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An analysis on the approximate controllability results for Caputo fractional hemivariational inequalities of order $1 < r < 2$ using sectorial operators

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Abstract. In this paper, we investigate the effect of hemivariational inequalities on the approximate controllability of Caputo fractional differential systems. The main results of this study are tested by using multivalued maps, sectorial operators of type (P, η, r, γ), fractional calculus, and the fixed point theorem. Initially, we introduce the idea of mild solution for fractional hemivariational inequalities. Next, the approximate controllability results of semilinear control problems were then established. Moreover, we will move on to the system involving nonlocal conditions. Finally, an example is provided in support of the main results we acquired.

Keywords: hemivariational inequalities, fractional derivative, approximate controllability, sectorial operators, mild solution, generalized Clarke's subdifferential, multivalued functions.

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1 Introduction

Fractional differential systems and fractional calculation have both played an important role in mathematics over the last few decades. Fractional calculus is thought-about as a generalization of classical calculus. Some fundamental problems cannot be solved using difference calculus of integer order, but they can be solved using fractional-order differential equations. There are numerous definitions for derivatives and integrals of arbitrary order. Even though in the beginning, fractional calculus was just a strictly mathematical idea, in modern times, its use has unfolded into many distinct fields of technological know-how such as mechanics, signal processing, control theory, fluid flow, biological engineering challenges, image processing, viscoelasticity, porous media, theology, and other fields have been significant. For further details, we refer to books and articles [8–11, 14, 19, 25, 31, 32]. Researchers discussed the Hölder regularity result for nonautonomous fractional evolution very recently [5]. The existence and singularity of global mild solutions for a class of nonlinear fractional reaction–diffusion equations with delay were developed in [34] by making use of the theory of resolvent operators, the Banach contraction mapping principle, the solution operator, and the convex-power condensing operator.

In 1981, Panagiotopoulos established the concept of hemivariational inequality, which he used to tackle nonconvex and nonsmooth superpotential problems [16,17]. After, in [2], the author investigated the optimization, nonsmooth analysis, and geometry problems. Using multivalued analysis and a surjectivity outcome for L-generalized pseudomonotone operators, the researcher examined the existence of solutions for parabolic hemivariational inequalities in [12]. The authors of [13] expressed the existence, uniqueness, and regularity results of the dynamical hemivariational inequality using evolution inclusions, contact problems, Clarke subdifferential, and quasistatic processes. A rising number of researchers have recently made major contributions to the field of hemivariational inequality. In [3, 7, 9, 9–11, 22, 23, 29], the authors used semigroup theory, cosine families, Hilfer fractional derivative, stochastic systems, Sobolev-type, fractional derivative of order $1 < r < 2$, and neutral systems to prove the existence of the functional evolution hemivariational inequality and its exact and approximate controllability results.

Controllability notation has shown to be a valuable resource for control system analysis and innovation. To solve these problems, fractional derivatives with variable significations can be used. They are employed in a wide range of fields such as economics, biology, power systems, chemical outgrowth control, electronics, transportation, space technology, engineering, physics, robotics, and chemistry. As indicated by the researchers' papers [7,9,10,23,33], resolving these types of challenges has become a major project for young academics. Many scholars have recently argued the approximate control problems defined as evolution inclusions, integro-differential equations, impulsive functional inclusions, neutral functional differential equations, and semilinear functional equations as evidenced by research publications [3, 7, 11, 20, 22, 29].

The authors [4, 33] established fractional differential systems of order $\alpha \in (1, 2)$ as well as control problems employing various fixed point theorems, cosine families, the measure of noncompactness, the Laplace transform, nonlocal conditions, and mild solutions. Furthermore, controllability results for Caputo fractional derivative with delay of order $1 < r < 2$, as well as integrodifferential equations, cosine operators, and fixed-point techniques, were examined using Laplace transform [23]. Many researchers have studied on fractional differential systems of order $r \in (1,2)$ employing cosine families, infinite delay, Volterra–Fredholm integrodifferential systems, hemivariational inequalities, various fixed point theorems, and generalized Clarke's subdifferential type [22]. In [3], researchers developed fractional delay stochastic differential inclusions of order $r \in (1,2)$ by referring to the Wiener process, Sobolev-type, integrodifferential systems, control systems, fixed point theorems, and cosine families.

Several investigators have lately made substantial developments in the area of fractional derivatives with sectorial operators. Using a sectorial operator of type (M,θ,α,μ), the researchers of [25] examined the existence results for impulsive fractional derivative. In addition, the researchers proved the impulsive fractional differential equations of order $0<\alpha<1$ and $\alpha \in (1,2)$ in [26] by utilizing the fixed point technique, fractional partial differential equations, and mild solutions. In [27], the authors looked into the existence and uniqueness of fractional differential equations of order $\alpha \in (1,2)$. In [20], Gronwall's inequality and sectorial operators are used to analyze optimal control outcomes for fractional evolution equations of order $(1,2)$. The authors of [28] utilized upper and lower solution techniques to find extremal solutions of fractional partial differential equations of order $\alpha \in (1,2)$. Hausdorff measure of noncompactness, sectorial operators, and Mittag-Leffler function were applied. The existence of positive mild solutions for Caputo fractional evolution systems of order $1 < \alpha < 2$ was also addressed by the authors in [24]. Very recently, in [21], the authors developed the existence and optimal control results for fractional differential equations and inclusions of order $1<r<2$ by referring to the sectorial operators of type (P,η,r,γ), Volterra–Fredholm-type integrodifferential systems, Sobolev-type, Lagrange problem (P), and different fixed point theorems.

Our article makes the following valuable contribution: semigroup theory, mild solutions, and other methods are commonly used to investigate fractional differential systems of order $0<r<1$ with or without delay. In our study, the main motivation is to evaluate the mild solution as well as results for approximate controllability for the given system by using sectorial operator of type (P,η,r,γ), Caputo fractional derivative of order $1<r<2$, generalized Clarke's subdifferential, hemivariational inequality, nonlocal conditions, control systems, and, in particular, fixed point theorem for multivalued maps.

Examine the approximate controllability results for the following Caputo fractional hemivariational inequality of order $1<r<2$ with sectorial operators in Hilbert space as motivated by the foregoing results:

$$\begin{aligned}&\bigl\langle -{}^{C}D_t^r z(t) + Az(t) + Bx(t),\, k\bigr\rangle_{\mathbb{Z}} + G^0\bigl(t, z(t); k\bigr) \geqslant 0,\\ &\quad t \in [0,T],\ \text{for all } k \in \mathbb{Z},\\ &z(0) = z_0, \qquad z'(0) = z_1.\end{aligned} \tag{1}$$

Here the state $z(\cdot)$ takes values in separable Hilbert space $\mathbb{Z}$ with the norm $|\cdot|$. $\langle\cdot,\cdot\rangle_{\mathbb{Z}}$ represents the scalar product of the separable Hilbert space. The Caputo fractional derivative of order $r \in (1,2)$ is expressed by $-{}^{C}D_t^r z(t)$; $A : D(A) \subset \mathbb{Z} \to \mathbb{Z}$ denotes sectorial operator of type (P,η,r,γ) on $\mathbb{Z}$. $G^0(t,z(\cdot);\cdot)$ is the generalized Clarke directional derivative [2] of a locally Lipschitz function $G(\cdot,\cdot):\mathbb{Z}\to\mathbb{R}$. The control function x is given in $L^2(0,T;\mathbb{H})$, where $\mathbb{H}$ stand for Hilbert space. Furthermore, the bounded linear operator B from $\mathbb{H}$ into $\mathbb{Z}$.

We list the significant of the derived main result of Caputo fractional hemivariatioal inequality of order $1<r<2$ as follows:

(i) For the first time in literature, the existence of mild solution for fractional hemivariatioal inequality of order $1<r<2$ with sectorial operator of type (P,η,r,γ) involving Caputo fractional derivative is investigated.
(ii) New set of sufficient conditions is established for approximate controllability results for fractional hemivariatioal inequality of order $1<r<2$ with sectorial operator of type (P,η,r,γ) in separable Hilbert spaces.
(iii) The properties of generalized Clarke subdifferentials is adopted to prove the existence and approximate controllability results for the given systems.
(iv) The fixed point theorem of multivalued maps is effectively used to establish the existence of mild solutions. Furthermore, we discussed the nonlocal fractional hemivariatioal inequality of order $1<r<2$.
(v) Obtained theoretical result is verified through an example.

