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Ruin probability for renewal risk models with neutral net profit condition

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Abstract. In ruin theory, the net profit condition intuitively means that the sizes of the incurred random claims are on average less than the premiums gained between the successive interoccurrence times. The breach of the net profit condition causes guaranteed ruin in few but simple cases when both the claims' interoccurrence time and random claims are degenerate. In this work, we give a simplified argumentation for the unavoidable ruin when the incurred claims are on average equal to the premiums gained between the successive interoccurrence times. We study the discrete-time risk model with $N \in \mathbb{N}$ periodically occurring independent distributions, the classical risk model, also known as the Cramér–Lundberg risk process, and the more general Sparre Andersen model.

Keywords: net profit condition, ruin probability, discrete-time risk model, classical risk model, Sparre Andersen risk model, random walk.

1 Introduction

In 1957, during the 15th International Congress of Actuaries, E. Sparre Andersen [16] proposed to use a *renewal risk model* to describe the behavior of the insurer's surplus. According to Sparre Andersen's proposed model, the insurer's surplus process W admits the following representation:

$$W_u(t) = u + ct - \sum_{i=1}^{\Theta(t)} X_i, \quad t \geqslant 0, \tag{1}$$

where

- $u \geqslant 0$ denotes the initial insurer's surplus, $W(0) = u$;
- $c > 0$ denotes the premium rate per unit of time;
- the cost of claims $X_1, X_2, \ldots$ are independent copies of a nonnegative random variable X;

- $\Theta(t) = \#\{n \geqslant 1,\ T_n \in [0,t]\}$ is the renewal process generated by the interoccurrence times $\theta_1, \theta_2, \ldots$, where $T_n = \theta_1 + \theta_2 + \cdots + \theta_n$;
- the interoccurrence times $\theta_1, \theta_2, \ldots$ between claims is a sequence of independent copies of a nonnegative random variable θ, which is not degenerate at zero, i.e., $\mathbf{P}(\theta = 0) < 1$;
- the sequences $\{X_1, X_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are supposed to be mutually independent.

The main critical characteristics of the defined renewal risk model (1) are the *time of ruin*

$$\tau_u = \begin{cases} \inf\{t \geqslant 0\colon\ W_u(t) < 0\}, \\ \infty \quad \text{if } W_u(t) \geqslant 0 \text{ for all } t \geqslant 0 \end{cases}$$

and the *ultimate time ruin probability* (or just the *ruin probability*)

$$\psi(u) = \mathbf{P}(\tau_u < \infty).$$

Model (1) and the definition of $\psi(u)$ imply that for all $u \geqslant 0$,

$$\begin{aligned} \psi(u) &= \mathbf{P}\biggl(\bigcup_{t \geqslant 0} \bigl\{W_u(t) < 0\bigr\}\biggr) \\ &= \mathbf{P}\left(\inf_{n \geqslant 1}\left\{u + cT_n - \sum_{i=1}^{n} X_i\right\} < 0\right) \\ &= \mathbf{P}\left(\sup_{n \geqslant 1} \sum_{k=1}^{n} (X_k - c\theta_k) > u\right). \end{aligned}$$

Thus, the ultimate time ruin probability $\psi(u)$ is nothing but the tail distribution of the random variable $\sup_{n\geqslant 1}\sum_{k=1}^{n}(X_k - c\theta_k)$. In ruin theory, the difference $\mathbf{E}X - c\mathbf{E}\theta$ describes the so-called *net profit condition*. It is well known that $\psi(u) = 1$ for any $u \geqslant 0$ if $\mathbf{E}X - c\mathbf{E}\theta > 0$, and this fact is easily implied by the strong law of large numbers; see [12, Prop. 7.2.3]. Also, $\psi(u) = 1$ for any $u \geqslant 0$ if $\mathbf{E}X - c\mathbf{E}\theta = 0$ (see [12, pp. 559–564]), except in some simple cases when both random variables X and θ are degenerate. Let us call the net profit condition *neutral* if $\mathbf{E}X - c\mathbf{E}\theta = 0$, and recall that the guaranteed ultimate time ruin is avoidable, i.e., $\psi(u) \not\equiv 1$ if $\mathbf{E}X - c\mathbf{E}\theta < 0$ only. If $\mathbf{E}X - c\mathbf{E}\theta < 0$, it is said that the net profit condition holds. In general, the fact that

$$\mathbf{E}X - c\mathbf{E}\theta = 0 \quad \Rightarrow \quad \psi(u) = 1 \tag{2}$$

for all $u \geqslant 0$ can be deduced from some deep study of random walk; see, for example, [6, 12, 18]. Therefore, the mathematical curiosity drives us to derive (2) by using simpler arguments.

