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Razumikhin and Krasovskii stability of impulsive stochastic delay systems via uniformly stable function method*

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Abstract. This paper generalizes Razumikhin-type theorem and Krasovskii stability theorem of impulsive stochastic delay systems. By proposing uniformly stable function (USF) in the form of impulse as a new tool, some properties about USF and some novel pth moment decay theorems are derived. Based on these new theorems, the stability theorems of impulsive stochastic linear delay system are acquired via the Razumikhin method and the Krasovskii method. The obtained results enhance the elasticity of the impulsive gain by comparing the previous results. Finally, numerical examples are given to demonstrate the effectiveness of theoretical results.

Keywords: stochastic delay systems, Razumikhin, Krasovskii, impulse, uniformly stable function.

1 Introduction

For the stability and the stabilization of time-delay systems, the Razumikhin approach and the Krasovskii approach are efficient methods to deal with the influence of time delay in the dynamic evolution process [5, 9, 18, 19]. However, time derivative of Razumikhin theorem and Krasovskii functional is usually required to be negative definite, which brings conservative in the application of theorems. To overcome this restriction, Maliso and

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Mazenc [16, 17] relaxed the restriction of negativity of the time derivative of Lyapunov function. Recently, in [31, 32], the authors proposed the notion of uniformly stable functions, which can effectively apply to stability analysis of time-delay system and weaken the negativity of time derivative of Razumikhin condition and Krasovskii functionals.

During the process of system information transmission, signals are often disturbed by random noises. Thereafter, stochastic system has been well applied to noise disturbances in engineering. Correspondingly, many results have been obtained in the stability and the dynamics of stochastic systems [13, 14, 23, 29]. The Razumikhin method and the Krasovskii functional method are two very effective methods to deal with stability analysis of stochastic systems. In [2], Chang firstly obtained Razumikhin-type asymptotic stability theorems of stochastic systems. Afterwards, some Razumikhin-type theorems and the Krasovskii functional methods on pth moment exponential stability and almost sure exponential stability of stochastic delay systems have been generalized in [1, 8, 15]. Impulse can effectively model the relationship between continuous phenomena and sudden changes at certain moment in nature. Therefore, impulsive input has been widely applied in nonlinear systems [10,24,28,30], stochastic systems [20,22,25] and networked systems [3,4,6,7,11,12,21,26,27]. However, many original results have been generalized to state the stability and stabilization of the impulsive control systems. The impulsive gains are somewhat inflexible, i.e., for a divergent system adding impulsive control, stabilizing impulse gains are required to stabilize the overall system (see [4, 6, 10, 12, 20, 24, 25]). Additionally, if an unstable system permits destabilizing impulses, the finite number of destabilizing impulses is required to ensure that the stability of the systems will not be destroyed (see [3]). It is obvious to see that the role of impulsive control is greatly limited.

Motivated by the above discussions, this paper generalizes Lyapunov stability of stochastic delay systems under impulsive model. By introducing a USF in the form of impulse, Razumikhin and Krasovskii stability theorems are deduced. The innovations are as follows:

(i) A USF in the form of impulse is proposed, which is an effective tool for dealing impulsive gains and stochastic delay systems. By applying the USF method, Razumikhin-type theorems and Krasovskii stability theorems of impulsive stochastic systems with time-varying delay are established.

(ii) Compared to the classical Razumikhin and Krasovskii theorems of stochastic systems, Lyapunov function may not require the negativity of the time derivative. By USF in the form of impulse, general Razumikhin and Krasovskii stability theorems of nonlinear stochastic systems are deduced, which relaxes the condition of traditional stability theorem.

(iii) For the divergent stochastic system without impulses, the conditions of impulses can be relaxed to destabilizing impulses and infinite number of destabilizing impulses. However, if stochastic system is convergent, the growth exponent of the impulsive gains product is less than some positive number to make the stability of impulsive stochastic systems, which implies that conditions of impulsive gains are more better than that of the previous theorems.

The rest of this paper is organized as follows. In Section 2, nonlinear impulsive stochastic delay systems and USF in the form of impulse are presented. Additionally, some lemmas on properties of uniformly stable functions are given. Section 3 is devoted to the Razumikhin stability theorem and the Krasovskii stability theorem of nonlinear impulsive stochastic systems via uniformly stable function method. In Section 4, these stability theorems will be applied to impulsive stochastic linear systems with delay. Section 5 provides two numerical examples to demonstrate the effectiveness of our results. Finally, some conclusions are given in Section 6.

Notations. Let I_n be the identity matrix with n dimension. $\mathcal{K}$ is the set of strictly increasing and continuous function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\theta(0) = 0$. $\mathcal{K}_\infty$ denotes the set of unbounded functions belonging to $\mathcal{K}$. $\mathcal{H}$ is the set of continuous functions that are strictly decreasing to 0 as $t \to \infty$. $\mathcal{KH}$ is the class of function $\gamma(s,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying that $\gamma(\cdot, t)$ is of class $\mathcal{K}$ for $t \geqslant 0$ and $\gamma(s, \cdot)$ is of class $\mathcal{H}$ for $s > 0$. $PC([-\tau, 0], \mathbb{R}^n)$ is the family of $\mathbb{R}^n$-valued piecewise continuous functions defined on $[-\tau, 0]$.

2 Model description and preliminaries

2.1 Model description

Consider impulsive stochastic systems with time delay

$$\begin{aligned} \mathrm{d}x(t) &= f\bigl(t, x(t), x_t\bigr)\,\mathrm{d}t + g\bigl(t, x(t), x_t\bigr)\,\mathrm{d}\omega(t), \quad t \geqslant 0,\ t \neq t_k, \\ x\bigl(t_k^+\bigr) &= h_k\bigl(t_k, x(t_k)\bigr), \quad k = 1, 2, \dots, \end{aligned} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$ is the state vector, $x_t = x(t+\zeta)$, $\zeta \in [-\tau, 0]$, $\tau > 0$, $f, g : [0, +\infty) \times \mathbb{R}^n \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$, $h_k : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots$, f, g and I_k satisfy Lipschitz condition with $f(t, 0, 0) = 0$, $g(t, 0, 0) = 0$ and $I_k(t, 0) = 0$. $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))^{\mathrm{T}}$ is an n-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $x(t_k^+) = \lim_{h\to 0+} x(t_k + h)$, $x(t_k) = \lim_{h\to 0-} x(t_k + h)$ is left-hand continuous at $t = t_k$. The initial condition of $x(t)$ is defined by $x(t) = \xi \in PC_{\mathcal{F}_t}([-\tau, 0], \mathbb{R}^n)$, where $PC_{\mathcal{F}_t}([-\tau, 0], \mathbb{R}^n)$ is the family of all $\mathcal{F}_t$-measurable, $PC([-\tau, 0], \mathbb{R}^n)$-value random variable ξ satisfying $\int_{-\tau}^{0} \mathbf{E}|\xi(\varsigma)|^2\,\mathrm{d}\varsigma < \infty$, $PC([-\tau, 0], \mathbb{R}^n)$ is the family of piecewise continuous functions ξ with the norm $\|\xi\| = \sup_{-\tau\leqslant\varsigma\leqslant 0} |\xi(\varsigma)|$.

Definition 1. The impulsive stochastic system (1) is said to be pth moment asymptotically stable (pMAS), if there exists a $\mathcal{KH}$ function γ such that

$$\mathbf{E}\bigl[\bigl|x(t)\bigr|^p\bigr] \leqslant \gamma\bigl(\mathbf{E}\bigl[\|\xi\|^p\bigr], t\bigr), \quad t \geqslant 0.$$

Moreover, if there exist $M > 0$ and $\lambda > 0$ such that for $t \geqslant 0$,

$$\mathbf{E}\bigl[\bigl|x(t)\bigr|^p\bigr] \leqslant M\mathrm{e}^{-\lambda t}\mathbf{E}\bigl[\|\xi\|^p\bigr],$$

system (1) is said to be pth moment exponentially stable (pMES).