This article has been divided into several sections. In Section 2, we review certain foundational concepts and the preparation process results. In Section 3, we look into the existence of mild solution for system (2) by applying the fixed point theorem of multivalued analysis and some essential properties. Further, in Section 4, we investigate the approximate controllability results for the considered fractional control system (2). Moreover, we develop our system (2) with nonlocal conditions in Section 5. Finally, an example is given for establishing the law based on the key results.

2 Preliminaries

In this section, we give some basic definitions related to some fundamental fractional calculus, approximate controllability, multivalued maps, and sectorial operators of type (P,η,r,γ), which are essential for the proof of our results.

We assume that X is a Banach space with the norm $\|\cdot\|_X$, X^* stands for its dual, and $(\cdot,\cdot)_X$ denotes the duality pairing between X^*. $C(0,T;X)$ denotes the Banach space of all continuous functions from $[0,T]$ into X with the norm $\|z\|_{C(0,T;X)}=\sup_{t\in[0,T]}\|z(t)\|$.

Definition 1. (See [31, Defs. 1.5, 1.6].) The Riemann–Liouville fractional integral of order $\beta\in\mathbb{R}^+$ with the lower limit zero for $g:[0,\infty)\to\mathbb{R}^+$ is given by

$$I^\beta g(t)=\frac{1}{\Gamma(\beta)}\int_0^t\frac{g(s)}{(t-s)^{1-\beta}}\,\mathrm{d}s,\quad t>0,$$

if the right-hand side is point-wise defined on $[0,\infty)$.

Definition 2. (See [31, Defs. 1.5, 1.6].) The Riemann–Liouville fractional derivative of order $\beta \in \mathbb{R}^+$ with the lower limit zero for g is defined by

$$^{L}D^{\beta} g(t) = \frac{1}{\Gamma(j-\beta)} \frac{\mathrm{d}^j}{\mathrm{d}t^j} \int_0^t g(s)(t-s)^{j-\beta-1}\, \mathrm{d}s, \quad t > 0,\ j-1 < \beta < j.$$

Definition 3. (See [31, Def. 1.8].) Caputo fractional derivative of order $\beta \in \mathbb{R}^+$ with the lower limit zero for g is given by

$$^{C}D^{\beta} g(t) = {}^{L}D^{\beta}\left(g(t) - \sum_{i=0}^{j-1} \frac{g^{(i)}(0)}{i!} t^i \right), \quad t > 0,\ j-1 < \beta < j,\ \beta \in \mathbb{R}^+.$$

Definition 4. (See [27, Def. 2.3].) Let $A : D \subseteq \mathbb{Z} \to \mathbb{Z}$ be a closed and linear operator. A is said to be sectorial operator of type (P, η, r, γ) if there exists $\gamma \in \mathbb{R}$, $0 < \eta < \pi/2$, and $P > 0$ such that the r-resolvent of A exists outside the sector

$$\gamma + \mathcal{S}_\eta = \left\{ \eta + \mu^r \colon\ \mu \in C(0, T; \mathbb{Z}),\ \left|\operatorname{Arg}\left(-\mu^r\right)\right| < \eta \right\}$$

and

$$\left\| \left(\mu^r I - A\right)^{-1} \right\| \leqslant \frac{P}{|\mu^r - \gamma|}, \quad \mu^r \notin \gamma + \mathcal{S}_\eta.$$

Further, if A is a sectorial operator of type (P, η, r, γ), then it is not difficult to see that A is the infinitesimal generator of a r-resolvent family $\{K_r(t)\}_{t \geqslant 0}$ in a Banach space, where

$$K_r(t) = \frac{1}{2\pi \mathrm{i}} \int_c \mathrm{e}^{\mu r} R\left(\mu^r, A\right) \mathrm{d}\mu,$$

c is a suitable path satisfying $\mu^r \notin \gamma + \mathcal{S}_\eta$ for $\mu \in c$.

Theorem 1. (*See* [27, *Thm.* 3.3].) *If A is a sectorial operator of type (P, η, r, γ), then the following estimates on $\|S_r(t)\|$ hold*:

(i) *If $\gamma \geqslant 0$, for $\psi \in (0, \pi)$, we get*

$$\begin{aligned} \left\|S_r(t)\right\| \leqslant{} & \frac{M_1(\eta, \psi) P \exp\{[M_1(\eta, \psi)(1 + \gamma t^r)][(1 + \frac{\sin\psi}{\sin(\psi - \eta)})^{1/r} - 1]\}}{\pi \sin^{1+1/r} \eta} \\ & \times \left(1 + \gamma t^r\right) \\ & + \frac{\Gamma(r) P}{\pi (1 + \gamma t^r) |\cos\frac{\pi - \psi}{r}|^r \sin\eta \sin\psi} \end{aligned}$$

for $t > 0$, where $M_1(\eta, \psi) = \max\{1, \sin\eta / \sin(\psi - \eta)\}$.

(ii) *If $\gamma < 0$, for $0 < \psi < \pi$, we get*

$$\left\|S_r(t)\right\| \leqslant \left(\frac{\mathrm{e} P[(1 + \sin\psi)^{1/r} - 1]}{\pi |\cos\psi|^{1+1/r}} + \frac{\Gamma(r) P}{\pi |\cos\psi| |\cos\frac{\pi - \psi}{r}|^r} \right) \frac{1}{1 + |\gamma| t^r}$$

for $t > 0$.

Theorem 2. (*See* [27, *Thm.* 3.4].) *If A is a sectorial operator of type (P, η, r, γ), then the following estimates on $\|K_r(t)\|$, $\|Q_r(t)\|$ hold*:

(i) *If $\gamma \geqslant 0$, for $\psi \in (0, \pi)$, we get*

$$\begin{aligned}\|K_r(t)\| \leqslant{}& \frac{P[(1+\frac{\sin\psi}{\sin(\psi-\eta)})^{1/r}-1]}{\pi\sin\eta}\left(1+\gamma t^r\right)^{1/r}t^{r-1} \\ &\times \exp\{[M_1(\eta,\psi)(1+\gamma t^r)]^{1/r}\} \\ &+ \frac{Pt^{r-1}}{\pi(1+\gamma t^r)|\cos\frac{\pi-\psi}{r}|^r\sin\eta\sin\psi}, \\ \|Q_r(t)\| \leqslant{}& \frac{P[(1+\frac{\sin\psi}{\sin(\psi-\eta)})^{1/r}-1]M_1(\eta,\psi)}{\pi\sin\eta^{(r+2)/r}}\left(1+\gamma t^r\right)^{(r-1)/r}t^{r-1} \\ &\times \exp\{[M_1(\eta,\psi)(1+\gamma t^r)]^{1/r}\} \\ &+ \frac{Pr\Gamma(r)}{\pi(1+\gamma t^r)|\cos\frac{\pi-\psi}{r}|^r\sin\eta\sin\psi}\end{aligned}$$

for $t > 0$, where $M_1(\eta, \psi) = \max\{1, \sin\eta/\sin(\psi-\eta)\}$.

(ii) *If $\gamma < 0$, for $\psi \in (0, \pi)$, we get*

$$\begin{aligned}\|K_r(t)\| &\leqslant \left(\frac{\mathrm{e}P[(1+\sin\psi)^{1/r}-1]}{\pi|\cos\psi|} + \frac{P}{\pi|\cos\psi||\cos\frac{\pi-\psi}{r}|}\right)\frac{t^{r-1}}{1+|\gamma|t^r}, \\ \|Q_r(t)\| &\leqslant \left(\frac{\mathrm{e}P[(1+\sin\psi)^{1/r}-1]t}{\pi|\cos\psi|^{1+2/r}} + \frac{r\Gamma(r)P}{\pi|\cos\psi||\cos\frac{\pi-\psi}{r}|}\right)\frac{1}{1+|\gamma|t^r}\end{aligned}$$

for $t > 0$.

We also introduce some basic definitions and results of multivalued maps. For more details on multivalued maps, we refer to book [6].

Definition 5. (See [9, p. 3].)

(i) A multivalued map $G : X \to 2^X \setminus \{\emptyset\} := \mathcal{P}(X)$ is convex-valued (closed-valued) if $G(z)$ is convex (closed) for every $z \in X$.

(ii) A multivalued map G is called upper semicontinuous on X if for any $z \in X$, $G(z)$ is a nonempty, closed subset of X and if for any open set M of X containing $G(z)$, there exists an open neighborhood J of z such that

$$G(J) \subseteq M.$$

(iii) A multivalued map G is said to be completely continuous if $G(M)$ is relatively compact for any bounded subset $M \subseteq X$.

