(1-s_1))^{\alpha-1}}{\Gamma\alpha} & \text{if } 0 \leqslant t_1 \leqslant s_1 \leqslant 1. \end{cases}$$

Let $\mathfrak{X} = \mathfrak{C}([0,1], \mathbb{R})$ be a metric spaces endowed with the metric

$$\rho\big(\zeta(t_1), \eta(t_1)\big) = \big\|\zeta(t_1) - \eta(t_1)\big\|_\infty = \sup_{t_1 \in [0,1]} \big|\zeta(t_1) - \eta(t_1)\big|$$

for all $\zeta(t_1), \eta(t_1) \in \mathfrak{X}$.

Define a membership function $\mathfrak{Z} : \mathfrak{X} \times \mathfrak{X} \times (0, +\infty) \to [0,1]$ by

$$\mathfrak{Z}(\zeta(t_1), \eta(t_1), s) = \mathrm{e}^{-\rho(\zeta(t_1), \eta(t_1))/s} \quad \text{for all } \zeta(t_1), \eta(t_1) \in \mathfrak{X} \text{ and } s > 0,$$

the ordered tripled $(\mathfrak{X}, \mathfrak{Z}, \circledast)$ is a 1-$\mathfrak{Z}$ fuzzy metric-like space endowed with the product t-norm, and also, space is a fuzzy metric.

Now we insert the main theorem of this section in the form of an application.

Theorem 4. *Assume that*

(i) *for all* $t_1 \in [0,1]$*,* $a, b \in \mathbb{R}$*,* $s > 0$*,* $\tau \in (0,1)$*, and* $q \in (0,1)$*,*

$$\mathfrak{Z}\big(\zeta(t_1), \mathcal{A}\zeta(t_1), s\big) > q \cdot \mathfrak{Z}\big(\zeta(t_1), \eta(t_1), s\big) \implies |\xi(t_1, a) - \xi(t_1, b)| \leqslant \tau \cdot |a - b|;$$

(ii) *a self-map* $\mathcal{A} : \mathfrak{C}([0,1], \mathbb{R}) \to \mathfrak{C}([0,1], \mathbb{R})$ *defined by*

$$\mathcal{A}\zeta(t_1) = \int_0^1 \mathcal{G}(t_1, s_1) \cdot \xi\big(s_1, \zeta(s_1)\big)\, \mathrm{d}s_1;$$

(iii) *for each* $t_1 \in [0,1]$ *and* $\zeta(t_1), \eta(t_1) \in \mathfrak{C}([0,1], \mathbb{R})$*, if* $\{\zeta_n(t_1)\}$ *is a sequence in* $\mathfrak{C}([0,1], \mathbb{R})$ *and* $\{\zeta_{m_k}(t_1)\}$*,* $\{\zeta_{n_k}(t_1)\}$ *be two subsequences of* $\{\zeta_n(t_1)\}$ *such that*

$$\lim_{k \to +\infty} \big|\zeta_{m_k}(t_1) - \zeta_{n_k}(t_1)\big| \to a(k) \in \mathbb{R}^+ \implies -\log_\mathrm{e} q + \big|\zeta_{m_k}(t_1) - \zeta_{n_k}(t_1)\big| > \big|\zeta_{m_k}(t_1) - \zeta_{m_{k+1}}(t_1)\big|$$

for $m_k > n_k > k$*,* $k \in \mathbb{N}$*, and* $q \in (0,1)$*;*

(iv) *for each* $t_1 \in [0,1]$*, if* $\{\zeta_n(t_1)\}$ *is a sequence in* $\mathfrak{C}([0,1], \mathbb{R})$ *such that*

$$\big\{\zeta_n(t_1)\big\} \to \zeta^*(t_1) \in \mathfrak{C}\big([0,1], \mathbb{R}\big) \implies -\log_\mathrm{e} q + \big|\zeta_n(t_1) - \zeta^*(t_1)\big| > \big|\zeta_n(t_1) - \zeta_{n+1}(t_1)\big|$$

for all $n \in \mathbb{N}$ *and* $q \in (0,1)$*.*

Then problem (6) *has at least one solution.*

Proof. We may see that $\zeta(t_1) \in \mathfrak{X}$ is solution of (6), if $\zeta^*(t_1)$ is the solution of the equation

$$\zeta(t_1) = \int_0^1 \mathcal{G}(t_1, s_1)\xi\big(s_1, \zeta(s_1)\big)\,\mathrm{d}s_1 \quad \text{for all } t_1 \in [0, 1].$$

