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Fractional SDEs with stochastic forcing: Existence, uniqueness, and approximation

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Abstract. In this article, we are interested in fractional stochastic differential equations (FSDEs) with stochastic forcing, i.e., to FSDE we add a stochastic forcing term. The conditions for the existence and uniqueness of solutions of such equations are obtained, and the convergence rate of the implicit Euler approximation scheme for them is established. Such types of equations can be applied to the consideration of FSDEs with a permeable wall.

Keywords: stochastic differential equations, stochastic forcing, fractional Brownian motion, Picard iteration, Vasicek process, p-variation.

1 Introduction

Stochastic differential equations (SDEs) are used as a modelling tool in many fields of science. Currently, a lot of research is done around models with fractional Brownian motion (fBm) $B^H = (B^H)_{t\geqslant 0}$, $0 < H < 1$, since fBm introduces a memory element, which provides new and promising modelling possibilities. SDEs driven by fBm

$$X_t = X_0 + \int_0^t f(s, X_s)\,\mathrm{d}s + \int_0^t g(s, X_s)\,\mathrm{d}B_s^H, \quad t \in [0, T],\ \frac{1}{2} < H < 1,$$

or mixed SDEs is the focus of many authors. Many authors have considered the problem of the existence and uniqueness of solutions to such equations [3,4,7–12,16–21,23,25,28]. Also, numerical methods for this type of equation are of interest (e.g., see [5, 6, 13–16] and references therein).

Interest in considering FSDEs with a stochastic forcing term was aroused by an article by Vojta et al. [24]. In this article, the authors add repulsive forces G to the recursion relation

$$x_{n+1} = x_n + \xi_n + G(x_n),$$

where x_n – process value, ξ_n – discrete fractional Gaussian noise. As an example of repulsive forces, exponential forces were taken with a wall w defined by the functions

$$G(x) = G_0 \exp\bigl\{-\lambda(x - w)\bigr\}$$

characterized by amplitude G_0 and decay constant λ. Such a model is called a „soft wall" model. It should be noted that the „soft wall" model has a permeable wall. The process may cross the wall w, but it is affected by the force of the selected quantity in the opposite direction. When the process is far from the wall, the force acts weakly. As it approaches or crosses the wall, the force acts stronger.

The first attempt to generalize the soft-wall model was made in an article by Kubilius and Medžiūnas [10]. In this article, a class of FSDEs with a soft wall was considered. Equations with constant and strictly positive diffusion coefficients belong to this class. To illustrate the behavior of process trajectories when a repulsive force is added to it, we considered the Vasicek process. For the convenience of the reader, we will repeat this example in the appendix with additional comments. This example will make it easier to understand what happens when a repulsive force is added.

In our article, we will consider a further generalization of the soft-wall model. We will introduce and study FSDEs with stochastic forcing defined by the equation

$$X_t = X_0 + \varPhi(X_t) - \varPhi(X_0) + \int_0^t f(s, X_s)\,\mathrm{d}s + \int_0^t g(s, X_s)\,\mathrm{d}B_s^H, \quad t \in [0, T], \quad (1)$$

where $1/2 < H < 1$, $\varPhi : \mathbb{R} \to \mathbb{R}$ is continuous function, $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions. The stochastic integral in Eq. (1) is a pathwise generalized Lebesgue–Stieltjes integral. Thus, we can use the pathwise approach to consider this FSDE. We call such equation FSDE with a stochastic forcing term $\varPhi$. As an example of a stochastic forcing term, we can take the repulsive force.

This paper aims to find conditions under which FSDE (1) has a unique solution and to study the convergence rate of implicit Euler approximation for it in the pathwise sense. The proof of the existence and uniqueness of the solution of Eq. (1) is based on the estimates obtained in the article by Nualart and Rǎşcanu [20] and the application of the implicit Picard iteration procedure.

The paper is organized in the following way. In Section 2, we present the paper's main results. Section 3 contains definitions of considered spaces of functions and a priori estimates for the Lebesgue–Stieltjes integral. In Section 4, we prove the existence and uniqueness of a solution for a deterministic differential equation and obtain a convergence rate for implicit Euler approximation. Finally, in the Appendix, the fractional Vasicek process with the soft wall is considered as a modeling example.

2 Main result

We will assume that the coefficients f, g are measurable and satisfy the following conditions with some nonrandom constants:

(H1) $g(t,x)$ is differentiable in x, and there exist some constants $0 < \beta, \delta \leqslant 1$, and for every $N \geqslant 0$, there exists $M_N > 0$ such that the following properties hold:

(i) Lipschitz continuity in x

$$\left|g(t,x) - g(t,y)\right| \leqslant M_0|x-y| \quad \forall x, y \in \mathbb{R},\ t \in [0,T];$$

(ii) local uniform Hölder continuity of the derivative in x

$$\left|g'_x(t,x) - g'_x(t,y)\right| \leqslant M_N|x-y|^\delta \quad \forall x, y \in [-N,N],\ \forall t \in [0,T];$$

(iii) Hölder continuity in t

$$\left|g(s,x) - g(t,x)\right| + \left|g'_x(s,x) - g'_x(t,x)\right| \leqslant M_0|t-s|^\beta \quad \forall x \in \mathbb{R},\ \forall t, s \in [0,T].$$

(H2) There exists $b_0 \in L^\rho(0,T))$, where $\rho \geqslant 2$, and for every $N \geqslant 0$, there exists $L_N > 0$ such that the following properties hold:

(i) local uniform Lipschitz continuity in x

$$\left|f(t,x) - f(t,y)\right| \leqslant L_N|x-y| \quad \forall x, y \in [-N,N],\ \forall t \in [0,T];$$

(ii) rate of growth

$$\left|f(t,x)\right| \leqslant L_0|x| + b_0(t) \quad \forall x \in \mathbb{R},\ \forall t \in [0,T].$$

(H3) Assume that

(i) function $D : \mathbb{R} \to \mathbb{R}$, where $D(x) := x - \Phi(x)$, is strictly monotonic and surjective;

(ii) there is a constant $d > 0$ such that

$$\left|D(x) - D(y)\right| \geqslant d|x-y|. \tag{2}$$

Remark 1. In the Appendix, the considered repulsive forces are examples satisfying assumptions (H3). More classes of functions satisfying (2) are given in [10].

We can now formulate our main result. Set

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}, \qquad \gamma_0 = 1 - \alpha_0. \tag{3}$$

Theorem 1. *Suppose that the functions* $f(t,x)$ *and* $g(t,x)$ *satisfy assumptions* (H1) *and* (H2) *with* $\rho = (1-H)^{-1}$, $1/H - 1 < \delta \leqslant 1$, $1-H < \beta \leqslant 1$, $1/2 < H < 1$. *Moreover, let assumptions* (H3) *hold. If* $\gamma \in (\gamma_0, H)$, *then there exists a unique stochastic process* $X \in C^\gamma(0,T)$ *satisfying FSDE* (1).

Let $\pi^n = \{t_k^n = kT/n,\ 1 \leqslant k \leqslant n\}$ be a sequence of uniform partitions of the interval $[0,T]$, and let $\Delta_n = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$, $\Delta_n < 1$. We define the implicit Euler approximations for solution of Eq. (1) as

$$\begin{aligned} Y^n(t_{k+1}^n) - \Phi\big(Y^n(t_{k+1}^n)\big) &= Y^n(t_k^n) - \Phi\big(Y^n(t_k^n)\big) + f\big(t_k^n, Y^n(t_k^n)\big)\Delta_n \\ &\quad + g\big(t_k^n, Y^n(t_k^n)\big)\big(B^H(t_{k+1}^n) - B^H(t_k^n)\big), \\ Y^n(0) &= X_0, \end{aligned}$$

and their continuous interpolations as

$$\begin{aligned} Y^n(t) - \Phi\big(Y^n(t)\big) = X_0 - \Phi(X_0) &+ \int_0^t f\big(\tau_s^n, Y^n(\tau_s^n)\big)\,\mathrm{d}s \\ &+ \int_0^t g\big(\tau_s^n, Y^n(\tau_s^n)\big)\,\mathrm{d}B_s^H, \end{aligned}$$

where $\tau_s^n = t_{k-1}^n$ and $Y^n(\tau_s^n) = Y^n(t_{k-1}^n)$ if $s \in [t_{k-1}^n, t_k^n)$, $1 \leqslant k \leqslant n$.

We introduce the symbol O_ω for simplicity of notation. Let (ξ_n) be a sequence of r.v.s, let ς be an a.s. nonnegative r.v., and let $(a_n) \subset (0,\infty)$ be a vanishing sequence. Then $\xi_n = O_\omega(a_n)$ means that $|\xi_n| \leqslant \varsigma \cdot a_n$ for all n.

For arbitrarily fixed $T > 0$, we use of the following assumptions on the coefficients of Eq. (1):

(A1) The function $g(t,x)$ is differentiable in x for each $t \in [0,T]$, and there exists some constants $M > 0$ and $0 < \delta, \beta \leqslant 1$ such that the following properties hold:

$$\begin{gathered} \big|g(t,x)\big| \leqslant M, \qquad \big|g_x'(t,x)\big| \leqslant M, \qquad \big|g_x'(t,x) - g_x'(t,y)\big| \leqslant M|x-y|^\delta, \\ \big|g(s,x) - g(t,x)\big| + \big|g_x'(s,x) - g_x'(t,x)\big| \leqslant M|t-s|^\beta \end{gathered}$$

for all $x, y \in \mathbb{R}$, $s,t \in [0,T]$, and for some constants $0 < \delta \leqslant 1$.