(iv) Let $(W, \mathcal{E})$ be a measurable space, and let (X, d) be a separable metric space. A multivalued map $G : [0, T] \to \mathcal{P}(X)$ is called measurable if for any closed set $K \subseteq X$, we have

$$G^{-1}(K) = \{t \in [0, T]\colon G(t) \cap K \neq \emptyset\} \in \mathcal{E}.$$

Now, let us proceed to the definition of the generalized gradient of Clarke's subdifferential for a locally Lipschitz functional $q : X \to \mathbb{R}$ (see [6]). We denote by $q^0(u, v)$ the Clarke generalized directional derivative of q at u in the direction v, i.e.,

$$q^0(u, v) := \lim_{\kappa \to 0^+} \sup_{\alpha \to z} \frac{q(\alpha + \kappa v) - q(\alpha)}{\kappa}.$$

The generalized Clarke subdifferential of q at u is the subset of X^* defined by

$$\partial q(u) := \{u^* \in X^*\colon q^0(u, v) \geqslant \langle u^*, v\rangle \text{ for every } v \in X\}.$$

Lemma 1. (*See* [2, *Prop*. 2.1.2].) *Assume that q is a locally Lipschiz of rank $\mathcal{J}$ close to u. Then*

(i) *A nonempty set $\partial q(u)$ is convex, weak*-compact subset of X^*, and $\|u^*\|_{X^*} \leqslant \mathcal{J}$ for every $u^* \in \partial q(u)$;*
(ii) *For any $v \in X$, one has $q^0(u, v) = \max\{\langle u^*, v\rangle$: for every $u^* \in \partial q(u)\}$.*

In the sequel, we will analyze the existence of mild solutions and approximate controllability results for the following semilinear inclusion:

$$\begin{aligned} &{}^C D_t^r z(t) \in Az(t) + Bx(t) + \partial G(t, z(t)), \quad t \in [0, T], \\ &z(0) = z_0, \qquad z'(0) = z_1, \end{aligned} \tag{2}$$

where ${}^C D_t^r z(t)$ stands for Caputo fractional derivative of order $r \in (1, 2)$; A denotes sectorial operator of type (P, η, r, γ) defined from the domain $D(A) \subset \mathbb{Z}$ into $\mathbb{Z}$; ∂G denotes the generalized Clarke subdirectional derivative [2] of a locally Lipschitz function $G(\cdot,\cdot) : \mathbb{Z} \to \mathbb{R}$; the control function x is given in $L^2(0, T; \mathbb{H})$, and the admissible controls set $\mathbb{H}$ denotes a Hilbert space; moreover B is the bounded linear operator from $\mathbb{H}$ into $\mathbb{Z}$.

We show that every solution to (2) is also a solution to (1). Based on the definition of solution for (2), if $z \in C(0, T; \mathbb{Z})$ is a solution of system (2), then there exists $g(t) \in \partial G(t, z(t))$ such that $g \in L^2(0, T; \mathbb{Z})$ and

$$\begin{aligned} &{}^C D_t^r z(t) = Az(t) + Bx(t) + g(t), \quad t \in [0, T], \\ &z(0) = z_0, \qquad z'(0) = z_1. \end{aligned}$$

Consequently,

$$\begin{aligned} &\langle -{}^C D_t^r z(t) + Az(t) + Bx(t),\, k\rangle_{\mathbb{Z}} + \langle g(t); k\rangle_{\mathbb{Z}} = 0, \\ &\quad t \in [0, T], \text{ for all } k \in \mathbb{Z}, \\ &z(0) = z_0, \qquad z'(0) = z_1. \end{aligned}$$

Hence, $g \in \partial G(t, z(t))$ and $\langle g(t); k\rangle_{\mathbb{Z}} \leqslant G^0(t, z(t); k)$, and we get

$$\begin{aligned}
&\bigl\langle -{}^C D_t^r z(t) + Az(t) + Bx(t),\, k\bigr\rangle_{\mathbb{Z}} + G^0\bigl(t, z(t); k\bigr) \geqslant 0,\\
&\quad t \in [0, T],\ \text{for all } k \in \mathbb{Z},\\
&z(0) = z_0, \qquad z'(0) = z_1.
\end{aligned}$$

Therefore, in order to study the hemivariational inequality (1), we only need to deal with the semilinear inclusion (2). According to the book [18], we may define a mild solution of problem (2) as follows.

Definition 6. A function $z \in C(0, T; \mathbb{Z})$ is called a mild solution of system (2) if there exists the function $g \in L^2(0, T; \mathbb{Z})$ such that $g(t) \in \partial G(t, z(t))$ for a.e. $t \in [0, T]$ and

$$z(t) = S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s,$$

where

$$\begin{aligned}
S_r(t) &= \frac{1}{2\pi\mathrm{i}} \int_c \mathrm{e}^{\mu r} \mu^{r-1} \mathscr{R}(\mu^r, A)\,\mathrm{d}\mu,\\
Q_r(t) &= \frac{1}{2\pi\mathrm{i}} \int_c \mathrm{e}^{\mu r} \mu^{r-2} \mathscr{R}(\mu^r, A)\,\mathrm{d}\mu,\\
K_r(t) &= \frac{1}{2\pi\mathrm{i}} \int_c \mathrm{e}^{\mu r} \mathscr{R}(\mu^r, A)\,\mathrm{d}\mu
\end{aligned}$$

with c being a suitable path such that $\mu^r \notin \gamma + \mathcal{S}_\eta$ for μ in c.

Set

$$\begin{aligned}
\mathcal{U}_T(G) = \bigl\{z(T) \in \mathbb{Z}\colon\ &z(\cdot) \text{ is a mild solution of system (2) corresponding}\\
&\text{to a control } x \in L^2(0, T; \mathbb{H}) \text{ with initial values } z_0, z_1 \in \mathbb{Z}\bigr\},
\end{aligned}$$

which is said to be reachable set of system (1). If $G = 0$, then this system is called the corresponding linear system of (1). Let $\mathcal{U}_T(0)$ represents the reachable set of the linear system.

Definition 7. (See [9, Def. 2.6].) System (1) is called approximately controllable on $[0, T]$ if $\overline{\mathcal{U}_T(G)} = \mathbb{Z}$, where $\overline{\mathcal{U}_T(G)}$ denotes the closure of $\mathcal{U}_T(G)$. Then the corresponding linear system is approximately controllable on $[0, T]$ if $\overline{\mathcal{U}_T(0)} = \mathbb{Z}$.

Theorem 3. (*See* [9, *Thm.* 2.8].) *Let X be a Banach space and $\Omega : X \to 2^X$ be a compact convex-valued, upper semicontinuous multivalued map such that there exists a closed neighborhood M of zero for which $\Omega(M)$ is a relatively compact set. If the set*

$$\Phi = \bigl\{z \in X\colon\ \varphi z \in \Omega(z)\ \textit{for some}\ \varphi > 1\bigr\}$$

is bounded, then Ω has a fixed point.

3 Existence results

The purpose of this section is to study the existence of mild solutions for the fractional differential system (2). Before starting and proving the main results of this section, we impose the following.

It is simple to show that they are bounded because of the estimations on $S_r(t)$, $Q_r(t)$, and $K_r(t)$ in Theorems 1 and 2.

(H1) The linear operator A, which is a sectorial accretive operator of type (P, η, r, γ), generates the compact r-resolvent families $S_r(t)$, $Q_r(t)$, and $K_r(t)$ for any $t \in [0, T]$, and there exists $\widehat{P} > 0$ such that

$$\sup_{0 \leqslant t \leqslant T} \|S_r(t)\| \leqslant \widehat{P}, \qquad \sup_{0 \leqslant t \leqslant T} \|Q_r(t)\| \leqslant \widehat{P}, \qquad \sup_{0 \leqslant t \leqslant T} \|K_r(t)\| \leqslant \widehat{P}.$$

(H2) The multivalued function $G : [0, T] \times \mathbb{Z} \to \mathbb{R}$ such that

(i) the function $t \mapsto G(t, z)$ is measurable for any $z \in \mathbb{Z}$;

(ii) the function $z \mapsto G(t, z)$ is locally Lipschitz for any $t \in [0, T]$;

(iii) there exists a function $\omega \in L^2([0, T], \mathbb{R}^+)$ and a constant $h > 0$ such that

$$\|\partial G(t, z)\|_{\mathbb{Z}} = \sup\{\|g\|_{\mathbb{Z}}\colon g \in \partial G(t, z)\} \leqslant \omega(t) + h\|z\|_{\mathbb{Z}}$$

for any $z \in \mathbb{Z}$ and for any $t \in [0, T]$.