Definition 2. Let $V : [-\tau, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ be a $\mathcal{C}_1^2$ function. The differential operator $\mathcal{L}V : (t_{k-1}, t_k] \times PC_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ associated with system (1) is defined as

$$\begin{aligned}\mathcal{L}V(t,\varphi) &= V_t\bigl(t,\varphi(0)\bigr) + V_x\bigl(t,\varphi(0)\bigr) f\bigl(t,\varphi(0),\varphi\bigr) \\ &\quad + \frac{1}{2}\operatorname{trace}\bigl[g^{\mathrm{T}}\bigl(t,\varphi(0),\varphi\bigr) V_{xx} g\bigl(t,\varphi(0),\varphi\bigr),\end{aligned}$$

where $\varphi = \varphi(t+\zeta)$, $\zeta \in [-\tau, 0]$.

2.2 Uniformly stable function

In the following, we propose the definition of the USF in the form of impulse and some properties of USF are introduced.

Definition 3. Let for $u_k > 0$, $k = 1, 2, \dots$, the following system

$$\begin{aligned}\dot{e}(t) &= \delta(t)e(t), \quad t \geqslant 0,\ t \neq t_k, \\ e(t_k^+) &= u_k e(t_k), \quad k = 1, 2, \dots,\end{aligned} \tag{2}$$

is globally uniformly exponentially stable. Then

$$\rho(t) = \prod_{0 \leqslant t_k < t} u_k \exp\Biggl(\int_0^t \delta(s)\,\mathrm{d}s\Biggr), \quad t \geqslant 0, \tag{3}$$

is said to be USF.

Indeed, from (2) and (3) we have $e(t) = \rho(t)e(0)$, $t \geqslant 0$, which shows that system (2) is globally uniformly exponentially stable if and only if there exist two constants $\alpha > 0$ and $\beta \geqslant 0$ such that

$$\ln \rho(t) \leqslant -\alpha t + \beta, \quad t \geqslant 0.$$

In the following, we give some testable criteria of the USF in the form of impulse. Let $\{\nu_i\}_{i=0}^{\infty}$ be an admissible sequence, i.e., $\{\nu_i\}_{i=0}^{\infty}$ is strictly increasing and tends to infinity, and there exists $\eta > 0$ such that $\eta_i = \nu_{i+1} - \nu_i \in (0, \eta]$, $i = 0, 1, \dots$.

Lemma 1. *If there exist an admissible sequence $\{\nu_i\}_{i=0}^{\infty}$ and two constants $a > 0$, b such that for $i = 0, 1, \dots,$*

$$\ln\Biggl(\prod_{\nu_i \leqslant t_k < \nu_{i+1}} u_k\Biggr) + \int_{\nu_i}^{\nu_{i+1}} \delta(s)\,\mathrm{d}s \leqslant -a$$

and

$$\ln\Biggl(\prod_{\nu_i \leqslant t_k < \nu_i + \varepsilon} u_k\Biggr) + \int_{\nu_i}^{\nu_i + \varepsilon} \delta(s)\,\mathrm{d}s \leqslant b \quad \forall \varepsilon \in [0, \eta_i],$$

then function $\rho(t)$ is a USF.

Proof. For $t \geqslant 0$, there exists an integer $j \geqslant 0$ such that $t \in [\nu_j, \nu_{j+1})$. Let $\varepsilon = t - \nu_j$, it yields that $\varepsilon \in [0, \eta_j)$. Thus

$$\begin{aligned}
\ln \rho(t) &= \sum_{i=0}^{j-1} \left[\ln\left(\prod_{\nu_i \leqslant t_k < \nu_{i+1}} d_k \right) + \int_{\nu_i}^{\nu_{i+1}} \delta(s)\,\mathrm{d}s \right] + \ln\left(\prod_{\nu_j \leqslant t_k < \nu_j + \varepsilon} d_k \right) + \int_{\nu_j}^{\nu_j+\varepsilon} \delta(s)\,\mathrm{d}s \\
&\leqslant -ja + b = -a \sum_{i=0}^{j-1} \frac{\nu_{i+1} - \nu_i}{\eta_i} + b \leqslant -\frac{a}{\eta}(\nu_j - \nu_0) + b \\
&\leqslant -\frac{a}{\eta} t + b + \frac{a}{\eta}\varepsilon \leqslant -\frac{a}{\eta} t + b + a.
\end{aligned}$$

We conclude that for $t \geqslant 0$, $\ln \rho(t) \leqslant -\alpha t + \beta$, where $\alpha = a/\eta$, $\beta = \max\{a+b,\, 0\}$. □

Lemma 2. *If there exist constants $\eta_k > 0$, $k = 0, 1, \dots$, $a > 0$, b such that for $k = 0, 1, \dots$,*

$$t_{k+1} - t_k \leqslant \eta_k,$$

$$\ln u_k + \int_{t_k}^{t_{k+1}} \delta(s)\,\mathrm{d}s \leqslant -a \quad \textit{and} \quad \ln u_k + \int_{t_k}^{t_k+\varepsilon} \delta(s)\,\mathrm{d}s \leqslant b \quad \forall \varepsilon \in [0, \eta_k],$$

then function $\rho(t)$ is a USF.

Lemma 3. *If there exist $i \in \{1, 2, \dots\}$ and $\theta > 0$, $a > 0$ such that*

$$t_{k+i} = t_k + \theta, \qquad u_{k+i} = u_k, \qquad \delta(t + \theta) = \delta(t)$$

and

$$\ln\left(\prod_{t \leqslant t_k < t+\theta} u_k \right) + \int_t^{t+\theta} \delta(s)\,\mathrm{d}s \leqslant -a \quad \forall t \geqslant 0,$$

then function $\rho(t)$ is a USF.

Proof. For $t > 0$, there exists an integer $j \geqslant 0$ such that $j\theta \leqslant t < (j+1)\theta$. Thus

$$\begin{aligned}
\ln \rho(t) &= \sum_{i=0}^{j-1} \left[\ln\left(\prod_{i\theta \leqslant t_k < (i+1)\theta} u_k \right) + \int_{i\theta}^{(i+1)\theta} \delta(s)\,\mathrm{d}s \right] + \ln\left(\prod_{j\theta \leqslant t_k < t} u_k \right) + \int_{j\theta}^{t} \delta(s)\,\mathrm{d}s \\
&\leqslant -ja + \max_{t \in [0,\theta]} \left\{ \left| \ln\left(\prod_{0 \leqslant t_k < t} u_k \right) \right| \right\} + \theta \max_{t \in [0,\theta]} \left\{ \left| \delta(t) \right| \right\} \\
&\leqslant -\frac{a}{\theta} t + a + \max_{t \in [0,\theta]} \left\{ \left| \ln\left(\prod_{0 \leqslant t_k < t} u_k \right) \right| \right\} + \theta \max_{t \in [0,\theta]} \left\{ \left| \delta(t) \right| \right\},
\end{aligned}$$

which means that $\ln \rho(t) \leqslant -\alpha t + \beta$, where $\alpha = a/\theta$ and

$$\beta = a + \max_{t\in[0,\theta]}\left\{\left|\ln\left(\prod_{0\leqslant t_k<t} u_k\right)\right|\right\} + \theta \max_{t\in[0,\theta]}\left\{\left|\delta(t)\right|\right\}. \qquad \square$$

Remark 1. From Lemma 3, if $\delta(t)$ is a periodic function with period θ and the impulsive effects t_k, u_k are periodic impulses with period θ, then $\rho(t)$ is a USF if and only if

$$\ln\left(\prod_{0\leqslant t_k<\theta} u_k\right) + \int_0^\theta \delta(s)\,\mathrm{d}s < 0.$$

Definition 4. For a uniformly stable function $\rho(t) = \prod_{0\leqslant t_k<t} u_k \exp(\int_0^t \delta(s)\,\mathrm{d}s)$, define the set by

$$\Lambda_\rho = \left\{r > 0\colon \sup_{t\geqslant 0}\left\{\ln\left(\prod_{t\leqslant t_k<t+r} u_k\right) + \int_t^{t+r} \delta(s)\,\mathrm{d}s\right\} < 0\right\}.$$

Moreover, for any $M > 0$, the overshoot of $\rho(t)$ is defined by

$$\psi_\rho(M) = \sup_{t\geqslant 0}\left\{\max_{\varsigma\in[0,M]}\left\{\ln\left(\prod_{t\leqslant t_k<t+\varsigma} d_k\right) + \int_t^{t+\varsigma} \delta(s)\,\mathrm{d}s\right\}\right\}.$$

Remark 2. For $M \geqslant 0$, if $\rho(t) = \prod_{0\leqslant t_k<t} u_k \exp(\int_0^t \delta(s)\,\mathrm{d}s)$ is a USF and $\psi_\rho(M)$ is the overshoot, we see that $\psi_\rho(M)$ is a nondecreasing function of M and $0 \leqslant \psi_\rho(M) \leqslant \beta$.