(A2) There exist some constants $L > 0$ and $0 < \beta \leqslant 1$ such that the following properties hold:

$$\begin{gathered} \big|f(t,x)\big| \leqslant L, \qquad \big|f(t,x) - f(t,y)\big| \leqslant L|x-y|, \\ \big|f(s,x) - f(t,x)\big| \leqslant L|t-s|^\beta \end{gathered}$$

for all $x, y \in \mathbb{R}$, $s,t \in [0,T]$.

Remark 2. Note that assumptions (A1)–(A2) are stronger then (H1)–(H2).

Theorem 2. *Let* $\gamma \in (\gamma_0, H)$, $1/2 < H < 1$, *and assume that there exist* $\varepsilon > 0$ *such that* $\gamma_0 < \gamma - \varepsilon < \gamma + \varepsilon < H$. *Let assumptions* (A1), (A2), *and* (H3) *be satisfied with* $\beta = 1$ *and* $1/H - 1 < \delta \leqslant 1$. *If* $\delta > \widetilde{\theta} := 1/(\gamma - \varepsilon) - 1$, *then Eq.* (1) *has a unique solution* $x \in C^\gamma(0,T)$, *and*

$$\|x - y^n\|_{1-\gamma,\infty} = O\big(\Delta_n^{2\gamma-1-\varepsilon}\big)$$

for any $0 < \varepsilon < (\gamma - \gamma_0) \wedge (H - \gamma)$, *where* γ_0 *is defined in* (3), *norm* $\|\cdot\|_{1-\gamma,\infty}$ *is defined in Section* 3.1.

The statements of Theorems 1 and 2 follow directly from the results for deterministic differential equations since for FSDE (1), we can apply the pathwise approach.

3 Preliminaries

3.1 Spaces of functions and norms

Let us now introduce some function spaces that will be used to analyse solutions of (1).

Denote by $L^p(0,T)$, $1 \leqslant p < \infty$, the usual space of Lebesgue measurable functions $f : [0,T] \to \mathbb{R}$ for which $\|f\|_{L^p(0,T)} < \infty$, where

$$\|f\|_{L^p(0,T)} = \left(\int_0^T \big|f(x)\big|^p \,\mathrm{d}x\right)^{1/p}.$$

Denote by $W_0^{\alpha,\infty}(0,T)$, $0 < \alpha < 1/2$, the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{s\in[0,T]}\left(\big|f(s)\big| + \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{1+\alpha}}\,\mathrm{d}u\right) < \infty.$$

The space $W_0^{\alpha,\infty}(0,T)$ is a Banach space with respect to the norm $\|f\|_{\alpha,\infty}$, and for $\lambda \geqslant 0$, the equivalent norm is defined by

$$\|f\|_{\alpha,\lambda} := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\left(\big|f(t)\big| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{1+\alpha}}\,\mathrm{d}s\right).$$

For any $\mu \in (0,1]$, denote by $C^\mu(0,T)$ the space of μ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$ equipped with a norm $\|f\|_\mu := |f|_\infty + |f|_\mu$, where

$$|f|_\mu := \sup_{0\leqslant s<t\leqslant T} \frac{|f(t) - f(s)|}{|s-t|^\mu}, \qquad |f|_\infty := \sup_{t\in[0,T]} \big|f(t)\big|.$$

Recall that $C^{\nu}(0,T) \subset C^{\mu}(0,T)$, $0<\mu<\nu\leqslant 1$. Moreover, $C^{1-\alpha}(0,T) \subset W_0^{\infty,\alpha}(0,T)$ for $0<\alpha<1/2$. Indeed, if $f \in C^{1-\alpha}(0,T)$ and $\lambda \geqslant 0$, then

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\left(\left|f(t)\right| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{1+\alpha}}\,\mathrm{d}s\right) \\
&\quad\leqslant \mathrm{e}^{-\lambda t}\left(|f|_\infty + |f|_{1-\alpha}\int_0^t \frac{1}{(t-s)^{2\alpha}}\,\mathrm{d}s\right) \leqslant \mathrm{e}^{-\lambda t}\left(|f|_\infty + |f|_{1-\alpha}\frac{t^{1-2\alpha}}{1-2\alpha}\right) \\
&\quad\leqslant \|f\|_{1-\alpha}\left(1+\frac{T^{1-2\alpha}}{1-2\alpha}\right).
\end{aligned}$$

Thus,

$$\|f\|_{\alpha,\lambda} \leqslant \|f\|_{1-\alpha}\left(1+\frac{T^{1-2\alpha}}{1-2\alpha}\right). \tag{4}$$

Denote by $W_T^{1-\alpha,\infty}(0,T)$, $0<\alpha<1/2$, the space of measurable functions $g\colon [0,T]\to\mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0\leqslant s<t<T}\left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}}\,\mathrm{d}y\right) < \infty.$$

Note that $W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T)$ (see [20]) and $W_T^{1-\alpha,\infty}(0,T) \subset W_T^{1-\widehat{\alpha},\infty}(0,T)$ for $\widehat{\alpha} > \alpha$. Indeed,

$$\begin{aligned}
\|g\|_{1-\widehat{\alpha},\infty,T} &= \sup_{0\leqslant s<t<T}\left(\frac{|g(t)-g(s)|(t-s)^{(\widehat{\alpha}-\alpha)}}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|(t-s)^{(\widehat{\alpha}-\alpha)}}{(y-s)^{2-\alpha}}\,\mathrm{d}y\right) \\
&\leqslant (T\vee 1)^{(\widehat{\alpha}-\alpha)} \sup_{0\leqslant s<t<T}\left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}}\,\mathrm{d}y\right) \\
&= (T\vee 1)^{(\widehat{\alpha}-\alpha)}\,\|g\|_{1-\alpha,\infty,T}\,.
\end{aligned}$$

We also denote by $W_0^{\alpha,1}(0,T)$, $0<\alpha<1/2$, the space of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha}\,\mathrm{d}s + \int_0^T\int_0^s \frac{|f(s)-f(y)|}{|s-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}s < \infty.$$

Fix $p \in (0,\infty)$. Let $\varkappa = \{\{t_0,\dots,t_n\}\colon 0=t_0<\dots<t_n=T,\ n\geqslant 1\}$ denotes a set of all possible partitions of $[0,T]$. For any $f:[0,T]\to\mathbb{R}$, define

$$v_p\bigl(f;[0,T]\bigr) = \sup_{\varkappa}\sum_{k=1}^n \left|f(t_k)-f(t_{k-1})\right|^p, \qquad V_p\bigl(f;[0,T]\bigr) = v_p^{1/p}\bigl(f;[0,T]\bigr).$$

Recall that v_p is called p-variation of f on $[0,T]$. Denote by $\mathcal{W}_p([a,b])$ (resp. $C\mathcal{W}_p([a,b])$) the class of (resp. continuous) functions on $[0,T]$ with bounded p-variation, $p \in (0,\infty)$.

Define $V_p(f) := V_p(f;[0,T])$, which is seminorm on $\mathcal{W}_p([0,T])$, and $V_p(f)$ is 0 if and only if f is a constant. For each f, $V_p(f)$ is a nonincreasing function of p, i.e., if $q<p$, then $V_p(f) \leqslant V_q(f)$. Thus, $\mathcal{W}_q([0,T]) \subseteq \mathcal{W}_p([0,T])$ if $1 \leqslant q < p < \infty$.

Let $p \geqslant 1$ and $V_{p,\infty}(f) := V_{p,\infty}(f;[0,T]) = V_p(f) + |f|_\infty$. Then $V_{p,\infty}(f)$ is a norm, and $\mathcal{W}_p([0,T])$ equipped with the p-variation norm is a Banach space.

3.2 A priori estimates

3.2.1 Riemann–Stieltjes integral

Let us first give the definition of the generalized Lebesgue–Stieltjes integral (see [27]). Consider two continuous functions f and h defined on $[a,b] \subset \mathbb{R}$. For $\alpha \in (0,1)$, define fractional derivatives

$$\mathcal{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{1+\alpha}}\,\mathrm{d}y\right)\mathbf{1}_{(a,b)}(x),$$

$$\mathcal{D}_{b-}^{1-\alpha} h(x) = \frac{(-1)^\alpha}{\Gamma(\alpha)}\left(\frac{h(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{h(x)-h(y)}{(y-x)^{2-\alpha}}\,\mathrm{d}y\right)\mathbf{1}_{(a,b)}(x).$$

Assume that $\mathcal{D}_{a+}^{\alpha} f \in L^p(a,b)$, $\mathcal{D}_{b-}^{1-\alpha} g \in L^q(a,b)$ for some $p \in (1, 1/\alpha)$, $q = p/(p-1)$. Under these assumptions (see [27]), the generalized Lebesgue–Stieltjes integral is defined as

$$\int_a^b f(x)\,\mathrm{d}h(x) = (-1)^\alpha \int_a^b \mathcal{D}_{a+}^{\alpha} f(x) \mathcal{D}_{b-}^{1-\alpha} h(x)\,\mathrm{d}x. \tag{5}$$

Notice (see [20]) that if $f \in W_0^{\alpha,1}(0,T)$ and $h \in W_T^{1-\alpha,\infty}(0,T)$, where $0<\alpha<1/2$, then the generalized Lebesgue–Stieltjes integral $\int_0^t f\,\mathrm{d}h$ exists for all $t \in [0,T]$. Furthermore, the integral $\int_0^t f\,\mathrm{d}h$ exists if $f \in W_0^{\alpha,\infty}(0,T)$.