Using [9, 11], we define for all $z \in L^2(V, \mathbb{Z})$ an operator $\mathcal{V} : L^2(V, \mathbb{Z}) \to 2^{L^2(V, \mathbb{Z})}$ as follows:

$$\mathcal{V}(z) = \{z \in L^2(V, \mathbb{Z})\colon z(t) \in \partial G(t; z(t)),\ \text{a.e. } t \in V = [0, T]\}.$$

Lemma 2. (*See* [12].) *If hypotheses* (H1)–(H2) *are satisfied, then for any* $z \in L^2(0, T; \mathbb{Z})$, *the set* $\mathcal{V}(z)$ *has nonempty, convex, and weakly compact values.*

Lemma 3. (*See* [13].) *If* (H1)–(H2) *are satisfied, the operator* $\mathcal{V}$ *fulfills*: *if* $z_i \to z \in L^2(0, T; \mathbb{Z})$, $y_i \to y$ *weakly in* $L^2(0, T; \mathbb{Z})$ *and* $y_i \in \mathcal{V}(z_i)$, *then we obtain* $y \in \mathcal{V}(z)$.

Theorem 4. *If* (H1)–(H2) *are satisfied, then* (2) *has a mild solution on* $[0, T]$.

Proof. For $x \in L^2(0, T; \mathbb{H})$ and for every $z \in C(0, T; \mathbb{Z}) \subset L^2(0, T; \mathbb{Z})$, from Definition 6 one can consider the multivalued map $\Omega : C(0, T; \mathbb{Z}) \to 2^{C(0, T; \mathbb{Z})}$ as follows:

$$\Omega(z) = \biggl\{m \in C(0, T; \mathbb{Z})\colon m(t) = S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s,\ g \in \mathcal{V}(z)\biggr\}.$$

Then our problem is reduced to find the fixed point of Ω. For this, we shall verify that Ω satisfies all the assumptions of Theorem 3. Now, $\Omega(z)$ is convex by the convexity of $\mathcal{V}(z)$. We divided our proof into the steps below for reader convenience.

Step 1. Ω maps bounded subsets into bounded subsets in $C(0,T;\mathbb{Z})$. As for every $z \in \mathcal{B}_p = \{z \in C(0,T;\mathbb{Z})\colon \|z\|_{\mathbb{C}} \leqslant p,\ p > 0\}$, $\lambda \in \Omega(z)$, we obtain $g \in \mathcal{V}(z)$ such that

$$\begin{aligned}\lambda(t) &= S_r(t)z_0 + Q_r(t)z_1 \\ &\quad + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s, \quad t \in [0,T]. \end{aligned}\tag{3}$$

Using (H2)(iii), we get

$$\begin{aligned}\big\|\lambda(t)\big\|_{\mathbb{Z}} &\leqslant \big\|S_r(t)z_0\big\|_{\mathbb{Z}} + \big\|Q_r(t)z_1\big\|_{\mathbb{Z}} \\ &\quad + \int_0^t \big\|K_r(t-s)g(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s + \int_0^t \big\|K_r(t-s)Bx(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^t \big\|g(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s + \widehat{P}\int_0^t \big\|Bx(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^t \big(\omega(s)+h\big\|z(s)\big\|_{\mathbb{Z}}\big)\,\mathrm{d}s + \widehat{P}\|B\|\int_0^t \big\|x(s)\big\|_{\mathbb{H}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\big[\|z_0\|_{\mathbb{Z}} + \|z_1\|_{\mathbb{Z}} + \sqrt{T}\big(\|\omega\|_{L^2([0,T],\mathbb{R}^+)} + \|B\|\|x\|_{L^2(0,T;\mathbb{H})}\big) + hpT\big].\end{aligned}$$

Therefore, $\Omega(\mathcal{B}_p)$ is bounded in $C(0,T;\mathbb{Z})$.

Step 2. $\{\Omega(z)\colon z \in \mathcal{B}_p\}$ is equicontinuous for every $p > 0$.

For any $z \in \mathcal{B}_p$, $\lambda \in \Omega(z)$, there exists $g \in \mathcal{V}(z)$ such that (3) holds true. Now, for every $\epsilon > 0$, $0 \leqslant t_1 < t_2 \leqslant T$, we get

$$\begin{aligned}&\big\|\lambda(t_2) - \lambda(t_1)\big\|_{\mathbb{Z}} \\ &\quad = \Bigg\|S_r(t_2)z_0 + Q_r(t_2)z_1 + \int_0^{t_2} K_r(t_2-s)g(s)\,\mathrm{d}s + \int_0^{t_2} K_r(t_2-s)Bx(s)\,\mathrm{d}s \\ &\qquad - S_r(t_1)z_0 + Q_r(t_1)z_1 + \int_0^{t_1} K_r(t_1-s)g(s)\,\mathrm{d}s + \int_0^{t_1} K_r(t_1-s)Bx(s)\,\mathrm{d}s\Bigg\|_{\mathbb{Z}} \\ &\quad \leqslant \big\|\big[S_r(t_2) - S_r(t_1)\big]z_0\big\| + \big\|\big[Q_r(t_2) - Q_r(t_1)\big]z_1\big\| \\ &\qquad + \int_0^{t_1} \big\|\big[K_r(t_2-s) - K_r(t_1-s)\big]g(s)\big\|\,\mathrm{d}s + \int_{t_1}^{t_2} \big\|K_r(t_2-s)g(s)\big\|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t_1} \big\| \big[K_r(t_2 - s) - K_r(t_1 - s)\big] Bx(s) \big\| \,\mathrm{d}s + \int_{t_1}^{t_2} \big\| K_r(t_2 - s) Bx(s) \big\| \,\mathrm{d}s \\
\leqslant{}& \big\| S_r(t_2) - S_r(t_1) \big\| \|z_0\| + \big\| Q_r(t_2) - Q_r(t_1) \big\| \|z_1\| \\
&+ \int_0^{t_1} \big\| K_r(t_2 - s) - K_r(t_1 - s) \big\| \big(\omega(s) + h\big\|z(s)\big\|_{\mathbb{Z}}\big) \,\mathrm{d}s \\
&+ \widehat{P} \int_{t_1}^{t_2} \big(\omega(s) + h\big\|z(s)\big\|_{\mathbb{Z}}\big) \,\mathrm{d}s + \|B\| \int_0^{t_1} \big\| K_r(t_2 - s) - K_r(t_1 - s) \big\| \big\|x(s)\big\| \,\mathrm{d}s \\
&+ \widehat{P} \|B\| \int_{t_1}^{t_2} \big\|x(s)\big\| \,\mathrm{d}s \\
\leqslant{}& \big\| S_r(t_2) - S_r(t_1) \big\| \|z_0\| + \big\| Q_r(t_2) - Q_r(t_1) \big\| \|z_1\| \\
&+ \sup_{s \in [t_1 - \epsilon]} \big\| K_r(t_2 - s) - K_r(t_1 - s) \big\| \big[\|\omega\|_{L^2([0,T],\mathbb{R}^+)} \sqrt{T} + phT\big] \\
&+ \widehat{P} \big[\|\omega\|_{L^2([0,T],\mathbb{R}^+)} (2\epsilon + \sqrt{t_2 - t_1}\,) + ph(2\epsilon + \sqrt{t_2 - t_1}\,)\big] \\
&+ \|B\| \sup_{s \in [t_1 - \epsilon]} \big\| K_r(t_2 - s) - K_r(t_1 - s) \big\| \|x\|_{L^2(0,T;\mathbb{H})} \sqrt{T} \\
&+ \widehat{P} \|B\| \|x\|_{L^2(0,T;\mathbb{H})} (2\epsilon + \sqrt{t_2 - t_1}\,).
\end{aligned}
$$

The right-hand side of the above stated inequality tends to zero independently of whether $z \in \mathcal{B}_p$ as $t_2 \to t_1$. Hence, the compactness of operators $S_r(t)$, $Q_r(t)$, and $K_r(t)$ for $t > 0$ [19] is similar to the continuity in the uniform operator topology. Therefore, $\{\Omega(z)\colon z \in \mathcal{B}_p\}$ is equicontinuous.

Step 3. We verify that Ω is completely continuous.

