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Asymptotic analysis of optimal control problems on the semiaxes for Carathéodory differential inclusions with fast oscillating coefficients*

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Abstract. We consider an optimal control problem for a differential inclusion of the Carathéodory type affine with respect to the control with a coercive cost functional on a semiaxis and with fast oscillating time-dependent coefficients. We prove that, when the small parameter converges to zero, the solution to this problem tends to some solution of the optimal control problem with averaged coefficients, where the averaging we understand in the sense of the Kuratowski upper limit.

Keywords: differential inclusions, optimal control, averaging.

1 Introduction

Starting from the pioneering works of Bogolubov [3], the averaging method is one of the most known and well-studied asymptotic methods in the theory of differential equations and has a lot of applications in classical [13] and modern problems [15]. This method allows us to overcome difficulties appearing due to the presence of (fast oscillating) functions like $f(t/\varepsilon, x)$ with a small parameter $\varepsilon > 0$, describing the dynamics of the system, replacing them with an averaged value

$$\bar{f}(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x)\,\mathrm{d}t. \tag{0}$$

The corresponding convergence results, known as Bogolubov theorems [3, 12], served as a basis for deep generalizations to many types of evolutionary problems. In particular,

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considerable efforts were devoted to the extensions of the averaging methods for differential inclusions [6].

Such generalizations are motivated both from practical point of view due to applications in control theory, hybrid dynamical systems, including impulsive ones [4, 8], and other evolutionary systems, where multivalued approach arises naturally; as well as due to purely mathematical reasons, for example, when the limit (0) does not exist.

Rather general results about asymptotic analysis of inclusions, which do not require existence of classical averaged value, were obtained in [7]. These results are based on the notion of the Kuratowski upper limit.

In this work, we essentially use this approach applying it to an optimal control problem for the differential inclusion

$$\dot{x}(t) \in f\left(\frac{t}{\varepsilon}, x(t)\right) + g\big(x(t)\big)u(t), \quad t \geqslant 0, \tag{1'}$$

where $f : [0, \infty) \times \mathbb{R}^d \to \operatorname{conv} \mathbb{R}^d$ is a given set-valued mapping, $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a given matrix-valued mapping, and $u : [0, \infty) \to \mathbb{R}^m$ is a control function. Here $\operatorname{conv} \mathbb{R}^d$ denotes the set of all nonempty, convex, compact subsets of $\mathbb{R}^d$ and $d, m \in \mathbb{N}$.

A lot of results of the averaging type exists for the case of control problems in case of differential inclusions like

$$\dot{x}(t) \in \varepsilon f\big(t, x(t), u(t)\big), \quad t \geqslant 0; \tag{2'}$$

see, for example, [7].

An extensive overview of the averaging theory for differential inclusions, including control problems for (2′), can be found in [7]. Asymptotic analysis for problems of the type (1′) was carried out in [10] for ordinary differential equations with Lipschitz function f both on finite and infinite time intervals. The authors of [10] assumed the existence of classical average $\bar{f}$ and used Krasnoselskii–Krein theorem. This approach can be extended naturally to systems with Lipschitz set-valued maps possessing an average $\bar{f}$ in the Hausdorff metric (see [11] for the set-valued analogon of the Krasnoselskii–Krein theorem). Some optimal control problems with perturbed coefficients were also considered in [9]. The situation becomes essentially more complicated if in (1′) the mapping f is not Lipschitz. In this paper, we consider the case when f in (1′) is a Caratheodory set-valued map, i.e., it is integrably bounded w.r.t. time variable and upper-semicontinuous w.r.t. phase variable. Following the approach of [6], we show that under assumptions needed for resolvability of the optimal control problem with inclusion (1′), the optimal process of ε-dependent problem converges in a suitable sense to an optimal process of the optimal control problem for the inclusion

$$\dot{x} \in \bar{f}(x) + g(x)u(t),$$

where $\bar{f}$ is the Kuratowski upper limit [2].

In the sequel, we will use the following notation: $\|x\|$ is a norm of $x \in \mathbb{R}^d$, $|u|$ is a norm of $u \in \mathbb{R}^m$, $\operatorname{conv} \mathbb{R}^d$ is the set of all nonempty, convex, compact subsets of $\mathbb{R}^d$.