Then the solution of problem (6) is equivalent to calculating $\zeta^*(t_1) \in \mathfrak{X}$ which will be a fixed point of self-map $\mathcal{A}$. With assumptions (i) and (ii) for all different $\zeta(t_1), \eta(t_1) \in \mathfrak{X}$ such that $\mathfrak{Z}(\zeta(t_1), \eta(t_1), s) > 0$ for all $t_1 \in [0, 1]$ and $s > 0$,

$$\begin{aligned}
\big|\mathcal{A}\zeta(t_1) - \mathcal{A}\eta(t_1)\big| &= \left|\int_0^1 \mathcal{G}(t_1, s_1)\xi\big(s_1, \zeta(s_1)\big)\,\mathrm{d}s_1 - \int_0^1 \mathcal{G}(t_1, s)\xi\big(s_1, \eta(s_1)\big)\,\mathrm{d}s_1\right| \\
&= \left|\int_0^1 \mathcal{G}(t_1, s_1)\big[\xi\big(s_1, \zeta(s_1)\big) - \xi\big(s_1, \eta(s_1)\big)\big]\,\mathrm{d}s_1\right| \\
&\leqslant \int_0^1 \mathcal{G}(t_1, s_1)\big|\xi\big(s_1, \zeta(s_1)\big) - \xi\big(s_1, \eta(s_1)\big)\big|\,\mathrm{d}s_1 \\
&\leqslant \int_0^1 \mathcal{G}(t_1, s_1)\,\mathrm{d}s_1 \cdot \tau\big|\zeta(s_1) - \eta(s_1)\big| \\
&\leqslant \int_0^1 \mathcal{G}(t_1, s_1)\,\mathrm{d}s_1 \cdot \tau \sup_{s_1\in[0,1]} \big|\zeta(s_1) - \eta(s_1)\big| \\
&\leqslant \tau\big\|\zeta(t_1) - \eta(t_1)\big\|_\infty
\end{aligned}$$

implies that

$$\begin{aligned}
&\big\|\mathcal{A}\zeta(t_1) - \mathcal{A}\eta(t_1)\big\|_\infty = \sup_{t_1\in[0,1]} \big|\mathcal{A}\zeta(t_1) - \mathcal{A}\eta(t_1)\big| \leqslant \tau\big\|\zeta(t_1) - \eta(t_1)\big\|_\infty \\
&\implies \quad \frac{\|\mathcal{A}\zeta(t_1) - \mathcal{A}\eta(t_1)\|_\infty}{s} \leqslant \tau\,\frac{\|\zeta(t_1) - \eta(t_1)\|_\infty}{s} \\
&\implies \quad -\tau\,\frac{\|\zeta(t_1) - \eta(t_1)\|_\infty}{s} \leqslant \frac{-\|\mathcal{A}\zeta(t_1) - \mathcal{A}\eta(t_1)\|_\infty}{s} \\
&\implies \quad \mathrm{e}^{-\tau\|\zeta(t_1)-\eta(t_1)\|_\infty/t} = \big(\mathrm{e}^{-\|\zeta(t_1)-\eta(t_1)\|_\infty/s}\big)^\tau \leqslant \mathrm{e}^{-\|\mathcal{A}\zeta(t_1)-\mathcal{A}\eta(t_1)\|_\infty/s} \\
&\implies \quad \big(\mathfrak{Z}\big(\zeta(t_1), \eta(t_1), s\big)\big)^\tau \leqslant \mathfrak{Z}\big(\mathcal{A}\zeta(t_1), \mathcal{A}\eta(t_1), s\big).
\end{aligned}$$

Consider a function $\Theta : [0, 1] \to [0, 1]$ defined by $\Theta(l) = l^\tau$, where $\tau \in (0, 1)$ for all $l \in [0, 1]$, which belongs to the class $\Theta_{\mathfrak{B}}$.

$$\Theta\big(\mathfrak{Z}\big(\zeta(t_1), \eta(t_1), s\big)\big) \leqslant \mathfrak{Z}\big(\mathcal{A}\zeta(t_1), \mathcal{A}\eta(t_1), s\big).$$

Therefore, $\mathcal{A}$ is Suzuki-type fuzzy Θ_1-contractive mapping.