Remark 3. If $\delta(t)$, t_k, u_k are periodic with period θ and there exist an interval $[m, n) \subset [0, \theta]$ such that

$$\delta(t) \geqslant 0, \quad t \in [m, n), \qquad \prod_{m\leqslant t_k<n} u_k \geqslant 1$$

and

$$\delta(t) \leqslant 0, \quad t \in [0, m) \cup [n, \theta], \qquad \prod_{0\leqslant t_k<m} u_k \leqslant 1, \qquad \prod_{n\leqslant t_k<\theta} u_k \leqslant 1,$$

then

$$\psi_\rho(\theta) = \ln\left(\prod_{m\leqslant t_k<n} u_k\right) + \int_m^n \delta(s)\,\mathrm{d}s.$$

Lemma 4. (*See* [16].) *For $M > 0$ and a function $z(t) \in PC([-M, +\infty); \mathbb{R}^+)$, if there exist $c \in (0, 1)$ and $d > 0$ such that*

$$z(t) \leqslant c \sup_{t-M\leqslant\zeta\leqslant t}\left\{z(\zeta)\right\} + d, \quad t \geqslant 0,$$

then

$$z(t) \leqslant \sup_{-M\leqslant\zeta\leqslant 0}\left\{z(\zeta)\right\}\mathrm{e}^{\ln(c)t/M} + \frac{1}{(1-c)^2}d.$$

Lemma 5. *For $v(t) \in PC([-\tau, +\infty); \mathbb{R}^+)$, if there exist a function $\delta(t)$ and constants $c \in (0, 1)$, $u_k > 0$, $k = 1, 2, \dots$, such that*

(i) $D^+v(t) \leqslant \delta(t)v(t)$ *if* $v(t + \zeta) \leqslant \chi(v(t))$, $t \in (t_{k-1}, t_k]$, $\zeta \in [-\tau, 0]$;
(ii) $v(t_k^+) \leqslant u_k v(t_k)$, $k = 1, 2, \dots$;
(iii) $\rho(t) = \prod_{0 \leqslant t_k < t} u_k \exp(\int_0^t \delta(s)\,\mathrm{d}s)$ *is a USF*;
(iv) $\chi(cs/\exp(\psi_\rho(M))) > s$, $s \geqslant 0$,

then

$$v(t) \leqslant \sup_{-M^* \leqslant s \leqslant 0} \bigl[v(M + s)\bigr] \exp\left(\frac{\ln c^*}{M^*}(t - M)\right), \quad t \geqslant M,$$

where χ is a $\mathcal{K}_\infty$-function, and $M \in \Lambda_\rho$, $M^* = M + \tau$,

$$c^* = \max\left\{\prod_{t - M \leqslant t_k < t} u_k \exp\left(\int_{t-M}^{t} \delta(s)\,\mathrm{d}s\right), c\right\}.$$

Proof. If

$$\sup_{\zeta \in [-\tau, 0]} \bigl\{v(s + \zeta)\bigr\} \leqslant \chi\bigl(v(s)\bigr), \quad s \in [t - M, t],\ t \geqslant M,\ \zeta \in [-\tau, 0], \tag{4}$$

by (i) and (ii), we have

$$v(t) \leqslant v(t - M) \prod_{t - M \leqslant t_k < t} u_k \exp\left(\int_{t-M}^{t} \delta(s)\,\mathrm{d}s\right).$$

If inequality (4) is not true for some $s \in [t - M,\ t]$, set

$$\hat{t} = \sup\left\{s \in [t - M, t]\colon \sup_{\zeta \in [-\tau, 0]} \bigl\{v(s + \zeta)\bigr\} > \chi\bigl(v(s)\bigr)\right\}.$$

Then we have two cases to prove the conclusion: (a) $\hat{t} < t$; (b) $\hat{t} = t$.

Case (a). If $\hat{t}$ is not an impulsive time, we have $\sup_{\zeta \in [-\tau, 0]}\{v(s + \zeta)\} \leqslant \chi(v(s))$, $s \in [\hat{t}, t]$. From (i) it yields that

$$\begin{aligned} v(t) &\leqslant v(\hat{t}) \prod_{\hat{t} \leqslant t_k < t} u_k \exp\left(\int_{\hat{t}}^{t} \delta(s)\,\mathrm{d}s\right) \\ &\leqslant \chi^{-1}\Bigl(\sup_{s \in [t - M, t]} \Bigl\{\sup_{\zeta \in [-\tau, 0]} \bigl\{v(s + \zeta)\bigr\}\Bigr\}\Bigr) \exp\bigl(\psi_\rho(M)\bigr). \end{aligned}$$

If $\hat{t}$ is an impulsive time, there exists $i = 1, 2, \dots$ such that $t_i = \hat{t} < t$. By (i) and (ii), we have

$$v(t) \leqslant v(t_i^+) \prod_{t_i < t_k < t} u_k \exp\left(\int_{t_i}^{t} \delta(s)\,\mathrm{d}s\right) \leqslant u_i v(t_i) \prod_{t_i < t_k < t} u_k \exp\left(\int_{t_i}^{t} \delta(s)\,\mathrm{d}s\right)$$

$$
\begin{aligned}
&= v(\bar{t}) \prod_{t_i \leqslant t_k < t} u_k \exp\Biggl(\int\limits_{t_i}^{t} \delta(s)\,\mathrm{d}s\Biggr) \\
&\leqslant \chi^{-1}\Bigl(\sup_{s\in[t-M,t]}\Bigl\{\sup_{\zeta\in[-\tau,0]}\bigl\{v(s+\zeta)\bigr\}\Bigr\}\Bigr)\exp\bigl(\psi_\rho(M)\bigr).
\end{aligned}
$$

Case (b). We have

$$
\begin{aligned}
v(t) = v(\hat{t}) &\leqslant \chi^{-1}\Bigl(\sup_{s\in[t-M,t]}\Bigl\{\sup_{\zeta\in[-\tau,0]}\bigl\{v(s+\zeta)\bigr\}\Bigr\}\Bigr) \\
&= \chi^{-1}\Bigl(\sup_{s\in[t-M,t]}\Bigl\{\sup_{\zeta\in[-\tau,0]}\bigl\{v(s+\zeta)\bigr\}\Bigr\}\Bigr)\exp\bigl(\psi_\rho(M)\bigr).
\end{aligned}
$$

Hence, for $M > 0$ and $t \geqslant M$, we have

$$
\begin{aligned}
v(t) &\leqslant \max\Biggl\{v(t-M)\prod_{t-M\leqslant t_k<t} u_k \exp\Biggl(\int\limits_{t-M}^{t}\delta(s)\,\mathrm{d}s\Biggr), \\
&\qquad \chi^{-1}\Bigl(\sup_{s\in[t-M,t]}\Bigl\{\sup_{\zeta\in[-\tau,0]}\bigl\{v(s+\zeta)\bigr\}\Bigr\}\Bigr)\exp\bigl(\psi_\rho(M)\bigr)\Biggr\} \\
&\leqslant \max\Biggl\{v(t-M)\prod_{t-M\leqslant t_k<t} u_k \exp\Biggl(\int\limits_{t-M}^{t}\delta(s)\,\mathrm{d}s\Biggr), \\
&\qquad \chi^{-1}\Bigl(\sup_{-M^*\leqslant s\leqslant 0}\bigl\{v(t+s)\bigr\}\Bigr)\exp\bigl(\psi_\rho(M)\bigr)\Biggr\}.
\end{aligned}
$$

By (iv), we see that $cs \geqslant \chi^{-1}(s)\exp(\psi_\rho(M))$ and

$$
\begin{aligned}
v(t) &\leqslant \sup_{-M^*\leqslant s\leqslant 0}\bigl\{v(t+s)\bigr\}\max\Biggl\{\prod_{t-M\leqslant t_k<t} u_k \exp\Biggl(\int\limits_{t-M}^{t}\delta(s)\,\mathrm{d}s\Biggr), c\Biggr\} \\
&= c^* \sup_{-M^*\leqslant s\leqslant 0}\bigl\{v(t+s)\bigr\},
\end{aligned}
$$

where $c^* = \max\{\prod_{t-M\leqslant t_k<t}\eta_k \mathrm{e}^{\delta T}, c\} < 1$, $M^* = M + \tau$. Thus, it follows from Lemma 4 that

$$
v(t) \leqslant \sup_{-M^*\leqslant s\leqslant 0}\bigl\{v(M+s)\bigr\}\exp\biggl(\frac{\ln c^*}{M^*}(t-M)\biggr), \quad t \geqslant M. \qquad \square
$$

Remark 4. In Lemma 5, if $\delta(t)$ is a negative function, (i) is the condition of classical Razumikhin theorem. However, in view of impulsive effects, $\delta(t)$ may be nonnegative function. Actually, we may adjust the impulsive gain u_k, $k = 1, 2, \dots$, such that USF $\rho(t)$ is a USF and the overshoot $\Psi_\rho(M)$ satisfies (iv). Moreover, if the bound β and decay index α of USF $\rho(t)$ can be determined, we need to choose M such that $M > \beta/\alpha$.