Let $f \in C^\nu(a,b)$, $\nu \in (0,1)$, and $h \in C^\mu(a,b)$, $\mu \in (0,1)$, with $\nu + \mu > 1$, then we can choose α such that $1-\mu < \alpha < \nu$, the generalized Lebesgue–Stieltjes integral exists, it is given by (5) and coincides with the Riemann–Stieltjes integral (see [27]).

From Young's Stieltjes integrability theorem [26] the Riemann–Stieltjes integral $\int_0^t f\,\mathrm{d}h$ can be defined for functions having bounded p-variation on $[0,T]$ (see [2]).

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $p>0$, $q>0$, $1/p+1/q>1$. If f and h have no common discontinuities, then the extended Riemann–Stieltjes integral $\int_a^b f\,\mathrm{d}h$ exists, and the Love–Young inequality

$$\left|\int_a^b f\,\mathrm{d}h - f(y)\big[h(b)-h(a)\big]\right| \leqslant C_{p,q} V_q\big(f;[a,b]\big) V_p\big(h;[a,b]\big) \tag{6}$$

holds for any $y \in [a,b]$, where $C_{p,q} = \zeta(p^{-1}+q^{-1})$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$. If the function $h \in C\mathcal{W}_p([a,b])$, then the indefinite integral $\int_a^t f\,\mathrm{d}h$, $t \in [a,b]$, is a continuous function. It is easily seen that for $f \in C^\nu(a,b)$, $\nu \in (0,1)$, and $h \in C^\mu(a,b)$, $\mu \in (0,1)$, with $\nu+\mu > 1$, the Riemann–Stieltjes integral $\int_0^t f\,\mathrm{d}h$ exists.

From (6) it follows that

$$V_p\left(\int_a^{\cdot} f\,\mathrm{d}h;[a,b]\right) \leqslant \left[C_{p,q}V_q\big(f;[a,b]\big) + |f|_\infty\right]V_p\big(h;[a,b]\big) \\ \leqslant C_{p,q}V_{q,\infty}\big(f;[a,b]\big)V_p\big(h;[a,b]\big). \tag{7}$$

3.2.2 *Estimation of the generalized Lebesgue–Stieltjes integrals*

From now on, we fix $0 < \alpha < 1/2$. For any function $u \in W_0^{\alpha,\infty}(0,T)$, define

$$F_t^{(f)}(u) = \int_0^t f(s,u_s)\,\mathrm{d}s, \tag{8}$$

where f satisfies assumptions (H2) with constant $\rho = 1/\alpha$.

Proposition 1. (*See* [20].) *If $u \in W_0^{\alpha,\infty}(0,T)$, then $F^{(f)}(u) \in C^{1-\alpha}(0,T)$, and*

$$\text{(i)}\quad \big\|F^{(f)}(u)\big\|_{1-\alpha} \leqslant c^{(1)}\big(1+|u|_\infty\big),$$

$$\text{(ii)}\quad \big\|F^{(f)}(u)\big\|_{\alpha,\lambda} \leqslant \frac{c^{(2)}}{\lambda^{1-2\alpha}}\big(1+\|u\|_{\alpha,\lambda}\big)$$

for all $\lambda \geqslant 1$, where $c^{(i)}$, $i \in \{1,2\}$, are positive constants depending only on α, T, L_0, and $\|b_0\|_{L^{1/\alpha}}$.

If $u, v \in W_0^{\alpha,\infty}(0,T)$ are such that $|u|_\infty \leqslant N$ and $|v|_\infty \leqslant N$, then

$$\big\|F^{(f)}(u) - F^{(f)}(v)\big\|_{\alpha,\lambda} \leqslant \frac{c_N}{\lambda^{1-\alpha}}\|u-v\|_{\alpha,\lambda}$$

for all $\lambda \geqslant 1$, where $c_N = C_{\alpha,T}L_N\Gamma(1-\alpha)$, $C_{\alpha,T} = T^\alpha + \alpha^{-1}$, depends only on α, T, and L_N from (H2).

Given two functions $h \in W_T^{1-\alpha,\infty}(0,T)$ and $u \in W_0^{\alpha,\infty}(0,T)$, we denote

$$G_t(u) = \int_0^t u_s\,\mathrm{d}h_s, \qquad G_t^{(g)}(u) = \int_0^t g(s,u_s)\,\mathrm{d}h_s, \tag{9}$$

where g satisfies assumptions (H1) with constant $\beta > \alpha$.

Proposition 2. (*See* [20].) *If* $u \in W_0^{\alpha,\infty}(0,T)$*, then* $G(u) \in C^{1-\alpha}(0,T)$*, and*

$$\left\|G(u)\right\|_{1-\alpha} \leqslant \Lambda_\alpha(h) c_{\alpha,T} \left\|u\right\|_{\alpha,\infty},$$

where $c_{\alpha,T} = T^{1-\alpha}/(1-\alpha) + \alpha T + \max\{1, \alpha T^\alpha\}/(1-\alpha)$*,*

$$\Lambda_\alpha(h) = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \left\|h\right\|_{1-\alpha,\infty,T},$$

$\Gamma(\cdot)$ *is a Gamma function.*

Proposition 3. (*See* [20].) *If* $u \in W_0^{\alpha,\infty}(0,T)$*, then* $G^{(g)}(u) \in C^{1-\alpha}(0,T)$*, and*

$$\text{(i)} \quad \left\|G^{(g)}(u)\right\|_{1-\alpha} \leqslant \Lambda_\alpha(h) C^{(1)}\bigl(1 + \|u\|_{\alpha,\infty}\bigr),$$

$$\text{(ii)} \quad \left\|G^{(g)}(u)\right\|_{\alpha,\lambda} \leqslant \frac{\Lambda_\alpha(h) C^{(2)}}{\lambda^{1-2\alpha}}\bigl(1 + \|u\|_{\alpha,\lambda}\bigr)$$

for all $\lambda \geqslant 1$*, where the constants* $C^{(1)}$ *and* $C^{(2)}$ *are independent of* λ*,* u*,* h (*they depend on* T *and the constants* $|g(0,0)|$*,* M_0*,* α*,* β *from* (H1)).

If $u, v \in W_0^{\alpha,\infty}(0,T)$ *are such that* $|u|_\infty \leqslant N$ *and* $|v|_\infty \leqslant N$*, then*

$$\left\|G^{(g)}(u) - G^{(g)}(v)\right\|_{\alpha,\lambda} \leqslant \frac{\Lambda_\alpha(h) C_N^{(3)}}{\lambda^{1-2\alpha}} \bigl(1 + \Delta(u) + \Delta(v)\bigr)\|u - v\|_{\alpha,\lambda}$$

for all $\lambda \geqslant 1$*, where*

$$\Delta(u) = \sup_{r\in[0,T]} \int_0^r \frac{|u_r - u_s|^\delta}{|r-s|^{1+\alpha}}\,\mathrm{d}s,$$

and the constant $C_N^{(3)}$ *is independent of* λ*,* u*,* v*,* h ($C_N^{(3)}$ *depends on* T *and the constants from* (H1)).

Remark 3. If $u \in C^{1-\alpha}(0,T)$ and $\delta/(1+\delta) > \alpha$, then

$$\Delta(u) \leqslant |u|_{1-\alpha} \sup_{r\in[0,T]} \int_0^r \frac{|r-s|^{(1-\alpha)\delta}}{|r-s|^{1+\alpha}}\,\mathrm{d}s = \frac{T^{\delta-\alpha(1+\delta)}}{\delta - \alpha(1+\delta)} |u|_{1-\alpha}.$$

3.2.3 Estimation of p-seminorm

Let $f \in \mathcal{W}_p([a,b])$ and $p_1 > p > 0$, then (see [2, p. 33])

$$V_{p_1}\bigl(f;[a,b]\bigr) \leqslant \operatorname{Osc}\bigl(f;[a,b]\bigr)^{(p_1-p)/p_1} V_p^{p/p_1}\bigl(f;[a,b]\bigr), \tag{10}$$

where $\operatorname{Osc}(f;[a,b]) = \sup\{|f(x) - f(y)|\colon x, y \in [a,b]\}$.

The following result can be proven in the same way as Theorem 2 in [1] or Lemma 2 in [7].