In [2], authors basically use Silverman–Toeplitz theorem to prove (2) for the discrete-time and classical risk models, and their proofs presented for both mentioned models are significantly simpler than those given in [6, 12, 18]. In this article, we show that the implication (2) can be simplified even further, however in some instances using the Pollaczek–Khinchine formula. The desired simplification of the proof can be achieved by defining the random vector (X^*, X), where X^* is the new random variable, which is arbitrarily close to X, and $\mathbf{P}(X^* \leqslant X) = 1$.[1] This way is similar to the probabilistic proof of Turan's graph theorem given in [1, p. 184, third proof].[2] For the defined random variable X^*, we make the neutral net profit condition $\mathbf{E}X - c\mathbf{E}\theta = 0$ satisfied $\mathbf{E}X^* - c\mathbf{E}\theta < 0$ and show that the known algorithms of the ruin probability calculation under the net profit condition imply $\psi(u) = 1$ for all $u \geqslant 0$ as X^* approaches to X. The related ideas can be met in various other probabilistic research papers too where the difference of certain numerical characteristics is studied when two (or more) random variables are close to each other under a certain metric; see, for instance, [11] or [19].

In Section 2, we derive (2) for the more general discrete-time risk model, which is described as follows. Let us consider model (1). Suppose $c \in \mathbb{N}$, $\theta \equiv 1$, the independent random variables $X_1, X_2, \ldots$ are nonnegative integer-valued and follow the N-seasonal pattern, i.e., $X_i \overset{\mathrm{d}}{=} X_{i+N}$ for all $i \in \mathbb{N}$ and some fixed $N \in \mathbb{N}$. In other words, we allow the random variables $X_1, X_2, \ldots$ in model (1) to be independent but not necessarily identically distributed, and obviously, if $N = 1$, we get that the random variables $X_1, X_2, \ldots$ are identically distributed. If these requirements are satisfied, the general Sparre Andersen's renewal risk model (1) becomes the discrete-time risk model

$$W_u(t) = u + ct - \sum_{i=1}^{\lfloor t \rfloor} X_i, \quad t \geqslant 0, \tag{3}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$. Then there is sufficient to consider (3) (in terms of $W_u(t) < 0$ for at least one $t \geqslant 0$) when $u \in \{0, 1, 2, \ldots\} =: \mathbb{N}_0$ and $t \in \mathbb{N}$ only. Then the ruin time and the ultimate time ruin probability have the following standard expressions:

$$\tau_u = \begin{cases} \min\{t \in \mathbb{N}\colon W_u(t) < 0\}, \\ \infty \quad \text{if } W_u(t) \geqslant 0 \text{ for all } t \in \mathbb{N}, \end{cases}$$

$$\psi(u) = \mathbf{P}(\tau_u < \infty) = \mathbf{P}\left(\sup_{k \geqslant 1} \sum_{i=1}^{k} (X_i - c) > u \right), \quad u \in \mathbb{N}_0. \tag{4}$$

If $\varphi := 1 - \psi$ denotes the ultimate time survival probability, then, according to (4),

$$\varphi(u) = \mathbf{P}\left(\sup_{k \geqslant 1} \sum_{i=1}^{k} (X_i - c) \leqslant u \right), \quad u \in \mathbb{N}_0. \tag{5}$$

[1] Originally, the idea was raised by the fourth-year student of Faculty of Mathematics and Informatics Justas Klimavičius in 2017.

[2] We thank Professor Eugenijus Manstavičius for pointing to this fact.

In [7] and various other papers, the survival probability is studied according to a slightly different definition than (5), i.e.,

$$\hat{\varphi}(u) = \mathbf{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k}(X_i - c) < u\right). \tag{6}$$

It is easy to see that the survival probabilities in (5) and (6) are related as $\varphi(u) = \hat{\varphi}(u+1)$ for all $u \in \mathbb{N}_0$.

In Section 3, we derive (2) for the classical risk model when $\Theta(t)$ in (1) is assumed to be a Poisson process with intensity $\lambda > 0$, i.e.,

$$\mathbf{P}\big(\Theta(t+s) - \Theta(s) = n\big) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

for all $n \in \mathbb{N}_0$ and $t, s > 0$.

In Section 4, we consider the most general Sparre Andersen's model (1) in terms of proving (2) by the known facts of ruin probability calculation under the net profit condition. More precisely, in Sections 2, 3, and 4 respectively, we formulate and give different proofs than the existing ones to the following three theorems.