Remark 5. If $\chi(s) = ds$, where $d > 0$, then condition (iv) of Lemma 5 can be replaced by $dc \geqslant \exp(\psi_\rho(M))$. Moreover, the condition can also be replaced by the following condition:

$$d > \exp\bigl(\psi_\rho(M)\bigr).$$

3 Impulsive stability theorems

In this section, by the above theory, some sufficient conditions of Razumikhin stability theorem and Krasovskii stability are obtained for impulsive stochastic delay systems via USF.

3.1 The Razumikhin stability theorem

Theorem 1. *Let $V(t,x)$ be $\mathcal{C}_1^2$ function. If there exist $\mathcal{K}_\infty$-functions χ_1, χ_2, χ and function $\delta(t)$ and constants $c \in (0,1)$, $u_k > 0$, $k = 1,2,\dots$, such that*

- (i) $\chi_1(\mathbf{E}[|x|^p]) \leqslant \mathbf{E}[V(t,x)] \leqslant \chi_2(\mathbf{E}[|x|^p])$ *for* $(t,x) \in [-\tau,+\infty) \times \mathbb{R}^n$;
- (ii) $\mathbf{E}[\mathcal{L}V(t,x(t))] \leqslant \delta(t)\mathbf{E}[V(t,x(t))]$ *if* $\mathbf{E}[V(t+\zeta,x(t+\zeta))] \leqslant \chi(\mathbf{E}[V(t,x(t))])$, $t \in (t_{k-1},t_k]$, $\zeta \in [-\tau,0]$;
- (iii) $V(t_k^+, h_k(t_k,x(t_k))) \leqslant u_k V(t_k,x(t_k))$, $k = 1,2,\dots$;
- (iv) $\rho(t) = \prod_{0\leqslant t_k<t} u_k \exp(\int_0^t \delta(s)\,\mathrm{d}s)$ *is a USF*;
- (v) *if* $\chi(cs/\exp(\psi_\rho(M))) > s$, $s \geqslant 0$, *where* $M \in \varLambda_\rho$, *then system* (1) *is pMAS.*

Furthermore, if there exist $d_1 > 0$, $d_2 > 0$, $l > 0$ such that $\chi_1(s) = d_1 s^l$, $\chi_2(s) = d_2 s^l$, system (1) *is pMES.*

Proof. For $t \in (t_{k-1},t_k]$, based on Itô formula, we have

$$\mathrm{d}V\bigl(t,x(t)\bigr) = \mathcal{L}V\bigl(t,x(t)\bigr)\,\mathrm{d}t + V_x\bigl(t,x(t)\bigr)g\bigl(t,x(t),x_t\bigr)\,\mathrm{d}\omega(t).$$

Let ϵ be small enough such that $t+\epsilon \in (t_{k-1},t_k)$. We have

$$\mathbf{E}\bigl[V\bigl(t+\epsilon,\, x(t+\epsilon)\bigr)\bigr] - \mathbf{E}\bigl[V\bigl(t,x(t)\bigr)\bigr] = \int_t^{t+\epsilon} \mathbf{E}\bigl[\mathcal{L}V\bigl(s,x(s)\bigr)\bigr]\,\mathrm{d}s.$$

Let $\epsilon \to 0$, we have

$$D^+\mathbf{E}\bigl[V\bigl(t,x(t)\bigr)\bigr] = \mathbf{E}\bigl[\mathcal{L}V\bigl(t,x(t)\bigr)\bigr], \quad t \in (t_{k-1},t_k].$$

Then if $\mathbf{E}[V(t+\zeta,\, x(t+\zeta))] \leqslant \chi(\mathbf{E}[V(t,x(t))])$, it yields that

$$D^+\mathbf{E}\bigl[V\bigl(t,x(t)\bigr)\bigr] \leqslant \delta(t)\mathbf{E}\bigl[V\bigl(t,x(t)\bigr)\bigr], \quad t \in (t_{k-1},t_k].$$

If for $s \in [t-M,\, t]$, $t \geqslant M$, $\zeta \in [-\tau,0]$,

$$\sup_{\zeta\in[-\tau,0]} \bigl\{\mathbf{E}\bigl[V\bigl(s+\zeta,\, x(s+\zeta)\bigr)\bigr]\bigr\} \leqslant \chi\bigl(\mathbf{E}\bigl[V\bigl(s,x(s)\bigr)\bigr]\bigr),$$

by the proof of Lemma 5, for $M > 0$, we have

$$\begin{aligned}
&\mathbf{E}\big[V\big(t, x(t)\big)\big] \\
&\quad \leqslant \max\Biggl\{\mathbf{E}\big[V\big(t-M,\, x(t-M)\big)\big] \prod_{t-M\leqslant t_k<t} u_k \exp\Biggl(\int\limits_{t-M}^{t} \delta(s)\,\mathrm{d}s\Biggr), \\
&\qquad \chi^{-1}\Bigl(\sup_{-M^*\leqslant s\leqslant 0}\bigl\{\mathbf{E}\big[V\big(t+s,\, x(t+s)\big)\big]\bigr\}\exp\bigl(\psi_\rho(M)\bigr)\Biggr\},
\end{aligned}$$

where $M^* = M + \tau$. In view of (v), the above inequality can be concluded as

$$\begin{aligned}
\mathbf{E}\big[V\big(t, x(t)\big)\big] &\leqslant \mathbf{E}\big[V\big(t+s,\, x(t+s)\big)\big] \max\Biggl\{\prod_{t-U\leqslant t_k<t} u_k \exp\Biggl(\int\limits_{t-U}^{t} \delta(s)\,\mathrm{d}s\Biggr),\, c\Biggr\} \\
&= c^*\mathbf{E}\big[V\big(t+s,\, x(t+s)\big)\big],
\end{aligned}$$

where $c^* = \max\{\prod_{t-M\leqslant t_k<t} u_k \exp(\int_{t-M}^{t} \delta(s)\,\mathrm{d}s), c\} < 1$. Therefore, by Lemma 4 and (i), we have

$$\begin{aligned}
\mathbf{E}\big[V\big(t, x(t)\big)\big] &\leqslant \sup_{-M^*\leqslant s\leqslant 0}\bigl\{\mathbf{E}\big[V\big(M+s,\, x(M+s)\big)\big]\bigr\}\exp\biggl(\frac{\ln c^*}{M^*}(t-M)\biggr) \\
&\leqslant \mathbf{E}\big[\chi_2\big(\|x_M\|\big)\big]\exp\biggl(\frac{\ln c^*}{M^*}(t-M)\biggr), \quad t \geqslant M.
\end{aligned}$$

By (i), it yields that

$$\mathbf{E}\big[\big|x(t)\big|^p\big] \leqslant \chi_1^{-1}\biggl(\mathbf{E}\big[\chi_2\big(\|x_M\|\big)\big]\exp\biggl(\frac{\ln c^*}{M^*}(t-M)\biggr)\biggr), \quad t \geqslant M.$$

Therefore, system (1) is pth moment asymptotically stable.