Lemma 1. *Let a constant θ be such that $\theta < \beta \wedge \delta \leqslant 1$. If function $Q(t,x)$ satisfies assumptions* (A1) *on $[a,b]$ and $f, g \in \mathcal{W}_p([a,b])$, $p \geqslant 1$, then*

$$\begin{aligned} &V_{p/\theta}\big(Q(t,f) - Q(t,g); [a,b]\big) \\ &\quad \leqslant M V_{p/\theta}\big(f - g; [a,b]\big) + 2M|f-g|_\infty \big[V_p^\theta\big(g; [a,b]\big) + (b-a)^\beta\big]. \end{aligned}$$

Proof. For any $a \leqslant s < t \leqslant b$, we have

$$\begin{aligned} &\big[Q(t,f_t) - Q(t,g_t)\big] - \big[Q(s,f_s) - Q(s,g_s)\big] \\ &\quad = \big[Q\big(t,\, g_t + (f_t - g_t)\big) - Q\big(t,\, g_t + (f_s - g_s)\big)\big] \\ &\qquad + \big[Q\big(t,\, g_t + (f_s - g_s)\big) - Q(t,g_t)\big] - \big[Q\big(s,\, g_s + (f_s - g_s)\big) - Q(s,g_s)\big] \\ &\quad = \big[Q\big(t,\, g_t + (f_t - g_t)\big) - Q\big(t,\, g_t + (f_s - g_s)\big)\big] \\ &\qquad + \int_0^{f_s - g_s} \big[Q'_x(t,\, g_t + v) - Q'_x(s,\, g_s + v)\big]\,\mathrm{d}v = I_1 + I_2. \end{aligned}$$

Under assumptions (A1), it follows that

$$|I_1| \leqslant M\big|(f_t - g_t) - (f_s - g_s)\big|.$$

Now we estimate I_2. From assumptions (A1) and inequality $\theta < \delta$ it follows that

$$\big|Q'_x(t,u) - Q'_x(t,v)\big| \leqslant 2M|u-v|^\theta. \tag{11}$$

Indeed,

$$\begin{aligned} \frac{|Q'_x(t,u) - Q'_x(t,v)|}{|u-v|^\theta} &= \left(\frac{|Q'_x(t,u) - Q'_x(t,v)|}{|u-v|^\delta}\right)^{\theta/\delta} \big|Q'_x(t,u) - Q'_x(t,v)\big|^{1-\theta/\delta} \\ &\leqslant M^{\theta/\delta} 2M^{1-\theta/\delta} \leqslant 2M. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\big|Q'_x(t, g_t + v) - Q'_x(s, g_s + v)\big| \\ &\quad \leqslant \big|Q'_x(t, g_t + v) - Q'_x(t, g_s + v)\big| + \big|Q'_x(t, g_s + v) - Q'_x(s, g_s + v)\big| \\ &\quad \leqslant 2M|g_t - g_s|^\theta + M|t-s|^\beta \end{aligned}$$

and

$$|I_2| \leqslant M\big[2|g_t - g_s|^\theta + |t-s|^\beta\big]|f_s - g_s|.$$

Consequently,

$$\begin{aligned} &\big|\big[Q(t,f_t) - Q(t,g_t)\big] - \big[Q(s,f_s) - Q(s,g_s)\big]\big| \\ &\quad \leqslant M\big|(f_t - g_t) - (f_s - g_s)\big| + M\big[2|g_t - g_s|^\theta + |t-s|^\beta\big]|f_s - g_s|. \end{aligned}$$

Note that for any partition $\{a = s_0 < s_1 < \cdots < s_n = b\}$ of the interval $[a, b]$, applying the Minkowski inequality, we obtain

$$\left(\sum_{i=1}^{n}\left|\left[Q(s_i, f_{s_i}) - Q(s_i, g_{s_i})\right] - \left[Q(s_{i-1}, f_{s_{i-1}}) - Q(s_{i-1}, g_{s_{i-1}})\right]\right|^{p/\theta}\right)^{\theta/p}$$
$$\leqslant M\left(\sum_{i=1}^{n}\left|(f_{s_i} - g_{s_i}) - (f_{s_{i-1}} - g_{s_{i-1}})\right|^{p/\theta}\right)^{\theta/p}$$
$$+ 2M\max_{1\leqslant i\leqslant n-1}\left|f_{s_i} - g_{s_i}\right|\left(\sum_{i=1}^{n}|g_{s_i} - g_{s_{i-1}}|^{p}\right)^{\theta/p}$$
$$+ M\max_{1\leqslant i\leqslant n-1}|f_{s_i} - g_{s_i}|\left(\sum_{i=1}^{n}(s_i - s_{i-1})^{\beta p/\theta}\right)^{\theta/p}.$$

The proof is now easy to complete as

$$\left(\sum_{i=1}^{n}(s_i - s_{i-1})^{\beta p/\theta}\right)^{\theta/p} \leqslant \left(\sum_{i=1}^{n}(s_i - s_{i-1})\right)^{\beta} = (b-a)^{\beta}. \qquad \square$$

3.2.4 *Integration with respect to fBm*

The trajectories of $B^H = (B_t^H)_{t\geqslant 0}$, $0 < H < 1$, are almost surely locally γ-Hölder continuous for all $\gamma \in (0, H)$. The pathwise generalized Lebesgue–Stieltjes integral with respect to one-dimensional fBm B^H can be defined as

$$\int_a^b f(s)\,\mathrm{d}B^H(s) = (-1)^{\alpha}\int_a^b \mathcal{D}_{a+}^{\alpha} f(s)\mathcal{D}_{b-}^{1-\alpha}B^H(s)\,\mathrm{d}s \tag{12}$$

if $\mathcal{D}_{a+}^{\alpha} f \in L^1(a, b)$ (see [22, p. 225], [16]).

Assume that $1/2 < H < 1$. Then we can choose α such that $1 - H < \alpha < 1/2$. An easy computation shows that almost all trajectories of B^H belong to the space $W_T^{1-\alpha,\infty}(0, T)$. If $f \in W_T^{\alpha,1}(0, T)$, then the pathwise generalized Lebesgue–Stieltjes integral $\int_0^t f_s\,\mathrm{d}B_s^H$ exists, and we can express it according to (12) (see [20]).

4 Deterministic differential equations

Let $1 - H < \alpha < 1/2$ be fixed, where $1/2 < H < 1$. Assume that $h \in W_T^{1-\alpha,\infty}(0, T)$. Consider the deterministic differential equation

$$x_t = x_0 + \Phi(x_t) - \Phi(x_0) + \int_0^t f(s, x_s)\,\mathrm{d}s + \int_0^t g(s, x_s)\,\mathrm{d}h_s, \quad t \in [0, T], \tag{13}$$

where $x_0 \in \mathbb{R}$.

In this section, we prove the existence and uniqueness of a solution for a deterministic differential equation (13) and obtain the convergence rate for the implicit Euler approximation.

4.1 Existence and uniqueness of the solution

First, note that

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\} > 1 - H$$

if $\delta > 1/H - 1$, $\beta > 1 - H$.

Theorem 3. *Suppose that functions $f(s,x)$ and $g(s,x)$ satisfy assumptions* (H1), (H2), *respectively, with $\rho = 1/\alpha$, $1/H - 1 < \delta \leqslant 1$, $1 - H < \beta \leqslant 1$. Moreover, let assumption* (H3) *hold. For any $\widehat{\alpha}$ such that $1 - H < \alpha < \widehat{\alpha} < \alpha_0$, Eq.* (13) *has a unique solution $x \in C^{1-\widehat{\alpha}}(0,T)$.*

Corollary 1. *Suppose that the functions $f(s,x)$ and $g(s,x)$ satisfy assumptions* (H1), (H2), *respectively, with $\rho = (1-H)^{-1}$, $1/H - 1 < \delta \leqslant 1$, $1 - H < \beta \leqslant 1$, and the function $\Phi(x)$ satisfies assumption* (H3). *If $\gamma \in (\gamma_0, H)$, $\gamma_0 = 1 - \alpha_0$, then Eq.* (13) *has a unique solution $x \in C^{\gamma}(0,T)$.*

The proof of Theorem 3 is based on applying the implicit Picard iterations method. We define the implicit Picard iterations sequence as follows. Let

$$x_t^{k+1} = x_0 + \Phi\big(x_t^{k+1}\big) - \Phi\big(x_0^{k+1}\big) + \int_0^t f\big(s, x_s^k\big)\,\mathrm{d}s + \int_0^t g\big(s, x_s^k\big)\,\mathrm{d}h_s, \tag{14}$$
$$x_t^0 = x_0, \qquad x_0^k = x_0, \quad k \geqslant 0.$$

Recall that $D(x) = x - \Phi(x)$, $x \in \mathbb{R}$. We then rewrite the implicit sequence of Picard iterations (14) as follows:

$$D\big(x_t^{k+1}\big) = D(x_0) + F_t^{(f)}\big(x^k\big) + G_t^{(g)}\big(x^k\big), \quad k \geqslant 0, \tag{15}$$

where $F_t^{(f)}(\cdot)$ and $G_t^{(g)}(\cdot)$ are defined in (8) and (9), and

$$F_t^{(f)}(x^0) := \int_0^t f\big(s, x_s^0\big)\,\mathrm{d}s = \int_0^t f(s, x_0)\,\mathrm{d}s,$$
$$G_t^{(g)}(x^0) := \int_0^t g\big(s, x_s^0\big)\,\mathrm{d}h_s = \int_0^t g(s, x_0)\,\mathrm{d}h_s.$$

The implicit Picard approximation scheme is correctly defined if $F_t^{(f)}(\cdot)$ and $G_t^{(g)}(\cdot)$ make sense. Indeed, since the function $D(x)$ is strictly monotonic and surjective, it is continuous. Thus, it has the inverse function $D^{-1}(x)$, which is strictly monotonic and continuous. Consequently, we can compute $x_t^k = D^{-1}(y_t^k)$, where $y_t^k := D(x_t^k)$ if $F_t^{(f)}(\cdot)$ and $G_t^{(g)}(\cdot)$ are correctly defined. We will prove it.