For some fixed $t \in [0, T]$, we prove that $U(t) = \{\lambda(t)\colon\ \lambda \in \Omega(\mathcal{B}_p)\}$ is relatively compact in $\mathbb{Z}$. For $t = 0$, this is trivial, hence $U(0) = \{z_0 + z_1\}$, this is compact. As a result, only $t > 0$ must be considered. Let $t \in (0, T]$ be fixed. For any $z \in \mathcal{B}_p$, $\lambda \in \Omega(z)$, we have $g \in \mathcal{V}(z)$ such that

$$
\begin{aligned}
\lambda(t) ={}& S_r(t) z_0 + Q_r(t) z_1 \\
&+ \int_0^t K_r(t - s) g(s) \,\mathrm{d}s + \int_0^t K_r(t - s) Bx(s) \,\mathrm{d}s, \quad t \in [0, T].
\end{aligned}
$$

For each $0 < \epsilon < t$, $z \in \mathcal{B}_p$, and we introduce the operator λ^ϵ by

$$
\begin{aligned}
\lambda^\epsilon(t) ={}& S_r(t) z_0 + Q_r(t) z_1 \\
&+ \int_0^{t-\epsilon} K_r(t - s) g(s) \,\mathrm{d}s + \int_0^{t-\epsilon} K_r(t - s) Bx(s) \,\mathrm{d}s.
\end{aligned}
$$

Hence $K(t)$, $t > 0$, is a compact operator, then $U^\epsilon(t) = \{\lambda^\epsilon(t)\colon \lambda \in \Omega(\mathcal{B}_p)\}$ is relatively compact in $\mathbb{Z}$, $0 < \epsilon < t$. Further, for every $\lambda \in \Omega(z)$, we get

$$\begin{aligned}
\|\lambda(t) - \lambda^\epsilon(t)\| &= \Bigg\| \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s \\
&\quad - \int_0^{t-\epsilon} K_r(t-s)g(s)\,\mathrm{d}s - \int_0^{t-\epsilon} K_r(t-s)Bx(s)\,\mathrm{d}s \Bigg\|_{\mathbb{Z}} \\
&\leqslant \int_{t-\epsilon}^t \|K_r(t-s)g(s)\|_{\mathbb{Z}}\,\mathrm{d}s + \int_{t-\epsilon}^t \|K_r(t-s)Bx(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\
&\leqslant \widehat{P}\int_{t-\epsilon}^t \|g(s)\|_{\mathbb{Z}}\,\mathrm{d}s + \widehat{P}\int_{t-\epsilon}^t \|Bx(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\
&\leqslant \widehat{P}\int_{t-\epsilon}^t \big(\omega(s) + h\|z(s)\|_{\mathbb{Z}}\big)\,\mathrm{d}s + \widehat{P}\|B\|\int_{t-\epsilon}^t \|x(s)\|_{\mathbb{H}}\,\mathrm{d}s \\
&\leqslant \widehat{P}\big[\big(\|\omega\|_{L^2([0,T],\mathbb{R}^+)} + \|B\|\|x\|_{L^2(0,T;\mathbb{H})}\big)\sqrt{\epsilon} + hp\epsilon\big].
\end{aligned}$$

If ϵ is small enough, it implies that there are relatively compact sets arbitrarily close to the set $U(t)$ for any $t \in (0,T]$. Then for any $t \in (0,T]$, $U(t)$ is relatively compact in $C(0,T;\mathbb{Z})$. Since it is compact at $t = 0$, we have the relatively compactness of $U(t)$ in $C(0,T;\mathbb{Z})$ for any $t \in (0,T]$.

From Arzelà–Ascoli theorem we clarify that Ω is a completely continuous.

Step 4. Now, we prove that Ω is upper semicontinuous. We begin by demonstrating that Ω has a closed graph.

Let $z_i \to z_* \in C(0,T;\mathbb{Z})$, $\lambda_i \in \Omega(z_i)$, and $\lambda_i \to \lambda_* \in C(0,T;\mathbb{Z})$. We prove that $\lambda_* \in \Omega(z_*)$. In fact, $\lambda_i \in \Omega(z_i)$ implies that there exists $g_i \in \mathcal{V}(z_i)$ such that

$$\begin{aligned}
\lambda_i(t) &= S_r(t)z_0 + Q_r(t)z_1 \\
&\quad + \int_0^t K_r(t-s)g_i(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s, \quad t \in [0,T]. \qquad (4)
\end{aligned}$$

Using (H2)(iii), $\{g_i\}_{i\geqslant 1} \subseteq L^2(0,T;\mathbb{Z})$ is bounded. Since we approach to a subsequence if necessary, we have

$$g_i \to g_*, \quad \text{weakly in } L^2(0,T;\mathbb{Z}). \qquad (5)$$

From (4), (5), and the compactness of the operator K_r we obtain that

$$\lambda_i(t) \to S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g_*(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{6}$$

Note that $\lambda_i \to \lambda_*$ in $C(0,T;\mathbb{Z})$ and $g_i \in \mathcal{V}(z_i)$. By Lemma 3 and equation (6), we obtain g_* in $\mathcal{V}(z_*)$. Since we obtain $\lambda_* \in \Omega(z_*)$, then Ω has a closed graph. From [12] Ω is u.s.c.

Step 5. A priori estimate.

From the above steps we get that Ω is u.s.c., compact convex-valued, and $\Omega(\mathcal{B}_p)$ is a relatively compact.

Based on Theorem 3, it is necessary to verity that the set

$$\Phi = \bigl\{z \in C(0,T;\mathbb{Z})\colon\ \varphi z \in \Omega(z),\ \varphi > 1\bigr\}$$

is bounded to clarity that Ω has a fixed point. For any $z \in \Phi$, there exists $g \in \mathcal{V}(z)$ such that

$$z(t) = \varphi^{-1}S_r(t)z_0 + \varphi^{-1}Q_r(t)z_1 + \varphi^{-1}\int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \varphi^{-1}\int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s.$$

Using (H2)(iii), we obtain

$$\begin{aligned}
\|z(t)\|_{\mathbb{Z}} &\leqslant \|S_r(t)z_0\|_{\mathbb{Z}} + \|Q_r(t)z_1\|_{\mathbb{Z}} + \int_0^t \|K_r(t-s)g(s)\|_{\mathbb{Z}}\,\mathrm{d}s + \int_0^t \|K_r(t-s)Bx(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\
&\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^t \|g(s)\|_{\mathbb{Z}}\,\mathrm{d}s + \widehat{P}\int_0^t \|Bx(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\
&\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^t \bigl(\omega(s) + h\|z(s)\|_{\mathbb{Z}}\bigr)\,\mathrm{d}s + \widehat{P}\|B\|\int_0^t \|x(s)\|_{\mathbb{H}}\,\mathrm{d}s \\
&\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\sqrt{T}\|\omega\|_{L^2([0,T],\mathbb{R}^+)} + \widehat{P}h\int_0^t \|z(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\
&\quad + \widehat{P}\|B\|\sqrt{T}\|x\|_{L^2(0,T;\mathbb{H})} \\
&\leqslant \nu + \widehat{P}h\int_0^t \|z(s)\|\,\mathrm{d}s,
\end{aligned} \tag{7}$$

where

$$\nu = \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\sqrt{T}\big[\|\omega\|_{L^2([0,T],\mathbb{R}^+)} + \|B\|\|x\|_{L^2(0,T;\mathbb{H})}\big].$$

From (7) by Gronwall inequality we easily conclude that

$$\big\|z(t)\big\|_{\mathbb{Z}} \leqslant \nu \mathrm{e}^{\widehat{P}ht}.$$

Therefore, Φ is bounded. Theorem 3 states that Ω has a fixed point, that is, system (2) has a mild solution on $[0,T]$. □

4 Approximate controllability results

The purpose of this section is to discuss the approximate controllability of fractional differential system (2). Suppose the following linear fractional differential system:

$$\begin{aligned} &{}^{C}D_t^r z(t) = Az(t) + Bx(t), \quad t \in [0,T], \\ &z(0) = z_0, \qquad z'(0) = z_1. \end{aligned} \tag{8}$$

We assume that two relevant operators associated with system (8) are as follows:

$$\Gamma_0^T = \int_0^T K_r(t-s)BB^*K_r^*(t-s)\,\mathrm{d}s : \mathbb{Z} \to \mathbb{Z},$$

$$R\big(\delta, \Gamma_0^T\big) = \big(\delta I + \Gamma_0^T\big)^{-1} : \mathbb{Z} \to \mathbb{Z},$$

where $K_r^*(t-s)$ and B^* are adjoint of $K_r(t-s)$ and B, respectively. We can easily deduce that the linear operator Γ_0^T is bounded.