Theorem 1. *Suppose the insurer's surplus process $W_u(t)$ varies according to the discrete-time risk model* (3) *with N periodically occurring independent discrete and integer-valued nonnegative random variables $X_k \overset{\mathrm{d}}{=} X_{k+N}$, $k \in \mathbb{N}$, and $\theta \equiv 1$. Let $S_N = X_1 + X_2 + \cdots + X_N$. If the net profit condition is neutral $cN - \mathbf{E}S_N = 0$ and $\mathbf{P}(S_N = cN) < 1$, the ultimate time ruin probability $\psi(u) = 1$ for all $u \in \mathbb{N}_0$.*

Theorem 2. *Let $W_u(t)$, $t \geqslant 0$, be a surplus process of the classical risk model generated by a random claim amount X, an exponentially distributed interoccurrence time θ with mean $\mathbf{E}\theta = 1/\lambda$, $\lambda > 0$, and a constant premium rate $c > 0$. If the net profit condition is neutral $\lambda\mathbf{E}X = c$, then $\psi(u) = 1$ for all $u \geqslant 0$.*

Theorem 3. *Let $W_u(t)$, $t \geqslant 0$, be a surplus process of Sparre Andersen model generated by a random claim amount X, interoccurrence time θ, and a constant premium rate $c > 0$. If the net profit condition is neutral $\mathbf{E}X/\mathbf{E}\theta = c$ and $\mathbf{P}(X = c\theta) < 1$, then $\psi(u) = 1$ for all $u \geqslant 0$.*

As mentioned, proving Theorems 2 and 3, we use the Pollaczek–Khinchine formula. This raises the need for the following little auxiliary statement.

Lemma 1. *Let $\eta_1, \eta_2, \ldots$ be independent identically distributed nonnegative random variables, which are not degenerate at zero. Then, for any $x \geqslant 0$,*

$$\sum_{n=1}^{\infty}\mathbf{P}(\eta_1 + \cdots + \eta_n \leqslant x) < \infty.$$

Proof. Let t be some small positive number, and say that the nonnegative random variables $\eta_1, \eta_2, \ldots$ are independent copies of η. Then, rearranging and using Markov's inequality, we obtain

$$\sum_{n=1}^{\infty}\mathbf{P}(\eta_1+\cdots+\eta_n\leqslant x)=\sum_{n=1}^{\infty}\mathbf{P}\bigl(\mathrm{e}^{-t(\eta_1+\cdots+\eta_n)}\geqslant \mathrm{e}^{-tx}\bigr)$$
$$\leqslant \mathrm{e}^{tx}\sum_{n=1}^{\infty}\bigl(\mathbf{E}\mathrm{e}^{-t\eta}\bigr)^n<\infty \tag{7}$$

since $\mathbf{E}\mathrm{e}^{-t\eta}<1$ under the considered conditions. □

Of course, the upper bound of the sum $\sum_{n=1}^{\infty}\mathbf{P}(\eta_1+\cdots+\eta_n\leqslant x)$ can be improved compared to the given one in (7); see, for instance, [10, Proof of Lemma 8], [20, Lemma 3.2], and other literature on concentration inequalities.

2 Discrete-time risk model

Proof of Theorem 1. We first demonstrate the proof for the most simplistic version of the homogeneous discrete-time risk model (3) when $c=1$ and $N=1$. Let $h_k=\mathbf{P}(X=k)$, $k\in\mathbb{N}_0$, and observe that conditions $\mathbf{E}X=1$ and $\mathbf{P}(X=1)=h_1<1$ imply $h_l>0$ for some $l\geqslant 2$. Indeed,

$$\mathbf{E}X=h_1+2h_2+3h_3+\cdots=1,$$

and $h_1<1$ means that at least one probability out of $h_2,h_3,\dots$ is positive. In addition, conditions $h_1<1$ and $\mathbf{E}X=1$ imply $h_0>0$. Indeed, if $h_0=0$, then $h_1+h_2+h_3+\cdots=1$, and

$$1=\mathbf{E}X=h_1+2h_2+3h_3+\cdots>h_1+h_2+h_3+\cdots=1$$

leads to the contradiction.

Let us choose $l\geqslant 2$ such that $h_l=\mathbf{P}(X=l)>0$, and define the distribution of an integer-valued random vector (X^*,X) by the following equalities:

$$\begin{aligned}
&\mathbf{P}(X^*=k,\ X=k)=h_k,\quad k\in\mathbb{N}_0,\ k\neq l,\\
&\mathbf{P}(X^*=l,\ X=l)=h_l-\frac{\varepsilon}{l},\qquad \mathbf{P}(X^*=0,\ X=l)=\frac{\varepsilon}{l},\\
&\mathbf{P}(X^*=k,\ X=m)=0,\quad \{k,m\}\in\mathbb{N}_0^2,\ \{k,m\}\neq\{0,l\},\ k\neq m,
\end{aligned}$$

where $\varepsilon\in(0,lh_l)$ is arbitrarily small.

Visually, the distribution of the vector (X^*,X) is presented in Table 1.