For $A \subset \mathbb{R}^d$, we write $\|A\|_+ := \sup_{a\in A}\|a\|$, $\operatorname{co} A$ is a convex hull of A, $\bar{A}$ is the closure of A. For $A, B \subset \mathbb{R}^d$, we denote

$$\operatorname{dist}(A,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|,$$
$$\operatorname{dist}_H(A,B) := \max\bigl\{\operatorname{dist}(A,B), \operatorname{dist}(B,A)\bigr\},$$

and use the notation $B_1 := \{x \mid \|x\| \leqslant 1\}$, $O_\delta(x_0) = \{x \mid \|x - x_0\| < \delta\}$. For $f : [0,\infty) \mapsto \operatorname{conv}\mathbb{R}^d$, we define

$$\int_0^T f(t)\,\mathrm{d}t := \left\{ \int_0^T l(t)\,\mathrm{d}t \;\middle|\; l \in L^1\bigl(0,T;\mathbb{R}^d\bigr),\ l(t) \in f(t) \text{ a.e.} \right\}.$$

We also denote $\|u\|_{L^2}^2 = \int_0^\infty |u(t)|^2\,\mathrm{d}t$.

2 Problem statement

We consider the following optimal control problem:

$$\dot{x}(t) \in f\left(\frac{t}{\varepsilon}, x(t)\right) + g\bigl(x(t)\bigr)u(t), \quad t \geqslant 0, \qquad x(0) = x_0, \tag{1}$$
$$u \in \mathcal{U} = \bigl\{u \in L^2\bigl(0,\infty;\mathbb{R}^m\bigr) \bigm| u(t) \in U \text{ a.e. on } (0,\infty)\bigr\} \tag{2}$$

is such that

$$J(x,u) = \int_0^\infty \bigl(\mathrm{e}^{-\gamma t}\varphi\bigl(x(t)\bigr) + \bigl|u(t)\bigr|^2\bigr)\,\mathrm{d}t \to \inf, \tag{3}$$

where $\varepsilon > 0$ is a small parameter, and f, g, φ satisfy the following:

(i) $f : [0,\infty) \times \mathbb{R}^d \to \operatorname{conv}\mathbb{R}^d$;
(ii) For all $x \in \mathbb{R}^d$, the map $f(\cdot, x)$ possesses a measurable selector;
(iii) For all $t \geqslant 0$, the map $f(t,\cdot)$ is upper semicontinuous;
(iv) There exists $M \geqslant 0$ such that for all $x \in \mathbb{R}^d$ and $t \geqslant 0$, $\|f(t,x)\|_+ \leqslant M$;
(v) $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is continuous and bounded, that is, there exists $N \geqslant 0$ with $\|g(x)\| \leqslant N$, $x \in \mathbb{R}^d$;
(vi) $U \subset \mathbb{R}^m$ is closed, convex, and $0 \in U$;
(vii) $\varphi : \mathbb{R}^d \to \mathbb{R}$ is continuous, and there are constants $c > 0$ and $p \geqslant 1$ with

$$\inf_{x\in\mathbb{R}^d}\varphi(x) \geqslant -c, \qquad \bigl|\varphi(x)\bigr| \leqslant c\bigl(1 + \|x\|^p\bigr).$$

For a given control function $u \in \mathcal{U}$, we understand solution of (1) as an absolutely continuous function x, which satisfies (1) almost everywhere (a.e.) on $[0,+\infty)$. In this case, we say that $\{x, u\}$ is an admissible pair for (1)–(3). An admissible pair $\{x^\varepsilon, u^\varepsilon\}$ is called an optimal pair (or solution) for (1)–(3) if for every admissible pair $\{x, u\}$, we have

$$J(x^\varepsilon, u^\varepsilon) \leqslant J(x, u).$$

The existence of an optimal solution $\{x^\varepsilon, u^\varepsilon\}$ is established in the next section.

Let us denote

$$J^\varepsilon := \inf J(x, u) = J(x^\varepsilon, u^\varepsilon).$$

Using approach of [6], we define the average function $\bar{f}$ basing on the notion of the Kuratowski upper limit [2]

$$\bar{f}(x) = \bigcap_{\delta>0} \bar{F}^\delta(x),$$

where $\bar{F}^\delta$ is the convex hull of the map

$$\Phi^\delta(x) = \limsup_{\theta \nearrow 1} \limsup_{T\to\infty} \frac{1}{(1-\theta)T} I(\theta T, T, x, \delta),$$

$$I(\theta T, T, x, \delta) = \left\{ \int_{\theta T}^{T} v(t)\,\mathrm{d}t \;\middle|\; v(\cdot) \in L^1_{\mathrm{loc}}(0, \infty; \mathbb{R}^d),\ v(t) \in f(t, y),\ y \in \overline{O_\delta(x)} \right\}.$$

It is proved in [6] that if there exists $\bar{F}(x) = \lim_{T\to\infty}(1/T)\int_0^T f(t, x)\,\mathrm{d}t$ in the sense of the Hausdorff distance dist_H and if $f(t, \cdot)$ is Lipschitz, then $\bar{f} = \bar{F}$.