Theorem 5. *The linear fractional control system* (8) *is approximately controllable on* $[0,T]$ *if and only if* $R(\delta, \Gamma_0^T) \to 0$ *as* $\delta \to 0^+$ *in the strong operator topology.*

Firstly, for every $z \in C(0,T;\mathbb{Z}) \subset L^2(0,T;\mathbb{Z})$, from Lemma 2 we know that $\mathcal{V}(z) \neq \emptyset$. Since, for every $\delta > 0$, let as start with the multivalued map $\Omega_\delta : C(V,\mathbb{Z}) \to 2^{C(V,\mathbb{Z})}$ as follows:

$$\Omega_\delta(z) = \Bigg\{ m \in C(0,T;\mathbb{Z})\colon\ m(t) = S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx_\delta(s)\,\mathrm{d}s,\ g \in \mathcal{V}(z) \Bigg\},$$

where

$$x_\delta(t) = B^*K_r^*(t-s)R\big(\delta, \Gamma_0^T\big) \times \Bigg[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g(\iota)\,\mathrm{d}\iota \Bigg].$$

Theorem 6. *Suppose that* (H1)*,* (H2)(i)*, and* (H2)(ii) *are satisfied. Then, if there exists* $\zeta \in L^2([0,T],\mathbb{R}^+)$ *such that*

$$\|\partial G(t,z)\|_{\mathbb{Z}} \leqslant \zeta(t) \quad \textit{for all } t \in [0,T],\ z \in \mathbb{Z},$$

then Ω_δ *has a fixed point on* $[0,T]$.

Proof. The verification is the same as that of Theorem 4. For completeness of our paper, a simple version of proof is given. Obviously, for every $z \in C(0,T;\mathbb{Z})$, $\Omega_\delta(z)$ is convex by the convexity of $\mathcal{V}(z)$. We divided our proof into the steps below for reader convenience.

Step 1. We verify that Ω_δ maps bounded subsets into bounded subsets in $C(0,T;\mathbb{Z})$. For every $z \in \mathcal{B}_\jmath = \{z \in C(0,T;\mathbb{Z})\colon \|z\|_C \leqslant \jmath\}$, $\jmath > 0$, $\lambda \in \Omega_\delta(z)$, we obtain $g \in \mathcal{V}(z)$ such that

$$\begin{aligned}\lambda(t) &= S_r(t)z_0 + Q_r(t)z_1 \\ &\quad + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx_\delta(s)\,\mathrm{d}s, \quad t \in [0,T]. \end{aligned} \tag{9}$$

In view of $\|\partial G(t,z)\|_{\mathbb{Z}} \leqslant \zeta(t)$, we get

$$\begin{aligned}&\|x_\delta(t)\|_{\mathbb{H}} \\ &\quad = \left\| B^*K^*(T-\iota)R\big(\delta,\Gamma_0^T\big)\left[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g(\iota)\,\mathrm{d}\iota\right]\right\|_{\mathbb{Z}} \\ &\quad \leqslant \|B^*\|\big\|K^*(T-\iota)\big\|_{\mathbb{Z}}\big\|R\big(\delta,\Gamma_0^T\big)\big\|_{\mathbb{Z}} \\ &\qquad \times \left\|\left[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g(\iota)\,\mathrm{d}\iota\right]\right\|_{\mathbb{Z}} \\ &\quad \leqslant \|B^*\|\widehat{P}\frac{1}{\delta}\left[\|z_T\|_{\mathbb{Z}} + \big\|S(T)z_0\big\|_{\mathbb{Z}} + \big\|Q(T)z_1\big\|_{\mathbb{Z}} + \int_0^T \big\|K_r(T-\iota)g(\iota)\big\|_{\mathbb{Z}}\,\mathrm{d}\iota\right] \\ &\quad \leqslant \frac{\widehat{P}\|B^*\|}{\delta}\left[\|z_T\|_{\mathbb{Z}} + \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^T \big\|g(\iota)\big\|_{\mathbb{Z}}\,\mathrm{d}\iota\right] \\ &\quad \leqslant \frac{\widehat{P}\|B^*\|}{\delta}\left[\|z_T\|_{\mathbb{Z}} + \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\sqrt{T}\|\zeta\|_{L^2([0,T],\mathbb{R}^+)}\right] \\ &\quad := \xi. \end{aligned} \tag{10}$$

Using the above equation (10), we get

$$\begin{aligned}\|\lambda(t)\|_{\mathbb{Z}} &\leqslant \|S_r(t)z_0\|_{\mathbb{Z}} + \|Q_r(t)z_1\|_{\mathbb{Z}} + \int_0^t \|K_r(t-s)g(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\ &\quad + \int_0^t \|K_r(t-s)Bx_\delta(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^t \|g(s)\|\,\mathrm{d}s + \widehat{P}\int_0^t \|Bx_\delta(s)\|_{\mathbb{Z}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\int_0^t \zeta(s)\,\mathrm{d}s + \widehat{P}\|B\|\int_0^t \|x_\delta(s)\|_{\mathbb{H}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\bigl[\|z_0\|_{\mathbb{Z}} + \|z_1\|_{\mathbb{Z}} + \sqrt{T}\|\zeta\|_{L^2([0,T],\mathbb{R}^+)} + \|B\|\xi T\bigr].\end{aligned}$$

As a result, we get that $\Omega_\delta(\mathcal{B}_\jmath)$ is bounded in $C(0,T;\mathbb{Z})$.

Step 2. We verify that $\{\Omega_\delta(z){:}\ z \in \mathcal{B}_\jmath\}$ is equicontinuous. Firstly, for every $z \in \mathcal{B}_\jmath$, $t \in \Omega_\delta$, there exists $g \in \mathcal{V}(z)$ such that

$$\begin{aligned}\lambda(t) &= S_r(t)z_0 + Q_r(t)z_1 \\ &\quad + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx_\delta(s)\,\mathrm{d}s, \quad t \in [0,T].\end{aligned}$$

From the value of $\|x_\delta(t)\|$, as (10) and thus Step 2 of Theorem 4, then $\{\Omega_\delta(z){:}\ z \in \mathcal{B}_\jmath\}$ is equicontinuous.

Step 3. Ω_δ is completely continuous.

Let $t \in [0,T]$ be fixed. Now, we prove that $U(t) = \{\lambda(t){:}\ \lambda \in \Omega_\delta(\mathcal{B}_\jmath)\}$ is relatively compact in $\mathbb{Z}$. This is trivial for $t = 0$, hence $U(0) = \{z_0 + z_1\}$ is compact. Hence, only $t > 0$ must be considered. Let $t \in (0,T]$ be fixed. For every $z \in \mathcal{B}_\jmath$, $\lambda \in \Omega_\delta(z)$, we have $g \in \mathcal{V}(z)$ such that

$$\begin{aligned}\lambda(t) &= S_r(t)z_0 + Q_r(t)z_1 \\ &\quad + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx_\delta(s)\,\mathrm{d}s, \quad t \in [0,T].\end{aligned}$$

Let $0 < \epsilon < t$, $z \in \mathcal{B}_\jmath$, and introduce the operator λ^ϵ by

$$\begin{aligned}\lambda^\epsilon(t) &= S_r(t)z_0 + Q_r(t)z_1 \\ &\quad + \int_0^{t-\epsilon} K_r(t-s)g(s)\,\mathrm{d}s + \int_0^{t-\epsilon} K_r(t-s)Bx_\delta(s)\,\mathrm{d}s.\end{aligned}$$

Hence $K(t)$, $t > 0$, is a compact operator, then $U^\epsilon(t) = \{\lambda^\epsilon(t)\colon \lambda \in \Omega(\mathcal{B}_\jmath)\}$ is relatively compact in $\mathbb{Z}$, $0 < \epsilon < t$. Further, for every λ in $\Omega(z)$, we get

$$\begin{aligned}\|\lambda(t) - \lambda^\epsilon(t)\| &= \Bigg\| \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx_\delta(s)\,\mathrm{d}s \\ &\quad - \int_0^{t-\epsilon} K_r(t-s)g(s)\,\mathrm{d}s - \int_0^{t-\epsilon} K_r(t-s)Bx_\delta(s)\,\mathrm{d}s \Bigg\|_{\mathbb{Z}} \\ &\leqslant \int_{t-\epsilon}^t \big\|K_r(t-s)g(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s + \int_{t-\epsilon}^t \big\|K_r(t-s)Bx_\delta(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s \\ &\leqslant \widehat{P} \int_{t-\epsilon}^t \big\|g(s)\big\|_{\mathbb{Z}}\,\mathrm{d}s + \widehat{P}\|B\| \int_{t-\epsilon}^t \big\|x_\delta(s)\big\|_{\mathbb{H}}\,\mathrm{d}s \\ &\leqslant \widehat{P}\epsilon\|\zeta\|_{L^2([0,T],\mathbb{R}^+)} + \widehat{P}\|B\|\epsilon\xi.\end{aligned}$$

If ϵ is small enough, this implies that there are relatively compact sets arbitrarily close to the set $U(t)$ for any $t \in (0,T]$. Then for any $t \in (0,T]$, $U(t)$ is relatively compact in $C(0,T;\mathbb{Z})$. Since it is compact at $t = 0$, we have the relatively compactness of $U(t)$ in $C(0,T;\mathbb{Z})$ for any $t \in (0,T]$.