It is easy to see that $\mathbf{E}X^*=1-\varepsilon<1$ and

$$\mathbf{P}(X^*\leqslant X)=\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\mathbf{P}(X^*=k,\ X=m)$$
$$=\mathbf{P}(X^*=0,\ X=l)+\sum_{k=0}^{\infty}\mathbf{P}(X^*=k,\ X=k)=1.$$

Table 1. Distribution of the vector (X^*, X).

X^*	X								
	0	1	2	$\dots$	$l-1$	l	$l+1$	$\dots$	Σ
0	h_0	0	0	$\dots$	0	ε/l	0	$\dots$	$h_0+\varepsilon/l$
1	0	h_1	0	$\dots$	0	0	0	$\dots$	h_1
2	0	0	h_2	$\dots$	0	0	0	$\dots$	h_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$l-1$	0	0	0	$\dots$	h_{l-1}	0	0	$\dots$	h_{l-1}
l	0	0	0	$\dots$	0	$h_l-\varepsilon/l$	0	$\dots$	$h_l-\varepsilon/l$
$l+1$	0	0	0	$\dots$	0	0	h_{l+1}	$\dots$	h_{l+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Σ	h_0	h_1	h_2	$\dots$	h_{l-1}	h_l	h_{l+1}	$\dots$	1

Let (X_j^*, X_j), $j \in \mathbb{N}$, be independent copies of random vector (X^*, X). We have $\mathbf{P}(X_j^* \leqslant X_j) = 1$ for each $j \in \mathbb{N}$. Therefore,

$$\begin{aligned}
&\mathbf{P}(X_1^* + X_2^* \leqslant X_1 + X_2)\\
&\quad= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \mathbf{P}(X_1^* + k \leqslant X_1 + l)\mathbf{P}(X_2^* = k,\ X_2 = l)\\
&\quad= \sum_{k=0,\ k\neq l}^{\infty} \mathbf{P}(X_1^* + k \leqslant X_1 + k)h_k\\
&\qquad+ \mathbf{P}(X_1^* + l \leqslant X_1 + l)\left(h_l - \frac{\varepsilon}{l}\right) + \mathbf{P}(X_1^* \leqslant X_1 + l)\frac{\varepsilon}{l} = 1
\end{aligned}$$

due to $\mathbf{P}(X_1^* \leqslant X_1) = 1$.

We now use the mathematical induction to show

$$\mathbf{P}\left(\sum_{k=1}^{n} X_k^* \leqslant \sum_{k=1}^{n} X_k\right) = 1, \quad n \in \mathbb{N}. \tag{8}$$

Indeed, if $\mathbf{P}(\sum_{k=1}^{n} X_k^* \leqslant \sum_{k=1}^{n} X_k) = 1$ up to some natural n, we conclude that

$$\begin{aligned}
\mathbf{P}\left(\sum_{k=1}^{n+1} X_k^* \leqslant \sum_{k=1}^{n+1} X_k\right) &= \sum_{m=0,\, m\neq l}^{\infty} \mathbf{P}\left(\sum_{k=1}^{n} X_k^* \leqslant \sum_{k=1}^{n} X_k\right)h_m\\
&\quad+ \mathbf{P}\left(\sum_{k=1}^{n} X_k^* \leqslant \sum_{k=1}^{n} X_k\right)\left(h_l - \frac{\varepsilon}{l}\right)\\
&\quad+ \mathbf{P}\left(\sum_{k=1}^{n} X_k^* \leqslant \sum_{k=1}^{n} X_k + l\right)\frac{\varepsilon}{l} = 1.
\end{aligned}$$

For $u \in \mathbb{N}_0$, equality (8) implies that

$$\begin{aligned}\psi(u) &= \mathbf{P}\left(\sup_{n\geqslant 1}\left\{\sum_{k=1}^{n} X_k - n\right\} > u\right) = \mathbf{P}\left(\bigcup_{n=1}^{\infty}\left\{\sum_{k=1}^{n} X_k > n+u\right\}\right)\\ &\geqslant \mathbf{P}\left(\bigcup_{n=1}^{\infty}\left\{\sum_{k=1}^{n} X_k^* > n+u\right\}\right) = \mathbf{P}\left(\sup_{n\geqslant 1}\left\{\sum_{k=1}^{n} X_k^* - n\right\} > u\right)\\ &=: \psi_\varepsilon^*(u)\end{aligned}$$

or, equivalently,

$$\varphi(u) \leqslant \varphi_\varepsilon^*(u) \tag{9}$$

for all $u \in \mathbb{N}_0$, where $\varphi = 1 - \psi$ and $\varphi_\varepsilon^* = 1 - \psi_\varepsilon^*$ are the model's survival probabilities.