Also we consider the optimal control problem

$$\dot{x} \in \bar{f}(x) + g(x)u(t), \qquad x(0) = x_0, \tag{4}$$

$$u \in \mathcal{U}, \tag{5}$$

$$J(x, u) \to \inf. \tag{6}$$

Our aim is to prove that for $\varepsilon \to 0$, it follows that

$$J^\varepsilon \to \bar{J} \quad \text{and} \quad \{x^\varepsilon, u^\varepsilon\} \to \{\bar{x}, \bar{u}\} \quad \text{in some sense,}$$

where $\{\bar{x}, \bar{u}\}$ is a solution of (4)–(6), $\bar{J} = J(\bar{x}, \bar{u})$.

3 Main results

Let us analyze problem (1). Under conditions (i)–(v), for every $u \in L^2(0, \infty; \mathbb{R}^m)$ and $\varepsilon > 0$, problem (1) has a solution [1, 5, 14] and the following estimations hold true:

$$\frac{\mathrm{d}}{\mathrm{d}x}\|x(t)\| \leqslant \|\dot{x}(t)\| \leqslant M + N|u(t)| \quad \text{a.e.,} \tag{7}$$

$$\|x(t)\| \leqslant \|x_0\| + (M+N)t + N\|u\|^2_{L^2}. \tag{8}$$

Moreover, for a fixed $\varepsilon > 0$, if x_n is the solution to (1) corresponding to the control u_n, $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \|u_n\|_{L^2} < \infty$, then up to a subsequence, the following convergences hold:

$$u_n \to u \quad \text{weakly in } L^2\big(0, \infty; \mathbb{R}^m\big), \tag{9}$$

$$x_n \to x \quad \text{in } C\big([0, T]; \mathbb{R}^d\big),\ T > 0, \tag{10}$$

where x is a solution of (1) with control u. Additionally, if $u_n \in \mathcal{U}$ for $n \in \mathbb{N}$, then $u \in \mathcal{U}$.

Lemma 1. *Under conditions* (i)–(vii)*, the optimal control problem* (1)–(3) *has a solution* $\{x^\varepsilon, u^\varepsilon\}$.

Proof. Let us fix some $\varepsilon > 0$, and let admissible pairs $\{x_n, u_n\}$, $n \in N$, be such that

$$J^\varepsilon = \inf J(x, u) = \lim_{n\to\infty} J(x_n, u_n).$$

Then, due to assumption (vii), in the problem statement section, we have for n large enough that

$$-\frac{c}{\gamma} + \int_0^\infty \big|u_n(t)\big|^2 \,\mathrm{d}t \leqslant J(x_n, u_n) \leqslant J^\varepsilon + 1.$$

Hence, up to a subsequence, for $n \to \infty$, it follows that $\{x_n, u_n\} \to \{x, u\}$ in the sense of (9), (10), and $u \in \mathcal{U}$. Let us prove that $\{x, u\}$ is an optimal pair in (1)–(3). Hence, we have that

$$\mathrm{e}^{-\gamma t}\varphi\big(x_n(t)\big) \to \mathrm{e}^{-\gamma t}\varphi\big(x(t)\big), \quad t \geqslant 0.$$

Moreover, using (8), we get the estimates

$$\begin{aligned}
&\mathrm{e}^{-\gamma t}\varphi(x_n(t)\\
&\quad\leqslant \mathrm{e}^{-\gamma t}c\big(1 + \big\|x_n(t)\big\|^p\big)\\
&\quad\leqslant \mathrm{e}^{-\gamma t}c\big(1 + \big(\|x_0\| + (M+N)t + N\|u\|_{L^2}^2\big)^p\big)\\
&\quad\leqslant \mathrm{e}^{-\gamma t}c\left(1 + \left(\|x_0\| + (M+N)t + N\left(J^\varepsilon + 1 + \frac{c}{\gamma}\right)^p\right)\right.\\
&\quad\leqslant \mathrm{e}^{-\gamma t}c\left(1 + 2^{p-1}(M+N)^p t^p + 2^{p-1}\left(\|x_0\| + N\left(J^\varepsilon + 1 + \frac{c}{\gamma}\right)\right)^p\right).
\end{aligned}$$