From Arzelà–Ascoli theorem we obtain that Ω_δ is a completely continuous.

Step 4. We check that Ω_δ is upper semicontinuous. We begin by demonstrating that Ω_δ has a closed graph.

Let $z_i \to z_* \in C(0,T;\mathbb{Z})$, $\lambda_i \in \Omega_\delta(z_i)$, and $\lambda_i \to \lambda_* \in C(0,T;\mathbb{Z})$. We will prove that $\lambda_* \in \Omega_\delta(z_*)$. In fact, $\lambda_i \in \Omega_\delta(z_i)$ means that there exists $g_i \in \mathcal{V}(z_i)$ such that

$$\begin{aligned}\lambda_i(t) &= S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g_i(s)\,\mathrm{d}s \\ &\quad + \int_0^t K_r(t-s)BB^*K_r^*(t-s)R\big(\delta, \Gamma_0^T\big) \\ &\qquad \times \Bigg[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g_i(\iota)\,\mathrm{d}\iota\Bigg]\,\mathrm{d}s. \qquad (11)\end{aligned}$$

Using $\|\partial G(t,z)\|_{\mathbb{Z}} \leqslant \zeta(t)$, $\{g_i\}_{i\geqslant 1} \subseteq L^2(0,T;\mathbb{Z})$ is bounded. Since we approach to a subsequence if necessary, we have

$$g_i \to g_* \quad \text{weakly in } L^2(0,T;\mathbb{Z}). \qquad (12)$$

From (11), (12), and the compactness of $K_r(t)$ we obtain for $t \in [0, T]$,

$$\lambda_i(t) \to S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g_*(s)\,\mathrm{d}s + \int_0^t K_r(t-s)BB^*K_r^*(t-s)R(\delta, \varGamma_0^T) \times \left[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g_*(\iota)\,\mathrm{d}\iota\right]\mathrm{d}s. \tag{13}$$

Note that $\lambda_i \to \lambda_*$ in $C(0,T;\mathbb{Z})$ and g_i in $\mathcal{V}(z_i)$. From Lemma 3 and (13) we obtain $g_* \in \mathcal{V}(z_*)$. Hence, we obtain $\lambda_* \in \varOmega_\delta(z_*)$, then $\varOmega_\delta$ has a closed graph. Then from [12] $\varOmega_\delta$ is upper semicontinuous.

Step 5. A priori estimate.

By the above steps, we get that $\varOmega_\delta$ is u.s.c., compact convex-valued, and $\varOmega_\delta(\mathcal{B}_\jmath)$ is a relatively compact set. It is necessary to verity that the set

$$\varPhi = \{z \in C(0,T;\mathbb{Z}):\ \varphi z \in \varOmega_\delta(z),\ \varphi > 1\}$$

is bounded to clarity that $\varOmega_\delta$ has a fixed point.

For any $z \in \varPhi$, there exists $g \in \mathcal{V}(z)$ such that

$$z(t) = \varphi^{-1}S_r(t)z_0 + \varphi^{-1}Q_r(t)z_1 + \varphi^{-1}\int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \varphi^{-1}\int_0^t K_r(t-s)B\left(B^*K_r^*(T-s)R(\delta, \varGamma_0^T) \times \left[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g(\iota)\,\mathrm{d}\iota\right]\right)\mathrm{d}s.$$

From (H2)(iii) we obtain

$$\|u(t)\|_{\mathbb{Z}} \leqslant \|S_r(t)z_0\|_{\mathbb{Z}} + \|Q_r(t)z_1\|_{\mathbb{Z}} + \int_0^t \|K_r(t-s)g(s)\|_{\mathbb{Z}}\,\mathrm{d}s + \int_0^t \|K_r(t-s)B\|_{\mathbb{Z}} \left\|\left(B^*K_r^*(T-\iota)R(\delta, \varGamma_0^T) \times \left[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g(\iota)\,\mathrm{d}\iota\right]\right)\right\|_{\mathbb{Z}}\mathrm{d}s.$$

From (9) we have

$$\begin{aligned}\|z(t)\|_{\mathbb{Z}} &= \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\sqrt{T}\|\zeta\|_{L^2([0,T],\mathbb{R}^+)} \\ &\quad + \frac{\widehat{P}^2\|B\|^2T}{\delta}\big[\|z_T\|_{\mathbb{Z}} + \widehat{P}\|z_0\|_{\mathbb{Z}} + \widehat{P}\|z_1\|_{\mathbb{Z}} + \widehat{P}\sqrt{T}\|\zeta\|_{L^2([0,T],\mathbb{R}^+)}\big].\end{aligned}$$

Therefore, Φ is bounded. From Theorem 3 we obtain that Ω_δ has a fixed point. We can now declare that the proof is complete. □

Then we will prove that the main results of this paper are correct.

Theorem 7. *Let the hypotheses of Theorem* 6 *are satisfied. Furthermore,* (1) *is approximately controllable on* $[0,T]$ *if* (2) *is approximately controllable on* $[0,T]$.

Proof. By Theorem 6, we prove that the operator Ω_δ has a fixed point in $C(0,T;\mathbb{Z})$ for any $\delta > 0$. Let z^δ be a fixed point of Ω_δ in $C(0,T;\mathbb{Z})$. It is easy to know that any fixed point of Ω_δ is a mild solution of (2) corresponding to x. Hence, there exists $g^\delta \in \mathcal{V}(z^\delta)$ such that for any $t \in [0,T]$,

$$\begin{aligned}z^\delta(t) &= S_r(t)z_0 + Q_r(t)z_1 + \int_0^t K_r(t-s)g^\delta(s)\,\mathrm{d}s \\ &\quad + \int_0^t K_r(t-s)BB^*K_r^*(T-\iota)R\big(\delta,\Gamma_0^T\big) \\ &\quad\quad \times \left[z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g(\iota)\,\mathrm{d}\iota\right]\mathrm{d}s.\end{aligned}$$

$I \to \Gamma_0^T R(\delta,\Gamma_0^T) = \delta R(\delta,\Gamma_0^T)$, and we obtain

$$z^\delta(T) = z_T + \delta R\big(\delta,\Gamma_0^T\big)\mathcal{S}\big(g^\delta\big),$$

where

$$\mathcal{S}\big(g^\delta\big) = z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g^\delta(\iota)\,\mathrm{d}\iota.$$

For $\|\partial G(t,z)\|_{\mathbb{Z}} \leqslant \zeta(t)$, we get

$$\int_0^T \big\|g^\delta(s)\big\|\,\mathrm{d}s \leqslant \|\zeta\|_{L^2([0,T],\mathbb{R}^+)}\sqrt{T}.$$

Further, the sequence $\{g^\delta\}$ is bounded in $L^2(0,T;\mathbb{Z})$. As a result, there is a subsequence, still stand for $\{g^\delta\}$, which converges weakly to $g \in L^2(0,T;\mathbb{Z})$. Denote

$$h = z_T - S_r(T)z_0 - Q_r(T)z_1 - \int_0^T K_r(T-\iota)g^\delta(\iota)\,\mathrm{d}\iota.$$