Let $s \in \mathbb{C}$ and $h_k^* = \mathbf{P}(X^* = k)$, $k \in \mathbb{N}_0$. Since $\mathbf{E}X^* = 1 - \varepsilon < 1$, Corollary 3.2 of [8] implies that the generating function of the survival probability φ_ε^* is

$$\varphi_\varepsilon^*(0) + \varphi_\varepsilon^*(1)s + \varphi_\varepsilon^*(2)s^2 + \cdots = \frac{1 - \mathbf{E}X^*}{G_{X^*}(s) - s} = \frac{\varepsilon}{G_{X^*}(s) - s}, \quad |s| < 1, \tag{10}$$

where $G_{X^*}(s)$ is the probability generating function of the random variable X^*, i.e.,

$$G_{X^*}(s) = h_0^* + h_1^* s + h_2^* s^2 + \cdots, \quad |s| \leqslant 1.$$

Inequality (9) and equation (10) imply that

$$0 \leqslant \varphi(0) \leqslant \frac{\varepsilon}{h_0^*} = \frac{\varepsilon}{h_0 + \varepsilon/l}, \qquad 0 \leqslant \varphi(1) \leqslant \frac{\varepsilon(1 - h_1)}{(h_0 + \varepsilon/l)^2}$$

and, in general,

$$0 \leqslant \varphi(n) \leqslant \varepsilon \cdot \frac{1}{n!} \lim_{s\to 0} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\frac{1}{G_{X^*}(s) - s}\right)$$

for all $n \in \mathbb{N}_0$. Since ε can be arbitrarily small, we conclude that $\varphi(u) = 0$ or, equivalently, $\psi(u) = 1$ for all $u \in \mathbb{N}_0$.

It is worth mentioning that, having $\varphi(0) = 0$, the equality $\varphi(u) = 0$ for all $u \in \mathbb{N}$ can be concluded from the following recurrence formula (see, for instance, [3, Sect. 6] and [4, 14, 15]):

$$\varphi(u) = \frac{1}{h_0}\left(\varphi(u-1) - \sum_{k=1}^{u} \varphi(u-k)h_k\right), \quad u \in \mathbb{N}. \tag{11}$$

Indeed, the recurrence (11) yields $\varphi(u)$, $u \in \mathbb{N}_0$, being the multiple of $\varphi(0) = 1 - \mathbf{E}X$. More precisely,

$$\varphi(u) = \alpha_u \varphi(0)$$

with

$$\alpha_0 = 1, \qquad \alpha_u = \frac{1}{h_0}\left(\alpha_{u-1} - \sum_{k=1}^{u} \alpha_{u-k}h_k\right), \quad u \in \mathbb{N}.$$

The latter expression can be verified by mathematical induction. So, the particular case with $c = 1$ and $N = 1$ in Theorem 1 is proved.

The general case when $c \in \mathbb{N}$ and $N \in \mathbb{N}$ in the discrete-time risk model (3) can be considered by the same means. Let us explain how.

Let us suppose model (3) is generated by $X_1, X_2, \dots, X_N$ periodically occurring independent nonnegative and integer-valued random variables, i.e., $X_{i+N} \stackrel{\mathrm{d}}{=} X_i$ for all $i \in \mathbb{N}$ and some fixed $N \in \mathbb{N}$. In such a case, we can choose any random variable from $\{X_1, X_2, \dots, X_N\}$ and define the random vector (X_j^*, X_j) such that $\mathbf{P}(X_j^* \leqslant X_j) = 1$, where $j \in \{1, 2, \dots, N\}$ is some fixed number. Obviously, the random vector (X_j^*, X_j) must be defined in a similar way as vector (X^*, X) before, where both random variables X_j^* and X_j attain the same values, the probability of some smaller value of X_j^* gets enlarged by some arbitrarily small value, and the probability of some larger value of X_j^* gets reduced by the same size. Note that conditions $\mathbf{P}(X_j \geqslant c) = 1$ and $\mathbf{P}(S_N = cN) < 1$ imply the estimate $cN - \mathbf{E}S_N < 0$, which is not the case under consideration. Hence, there always exists at least one value in the set $\{0, 1, \dots, c-1\}$ of the values of X_j, which can be chosen to enlarge its probability defining X_j^*. We then achieve

$$\varepsilon := cN - \mathbf{E}S_N^* > cN - \mathbf{E}S_N = 0,$$

where $S_N^* = X_1 + \dots + X_j^* + \dots + X_N$. By the same arguments as deriving inequality (9), we get that $0 \leqslant \varphi(0) \leqslant \varphi_\varepsilon^*(0)$, where $\varphi_\varepsilon^*(0)$ is the ultimate time survival probability at $u = 0$ for the model in which the random variable X_j^* replaces X_j for some $j \in \{1, 2, \dots, N\}$. According to [7, Thm. 4], we obtain