Furthermore, if $\chi_1(s) = d_1 s^l$, $\chi_2(s) = d_2 s^l$, we can obtain that

$$\mathbf{E}\big[\big|x(t)\big|^p\big] \leqslant \frac{1}{d_1^{1/l}}\mathbf{E}\big[\|x_M\|\big] d_2^{1/l}\exp\biggl(\frac{\ln c^*}{M^*}(t-M)\biggr), \quad t \geqslant M,$$

which yields that system (1) is pMES. □

3.2 The Krasovskii stability theorem

Theorem 2. *Let $V(t,\phi)$ be $\mathcal{C}_1^2$ function, where $\phi \in PC([-\tau,0],\mathbb{R}^n)$. If there exist $\mathcal{K}_\infty$-functions χ_1, χ_2, χ and function $\delta(t)$ and constants $c \in (0,1)$, $u_k > 0$, $k = 1,2,\dots,$ such that*

(i) $\chi_1(\mathbf{E}[|\phi(0)|^p]) \leqslant \mathbf{E}[V(t,\phi)] \leqslant \chi_2(\mathbf{E}[\|\phi\|^p])$ *for* $t \geqslant 0$, $\phi \in PC([-\tau,0],\mathbb{R}^n)$;
(ii) $\mathcal{L}V(t,x_t) \leqslant \delta(t)V(t,x_t)$, $t \in (t_{k-1},t_k]$;

(iii) $V(t_k^+, h_k(t_k, x(t_k))) \leqslant u_k V(t_k, x(t_k))$, $k = 1, 2, \dots$;
(iv) $\rho(t) = \prod_{0 \leqslant t_k < t} u_k \exp(\int_0^t \delta(s)\, \mathrm{d}s)$ *is a USF*;
(v) *if* $\chi(cs/\exp(\psi_\theta(U))) > s$, $s \geqslant 0$, *where* $U \in \Lambda_\rho$, *then system* (1) *is pMAS.*

Furthermore, if there exist $d_1 > 0$, $d_2 > 0$, $l > 0$ *such that* $\chi_1(s) = d_1 s^l$, $\chi_2(s) = d_2 s^l$, *system* (1) *is pMES.*

Proof. Let $V(t) = V(t, x_t)$. By the proof of Theorem 1 and Itô formula, we have

$$D^+\mathbf{E}\bigl[V(t)\bigr] \leqslant \delta(t)\mathbf{E}\bigl[V(t)\bigr], \quad t \in (t_{k-1}, t_k]$$

and

$$V\bigl(t_k^+\bigr) \leqslant u_k V(t_k).$$

Thus, by (iv), there exist $\alpha > 0$ and $\beta \geqslant 0$ such that

$$\ln \frac{\mathbf{E}[V(t)]}{\mathbf{E}[V(0)]} = \int_0^t \mathrm{d} \ln \mathbf{E}\bigl[V(s)\bigr] = \int_0^t \frac{D^+\mathbf{E}[V(s)]}{\mathbf{E}[V(s)]\, \mathrm{d}s} \leqslant \ln \rho(t) \leqslant -\alpha t + \beta, \quad t \geqslant 0.$$

Furthermore, from (i), it yields that

$$\mathbf{E}\bigl[V(t, x_t)\bigr] \leqslant \exp(-\alpha t + \beta)\mathbf{E}\bigl[V(0, x_0)\bigr] \leqslant \exp(-\alpha t + \beta)\chi_2\bigl(\mathbf{E}\bigl[\|x_0\|^p\bigr]\bigr).$$

Hence, we can get

$$\mathbf{E}\bigl[\bigl|x(t)\bigr|^p\bigr] \leqslant \chi_1^{-1}\bigl(\mathbf{E}\bigl[V(t, x_t)\bigr]\bigr) \leqslant \chi_1^{-1}\bigl(\exp(-\alpha t + \beta)\chi_2\bigl(\mathbf{E}\bigl[\|x_0\|^p\bigr]\bigr)\bigr),$$

which means that system (1) is pth moment asymptotically stable. □

4 The stability of impulsive stochastic linear system with delay

Consider the following impulsive stochastic delay system:

$$\begin{aligned} \mathrm{d}x(t) &= \bigl[A(t)x(t) + B(t)x\bigl(t - \tau(t)\bigr)\bigr]\, \mathrm{d}t \\ &\quad + \bigl[C(t)x(t) + D(t)x\bigl(t - \tau(t)\bigr)\bigr]\, \mathrm{d}\omega(t), \quad t \geqslant 0,\ t \neq t_k, \\ x\bigl(t_k^+\bigr) &= u_k x(t_k), \quad k = 1, 2, \dots, \end{aligned} \tag{5}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$, $A(t), B(t), C(t), D(t) \in PC([0, +\infty), \mathbb{R}^{n\times n})$, $0 < \tau(t) \leqslant \tau$, $\tau'(t) \leqslant \sigma < 1$, t_k, $k = 0, 1, \dots$, are impulsive moments satisfying $0 = t_0 < t_1 < \dots < t_k < t_{k+1} < \cdots$, $\lim_{k\to\infty} t_k = +\infty$, $\sup_{k\geqslant 1}\{\Delta_k\} < +\infty$, where $\Delta_k = t_k - t_{k-1}$.

4.1 Stability theorem via the Razumikhin method

Theorem 3. *System* (5) *is pMES if there exist function* $\alpha_1(t)$ *and constant* q_1 *such that*

(i) $pA(t) + (p-1)|B(t)|I_n + p|C(t)|^2 I_n + (p-2)|D(t)|^2 I_n \leqslant \alpha_1(t) I_n$;

(ii) $\rho_1(t) = \prod_{0 \leqslant t_k < t} |u_k|^p \exp(\int_0^t \delta_1(s)\, \mathrm{d}s)$ *is a USF*;

(iii) $q_1 > \exp(\psi_{\rho_1}(M))$, *where* $\delta_1(t) = \alpha_1(t) + q_1(|B(t)| + |D(t)|^2)$, $M \in \Lambda_{\rho_1}$.

Proof. Let $V(t, x(t)) = |x(t)|^p$. For $t \in (t_{k-1}, t_k]$, we have

$$\begin{aligned}
&\mathcal{L}V\big(t, x(t)\big) \\
&\quad = V_t\big(t, x(t)\big) + V_x\big(t, x(t)\big)\big[A(t)x(t) + B(t)x\big(t - \tau(t)\big)\big] \\
&\qquad + \frac{1}{2}\,\mathrm{trace}\big\{\big[C(t)x(t) + D(t)x\big(t-\tau(t)\big)\big]^{\mathrm{T}}\big[C(t)x(t) + D(t)x\big(t-\tau(t)\big)\big]\big\} \\
&\quad = p\big|x(t)\big|^{p-2} x^{\mathrm{T}}(t)\big[A(t)x(t) + B(t)x\big(t-\tau(t)\big)\big] \\
&\qquad + \frac{p}{2}\big|x(t)\big|^{p-2}\,\mathrm{trace}\big\{\big[C(t)x(t) + D(t)x\big(t-\tau(t)\big)\big]^{\mathrm{T}} \\
&\qquad \times \big[C(t)x(t) + D(t)x\big(t-\tau(t)\big)\big]\big\} \\
&\quad \leqslant p\big|x(t)\big|^{p-2} x^{\mathrm{T}}(t)A(t)x(t) + p\big|B(t)\big|\big|x(t)\big|^{p-1}\big|x\big(t-\tau(t)\big)\big| \\
&\qquad + p\big|C(t)\big|^2\big|x(t)\big|^p + p\big|D(t)\big|^2\big|x(t)\big|^{p-2}\big|t-\tau(t)\big|^2.
\end{aligned}$$

In view of inequality

$$xy \leqslant \frac{x^m}{m} + \frac{y^n}{n}, \quad x, y \geqslant 0,\ m, n > 1,\ \frac{1}{m} + \frac{1}{n} = 1,$$

it yields that

$$\big|x(t)\big|^{p-1}\big|x\big(t-\tau(t)\big)\big| \leqslant \frac{p-1}{p}\big|x(t)\big|^p + \frac{1}{p}\big|x\big(t-\tau(t)\big)\big|^p \tag{6}$$

and

$$\big|x(t)\big|^{p-2}\big|x\big(t-\tau(t)\big)\big|^2 \leqslant \frac{p-2}{p}\big|x(t)\big|^p + \frac{2}{p}\big|x\big(t-\tau(t)\big)\big|^p. \tag{7}$$