Since the linear system (8) is approximately controllable, referring to Theorem 5, we obtain

$$\delta R(\delta, \Gamma_0^T) \to 0 \quad \text{as } \delta \to 0.$$

Then

$$\begin{aligned}\left\|\mathcal{S}(g^\delta) - h\right\| &= \left\| \int_0^T K_r(T-\iota)\left[g^\delta(\iota) - g(\iota)\right] \mathrm{d}\iota \right\| \\ &\leqslant \sup_{t\in[0,T]} \left\| \int_0^t K_r(t-\iota)\left[g^\delta(\iota) - g(\iota)\right] \mathrm{d}\iota \right\| \to 0\end{aligned}$$

as $\delta \to 0^+$ due to the compactness of function

$$g \to \int_0^{\cdot} K_r(\cdot - \iota) g(\iota)\,\mathrm{d}\iota : L^1([0,T],\mathbb{Z}) \to C(0,T;\mathbb{Z}).$$

Therefore, we get the previous arguments

$$\begin{aligned}\left\|z^\delta(T) - z_T\right\| &\leqslant \left\|\delta R(\delta, \Gamma_0^T)\mathcal{S}(g^\delta)\right\| \\ &\leqslant \left\|\delta R(\delta, \Gamma_0^T)(h)\right\| + \left\|\delta R(\delta, \Gamma_0^T)\left[\mathcal{S}(g^\delta) - h\right]\right\| \\ &\leqslant \left\|\delta R(\delta, \Gamma_0^T)(h)\right\| + \left\|\mathcal{S}(g^\delta) - h\right\| \quad \text{as } \delta \to 0^+.\end{aligned}$$

As a result, system (2) is approximately controllable on $[0,T]$. □

5 Nonlocal conditions

Physical problems contributed to the idea of nonlocal conditions. In [1], Byszewski proved existence and uniqueness results for nonlocal functional differential systems. In [15], the researchers proposed the concept of Caputo fractional derivative using fixed point theorems and mild solutions. The authors recently developed fractional differential systems with nonlocal conditions by utilizing nondense domains, semigroups, cosine families, various fixed point procedures, and measure of noncompactness. For more details, see the articles [21, 24, 27, 30]. Consider the following nonlocal fractional differential systems of order $1 < r < 2$ with sectorial operators of type (P, η, r, γ):

$$\begin{aligned}&{}^C D_t^r z(t) \in Az(t) + Bx(t) + \partial G\big(t, z(t)\big), \quad t \in [0,T], \\ &z(0) = z_0 + j(z), \qquad z'(0) = z_1,\end{aligned} \tag{14}$$

where $j : C(0,T;\mathbb{Z}) \to \mathbb{Z}$ is an appropriate function that meets the following requirement:

(H3) The function $j : C(0,T;\mathbb{Z}) \to \mathbb{Z}$ is continuous, and there exists $c, d > 0$ such that

$$\left\|j(z)\right\| \leqslant c\|z\|_C + d \quad \text{for all } z \in C(0,T;\mathbb{Z}).$$

Definition 8. A function $z \in C(0,T;\mathbb{Z})$ is called a mild solution of system (14) if there exists $g \in L^2(0,T;\mathbb{Z})$ such that $g(t) \in \partial G(t, z(t))$ for a.e. $t \in [0,T]$ and

$$z(t) = S_r(t)\big[z_0 + j(z)\big] + Q_r(t)z_1 \\ + \int_0^t K_r(t-s)g(s)\,\mathrm{d}s + \int_0^t K_r(t-s)Bx(s)\,\mathrm{d}s.$$

Theorem 8. *If hypotheses* (H1)–(H3) *are satisfied, then* (14) *has at least one mild solution on* $[0,T]$.

Proof. As a result, we considered the argument of this theorem, which is equivalent to the arguments of Theorems 6 and 7. $\square$

6 Application

Consider the following fractional differential system:

$$\begin{aligned} &\frac{\partial^r}{\partial t^r} z(t,\rho) = \frac{\partial^2}{\partial \rho^2} z(t,\rho) + J(t,\rho) + Bx(t,\rho), \quad t \in [0,1],\ 0 \leqslant \rho \leqslant \pi, \\ &z(t,0) = z(t,1) = 0, \quad t \in [0,T], \\ &z(0,\rho) = z_0(\rho), \quad z'(0,\rho) = z_1(\rho), \quad \rho \in [0,\pi], \end{aligned} \tag{15}$$

where $\partial^r/\partial t^r$ means fractional partial derivative of $r = 3/2$. Let

$$J = \overline{J} + \overline{\overline{J}},$$

where $\overline{\overline{J}}$ is provided, and $\overline{J}$ is a well-known temperature function of the form

$$-\overline{J}(t,\rho) \in \partial G\big(t,\rho,z(t,\rho)\big), \quad (t,\rho) \in [0,1] \times [0,\pi],$$

where $G = G(t,\rho,\zeta)$ is a locally Lipschitz energy function, which is generally nonsmooth and nonconvex. In the third variable ζ, ∂G stands for the generalized Clarke's gradient [2]. The following is a basic example of G that fulfills assumptions (H2):

$$G(\zeta) = \min\big\{h_1(\zeta), h_2(\zeta)\big\},$$

where h_i maps from $\mathbb{R}$ into itself ($i = 1,2$) are convex quadratic functions [13]. The functions $z(t)(\rho) = z(t,\rho)$, $Bx(t)(\rho) = Bx(t,\rho)$.

Consider $\mathbb{H} = L^2([0,\pi])$, suppose maps A from $D(A)$ into $\mathbb{Z}$ is presented as $Az = \partial^2 z/\partial \rho^2$ along with domain

$$D(A) = \left\{ z \in \mathbb{H},\ \frac{\partial z}{\partial \rho}, \frac{\partial^2 z}{\partial \rho^2} \in \mathbb{H} \right\}.$$

In addition, A can be expressed as

$$Az = \sum_{k=1}^{\infty} k^2 (z, \psi_k)\psi_k, \quad z \in D(A),$$

where $\{\psi_k\}_{k=1}^{\infty}(\rho) = \sqrt{(2/\pi)} \sin k(\rho)$ for any $k \in \mathbb{N}$ form an orthonormal basis of $\mathbb{Z}$. Then A is the infinitesimal generator of a compact semigroup $K(t)$ for $t \geqslant 0$ in $\mathbb{Z}$ given by

$$K(t)z = \sum_{k=1}^{\infty} \mathrm{e}^{-k^2 t}(z, \psi_k)\psi_k, \quad z \in \mathbb{Z}, \qquad \bigl\|K(t)\bigr\| \leqslant \mathrm{e}^{-1} < 1.$$

Consider infinite dimensional Hilbert space $\mathbb{H}$ defined by

$$\mathbb{H} := \Biggl\{x\colon\ x = \sum_{i=2}^{\infty} x_i e_i,\ \sum_{i=2}^{\infty} x_i^2 < \infty\Biggr\}.$$

The norm in $\mathbb{H}$ is presented as $\|x\|_{\mathbb{H}} = (\sum_{i=2}^{\infty} x_i^2)^{1/2}$. Determine a mapping $B \in \mathcal{L}([0,T], \mathbb{H})$ in the following way:

$$Bx = 2x_2 e_1 + \sum_{k=2}^{\infty} x_k e_k \quad \text{for } x = \sum_{k=2}^{\infty} x_k e_k \in \mathbb{H},$$

also, $y = \sum_{k=1}^{\infty} y_k e_k \in \mathbb{H}$, inner product $\langle Bx, y\rangle = \langle x, B^* y\rangle$, thus

$$B^* y = (2y_1 + y_2)e_2 + \sum_{k=3}^{\infty} y_k e_k,$$

and

$$B^* K^*(t) z = \bigl(2z_1 \mathrm{e}^{-t} + z_2 \mathrm{e}^{-4t}\bigr)e_2 + \sum_{k=3}^{\infty} \mathrm{e}^{-k^2 t} z_k e_k.$$

It follows that $\|BK^*(t)z\|_{\mathbb{H}} = 0$ for some $t \in [0,T]$ imply $z = 0$. Hence, the linear portion of (15) is approximate controllable on $[0,T]$. Thus, all the assumptions of Theorem 7 are satisfied. Hence, (15) is approximately controllable on $[0,T]$.

7 Conclusion

This paper investigates the effect of hemivariational inequalities on the approximate controllability of fractional differential systems of order $r \in (1,2)$. The main results of this article are tested utilizing fractional calculations, multivalued functions, sectorial operator, and the fixed point theorem. The existence of a mild solution for the system class was initially introduced. Next, we developed the approximate controllability results for semilinear fractional differential system. Following that, we will look to systems with nonlocal conditions. An example is provided to illustrate the application of the obtained theory. For fractional hemivariational inequalities, the exact controllability results with delay will be discussed in the future.

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