Lemma 2. *Let the assumptions of Theorem* 3 *be fulfilled. Then* $x^k \in C^{1-\alpha}(0,T)$ *for all* $k \geqslant 1$*, and* $F^{(f)}(x^k)$ *and* $G^{(g)}(x^k)$ *are correctly defined.*

Proof. We first assume that $F^{(f)}(x^k), G^{(g)}(x^k) \in C^{1-\alpha}(0,T)$. From (15) it follows that we can compute x^{k+1}. Now we prove that $x^{k+1} \in C^{1-\alpha}(0,T)$. Indeed, applying (H3)(ii), we get

$$d\left|x_t^k - x_0\right| \leqslant \left|D\left(x_t^k\right) - D(x_0)\right|,$$
$$d\left|x_t^k - x_s^k\right| \leqslant \left|D\left(x_t^k\right) - D\left(x_s^k\right)\right|.$$

Thus,

$$\begin{aligned}
&\left|x_t^{k+1}\right| - |x_0| + \frac{|x_t^{k+1} - x_s^{k+1}|}{(t-s)^{1-\alpha}} \\
&\quad\leqslant \left|x_t^{k+1} - x_0\right| + \frac{|x_t^{k+1} - x_s^{k+1}|}{(t-s)^{1-\alpha}} \\
&\quad\leqslant d^{-1}\left(\left|D(x_t^{k+1}) - D(x_0)\right| + \frac{|D(x_t^{k+1}) - D(x_s^{k+1})|}{(t-s)^{1-\alpha}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|x_t^{k+1}\right| + \frac{|x_t^{k+1} - x_s^{k+1}|}{(t-s)^{1-\alpha}} \\
&\quad\leqslant |x_0| + d^{-1}\left(\left|D(x_t^{k+1}) - D(x_0)\right| + \frac{|D(x_t^{k+1}) - D(x_s^{k+1})|}{(t-s)^{1-\alpha}}\right) \\
&\quad\leqslant |x_0| + d^{-1}\left(\left|F_t^{(f)}(x^k)\right| + \left|G_t^{(g)}(x^k)\right|\right. \\
&\qquad\left. + \frac{|F_t^{(f)}(x^k) - F_s^{(f)}(x^k)| + |G_t^{(g)}(x^k) - G_s^{(g)}(x^k)|}{(t-s)^{1-\alpha}}\right) \\
&\quad\leqslant |x_0| + d^{-1}\left(\|F^{(f)}(x^k)\|_{1-\alpha} + \|G^{(g)}(x^k)\|_{1-\alpha}\right).
\end{aligned}$$

Consequently,

$$\left\|x^{k+1}\right\|_{1-\alpha} \leqslant |x_0| + d^{-1}\left(\left\|F^{(f)}\left(x^k\right)\right\|_{1-\alpha} + \left\|G^{(g)}\left(x^k\right)\right\|_{1-\alpha}\right) \tag{16}$$

and $x^{k+1} \in C^{1-\alpha}(0,T)$.

It remains to prove that $F^{(f)}(x^k), G^{(g)}(x^k) \in C^{1-\alpha}(0,T)$. We first prove that $F^{(f)}(x^0) \in C^{1-\alpha}(0,T)$. From assumption (H2) it follows that

$$\begin{aligned}\left|F_t^{(f)}\left(x^0\right)\right| &\leqslant \int_0^t \left|f(u,x_0\right| \mathrm{d}u \leqslant L_0|x_0|t + \int_0^t \left|b_0(s)\right| \mathrm{d}s \\ &\leqslant L_0|x_0|T + T^{1-\alpha}\left(\int_0^T |b_0(s)|^{1/\alpha}\,\mathrm{d}s\right)^\alpha \\ &= L_0|x_0|T + T^{1-\alpha}\,\|b_0\|_{L^{1/\alpha}}\end{aligned}$$

and

$$\left|F_t^{(f)}\left(x^0\right) - F_s^{(f)}\left(x^0\right)\right| \leqslant L_0|x_0|(t-s) + (t-s)^{1-\alpha}\,\|b_0\|_{L^{1/\alpha}}.$$

Thus,

$$\left\|F^{(f)}(x^0)\right\|_{1-\alpha} \leqslant L_0|x_0|\left(T+T^\alpha\right) + \left(1+T^{1-\alpha}\right)\|b_0\|_{L^{1/\alpha}}$$

and $F^{(f)}(x^0) \in C^{1-\alpha}(0,T)$.

To prove that $G^{(g)}(x^0) \in C^{1-\alpha}(0,T)$, it is enough to prove that $g(\cdot,x_0) \in W_0^{\alpha,\infty}(0,T)$. Indeed, then, applying Proposition 2, we will get the desired result.

Note that $\beta > \alpha$ since $\alpha < \alpha_0$. From assumption (H1) it follows that

$$\begin{aligned}&\left|g(t,x_0)\right| + \int_0^t \frac{|g(t,x_0)-g(s,x_0)|}{(t-s)^{1+\alpha}}\,\mathrm{d}s \\ &\quad\leqslant \left|g(t,x_0)-g(0,0)\right| + \left|g(0,0)\right| + M_0\int_0^t \frac{|t-s|^\beta}{(t-s)^{1+\alpha}}\,\mathrm{d}s \\ &\quad\leqslant M_0t^\beta + M_0|x_0| + \left|g(0,0)\right| + M_0\int_0^t (t-s)^{\beta-1-\alpha}\,\mathrm{d}s \\ &\quad= M_0t^\beta + M_0|x_0| + \left|g(0,0)\right| + \frac{M_0}{\beta-\alpha}\,t^{\beta-\alpha}.\end{aligned}$$

Thus, we proved $g(\cdot,x_0) \in W_0^{\alpha,\infty}(0,T)$. Consequently, $G^{(g)}(x^0) \in C^{1-\alpha}(0,T)$.

From above it follows that $x^1 \in C^{1-\alpha}(0,T)$. Applying Propositions 1 and 3, we conclude that $F^{(f)}(x^1), G^{(g)}(x^1) \in C^{1-\alpha}(0,T)$. Thus, $x^2 \in C^{1-\alpha}(0,T)$. Repeating the previous proof, we get that $F_t^{(f)}(x^k), G_t^{(g)}(x^k) \in C^{1-\alpha}(0,T)$ for all $k \geqslant 0$. □

Now we will prove that $\sup_n \|x^n\|_{1-\alpha} < \infty$. We will do it in two steps.

Lemma 3. *Let the assumptions of Theorem* 3 *be fulfilled. Then there are* $\lambda_0 > 0$ *and a constant* C *such that*

$$\sup_k \left\|x^k\right\|_{\alpha,\lambda_0} \leqslant 1 + 2|x_0| + C.$$

Proof. From Lemma 2 we have that $x^k \in W_0^{\alpha,\infty}(0,T)$ and $F^{(f)}(x^k), G^{(g)}(x^k) \in W_0^{\alpha,\infty}(0,T)$, $k \geqslant 0$. Thus, norms $\|\cdot\|_{\alpha,\lambda}$ of x^k, $F^{(f)}(x^k)$, and $G^{(g)}(x^k)$ are finite. Similarly as in (16), we obtain

$$\|x^{k+1}\|_{\alpha,\lambda} \leqslant |x_0| + d^{-1}\big[\|F^{(f)}(x^k)\|_{\alpha,\lambda} + \|G_t^{(g)}(x^k)\|_{\alpha,\lambda}\big],$$

and from Propositions 1 and 3 we conclude that

$$\begin{aligned}\|x^{k+1}\|_{\alpha,\lambda} &\leqslant |x_0| + d^{-1}\left(\frac{c^{(2)}}{\lambda^{1-2\alpha}}\big(1+\|x^k\|_{\alpha,\lambda}\big) + \frac{\varLambda_\alpha(h)C^{(2)}}{\lambda^{1-2\alpha}}\big(1+\|x^k\|_{\alpha,\lambda}\big)\right)\\ &\leqslant |x_0| + \frac{c^{(2)}+\varLambda_\alpha(h)C^{(2)}}{d}\,\frac{1}{\lambda^{1-2\alpha}}\big(1+\|x^k\|_{\alpha,\lambda}\big).\end{aligned}$$

Assume that $\lambda = \lambda_0$ is large enough and such that

$$\frac{c^{(2)}+\varLambda_\alpha(h)C^{(2)}}{d}\,\frac{1}{\lambda_0^{1-2\alpha}} \leqslant \frac{1}{2}.$$

Then for $k \geqslant 1$, we get

$$\begin{aligned}\|x^{k+1}\|_{\alpha,\lambda_0} &\leqslant \frac{1}{2}\big(1+\|x^k\|_{\alpha,\lambda_0}\big) + |x_0|\\ &\leqslant \frac{1}{2}\left(1+\frac{1}{2}\big(1+\|x^{k-1}\|_{\alpha,\lambda_0}\big)+|x_0|\right) + |x_0|\\ &\leqslant \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right) + \frac{1}{8}\|x^{k-2}\|_{\alpha,\lambda_0} + \left(1+\frac{1}{2}+\frac{1}{4}\right)|x_0|\\ &\leqslant 1 + \frac{1}{2^k}\|x^1\|_{\alpha,\lambda_0} + 2|x_0|.\end{aligned}$$

From (4) and (16) it follows that

$$\|x^1\|_{\alpha,\lambda_0} \leqslant \left(1+\frac{T^{1-2\alpha}}{1-2\alpha}\right)\big(|x_0| + d^{-1}\big(\|F^{(f)}(x^0)\|_{1-\alpha} + \|G^{(g)}(x^0)\|_{1-\alpha}\big)\big).$$

Since $\|F^{(f)}(x^0)\|_{1-\alpha}$ and $\|G^{(g)}(x^0)\|_{1-\alpha}$ are bounded, then there exists a constant C such that $\|x^1\|_{\alpha,\lambda_0} \leqslant C$. Thus,

$$\|x^{k+1}\|_{\alpha,\lambda_0} \leqslant 1 + 2|x_0| + C. \qquad \square$$

Lemma 4. *Let the assumptions of Theorem* 3 *be fulfilled. Then* $\sup_k \|x^k\|_{1-\alpha} < \infty$.