Submitting (6) and (7) into (5), for $t \in (t_{k-1}, t_k]$, we have

$$\begin{aligned}
&\mathcal{L}V\big(t, x(t)\big) \\
&\quad \leqslant p\big|x(t)\big|^{p-2}x^{\mathrm{T}}(t)A(t)x(t) + (p-1)\big|B(t)\big|\big|x(t)\big|^p + p\big|C(t)\big|^2\big|x(t)\big|^p \\
&\qquad + (p-2)\big|D(t)\big|^2\big|x(t)\big|^p + \big|B(t)\big|\big|x\big(t-\tau(t)\big)\big|^p + \big|D(t)\big|^2\big|x\big(t-\tau(t)\big)\big|^p \\
&\quad \leqslant \alpha_1(t)\big|x(t)\big|^p + \big(\big|B(t)\big| + \big|D(t)\big|^2\big)\big|x\big(t-\tau(t)\big)\big|^p.
\end{aligned}$$

If $\mathbf{E}[V(t+\varsigma,\ x(t+\varsigma))] \leqslant q_1\mathbf{E}[V(t, x(t))]$, $\varsigma \in [-\tau, 0]$, it follows that for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned}
\mathbf{E}\big[\mathcal{L}V\big(t, x(t)\big)\big] &\leqslant \big[\alpha_1(t) + q_1\big(\big|B(t)\big| + v|D(t)\big|^2\big)\big]\mathbf{E}\big[\big|x(t)\big|^p\big] \\
&= \delta_1(t)\mathbf{E}\big[V\big(t, x(t)\big)\big].
\end{aligned}$$

For $t = t_k$, we have

$$V\big(t_k^+, x\big(t_k^+\big)\big) = \big|x\big(t_k^+\big)\big|^p = |u_k|^p V\big(t_k, x(t_k)\big).$$

Since $\rho_1(t) = \prod_{t_0 \leqslant t_k < t} |u_k|^p \exp(\int_0^t \delta_1(s)\,\mathrm{d}s)$ is a uniformly stable function and all the conditions in Theorem 1 hold, system (5) is pMES. □

Theorem 4. *System* (5) *is exponentially stable in the mean square if there exist* $P(t) = P^{\mathrm{T}}(t) \in C^1([0, +\infty), \mathbb{R}^{n\times n})$*, constants* $d_1 > 0$*,* $d_2 > 0$*,* $d_3 > 0$*,* $q_2 > 0$ *and function* $\alpha_2(t)$ *such that*

(i) $d_1 I_n \leqslant P(t) \leqslant d_2 I_n$;
(ii) $A^{\mathrm{T}}(t)P(t) + P(t)A(t) + \dot{P}(t) + 2d_2|C(t)|^2 I_n + d_3 P^2(t) \leqslant \alpha_2(t) I_n$;
(iii) $\rho_2(t) = \prod_{0 \leqslant t_k < t} u_k^2 \exp(\int_0^t \delta_2(s)\,\mathrm{d}s)$ *is a USF*;
(iv) $q_2 > \exp(\psi_{\rho_2}(M))$*, where* $\delta_2(t) = \alpha_2(t) + q_2(|B(t)|^2/(d_3 d_1) + 2d_2|D(t)|^2/d_1)$*,* $M \in \Lambda_{\rho_2}$.

Proof. Choosing the Lyapunov function $V(t, x(t)) = x^{\mathrm{T}}(t)P(t)x(t)$, for $t \in (t_{k-1}, t_k]$, we can obtain

$$\begin{aligned}
&\mathcal{L}V\big(t, x(t)\big) \\
&\quad = x^{\mathrm{T}}(t)\dot{P}(t)x(t) + 2x^{\mathrm{T}}(t)P(t)\big[A(t)x(t) + B(t)x\big(t-\tau(t)\big)\big] \\
&\qquad + \operatorname{trace}\big\{\big[C(t)x(t) + D(t)x\big(t-\tau(t)\big)\big]^{\mathrm{T}} P(t)\big[C(t)x(t) + D(t)x\big(t-\tau(t)\big)\big]\big\} \\
&\quad \leqslant x^{\mathrm{T}}(t)\big[A^{\mathrm{T}}(t)P(t) + P(t)A(t) + \dot{P}(t) + 2d_2\big|C(t)\big|^2 I_n\big]x(t) \\
&\qquad + 2x^{\mathrm{T}}(t)P(t)B(t)x\big(t-\tau(t)\big)\big] + 2d_2\big|D(t)\big|^2 x^{\mathrm{T}}\big(t-\tau(t)\big)x\big(t-\tau(t)\big).
\end{aligned}$$

There exists $d_3 > 0$ such that

$$\begin{aligned}
&2x^{\mathrm{T}}(t)P(t)B(t)x\big(t-\tau(t)\big) \\
&\quad \leqslant d_3 x^{\mathrm{T}}(t)P^2(t)x(t) + d_3^{-1}x^{\mathrm{T}}\big(t-\tau(t)\big)B^{\mathrm{T}}(t)B(t)x\big(t-\tau(t)\big).
\end{aligned}$$

Together with (i), it yields that

$$\begin{aligned}
&\mathcal{L}V\big(t, x(t)\big) \\
&\quad \leqslant \alpha_2(t)x^{\mathrm{T}}(t)P(t)x(t) \\
&\qquad + \left(\frac{1}{d_1 d_3}|B(t)|^2 + \frac{2d_2}{d_1}|D(t)|^2\right)x^{\mathrm{T}}\big(t-\tau(t)\big)P(t)x\big(t-\tau(t)\big) \\
&\quad = \alpha_2 V\big(t, x(t)\big) + \left(\frac{1}{d_1 d_3}|B(t)|^2 + \frac{2d_2}{d_1}|D(t)|^2\right)V\big(t-\tau(t),\, x\big(t-\tau(t)\big)\big).
\end{aligned}$$

If $\mathbf{E}[V(t+\varsigma,\, x(t+\varsigma))] \leqslant q_2\mathbf{E}[V(t, x(t))]$, $\varsigma \in [-\tau, 0]$, then we can see that for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned}
\mathbf{E}\big[\mathcal{L}V\big(t, x(t)\big)\big] &\leqslant \left[\alpha_2(t) + q_2\left(\frac{1}{d_1 d_3}\big|B(t)\big|^2 + \frac{2d_2}{d_1}\big|D(t)\big|^2\right)\right]\mathbf{E}\big[V\big(t, x(t)\big)\big] \\
&= \delta_2(t)\mathbf{E}\big[V\big(t, x(t)\big)\big].
\end{aligned}$$

For $t = t_k$,

$$V\big(t_k^+, x\big(t_k^+\big)\big) = x^{\mathrm{T}}\big(t_k^+\big)P\big(t_k^+\big)x\big(t_k^+\big) = u_k^2 x^{\mathrm{T}}(t_k)P(t_k)x(t_k) = u_k^2 V\big(t_k, x(t_k)\big).$$

Then it follows from Theorem 1 that the result is true. □

4.2 Stability theorem via the Krasovskii method

Theorem 5. *System* (5) *is pMES if there exist function* $\alpha_1(t)$ *and constant* $l > 0$ *such that*

(i) $pA(t) + (p-1)|B(t)|I_n + p|C(t)|^2 I_n + (p-2)|D(t)|^2 I_n \leqslant \alpha_1(t)I_n$;
(ii) $t_k - t_{k-1} \geqslant l$;
(iii) $\rho_3(t) = \prod_{0\leqslant t_k<t}[|u_k|^p + (\int_{t_{k-1}}^{t_k} |B(s)| + |D(s)|^2\, \mathrm{d}s)/(1-\sigma)] \exp(\int_0^t \delta_3(s)\, \mathrm{d}s)$ *is a USF, where* $\delta_3(t) = \max\{\alpha_1(t) + (|B(t)| + |D(t)|^2)/(1-\sigma),\ 0\}$.