Choosing a sufficiently large $c_1 = c_1(\varepsilon)$, we have

$$\begin{aligned}
&c\left(1 + 2^{p-1}(M+N)^p t^p + 2^{p-1}\left(\|x_0\| + N\left(J^\varepsilon + 1 + \frac{c}{\gamma}\right)\right)^p\right)\\
&\quad\leqslant c_1(\varepsilon)\mathrm{e}^{\gamma t/2},
\end{aligned}$$

so that finally, we conclude that

$$\begin{aligned}\mathrm{e}^{-\gamma t}\varphi\big(x_n(t)\big) &\leqslant \mathrm{e}^{-\gamma t}c\left(1+\left(\|x_0\|+(M+N)t+N\left(J^\varepsilon+1+\frac{c}{\gamma}\right)\right)^p\right)\\ &\leqslant c_1(\varepsilon)\mathrm{e}^{-\gamma t/2}.\end{aligned}$$

Then the Lebesgue dominated convergence theorem implies

$$\int_0^\infty \mathrm{e}^{-\gamma t}\varphi\big(x_n(t)\big)\,\mathrm{d}t \to \int_0^\infty \mathrm{e}^{-\gamma t}\varphi\big(x(t)\big)\,\mathrm{d}t$$

and hence

$$\begin{aligned}J^\varepsilon &= \varliminf_{n\to\infty} J(x_n,u_n)\\ &\geqslant \varliminf_{n\to\infty}\int_0^\infty \mathrm{e}^{-\gamma t}\varphi\big(x_n(t)\big)\,\mathrm{d}t + \varliminf_{n\to\infty}\int_0^\infty \big|u_n(t)\big|^2\,\mathrm{d}t\\ &\geqslant J(x,u),\end{aligned}$$

so that $\{x,u\}$ is a solution of (1)–(3). □

The following result is a slight generalization of [6, Thm. 41] and [16, Thm. 1.1]. It shows that for any set-valued function f satisfying assumptions (i)–(iv), there exists a sequence of set-valued functions monotone in the sense of set inclusions and containing the set $f(t,x)$ for any $t\geqslant 0$, $x\in\mathbb{R}^d$.

Lemma 2. *Let* $f:[0,\infty)\times\mathbb{R}^d\to\operatorname{conv}\mathbb{R}^d$ *satisfies* (i)–(iv). *Then there exists a sequence of locally Lipschitz maps* $f^k:[0,\infty)\times\mathbb{R}^d\to\operatorname{conv}\mathbb{R}^d$ *satisfying* (i)–(iv) *for* $k\in\mathbb{N}$ *with*

$$f(t,x)\subset\cdots\subset f^{k+1}(t,x)\subset f^k(t,x),\quad t\geqslant 0,\ x\in\mathbb{R}^d,\ k\in\mathbb{N},\tag{11}$$

and for each $k\in\mathbb{N}$ *and* $x\in\mathbb{R}^d$, *there exist* $l_k>0$ *and* $\delta_k>0$ *such that*

$$\operatorname{dist}_H\big(f^k(t,x'),f^k(t,x'')\big)\leqslant l_k\|x'-x''\|,\quad x',x''\in O_{\delta_k}(x),\ t\geqslant 0.\tag{12}$$

Moreover, for any $\varepsilon>0$, $t\geqslant 0$, $x\in\mathbb{R}^d$, *there is* $K=K(\varepsilon,t,x)$ *with*

$$f^k(t,x)\subset\overline{\operatorname{co}}f\big(t,O_\varepsilon(x)\big),\quad k\geqslant K.\tag{13}$$

Proof. Let $\{O_{r_k}(y_i^k)\}_{i=1}^\infty$ be a locally finite covering of $\mathbb{R}^d$, where $r_k:=1/3^{k-1}$, $k\in\mathbb{N}$. Let $\{\psi_i^k\}_{i=1}^\infty$ be a partition of unity subordinated to this covering and consisting of locally Lipschitz functions $\operatorname{supp}\psi_i^k\subset O_{r_k}(y_i^k)$; see [8]. For any $k\in\mathbb{N}$ and $x\in\mathbb{R}^d$, there exist $\delta_k=\delta_k(x)>0$ and $l_k(x)>0$ such that $\delta_k\to 0$, $k\to\infty$, and

$$N(k,x):=\big\{i\in\mathbb{N}\ \big|\ O_{r_k}\big(y_i^k\big)\cap O_{\delta_k}(x)\neq\emptyset\big\}$$

is finite, and for any $x,x''\in O_{\delta_k}(x)$,

$$\sum_{i\in N(k,x)}\big|\psi_i^k(x')-\psi_i^k(x'')\big|\leqslant l_k(x)\|x'-x''\|.$$