Proof. Let $f \in W_0^{\alpha,\infty}(0,T)$. Clearly,

$$\mathrm{e}^{-\lambda T}|f|_\infty \leqslant \mathrm{e}^{-\lambda T}\|f\|_{\alpha,\infty} \leqslant \|f\|_{\alpha,\lambda}$$

and

$$|f|_\infty \leqslant \mathrm{e}^{\lambda T}\|f\|_{\alpha,\lambda}. \tag{17}$$

From (16), Propositions 1–3, and Lemma 3 it follows that

$$\begin{aligned}\|x^{k+1}\|_{1-\alpha} &\leqslant |x_0| + d^{-1}\big[\|F^{(f)}(x^k)\|_{1-\alpha} + \|G^{(g)}(x^k)\|_{1-\alpha}\big] \\ &\leqslant |x_0| + d^{-1}\big[c^{(1)}\big(1+|x^k|_\infty\big) + \Lambda_\alpha(h)C^{(1)}\big(1+\|x^k\|_{\alpha,\infty}\big)\big] \\ &\leqslant |x_0| + d^{-1}\big[c^{(1)}\big(1+\mathrm{e}^{\lambda_0 T}\|x^k\|_{\alpha,\lambda_0}\big) \\ &\quad + \Lambda_\alpha(h)C^{(1)}\big(1+\mathrm{e}^{\lambda_0 T}\|x^k\|_{\alpha,\lambda_0}\big)\big] \\ &\leqslant |x_0| + d^{-1}\big[c^{(1)} + \Lambda_\alpha(h)C^{(1)}\big]\big(1+\big(1+2|x_0|+C\big)\mathrm{e}^{\lambda_0 T}\big). \qquad \square\end{aligned}$$

We can now prove our main result.

Proof of Theorem 3. *Existence of the solution.* Since $\sup_n \|x^n\|_{1-\alpha} < \infty$ and $\widehat{\alpha} > \alpha$, then the sequence of functions (x^n) is relatively compact in $C^{1-\widehat{\alpha}}(0,T)$. Thus, we can choose a subsequence x^{n_k}, which converges in $C^{1-\widehat{\alpha}}(0,T)$ to a limit $x \in C^{1-\widehat{\alpha}}(0,T)$, i.e.,

$$\|x^{n_k} - x\|_{1-\widehat{\alpha}} \underset{n_k\to\infty}{\longrightarrow} 0.$$

We show that x is a solution of Eq. (13). For simplicity of notation, write n instead of n_k. Recall that

$$D\big(x_t^n\big) = D(x_0) + \int_0^t f\big(s, x_s^{n-1}\big)\,\mathrm{d}s + \int_0^t g\big(s, x_s^{n-1}\big)\,\mathrm{d}h_s.$$

Thus,

$$\begin{aligned}&\left|D(x_\cdot) - D(x_0) - \int_0^\cdot f(s,x_s)\,\mathrm{d}s - \int_0^\cdot g(s,x_s)\,\mathrm{d}h_s\right|_\infty \\ &\quad\leqslant \big|D(x_\cdot) - D\big(x_\cdot^n\big)\big|_\infty + \left|\int_0^\cdot \big[f\big(s,x_s^{n-1}\big) - f(s,x_s)\big]\,\mathrm{d}s\right|_\infty \\ &\qquad + \left|\int_0^\cdot \big[g\big(s,x_s^{n-1}\big) - g(s,x_s)\big]\,\mathrm{d}h_s\right|_\infty.\end{aligned} \tag{18}$$

Since $|x - x^n|_\infty \to 0$ and the function D is continuous, then the first term converges to zero. It remains to prove that the second and third terms also converge to zero.

First, observe that there exists a constant N such that $\sup_n \|x^n\|_{1-\widehat{\alpha}} \leqslant N$ and $\|x\|_{1-\widehat{\alpha}} \leqslant N$. It follows from Lemma 4.

We next turn to estimate the second term in (18). Applying (17), (4), and Proposition 1 for any $\lambda > 0$, it follows that

$$\begin{aligned}
&\left| \int_0^{\cdot} \left[f\left(s, x_s^{n-1}\right) - f(s, x_s)\right] \mathrm{d}s \right|_\infty \\
&\quad \leqslant \mathrm{e}^{\lambda T} \left\| \int_0^{\cdot} \left[f(s, x_s) - f\left(s, x_s^{n-1}\right)\right] \mathrm{d}s \right\|_{\widehat{\alpha},\lambda} \\
&\quad \leqslant \mathrm{e}^{\lambda T} \frac{c_N}{\lambda^{1-\widehat{\alpha}}} \left(1 + \frac{T^{1-2\widehat{\alpha}}}{1-2\widehat{\alpha}}\right) \left\|x - x^{n-1}\right\|_{1-\widehat{\alpha}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

To estimate the third term in (18), we note that $h \in W_T^{1-\widehat{\alpha},\infty}(0,T)$ for $\widehat{\alpha} > \alpha$ and $G^{(g)}(x^n), G^{(g)}(x) \in C^{1-\widehat{\alpha}}(0,T)$. Applying Proposition 3 and Remark 3, we obtain

$$\left\| \int_0^{\cdot} \left[g(s, x_s) - g\left(s, x_s^{n-1}\right)\right] \mathrm{d}h_s \right\|_{\widehat{\alpha},\lambda} \leqslant \frac{\Lambda_{\widehat{\alpha}}(h) C_N^{(3)}}{\lambda^{1-2\widehat{\alpha}}} \left(1 + C^{(4)}\right) \left\|x - x^{n-1}\right\|_{\widehat{\alpha},\lambda},$$

where

$$C^{(4)} := \frac{T^{\delta-\widehat{\alpha}(1+\delta)}}{\delta - \widehat{\alpha}(1+\delta)} \left(\sup_n \left\|x^{n-1}\right\|_{1-\widehat{\alpha}} + \|x\|_{1-\widehat{\alpha}}\right) \leqslant 2N \frac{T^{\delta-\widehat{\alpha}(1+\delta)}}{\delta - \widehat{\alpha}(1+\delta)}. \tag{19}$$

From definition of α_0 it follows that $\delta - \widehat{\alpha}(1+\delta) > 0$.

Hence

$$\begin{aligned}
&\left| \int_0^{\cdot} \left[g(s, x_s) - g\left(s, x_s^{n-1}\right)\right] \mathrm{d}h_s \right|_\infty \\
&\quad \leqslant \mathrm{e}^{\lambda T} \left\| \int_0^{\cdot} \left[g(s, x_s) - g\left(s, x_s^{n-1}\right)\right] \mathrm{d}h_s \right\|_{\widehat{\alpha},\lambda} \\
&\quad \leqslant \frac{\Lambda_{\widehat{\alpha}}(h) C_N^{(3)} \mathrm{e}^{\lambda T}}{\lambda^{1-2\widehat{\alpha}}} \left(1 + C^{(4)}\right) \left(1 + \frac{T^{1-2\widehat{\alpha}}}{1-2\widehat{\alpha}}\right) \left\|x - x^{n-1}\right\|_{1-\widehat{\alpha}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

It follows from (17) and (4). The proof is completed.