$$\varphi_\varepsilon^*(0) = \frac{m_0^{*(1)}}{\mathbf{P}(S_N^* = 0)} \tag{12}$$

if $\mathbf{P}(S_N^* = 0) > 0$, where $m_0^{*(1)}$ is the first component of the solution of the following system of linear equations:

$$M_{cN\times cN} \times \begin{pmatrix} m_0^{*(1)} \\ m_1^{*(1)} \\ \vdots \\ m_{c-1}^{*(1)} \\ m_0^{*(2)} \\ m_1^{*(2)} \\ \vdots \\ m_{c-1}^{*(2)} \\ \vdots \\ m_0^{*(N)} \\ m_1^{*(N)} \\ \vdots \\ m_{c-1}^{*(N)} \end{pmatrix}_{cN\times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ cN - \mathbf{E}S_N^* \end{pmatrix}_{cN\times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varepsilon \end{pmatrix}_{cN\times 1}, \tag{13}$$

where $M_{cN\times cN}$ is a certain matrix with elements related to the roots of equation $G_{S_N^*}(s) = s^{cN}$, $|s| \leqslant 1$; see [7, Sect. 3]. Letting $\varepsilon \to 0^+$, we derive from system (13) that $\varphi_\varepsilon^*(0) \to 0$ because of (12). Consequently, $\varphi(0) = 0$ due to the estimate $0 \leqslant \varphi(0) \leqslant \varphi_\varepsilon^*(0)$ provided for an arbitrary $\varepsilon > 0$. It should be noted that the requirement $\mathbf{P}(S_N^* = 0) > 0$ for equality (12) does not reduce generality because $\mathbf{P}(S_N^* = 0)$ can be replaced by the probability of the smallest value of S_N^* if $\mathbf{P}(S_N^* = 0) = 0$; see the comments in [7, Sect. 4]. In addition, the nonsingularity of the matrix $M_{cN\times cN}$ in (13) is not known in general; see [7, Sect. 4] and [8], also [9]. On the other hand, if $c \in \mathbb{N}$, $N = 1$, and the roots of $G_{X^*} = s^c$ are simple, the solution of (13) admits the closed-form expression, and obviously, $m_0^{*(1)}$ is the multiple of $c - \mathbf{E}X^* = \varepsilon$; see [8]. In cases when the nonsingularity of the matrix $M_{cN\times cN}$ in (13) remains questionable, we can refer to [7, Thm. 3] for the different proof that $\varphi(0) = 0$ if the net profit condition is neutral $\mathbf{E}S_N = cN$ and $\mathbf{P}(S_N = cN) < 1$.

Having $\varphi(0) = 0$, the remaining values $\varphi(u) = 0$, $u \in \mathbb{N}$, can be obtained by the recurrence relation

$$\varphi(u) = \sum_{\substack{i_1 \leqslant u+c \\ i_1+i_2 \leqslant u+2c \\ \cdots \\ i_1+i_2+\cdots+i_N \leqslant u+cN}} \mathbf{P}(X_1 = i_1)\mathbf{P}(X_2 = i_2)\cdots\mathbf{P}(X_N = i_N)\,\varphi\left(u + cN - \sum_{j=1}^N i_j\right)$$

(see [7, Eq. (5)]) or by the following expression of survival probability generating function (see [7, Thm. 2]):

$$\varphi_\varepsilon^*(0) + \varphi_\varepsilon^*(1)s + \varphi_\varepsilon^*(2)s^2 + \cdots = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{v}}{G_{S_N^*}(s) - s^{cN}},$$

where, having in mind that some X_j from $\{X_1, \ldots, X_n\}$ is replaced by X_j^*,

$$\mathbf{u} = \begin{pmatrix} s^{c(N-1)} \\ s^{c(N-2)}G_{S_1^*}(s) \\ s^{c(N-3)}G_{S_2^*}(s) \\ \vdots \\ s^c G_{S_{N-2}^*}(s) \\ G_{S_{N-1}^*}(s) \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \sum_{i=0}^{c-1} m_k^{*(2)} \sum_{k=i}^{c-1} s^k F_{X_1}(k-i) \\ \sum_{i=0}^{c-1} m_i^{*(3)} \sum_{k=i}^{c-1} s^k F_{X_2}(k-i) \\ \vdots \\ \sum_{i=0}^{c-1} m_i^{*(j+1)} \sum_{k=i}^{c-1} s^k F_{X_j^*}(k-i) \\ \vdots \\ \sum_{i=0}^{c-1} m_i^{*(N)} \sum_{k=i}^{c-1} s^k F_{X_{N-1}}(k-i) \\ \sum_{i=0}^{c-1} m_i^{*(1)} \sum_{k=i}^{c-1} s^k F_{X_N}(k-i) \end{pmatrix},$$

and

$$G_{S_l^*}(s), \quad |s| \leqslant 1,\ l \in \{1, 2, \ldots, N-1\}$$

is the probability-generating function of the random variable

$$S_l^* = \begin{cases} X_1 + \cdots + X_l & \text{if } l < j, \\ X_1 + \cdots + X_{j-1} + X_j^* + X_{j+1} + \cdots + X_l & \text{if } l \geqslant j, \end{cases}$$