Then the map

$$f^k(t,x) = \sum_{i=1}^{\infty} \psi_i^k(x) \cdot \overline{\mathrm{co}} f\bigl(t, O_{2r_k}\bigl(y_i^k\bigr)\bigr) = \sum_{i \in N(k,x)} \psi_i^k(x) \cdot \overline{\mathrm{co}} f\bigl(t, O_{2r_k}\bigl(y_i^k\bigr)\bigr)$$

satisfies (i)–(iv), (11), (12); see [16] for details. Finally, for any $\varepsilon > 0$, choose $K = K(\varepsilon, t, x)$ such that $\delta_k + r_k < \varepsilon/3$, $k \geqslant K$. Then for any $i \in N(k,x)$ and any $z \in O_{2r_k}(y_i^k)$, we get

$$\|z - x\| \leqslant \bigl\|z - y_i^k\bigr\| + \bigl\|y_i^k - x\bigr\| \leqslant 2r_k + \delta_k + r_k < \varepsilon.$$

Therefore, $f(t, O_{2r_k}(y_i^k)) \subset f(t, O_\varepsilon(x))$ and $f^k(t,x) \subset \overline{\mathrm{co}} f(t, O_\varepsilon(x))$. The lemma is proved. □

Lemma 3. *Let $\varepsilon_n \to 0$ as $n \to \infty$, and let x_n be a solution of* (1) *with control u_n. Let $\{x_n, u_n\} \to \{x, u\}$ as $n \to \infty$ in the sense of* (9), (10)*. Then x is a solution of* (4) *with control u.*

Proof. Let f^k be a Lipschitz map from Lemma 2. Then for any $k \in \mathbb{N}$, we have

$$\dot{x}_n(t) \in f^k\left(\frac{t}{\varepsilon_n}, x_n(t)\right) + g\bigl(x_n(t)\bigr)u_n(t) \quad \text{a.e.}$$

We fix τ_1 such that the derivative $\dot{x}(\tau_1)$ exists, and let τ_1 be a Lebesgue point of $g(x(\cdot))u(\cdot)$. Then for any $n \geqslant N(x(\tau_1))$ and for sufficiently small $|s - \tau_1|$, we have $\|x_n(s) - x(\tau_1)\| < \delta_k$ and

$$x_n(\tau_2) - x_n(\tau_1) \in \int_{\tau_1}^{\tau_2} \left[f^k\left(\frac{s}{\varepsilon_n}, x(\tau_1)\right) + l_k \bigl\|x_n(s) - x(\tau_1)\bigr\| \cdot B_1 \right] \mathrm{d}s + \int_{\tau_1}^{\tau_2} g\bigl(x_n(s)\bigr)u_n(s)\,\mathrm{d}s,$$

where l_k is taken from (12), $B_1 = \{x \mid \|x\| \leqslant 1\}$. Then

$$\begin{aligned} &x_n(\tau_2) - x_n(\tau_1) \\ &\quad\in \int_{\tau_1}^{\tau_2} f^k\left(\frac{s}{\varepsilon_n}, x(\tau_1)\right) \mathrm{d}s \\ &\qquad + l_k \int_{\tau_1}^{\tau_2} \bigl\|x_n(s) - x_n(\tau_1)\bigr\| \,\mathrm{d}s \cdot B_1 + l_k \int_{\tau_1}^{\tau_2} \bigl\|x_n(\tau_1) - x(\tau_1)\bigr\| \cdot B_1 \\ &\qquad + \int_{\tau_1}^{\tau_2} \bigl(g(x_n(s)\bigr)u_n(s) - g\bigl(x(s)\bigr)u(s)\bigr)\,\mathrm{d}s + \int_{\tau_1}^{\tau_2} g\bigl(x(s)\bigr)u(s)\,\mathrm{d}s. \end{aligned}$$