Uniqueness of the solution. Let x and $\widetilde{x}$ be two solutions belonging to $C^{1-\widehat{\alpha}}(0,T) \subset W_0^{\widehat{\alpha},\infty}(0,T)$, and choose N such that $\|x\|_{\widehat{\alpha},\lambda} \leqslant N$ and $\|\widetilde{x}\|_{\widehat{\alpha},\lambda} \leqslant N$. The existence of such N follows from inequality (4). The existence of such N follows from inequality (4). Then from Propositions 1 and 3 it follows that

$$\begin{aligned}
\|x - \widetilde{x}\|_{\widehat{\alpha},\lambda} &\leqslant d^{-1} \left(\left\|F^{(f)}(x) - F^{(f)}(\widetilde{x})\right\|_{\widehat{\alpha},\lambda} + \left\|G^{(g)}(x) - G^{(g)}(\widetilde{x})\right\|_{\widehat{\alpha},\lambda} \right) \\
&\leqslant d^{-1} \left(\frac{c_N}{\lambda^{1-\widehat{\alpha}}} \|x - \widetilde{x}\|_{\widehat{\alpha},\lambda} + \frac{\Lambda_{\widehat{\alpha}}(h) C_N^{(3)} \left(1 + C^{(4)}\right)}{\lambda^{1-2\widehat{\alpha}}} \|x - \widetilde{x}\|_{\widehat{\alpha},\lambda} \right),
\end{aligned}$$

where $C^{(4)}$ defined in (19). Assume that $\lambda = \lambda_1$ is sufficiently large such that

$$d^{-1}\left(\frac{c_N}{\lambda_1^{1-\widehat{\alpha}}} + \frac{\Lambda_{\widehat{\alpha}}(h) C_N^{(3)}(1 + C^{(4)})}{\lambda_1^{1-2\widehat{\alpha}}}\right) \leqslant \frac{1}{2}.$$

Then

$$\|x - \widetilde{x}\|_{\widehat{\alpha},\lambda_1} \leqslant \frac{1}{2}\,\|x - \widetilde{x}\|_{\widehat{\alpha},\lambda_1}.$$

Thus, $|x - \widetilde{x}|_\infty = 0$ and, in consequence, $x = \widetilde{x}$. □

Proof of Corollary 1. Fix $\gamma_0 < \gamma < H$. Assume that there exist $\varepsilon > 0$ such that $\gamma + \varepsilon < H$. Let $\alpha = 1 - \gamma - \varepsilon$ and $\widehat{\alpha} = 1 - \gamma$. Then $1 - H < \alpha < \widehat{\alpha} < \alpha_0 = 1 - \gamma_0$ and $L^{1/(1-H)}(0,T) \subset L^{1/\alpha}(0,T)$. Thus, $b_0 \in L^{1/\alpha}(0,T)$. This completes the proof. □

4.2 The implicit Euler approximations scheme

Recall that $1 - H < \alpha < 1/2$ and $h \in W_T^{1-\alpha,\infty}(0,T)$. For Eq. (13), we define the implicit Euler approximations as

$$\begin{aligned} D\big(y^n\big(t_{k+1}^n\big)\big) &= D\big(y^n\big(t_k^n\big)\big) + f\big(t_k^n, y^n\big(t_k^n\big)\big)\Delta_n + g\big(t_k^n, y^n\big(t_k^n\big)\big)\big(h\big(t_{k+1}^n\big) - h\big(t_k^n\big)\big), \\ y^n(0) &= x_0 \end{aligned} \tag{20}$$

and their continuous interpolations as

$$D\big(y_t^n\big) = D(x_0) + F_t^{(f,\tau^n)}\big(y^n\big) + G_t^{(g,\tau^n)}\big(y^n\big),$$

where $\tau_s^n = t_{k-1}^n$ and $y^n(\tau_s^n) = y^n(t_{k-1}^n)$ if $s \in [t_{k-1}^n, t_k^n)$, $1 \leqslant k \leqslant n$,

$$F_t^{(f,\tau^n)}\big(y^n\big) = \int_0^t f\big(\tau_s^n, y^n\big(\tau_s^n\big)\big)\,\mathrm{d}s, \qquad G_t^{(g,\tau^n)}\big(y^n\big) = \int_0^t g\big(\tau_s^n, y^n\big(\tau_s^n\big)\big)\,\mathrm{d}h_s.$$

The function $G^{(g,\tau^n)}(y^n)$ is continuous since h is continuous.

We fix $\widehat{\alpha}$ such that $1 - H < \alpha < \widehat{\alpha} < 1/2$, and throughout this section, we denote $\theta = (1 - \widehat{\alpha} - \varepsilon)^{-1} - 1$, where $\varepsilon > 0$ is such that $\widehat{\alpha} + \varepsilon < 1/2$. Note that

$$\frac{1}{H} - 1 < \theta < 1, \qquad \frac{\theta}{1+\theta} = \widehat{\alpha} + \varepsilon > 1 - H.$$

Set

$$\widehat{\alpha}_0 = \min\left\{\frac{1}{2}, \beta, \frac{\theta}{1+\theta}\right\}.$$

It is easy to check that $\widehat{\alpha}_0 \leqslant \alpha_0 \leqslant 1/2$ if $\beta > 1 - H$ and $\delta > \theta$, where α_0 is defined in (3).

Theorem 4. *Let assumptions* (A1), (A2), (H3) *be satisfied with* $\beta = 1$ *and* $1/H - 1 < \delta \leqslant 1$. *If* $\delta > \theta$, *then Eq.* (13) *has a unique solution* $x \in C^{1-\widehat{\alpha}}(0,T)$, *and*

$$\|x - y^n\|_{\widehat{\alpha},\infty} = O\big(\Delta_n^{(1-\widehat{\alpha})(1-\theta)}\big)$$

for any fixed $\widehat{\alpha}$ *such that* $1 - H < \alpha < \widehat{\alpha} < 1/2$.

Corollary 2. *Fix* $\gamma \in (\gamma_0, H)$, $\gamma_0 = 1 - \alpha_0$, *and assume that there exist* $\varepsilon > 0$ *such that* $\gamma_0 < \gamma - \varepsilon < \gamma + \varepsilon < H$. *Let assumptions* (A1), (A2), *and* (H3) *be satisfied with* $\beta = 1$ *and* $1/H - 1 < \delta \leqslant 1$. *If* $\delta > \widetilde{\theta} := 1/(\gamma - \varepsilon) - 1$, *then Eq.* (13) *has a unique solution* $x \in C^{\gamma}(0,T)$, *and*

$$\|x - y^n\|_{1-\gamma,\infty} = O\big(\Delta_n^{2\gamma-1-\varepsilon}\big)$$

for any $0 < \varepsilon < (\gamma - \gamma_0) \wedge (H - \gamma)$.

It is easy to check that Eq. (13) has a unique solution $x \in C^{1-\widehat{\alpha}}(0,T)$ under the assumptions of Theorem 4. First, note that assumption (H1)(ii) holds for the constant θ. This follows from (11). Thus, assumptions (A1) and (A2) with the constants $\beta = 1$ and θ imply the conditions of Theorem 3 and for any $\widetilde{\alpha}$ such that $1 - H < \alpha < \widetilde{\alpha} < \widehat{\alpha}_0$, Eq. (13) has a unique solution $x \in C^{1-\widetilde{\alpha}}(0,T)$. Since $\widehat{\alpha}_0 = \widehat{\alpha} + \varepsilon$, then for a given $\widehat{\alpha}$, Eq. (13) has a unique solution $x \in C^{1-\widehat{\alpha}}(0,T)$.

Next, we consider the convergence rate of the implicit Euler approximation y^n to the solution x. Since $h \in W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T)$, then $h \in C\mathcal{W}_p([0,T])$ and

$$V_p\big(h;[s,t]\big) \leqslant h_{1-\alpha}(t-s)^{1-\alpha}, \tag{21}$$

where $p = (1-\alpha)^{-1}$. From now on, we assume that $p = (1-\alpha)^{-1}$.

The implicit Euler approximations scheme is correctly defined. From the recursive expression (20) we calculate $D(y^n(t_{k+1}^n))$. The properties of the function $D(x)$ give us a single value of $y^n(t_{k+1}^n)$. Since $D(y_t^n)$ is a continuous function, then y^n is a continuous function. Indeed, since $D^{-1}(x)$ and $D(y_t^n)$ are continuous functions, then y^n is continuous function.

Now we prove that $y^n \in C\mathcal{W}_p([0,T])$ for any $n \geqslant 1$. First, observe that the function $g(\tau_\cdot^n, y^n(\tau_\cdot^n))$ has bounded variation for any fixed n. Thus, $g(\tau_\cdot^n, y^n(\tau_\cdot^n)) \in \mathcal{W}_1([0,T]) \subset \mathcal{W}_q([0,T])$, $q \geqslant 1$, and from inequality (7) it follows that

$$V_p\big(G^{(g,\tau^n)}\big(y^n\big);[0,T]\big) \leqslant \big[C_{1,p}V_1\big(g\big(\tau^n, y^n\big(\tau^n\big)\big);[0,T]\big) + M\big]V_p\big(h;[0,T]\big).$$

It is easy to check that

$$V_1\big(F^{(f,\tau^n)}\big(y^n\big);[0,T]\big) \leqslant \int_0^T \big|f\big(\tau_s^n, y^n\big(\tau_s^n\big)\big)\big|\,\mathrm{d}s \leqslant LT. \tag{22}$$

Consequently, $V_p(D(y^n);[0,T]) < \infty$, and from (H3) it follows that

$$V_p\big(y^n;[0,T]\big) \leqslant d^{-1}V_p\big(D\big(y^n\big);[0,T]\big). \tag{23}$$

We have thus proved $y^n \in C\mathcal{W}_p([0,T])$.

The proof of Theorem 4 requires two auxiliary lemmas.