F_{X_l} is the distribution function of X_l, and the collection

$$\{m_0^{*(1)}, m_1^{*(1)}, \dots, m_{c-1}^{*(1)}, m_0^{*(2)}, m_1^{*(2)}, \dots, m_{c-1}^{*(2)}, \dots, m_0^{*(N)}, m_1^{*(N)}, \dots, m_{c-1}^{*(N)}\}$$

satisfies system (13) being the multiple of $cN - \mathbf{E}S_N^* = \varepsilon$. ☐

3 Classical risk model

Proof of Theorem 2. Since the random variable X in model (1) is nonnegative and $X \equiv 0$ is out of options for the considered stochastic process, then $\mathbf{E}X > 0$, and there exists $a > 0$ such that $\mathbf{P}(X > a) > 0$. Similarly as proving Theorem 1, we now define the pair of dependent random variables (X^*, X), where X^* for any $\varepsilon \in (0, a)$ is

$$X^* = \begin{cases} X - \varepsilon & \text{if } X > a, \\ X & \text{if } X \leqslant a. \end{cases}$$

Then, obviously, $\mathbf{E}X^* = \mathbf{E}X - \varepsilon\mathbf{P}(X > a) < \mathbf{E}X$ and $\mathbf{P}(X^* \leqslant X) = 1$. Let (X_j^*, X_j), $j = 1, 2, \dots$, be independent copies of (X^*, X). Then

$$\mathbf{P}(X_j^* \leqslant X_j) = 1 \quad \text{for all } j \in \mathbb{N},$$

$$\mathbf{P}\left(\sum_{j=1}^n X_j^* \leqslant \sum_{j=1}^n X_j\right) = 1 \quad \text{for all } n \in \mathbb{N},$$

$$\mathbf{P}\left(\sum_{j=1}^n (X_j^* - c\theta_j) \leqslant \sum_{j=1}^n (X_j - c\theta_j)\right) = 1 \quad \text{for all } n \in \mathbb{N},$$

$$\mathbf{P}\left(\sup_{n\geqslant 1}\sum_{j=1}^n (X_j^* - c\theta_j) \leqslant \sup_{n\geqslant 1}\sum_{j=1}^n (X_j - c\theta_j)\right) = 1,$$

and, by similar arguments as in (9),

$$\mathbf{P}\left(\sup_{n\geqslant 1}\sum_{j=1}^n (X_j^* - c\theta_j) > u\right) = \psi_\varepsilon^*(u) \leqslant \psi(u) = \mathbf{P}\left(\sup_{n\geqslant 1}\sum_{j=1}^n (X_j - c\theta_j) > u\right) \leqslant 1$$

for all $u \geqslant 0$. Conditions

$$\mathbf{E}X^* = \mathbf{E}X - \varepsilon\mathbf{P}(X > a), \qquad \frac{\lambda}{c}\mathbf{E}X = 1$$

and well-known formula for $\psi_\varepsilon^*(0)$ (see, for example, [13] or many other sources for the Pollaczek–Khinchine formula) imply that

$$\psi_\varepsilon^*(0) = \frac{\lambda\mathbf{E}X^*}{c} = 1 - \frac{\lambda\varepsilon\mathbf{P}(X > a)}{c} \leqslant \psi(0) \leqslant 1.$$

By letting $\varepsilon \to 0^+$ in the last inequalities, we get $\psi(0) = 1$ or, equivalently, $\varphi(0) = 0$. Then $\psi(u) = 1$ for all $u \geqslant 0$ is implied by the same Pollaczek–Khinchine formula observing $\varphi^*_\varepsilon(u)$ being the multiple of $\varphi^*_\varepsilon(0)$. Indeed,

$$\varphi^*_\varepsilon(u) = \left(1 - \frac{\lambda \mathbf{E}X^*}{c}\right)\left(1 + \sum_{n=1}^{\infty}\left(\frac{\lambda \mathbf{E}X^*}{c}\right)^n F_I^{*n}(u)\right)$$
$$= \varphi^*_\varepsilon(0)\left(1 + \sum_{n=1}^{\infty}\left(\psi^*_\varepsilon(0)\right)^n F_I^{*n}(u)\right), \quad u \geqslant 0,$$

where

$$F_I(u) = \frac{1}{\mathbf{E}X^*}\int_0^u \mathbf{P}(X^* > x)\,\mathrm{d}x,$$

and F_I^{*n} denotes the n-fold convolution of F_I. Here

$$\sum_{n=1}^{\infty}\left(\psi^*_\varepsilon(0)\right)^n F_I^{*n}(u) \leqslant \sum_{n=1}^{\infty} F_I^{*n}(u) = \sum_{n=1}^{\infty}\mathbf{P}(\eta_1 + \cdots + \eta_n \leqslant u) < \infty$$

because of Lemma 1, where the nonnegative independent and identically distributed random variables $\eta_1, \eta_2, \ldots$ are described by the distribution function F_I. □