Let $\eta > 0$ be arbitrary small. Then for sufficiently small $|\tau_2 - \tau_1|$ and sufficiently large $n \in \mathbb{N}$, due to (7), we get

$$
\begin{aligned}
l_k \int_{\tau_1}^{\tau_2} \big\|x_n(s) - x_n(\tau_1)\big\| \,\mathrm{d}s &\leqslant l_k \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{s} \big(M + N\big|u_n(t)\big|\big) \,\mathrm{d}t \,\mathrm{d}s \\
&\leqslant l_k \int_{\tau_1}^{\tau_2} \big(M(\tau_2 - \tau_1) + N\sqrt{\tau_2 - \tau_1}\|u_n\|_{L^2}\big) \,\mathrm{d}t \\
&< \frac{\eta}{2}(\tau_2 - \tau_1).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{x_n(\tau_2) - x_n(\tau_1)}{\tau_2 - \tau_1} \\
&\quad \in \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f^k\left(\frac{s}{\varepsilon_n}, x(\tau_1)\right) \mathrm{d}s + \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} g\big(x_n(t)\big) u_n(t) \,\mathrm{d}t \\
&\qquad + \eta B_1 \frac{x_n(\tau_2) - x_n(\tau_1)}{\tau_2 - \tau_1} \\
&\quad \in \frac{\varepsilon_n}{\tau_2 - \tau_1} \int_{\tau_1/\varepsilon_n}^{\tau_2/\varepsilon_n} f^k\big(s, x(\tau_1)\big) \,\mathrm{d}s + \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} g\big(x_n(t)\big) u_n(t) \,\mathrm{d}t + \eta B_1,
\end{aligned}
$$

and passing to the limit for $n \to \infty$, we obtain

$$
\begin{aligned}
\frac{x(\tau_2) - x(\tau_1)}{\tau_2 - \tau_1} \in \limsup_{T \to \infty} \frac{1}{(1 - \frac{\tau_1}{\tau_2})T} \int_{\tau_1/\tau_2 T}^{T} f^k\big(s, x(\tau_1)\big) \,\mathrm{d}s \\
+ \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} g\big(x(t)\big) u(t) \,\mathrm{d}t + \eta B_1.
\end{aligned}
$$

Next, passing to the limit for $\tau_2 \to \tau_1$ and since η is arbitrary small, we get

$$
\dot{x}(\tau_1) \in \limsup_{\theta \to 1} \limsup_{T \to \infty} \frac{1}{(1 - \theta)T} \int_{\theta T}^{T} f^k\big(s, x(\tau_1)\big) \,\mathrm{d}s + g\big(x(\tau_1)\big) u(\tau_1). \tag{14}
$$

By the convexity of the integral [7] and due to (13), we have that for any $\delta > 0$ and $x \in \mathbb{R}^d$, there exists $K = K(\delta, x)$ with

$$
\limsup_{\theta \to 1} \limsup_{T \to \infty} \frac{1}{(1 - \theta)T} \int_{\theta T}^{T} f^k(s, x) \,\mathrm{d}s \subset \bar{F}^\delta(x), \quad k \geqslant K.
$$

Finally, from (14) we have for any $\delta > 0$ that

$$\dot{x}(\tau_1) \in \bar{F}^{\delta}\big(x(\tau_1)\big) + g\big(x(\tau_1)\big)u(\tau_1)$$

and hence

$$\dot{x}(\tau_1) \in \bar{f}\big(x(\tau_1)\big) + g\big(x(\tau_1)\big)u(\tau_1),$$

which proves the lemma. □

Now we are in a position to prove our main result.

Theorem 1. *Assume that conditions* (i)–(vii) *are satisfied. Assume that for any* $u \in \mathcal{U}$*, problem* (4) *has a unique solution. Let* $\{x^{\varepsilon}, u^{\varepsilon}\}$ *be an optimal pair in* (1)–(3)*,* $J^{\varepsilon} = J(x^{\varepsilon}, u^{\varepsilon})$*. Then*

$$J^{\varepsilon} \to \bar{J} \quad \text{for } \varepsilon \to 0, \tag{15}$$

and for $\varepsilon_n \to 0$*, it holds that*

$$x^{\varepsilon_n} \to \bar{x} \quad \text{in } C\big([0,T];\mathbb{R}^d\big),\ T > 0, \tag{16}$$

$$u^{\varepsilon_n} \to \bar{u} \quad \text{weakly in } L^2\big(0,\infty;\mathbb{R}^m\big), \tag{17}$$

where $\{\bar{x}, \bar{u}\}$ *is an optimal pair in* (4)–(6)*,* $\bar{J} = J(\bar{x}, \bar{u})$*.*