Proof. Consider a Lyapunov–Krasovskii functional

$$V\big(t, x_t\big) = V_1(t) + V_2(t), \tag{8}$$

where $V_1(t) = |x(t)|^p$, $V_2(t) = \int_{t-\tau(t)}^{t}(|B(s)| + |D(s)|^2/(1-\sigma))|x(s)|^p\, \mathrm{d}s$. For $t \in (t_{k-1}, t_k]$, we have

$$\mathcal{L}V_1(t) \leqslant \alpha_1(t)V_1(t) + \big(\big|B(t)\big| + \big|D(t)\big|^2\big)V_1\big(t-\tau(t)\big).$$

For $t \in (t_{k-1}, t_k]$, we get

$$\mathcal{L}V_2(t) \leqslant \frac{|B(t)| + |D(t)|^2}{1-\sigma}V_1(t) - \big(\big|B(t)\big| + \big|D(t)\big|^2\big)V_1\big(t-\tau(t)\big).$$

Then

$$\mathbf{E}\big[\mathcal{L}V(t)\big] \leqslant \left(\alpha_1(t) + \frac{|B(t)| + |D(t)|^2}{1-\sigma}\right)\mathbf{E}\big[V_1(t)\big] \leqslant \delta_3(t)\mathbf{E}\big[V(t)\big].$$

Thus, for $t \in (t_{k-1}, t_k]$,

$$\mathbf{E}\big[V(t)\big] \leqslant \mathbf{E}\big[V\big(t_{k-1}^+\big)\big] \exp\left(\int_{t_{k-1}}^{t} \delta_3(s)\, \mathrm{d}s\right). \tag{9}$$

For $t = t_k$, by (ii), we have

$$\begin{aligned}\mathbf{E}\big[V_1\big(t_k^+\big)\big] &\leqslant |u_k|^p\mathbf{E}\big[V_1(t_k)\big] \leqslant |u_k|^p\mathbf{E}\big[V(t_k)\big] \\ &\leqslant |u_k|^p \exp\left(\int_{t_{k-1}}^{t_k} \delta_3(s)\, \mathrm{d}s\right)\mathbf{E}\big[V\big(t_{k-1}^+\big)\big],\end{aligned} \tag{10}$$

and there exists a $\bar{t}_k \in (t_{k-1}, t_k)$ such that

$$V_2(t_k^+) = \int_{t_k-\tau(t_k)}^{t_k} \frac{|B(s)| + |D(s)|^2}{1-\sigma} V_1(s)\,\mathrm{d}s \leqslant \int_{t_{k-1}}^{t_k} \frac{|B(s)| + |D(s)|^2}{1-\sigma} V_1(s)\,\mathrm{d}s$$
$$= \frac{1}{1-\sigma} V_1(\bar{t}_k) \int_{t_{k-1}}^{t_k} \left(\left|B(s)\right| + \left|D(s)\right|^2\right) \mathrm{d}s. \tag{11}$$

It follows from (9) and the above inequality that

$$\begin{aligned}
\mathbf{E}\left[V_2\left(t_k^+\right)\right] &\leqslant \frac{1}{1-\sigma} \int_{t_{k-1}}^{t_k} \left(\left|B(s)\right| + \left|D(s)\right|^2\right) \mathrm{d}s\, \mathbf{E}\left[V_1(\bar{t}_k)\right] \\
&\leqslant \frac{1}{1-\sigma} \int_{t_{k-1}}^{t_k} \left(\left|B(s)\right| + \left|D(s)\right|^2\right) \mathrm{d}s\, \mathbf{E}\left[V(\bar{t}_k)\right] \\
&\leqslant \frac{1}{1-\sigma} \int_{t_{k-1}}^{t_k} \left(\left|B(s)\right| + \left|D(s)\right|^2\right) \mathrm{d}s \exp\left(\int_{t_{k-1}}^{t_k} \delta_3(s)\,\mathrm{d}s\right) \mathbf{E}\left[V\left(t_{k-1}^+\right)\right].
\end{aligned} \tag{12}$$

Submitting (10) and (12) into (8), we have

$$\begin{aligned}
\mathbf{E}\left[V\left(t_k^+\right)\right] &= \mathbf{E}\left[V_1\left(t_k^+\right)\right] + \mathbf{E}\left[V_2\left(t_k^+\right)\right] \leqslant |u_k|^p \exp\left(\int_{t_{k-1}}^{t_k} \delta_3(s)\,\mathrm{d}s\right) \mathbf{E}\left[V\left(t_{k-1}^+\right)\right] \\
&\quad + \frac{1}{1-\sigma} \int_{t_{k-1}}^{t_k} \left(\left|B(s)\right| + \left|D(s)\right|^2\right) \mathrm{d}s \exp\left(\int_{t_{k-1}}^{t_k} \delta_3(s)\,\mathrm{d}s\right) \mathbf{E}\left[V\left(t_{k-1}^+\right)\right] \\
&\leqslant \left(|u_k|^p + \frac{1}{1-\sigma} \int_{t_{k-1}}^{t_k} \left|B(s)\right| + \left|D(s)\right|^2 \mathrm{d}s\right) \exp\left(\int_{t_{k-1}}^{t_k} \delta_3(s)\,\mathrm{d}s\right) \\
&\quad \times \mathbf{E}\left[V\left(t_{k-1}^+\right)\right].
\end{aligned} \tag{13}$$

Therefore, it yields that

$$\begin{aligned}
\mathbf{E}\left[V\left(t_k^+\right)\right] &\leqslant \prod_{0 \leqslant t_i \leqslant t_k} \left(|u_i|^p + \frac{1}{1-\sigma} \int_{t_{i-1}}^{t_i} \left|B(s)\right| + \left|D(s)\right|^2 \mathrm{d}s\right) \exp\left(\int_0^{t_k} \delta_3(s)\,\mathrm{d}s\right) \\
&\quad \times \mathbf{E}\Big[\sup_{-\tau \leqslant \varsigma \leqslant 0} V(\varsigma)\Big].
\end{aligned} \tag{14}$$

For $t \in (t_k, t_{k+1}]$, by (9)–(14), we see that

$$\mathbf{E}\big[V(t)\big] \leqslant \prod_{0\leqslant t_i\leqslant t}\left(|u_i|^p + \frac{1}{1-\sigma}\int\limits_{t_{i-1}}^{t_i} \big|B(s)\big| + \big|D(s)\big|^2\,\mathrm{d}s\right)\exp\left(\int\limits_0^t \delta_3(s)\,\mathrm{d}s\right) \times \mathbf{E}\Big[\sup_{-\tau\leqslant\varsigma\leqslant 0} V(\varsigma)\Big].$$

In view of (iii), we see that there exist constants $\vartheta_1 > 0$ and $\vartheta_2 > 0$ such that

$$\prod_{0\leqslant t_i\leqslant t}\left(|u_i|^p + \frac{1}{1-\sigma}\int\limits_{t_{i-1}}^{t_i} \big|B(s)\big| + \big|D(s)\big|^2\,\mathrm{d}s\right)\exp\left(\int\limits_0^t \delta_3(s)\,\mathrm{d}s\right) \leqslant \vartheta_2\mathrm{e}^{-\vartheta_1 t}$$

for $t \geqslant 0$, which implies that

$$\mathbf{E}\big[\big|x(t)\big|^p\big] \leqslant \mathbf{E}\big[V(t)\big] \leqslant \vartheta_2\mathrm{e}^{-\vartheta_1 t}\mathbf{E}\big[\|\xi\|^p\big], \quad t \geqslant 0. \qquad \square$$

Remark 6. In recent years, some stability results of divergent systems with impulsive effects have some strong restriction on impulsive gains. For instance, in [6, 10, 12, 20, 24, 25], all of impulsive gains were required to be stabilizing impulses. However, in our obtained theorems, it may admit a certain number of destabilizing impulsive gains.

Remark 7. In the convergent behavior of some divergent systems, most impulsive controllers allow the finite number of destabilizing impulses [4, 12]. However, the conditions of the obtained theorems can permit the infinite number of destabilizing impulses. By utilizing USF, we reduce these conservative.

5 Numerical simulations

This section provides two numerical examples to test the theoretic analysis.

Example 1. Consider the following system:

$$\begin{aligned}\mathrm{d}x(t) &= \left[-x(t) + B(t)x\left(t-\frac{1}{2}\right)\right]\mathrm{d}t \\ &\quad + \left[\frac{1}{2}x(t) + D(t)x\left(t-\frac{1}{2}\right)\right]\mathrm{d}\omega(t), \quad t \geqslant 0,\ t \neq t_k, \\ x\big(t_k^+\big) &= u_k x(t_k), \quad k = 1, 2, \dots,\end{aligned} \tag{15}$$

where $B(t)$, $D(t)$ are periodic function with period $\theta = 1$, and

$$B(t) = \begin{cases} 0, & t \in [0, h), \\ b, & t \in [h, 1], \end{cases} \qquad D(t) = \begin{cases} 0, & t \in [0, h), \\ d, & t \in [h, 1]. \end{cases}$$

The impulsive effects with period 1 are defined by $t_{k+2} = t_k + 1$, $t_1 = 0.2$, $u_{2k} = 0.4$, $u_{2k+1} = 1.2$, $k = 0, 1, \dots$. By Theorem 4, system (15) is exponentially stable in the mean square if b, d and h satisfy one of the following conditions:

(I) If $h \leqslant 0.2$, $\exp(-\ln 0.2304 + 0.5h) < (-\ln 0.2304 + 0.5h)/(1.44(1-h) \times (b^2 + 2d^2))$;

(II) If $h > 0.2$, $\exp(-\ln 0.2304 + 0.5h) < (-\ln 0.2304 + 0.5h)/((1-h)(b^2 + 2d^2))$.