4 Sparre Andersen's model

Proof of Theorem 3. Arguing the same as proving Theorem 2 in Section 3, we can define the random vector (X^*, X), its independent copies $(X_1^*, X_1), (X_2^*, X_2), \ldots$ and show that $\psi^*_\varepsilon(u) \leqslant \psi(u) \leqslant 1$ for all $u \geqslant 0$. Let $S_n^* = \sum_{k=1}^n (X_k^* - c\theta_k)$ and $S_n = \sum_{k=1}^n (X_k - c\theta_k)$ for all $n \in \mathbb{N}$. Then (see [5, Eq. (10)])

$$\psi^*_\varepsilon(0) = 1 - \exp\left\{-\sum_{n=1}^{\infty}\frac{\mathbf{P}(S_n^* > 0)}{n}\right\}$$

because of the net profit condition $\mathbf{E}X^* - c\mathbf{E}\theta = -\varepsilon\mathbf{P}(X > a) < 0$.

It is known that (see [17, Thm. 4.1]) $\mathbf{E}(X^* - c\theta) < 0$ implies

$$\sum_{n=1}^{\infty}\frac{\mathbf{P}(S_n^* > 0)}{n} < \infty,$$

while $\mathbf{E}(X - c\theta) = 0$ implies

$$\sum_{n=1}^{\infty}\frac{\mathbf{P}(S_n > 0)}{n} = \infty.$$

Therefore,

$$\varphi(0) \leqslant \varphi_\varepsilon^*(0) \leqslant \exp\left\{-\sum_{n=1}^{M} \frac{\mathbf{P}(S_n^* > 0)}{n}\right\} \tag{14}$$

for any $M \in \mathbb{N}$. Since

$$\lim_{\varepsilon \to 0^+} \mathbf{P}(S_n^* = S_n) = 1$$

for all $n \in \mathbb{N}$, by letting $\varepsilon \to 0^+$ in (14), we obtain

$$\varphi(0) \leqslant \exp\left\{-\sum_{n=1}^{M} \frac{\mathbf{P}(S_n > 0)}{n}\right\},$$

and consequently, $\varphi(0) = 0$ as M can be arbitrarily large, and the series

$$\sum_{n=1}^{\infty} \frac{\mathbf{P}(S_n > 0)}{n}$$

diverges. The equality $\psi(u) = 1$ for all $u \geqslant 0$ is implied by the fact that $\varphi_\varepsilon^*(u)$ is the multiple of $\varphi_\varepsilon^*(0)$. Indeed, by the Pollaczek–Khinchine formula (see [5, Eq. (10)])

$$\begin{aligned}\varphi_\varepsilon^*(u) &= \mathrm{e}^{-A}\left(1 + \sum_{n=1}^{\infty} \left(1 - \mathrm{e}^{-A}\right)^n H^{*n}(u)\right) \\ &= \varphi_\varepsilon^*(0)\left(1 + \sum_{n=1}^{\infty} \left(\psi_\varepsilon^*(0)\right)^n H^{*n}(u)\right), \quad u \geqslant 0,\end{aligned}$$

where

$$A = \sum_{n=1}^{\infty} \frac{\mathbf{P}(S_n^* > 0)}{n}, \qquad H(u) = \frac{F_+(u)}{F_+(\infty)},$$

$$F_+(u) = \mathbf{P}(S_{N^+}^* \leqslant u), \qquad N^+ = \inf\{n \geqslant 1\colon S_n^* > 0\},$$

and H^{*n} denotes the n-fold convolution of H. The proof of the considered theorem follows according to the comments at the end of the proof of Theorem 2. □

5 Concluding remarks

There are many statements in mathematics that can be proved differently. Usually, the simplest proofs attain the most attention. On the other hand, the variety or just the number of different derivations of certain facts is of interest too, one may reference to the already mentioned source [1]. In this paper, we managed to prove differently the known fact that

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty} \left\{\sum_{k=1}^{n} X_k < u\right\}\right) = 0 \tag{15}$$

for any $u \geqslant 0$ if $\mathbf{E}X = 0$ and $X_1, X_2, \dots$ are independent copies of the random variable X. On top of that, we demonstrated that equality (15) is implied by similar argumentation under some other than i.i.d. assumptions on the underlying random walk $\{\sum_{k=1}^{n} X_k,\, n \in \mathbb{N}\}$.

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