Proof. Let for $\varepsilon_n \to 0$, $\{x^{\varepsilon_n}, u^{\varepsilon_n}\}$ be an optimal pair for (1)–(3). From the optimality of u_n^{ε} it follows that

$$J\big(x^{\varepsilon_n}, u_n^{\varepsilon}\big) \leqslant J(x_n, 0),$$

where x_n is a solution of (1) with $\varepsilon = \varepsilon_n$, $u = 0$. Then, due to (8),

$$\begin{aligned} -\frac{c}{\gamma} + \|u_n^{\varepsilon}\|^2 &\leqslant \int_0^{\infty} \mathrm{e}^{-\gamma t}\varphi\big(x_n(t)\big) \\ &\leqslant \int_0^{\infty} \mathrm{e}^{-\gamma t} c\big(1 + \big(\|x_0\| + (M+N)t\big)^p\big)\,\mathrm{d}t \\ &\leqslant C_1, \end{aligned} \tag{18}$$

where C_1 does not depend on n. Additionally, from (7) we have

$$\big\|x_n^{\varepsilon}(t) - x_n^{\varepsilon}(s)\big\| \leqslant M|t-s| + N|t-s|^{1/2}\big\|u_n^{\varepsilon}\big\|_{L^2}. \tag{19}$$

Estimations (18), (19) and the Arzelà–Ascoli theorem imply that some subsequence $\{x^{\varepsilon_n}, u^{\varepsilon_n}\}$, $n \in \mathbb{N}$, converges to some $\{\bar{x}, \bar{u}\}$ in the sense of (16), (17). Hence, from Lemma 3 we deduce that $\bar{x}$ is a solution of (4) with control $u \in \mathcal{U}$. Let us prove that $\{\bar{x}, \bar{u}\}$ is an optimal pair.

For every $u \in \mathcal{U}$ and the corresponding solution x_n to (1), we have

$$J\big(x^{\varepsilon_n}, u^{\varepsilon_n}\big) \leqslant J(x_n, u). \tag{20}$$

Arguing as in the proof of Lemma 1, we get from (20) after passing to the limit:

$$J(\bar{x},\bar{u}) \leqslant \liminf_{n\to\infty} J\bigl(x^{\varepsilon_n}, u^{\varepsilon_n}\bigr) \leqslant \liminf_{n\to\infty} J(x_n, u). \tag{21}$$

Due to (19) with u_n^ε replaced with u, we have that $x_n \to x$ in the sense of (16). By Lemma 3 it follows that x is a unique solution of (4) with control u. So, from (21) it follows

$$J(\bar{x},\bar{u}) \leqslant \liminf_{n\to\infty} J(x_n, u) = J(x, u).$$

This inequality means that $\{\bar{x},\bar{u}\}$ is an optimal pair.

Applying previous arguments with $u = \bar{u}$, we get

$$J(\bar{x},\bar{u}) \leqslant \liminf_{n\to\infty} J^{\varepsilon_n} \leqslant \limsup_{n\to\infty} J^{\varepsilon_n} \leqslant \lim_{n\to\infty} J(x_n,\bar{u}) = J(\bar{x},\bar{u}),$$

where we write $J^{\varepsilon_n} := J(x^{\varepsilon_n}, u^{\varepsilon_n})$ for short.

This means that there exists $\lim_{n\to\infty} J^{\varepsilon_n} = J(\bar{x},\bar{u})$. Because of arbitrariness of $\varepsilon_n \to 0$, we get (15). Theorem is proved. □

4 Example

Consider the following optimal control problem:

$$\begin{aligned}
&\dot{x} \in \begin{cases} \psi_1(\frac{t}{\varepsilon}), & x<0,\\ \bigl[\psi_2(\frac{t}{\varepsilon}), \psi_1(\frac{t}{\varepsilon})\bigr], & x=0,\\ \psi_2(\frac{t}{\varepsilon}), & x>0, \end{cases} \quad + u(t),\\
&x(0) = 0,\\
&u \in \mathcal{U}, \qquad J(x,u) = \int_0^\infty \bigl(\mathrm{e}^{-t}\varphi(x) + u^2(t)\bigr)\,\mathrm{d}t \to \inf,
\end{aligned} \tag{22}$$

where functions ψ_1, ψ_2 are bonded and chosen such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \psi_1(t) = \psi_1 > 0, \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T \psi_2(t) = \psi_2 < 0.$$

It is easy to see that the set-valued map

$$f(t,x) = \begin{cases} \psi_1(t), & x<0,\\ \bigl[\psi_2(t), \psi_1(t)\bigr], & x=0,\\ \psi_2(t), & x>0, \end{cases}$$

as well as $g = 1$, $\mathcal{U}$, φ satisfy conditions (i)–(vii). We note that $f(t,\cdot)$ is not a Lipschitz map.