Next, we verify that the conditions of Theorem 4 hold under (I) or (II). Choosing $P(t) = 1$, $d_1 = d_2 = d_3$, we see that conditions (i) and (ii) are satisfied with $\alpha_2(t) = -1/2$. If (I) holds, there exists $q_2 > 0$ such that

$$\ln 0.2304 - 0.5 + q_2(1-h)\bigl(b^2 + 2d^2\bigr) < 0$$

and

$$q_2 > \exp\bigl(\ln 1.44 - 0.5 + q_2(1-h)\bigl(b^2 + 2d^2\bigr) + 0.5h\bigr).$$

By computation, we have

$$\delta_2(t) = -\frac{1}{2} + q_2\bigl(\bigl|B(t)\bigr|^2 + 2\bigl|D(t)\bigr|^2\bigr)$$

and

$$\ln\biggl(\prod_{0\leqslant t_k<1} u_k\biggr) + \int_0^1 \delta_2(t)\,\mathrm{d}t = \ln 0.24 - \frac{1}{2} + q_2(1-h)\bigl(b^2 + 2d^2\bigr) < 0.$$

It follows from Lemma 3 that $\rho_2(t) = \prod_{0\leqslant t_k<t} u_k^2 \exp(\int_0^t \delta_2(s)\,\mathrm{d}s)$ is a uniformly function. Furthermore, we can obtain that

$$\begin{aligned}\psi_{\rho_2}(\theta) &= \ln\biggl(\prod_{h\leqslant t_k<1} u_k^2\biggr) + \int_h^1 \delta_2(t)\,\mathrm{d}t\\ &= \ln 1.44 - 0.5 + q_2(1-h)\bigl(b^2 + 2d^2\bigr) + 0.5h,\end{aligned}$$

which yields that $q_2 > \exp(\psi_{\rho_2}(\theta))$. By Theorem 4, system (15) is exponentially stable in the mean square. Figure 1 depicts impulsive sequence with period 1. Figure 2 depicts state trajectory $x(t)$ of system (15) with $b = 1.2$, $d = 0.5$ and $h = 0.1$.

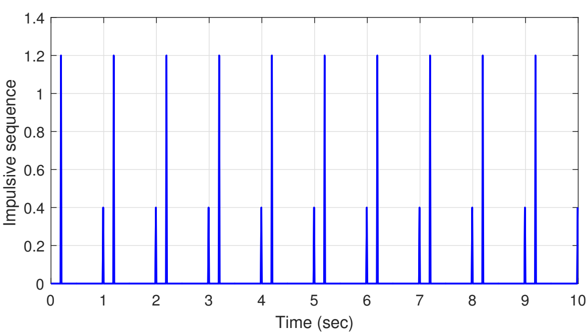

Figure 1. Impulsive sequence with period 1.

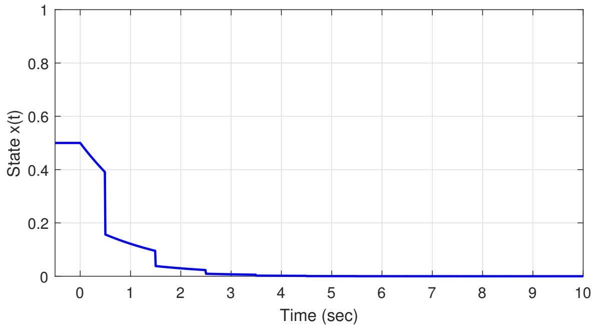

Figure 2. The state trajectory $x(t)$ with $b = 1.2$, $d = 0.5$, $h = 0.1$ and initial value 0.5.

Example 2. Consider 2-dimensional system

$$\begin{aligned} \mathrm{d}x(t) &= \left[Ax(t) + Bx(t-1)\right]\mathrm{d}t \\ &\quad + \left[Cx(t) + Dx(t-1)\right]\mathrm{d}\omega(t), \quad t \geqslant 0,\ t \neq t_k, \\ x\left(t_k^+\right) &= u_k x(t_k), \quad k = 1, 2, \dots, \end{aligned} \tag{16}$$

where $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$, $\tau(t) = 1$, impulsive moments t_k and impulsive gains satisfying $t_{k+2} = t_k + 0.2$, $t_1 = 0.1$, $u_{2k} = 1.2$, $u_{2k-1} = 0.1$, $k = 1, 2, \dots$, coefficient matrices are defined as follows:

$$A = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0.5 \\ -1 & 0.5 \end{pmatrix},$$
$$C = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}.$$

We can verify that conditions (i) and (ii) of Theorem 5 are satisfied with $\alpha_1(t) = 0.55$, $\tau = 0.2$. By computation, we see that $\delta_3(t) = 3.05$. Since the impulsive effects have a period 0.2, it follows that

$$\begin{aligned} &\ln\left(\prod_{0 \leqslant t_k < 0.2}\left[|u_k|^p + \frac{1}{1-\sigma}\left(\int_{t_{k-1}}^{t_k} \left|B(s)\right| + \left|D(s)\right|^2 \mathrm{d}s\right)\right]\right) + \int_0^{0.2} \delta_3(s)\,\mathrm{d}s \\ &\quad = \ln 0.378125 + 0.305 < 0. \end{aligned}$$

In view of Lemma 3, we see that

$$\rho_3(t) = \prod_{0 \leqslant t_k < t}\left[|u_k|^p + \frac{1}{1-\sigma}\left(\int_{t_{k-1}}^{t_k} \left|B(s)\right| + \left|D(s)\right|^2 \mathrm{d}s\right)\right] \exp\left(\int_0^t \delta_3(s)\,\mathrm{d}s\right)$$

is a uniformly stable function. By Theorem 5, system (16) is exponentially stable in the mean square. Figure 3 depicts impulsive sequence with period 0.2. Figure 4 depicts the trajectory $(x_1(t), x_2(t))$ of system (16).

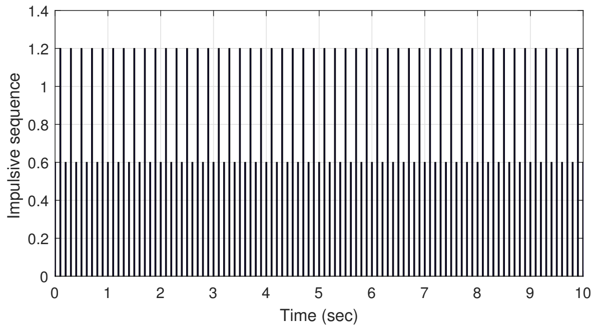

Figure 3. Impulsive sequence with period 0.2.

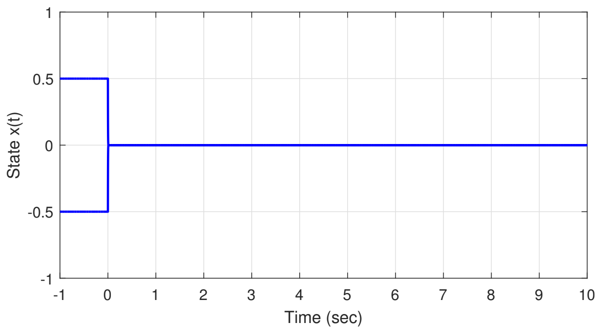

Figure 4. The state trajectory $(x_1(t), x_2(t))$ of system (16).

6 Conclusions

In this paper, USF in the form of impulse is proposed to achieve Razumikhin and Krasovskii stability of impulsive stochastic delay systems. These obtained results nicely address the relationship between time delay, impulsive effects and stochastic perturbations, which reduces the restrictions of impulsive gains. In the future, we expect to apply the USF method and impulsive control technique to dynamics of complex networks and consensus of multi-agent systems.

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