For a sequence $\{\zeta_n(t_1)\}$ in $\mathfrak{C}([0,1],\mathbb{R})$ and $\{\zeta_{m_k}(t_1)\}$, $\{\zeta_{n_k}(t_1)\}$ be two subsequences of $\{\zeta_n(t_1)\}$ such that $m_k > n_k > k$, $k \in \mathbb{N}$, by using assumption (iii)

$$\begin{aligned}\lim_{k\to+\infty} \mathfrak{Z}\big(\zeta_{m_k}(t_1), \zeta_{n_k}(t_1), s\big) &= \lim_{k\to+\infty} \mathrm{e}^{-|\zeta_{m_k}(t_1)-\zeta_{n_k}(t_1)|/s} \\ &= \lim_{k\to+\infty} \mathrm{e}^{-a(k)/s} = a(s) \in (0,1]\end{aligned}$$

implies that

$$\begin{aligned}\mathfrak{Z}\big(\zeta_{m_k}(t_1), \zeta_{m_k+1}(t_1), s\big) &= \mathrm{e}^{-|\zeta_{m_k}(t_1)-\zeta_{m_{k+1}}(t_1)|/s} \\ &> q \cdot \mathrm{e}^{-|\zeta_{m_k}(t_1)-\zeta_{n_k}(t_1)|/s} \\ &= q \cdot \mathfrak{Z}\big(\zeta_{m_k}(t_1), \zeta_{n_k}(t_1), s\big)\end{aligned}$$

Therefore, property $\mathcal{M}_{k_1}$ holds true.

If a sequence $\{\zeta_n(t_1)\}$ in $\mathfrak{C}([0,1],\mathbb{R})$ such that $\zeta_n(t_1) \to \zeta(t_1)$ in $\mathfrak{C}([0,1],\mathbb{R})$ by using assumption (iv),

$$\begin{aligned}\mathfrak{Z}\big(\zeta_n(t_1), \zeta_{n+1}(t_1), s\big) &= \mathrm{e}^{-|\zeta_n(t_1)-\zeta_{n+1}(t_1)|/s} \\ &> q \cdot \mathrm{e}^{-|\zeta_n(t_1)-\zeta(t_1)|/s} \\ &= q \cdot \mathfrak{Z}\big(\zeta_n(t_1), \zeta(t_1), s\big).\end{aligned}$$

Therefore, property $\mathcal{M}_{k_2}$ holds true. Hence all the conditions of Theorem 1 are satisfied. Thus we conclude that there exists $\zeta^*(t_1) \in \mathfrak{C}([0,1],\mathbb{R})$ such that $\mathcal{A}\zeta^*(t_1) = \zeta^*(t_1)$ and $\zeta^*(t_1)$ is the solution of (6). This completes the proof. □

5 Conclusions

The concepts of fuzzy mathematics provide a facility to convert existing results from the literature of fixed point theory into fuzzy settings, and it is very difficult to interpret crisp metric contraction conditions into fuzzy contractions. In spite of this, fuzzy fixed point theory becomes more generalized as compared to crisp metric fixed point theory. In addition, symmetry in fuzzy metric spaces and their generalized spaces provides us with the facility to ensure the existence of solutions to the various mathematical problems that arise through researchers. In this article, we have introduced various independent Suzuki-type fuzzy contractive conditions, which are generalizations of existing results in the literature. The paper includes a comprehensive set of examples to show the importance of the theorems and illustrate that our coding technique for inequalities is powerful. Further hope with this technique is that researchers can extend more results in terms of Suzuki-type views with applications in crisp distance spaces. Further, it will be interesting to apply these concepts in more fuzzy spaces, which are also applicable for set-valued mappings, multi-valued mappings, for cyclic maps, etc.

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