The averaged problem has the form

$$
\begin{aligned}
&\dot{x} \in \begin{cases} \psi_1, & x < 0, \\ [\psi_1, \psi_2], & x = 0, \\ \psi_2, & x > 0, \end{cases} \quad + u(t), \\
&x(0) = 0, \\
&u \in \mathcal{U}, \qquad J(x, u) \to \inf,
\end{aligned} \tag{23}
$$

where the Cauchy problem has a unique solution for every $u \in \mathcal{U}$ [9]. According to Theorem 1, the sequence of optimal pairs $\{x^\varepsilon, u^\varepsilon\}$ of (22) converges to $\{\bar{x}, \bar{u}\}$, where $\{\bar{x}, \bar{u}\}$ is an optimal pair of (23).

References

1. J.-P. Aubin, A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Grundlehren Math. Wiss., Vol. 264, Springer, Berlin, 1984, https://doi.org/10.1007/978-3-642-69512-4.
2. J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Mod. Birkhäuser Class., Birkhäuser, Boston, MA, 2009, https://doi.org/10.1007/978-0-8176-4848-0. Reprint of the 1990 edition.
3. N.N. Bogoliubov, Y.A. Mitropol'skiĭ, *Asymptotic Methods in the Theory Of Non-linear Oscillations*, Int. Monogr. Adv. Math. Phys., Hindustan Publishing Corp., Delhi, Gordon and Breach, New York, 1961. Translated from the Russian 2nd ed.
4. S. Dashkovskiy, P. Feketa, Asymptotic properties of Zeno solutions, *Nonlinear Anal. Hybrid Syst.*, **30**:256–265, 2018, https://doi.org/10.1016/j.nahs.2018.06.005.
5. A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Math. Appl., Sov. Ser., Vol. 18, Kluwer, Dordrecht, 1988, https://doi.org/10.1007/978-94-015-7793-9. Translated from the Russian.
6. R. Gama, A. Guerman, G. Smirnov, On the asymptotic stability of discontinuous systems analysed via the averaging method, *Nonlinear Anal., Theory Methods Appl.*, **74**(4):1513–1522, 2011, https://doi.org/10.1016/j.na.2010.10.024.
7. R. Gama, G. Smirnov, Stability and optimality of solutions to differential inclusions via averaging method, *Set-Valued Var. Anal.*, **22**(2):349–374, 2014, https://doi.org/10.1007/s11228-013-0261-4.
8. R. Goebel, R.G. Sanfelice, A.R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*, Princeton Univ. Press, Princeton, NJ, 2012, https://doi.org/10.1515/9781400842636.
9. O.V. Kapustyan, O.A. Kapustyan, A.V. Sukretna, Approximate stabilization for a nonlinear parabolic boundary-value problem, *Ukr. Math. J.*, **63**(5):759–767, 2011, https://doi.org/10.1007/s11253-011-0540-x.
10. O. Kichmarenko, O. Stanzhytskyi, Optimal control problems for some classes of functional-differential equations on the semi-axis, *Miskolc Math. Notes*, **20**(2):1021–1037, 2019, https://doi.org/10.18514/MMN.2019.2739.

11. N.V. Plotnikova, The Krasnosel'skii-Krein theorem for differential inclusions, *Differ. Equ.*, **41**(7):1049–1053, 2005, https://doi.org/10.1007/s10625-005-0248-5. Translation from *Differ. Uravn.*, **41**(7):997–1000, 2005.
12. A.M. Samoĭlenko, A.N. Stanzhitskiĭ, On averaging differential equations on an infinite interval, *Differ. Equ.*, **42**(4):505–511, 2006, https://doi.org/10.1134/S0012266106040070. Translation from *Differ. Uravn.*, **42**(4):476–482, 2006.
13. J.A. Sanders, F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*, Appl. Math. Sci.,Vol. 59, Springer, New York, 1985, https://doi.org/10.1007/978-1-4757-4575-7.
14. G.V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Grad. Stud. Math., Vol. 41, AMS, Providence, RI, 2002, https://doi.org/10.1090/gsm/041.
15. R. Suttner, S. Dashkovskiy, Exponential stability for extremum seeking control systems, *IFAC PapersOnLine*, **50**(1):15464–15470, 2017, https://doi.org/10.1016/J.IFACOL.2017.08.2106.
16. A.A. Tolstonogov, Ya.I. Umanskii, On solutions of evolution inclusions. II, *Sib. Math. J.*, **33**: 693–702, 1992, https://doi.org/10.1007/BF00971135.