Lemma 5. *Let assumptions* (A1), (A2), *and* (H3) *be satisfied with* $\beta = 1$ *and* $1/H - 1 < \delta \leqslant 1$. *For* $\theta := (1 - \widehat{\alpha} - \varepsilon)^{-1} - 1 < \delta$, *we get*

$$V_p\big(y^n; [0,T]\big) \leqslant d^{-1/(1-\theta)}\big[2MC_{p,p/\theta}h_{1-\alpha}T^{1-\alpha}\big]^{1/(1-\theta)} + (1-\theta)^{-1}d^{-1}\big[TL + M\big(2TC_{p,p/\theta} + 1\big)h_{1-\alpha}T^{1-\alpha}\big].$$

Proof. From (23) it follows that

$$V_p\big(y^n; [0,T]\big) \leqslant d^{-1}V_p\big(F^{(f,\tau^n)}\big(y^n\big); [0,T]\big) + d^{-1}V_p\big(G^{(g,\tau^n)}\big(y^n\big); [0,T]\big). \tag{24}$$

An easy computation shows that for any constant $\theta < \delta$, we get

$$V_{p/\theta}\big(g\big(\tau^n, y^n\big(\tau^n\big)\big); [0,T]\big) \leqslant 2M\big(T + V_p^\theta\big(y^n; [0,T]\big)\big). \tag{25}$$

Indeed, let $\varkappa = \{s_i,\ i = 0, 1, \dots, m\}$ be any partition of the interval $[0,T]$. Recall that $\theta < 1$. Then from assumptions (A1), inequality (11) for the function $g(t,x)$, and the Minkowski inequality we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^m \big|g\big(\tau^n(s_i), y^n\big(\tau^n(s_i)\big)\big) - g\big(\tau^n(s_{i-1}), y^n\big(\tau^n(s_{i-1})\big)\big)\big|^{p/\theta}\right)^{\theta/p}\\
&\quad\leqslant \left(\sum_{i=1}^m \big|g\big(\tau^n(s_i), y^n\big(\tau^n(s_i)\big)\big) - g\big(\tau^n(s_{i-1}), y^n\big(\tau^n(s_i)\big)\big)\big|^{p/\theta}\right)^{\theta/p}\\
&\qquad + \left(\sum_{i=1}^m \big|g\big(\tau^n(s_{i-1}), y^n\big(\tau^n(s_i)\big)\big) - g\big(\tau^n(s_{i-1}), y^n\big(\tau^n(s_{i-1})\big)\big)\big|^{p/\theta}\right)^{\theta/p}\\
&\quad\leqslant M\left(\sum_{i=1}^m \big|\tau^n(s_i) - \tau^n(s_{i-1})\big|^{p/\theta}\right)^{\theta/p}\\
&\qquad + 2M\left(\sum_{i=1}^m \big|y^n\big(\tau^n(s_i)\big) - y^n\big(\tau^n(s_{i-1})\big)\big|^{p}\right)^{\theta/p}\\
&\quad\leqslant M\sum_{i=1}^m \big|\tau^n(s_i) - \tau^n(s_{i-1})\big| + 2M\left(\sum_{i=1}^m \big|y^n\big(\tau^n(s_i)\big) - y^n\big(\tau^n(s_{i-1})\big)\big|^{p}\right)^{\theta/p}\\
&\quad\leqslant MT + 2MV_p^\theta\big(y^n\big(\tau^n\big); [0,T]\big) \leqslant 2M\big(T + V_p^\theta\big(y^n; [0,T]\big)\big).
\end{aligned}$$

This finishes the proof of (25).

Since $p^{-1} + \theta p^{-1} = (1+\theta)(1-\alpha) > 1$, we can apply inequality (25) to the second term of inequality (24). From (7) it follows that

$$V_p\big(G^{(g,\tau^n)}\big(y^n\big); [0,T]\big) \leqslant M\big[2C_{p,p/\theta}\big[T + V_p^\theta\big(y^n; [0,T]\big)\big] + 1\big]V_p\big(h; [0,T]\big). \tag{26}$$

By (24), (22), (26), and W.H. Young's inequality

$$\left|ab\right| \leqslant \theta|a|^{1/\theta} + (1-\theta)|b|^{1/(1-\theta)}$$

it is obvious that

$$\begin{aligned}
&V_p\left(y^n;[0,T]\right)\\
&\quad\leqslant d^{-1}2MC_{p,p/\theta}V_p^\theta\left(y^n;[0,T]\right)V_p\left(h;[0,T]\right)\\
&\qquad + d^{-1}\left[TL + M(2TC_{p,p/\theta}+1)V_p\left(h;[0,T]\right)\right]\\
&\quad\leqslant \theta V_p\left(y^n;[0,T]\right) + (1-\theta)d^{-1/(1-\theta)}\left[2MC_{p,p/\theta}V_p\left(h;[0,T]\right)\right]^{1/(1-\theta)}\\
&\qquad + d^{-1}\left[TL + M(2TC_{p,p/\theta}+1)V_p\left(h;[0,T]\right)\right].
\end{aligned}$$

From this inequality and (21) follows our result. □

Lemma 6. *Let the assumptions of Lemma 5 be satisfied. Then $y^n \in C^{1-\alpha}(0,T)$ for all $n \geqslant 1$.*

Proof. First, we prove that $F^{(f,\tau^n)}(y^n), G^{(g,\tau^n)}(y^n) \in C^{1-\alpha}(0,T)$. Note that for $s<t$,

$$\left|F_t^{(f,\tau^n)}\left(y^n\right) - F_s^{(f,\tau^n)}\left(y^n\right)\right| = \left|\int_s^t f\left(\tau_u^n, y^n\left(\tau_u^n\right)\right)\mathrm{d}u\right| \leqslant L(t-s).$$

Thus, $F^{(f,\tau^n)}(y^n) \in C^1(0,T) \subset C^{1-\alpha}(0,T)$ for all $n \geqslant 1$.

Assume that $s \in [t_k^n, t_{k+1}^n)$ for some $0 \leqslant k \leqslant n-1$ and $t \leqslant t_{k+1}^n$. Then

$$\left|G_t^{(g,\tau^n)}\left(y^n\right) - G_s^{(g,\tau^n)}\left(y^n\right)\right| = \left|\int_s^t g\left(\tau_u^n, y^n\left(\tau_u^n\right)\right)\mathrm{d}h_u\right| \leqslant M|h|_{1-\alpha}(t-s)^{1-\alpha}.$$

If $t > t_{k+1}^n$, then

$$\begin{aligned}
\left|G_t^{(g,\tau^n)}\left(y^n\right) - G_s^{(g,\tau^n)}\left(y^n\right)\right| &= \left|\int_s^t g\left(\tau_u^n, y^n\left(\tau_u^n\right)\right)\mathrm{d}h_u\right|\\
&\leqslant \left|\int_s^{t_{k+1}^n} g\left(\tau_u^n, y^n\left(\tau_u^n\right)\right)\mathrm{d}h_u\right| + \left|\int_{t_{k+1}^n}^t g\left(\tau_u^n, y^n\left(\tau_u^n\right)\right)\mathrm{d}h_u\right|\\
&= I_1 + I_2
\end{aligned}$$

and

$$\begin{aligned}
I_1 &\leqslant \left|g\left(\tau^n, y^n\left(\tau^n\right)\right)\right|_\infty \left|h\left(t_{k+1}^n\right) - h(s)\right|\\
&\leqslant \left|g\left(\tau^n, y^n\left(\tau^n\right)\right)\right|_\infty \left(\left|h\left(t_{k+1}^n\right) - h(s)\right|^p + \left|h(t) - h\left(t_{k+1}^n\right)\right|^p\right)^{1/p}\\
&\leqslant \left|g\left(\tau^n, y^n\left(\tau^n\right)\right)\right|_\infty |h|_{1-\alpha}(t-s)^{1-\alpha}.
\end{aligned}$$

This follows from inequality (21). Under the Love–Young inequality, (21), and (25) it follows that

$$\begin{aligned}I_2 &\leqslant C_{p,p} V_{p,\infty}\big(g\big(\tau^n, y^n(\tau^n)\big); \big[t^n_{k+1}, t\big]\big) V_p\big(h; \big[t^n_{k+1}, t\big]\big)\\ &\leqslant C_{p,p} V_{p,\infty}\big(g\big(\tau^n, y^n(\tau^n)\big); [0,T]\big) V_p\big(h; [s,t]\big)\\ &\leqslant M\big[2C_{p,p}\big[T + V_p\big(y^n; [0,T]\big)\big] + 1\big]|h|_{1-\alpha}(t-s)^{1-\alpha}.\end{aligned}$$

Thus

$$\big|G^{(g,\tau^n)}_t\big(y^n\big) - G^{(g,\tau^n)}_s(y^n)\big| \leqslant 2MC_{p,p}\big[T + V_p\big(y^n; [0,T]\big) + 1\big] h_{1-\alpha}(t-s)^{1-\alpha}.$$

Lemma 5 shows that there is a constant C independent of n such that $V_p(y^n; [0,T]) \leqslant C$. Thus, $G^{(g,\tau^n)}(y^n) \in C^{1-\alpha}(0,T)$ for all $n \geqslant 1$. Consequently, $D(y^n) \in C^{1-\alpha}(0,T)$ and $y^n \in C^{1-\alpha}(0,T)$ for all $n \geqslant 1$. Indeed, there is a constant $\widehat{C}$ independent of n such that

$$\big|y^n_t - y^n_s\big| \leqslant d^{-1}\big|D\big(y^n_t\big) - D\big(y^n_s\big)\big| \leqslant \widehat{C}(t-s)^{1-\alpha}. \qquad \square$$

Proof of Theorem 4. From the conditions of the theorem, as already mentioned at the beginning of the subsection, there is a solution to Eq. (13) such that $x \in C^{1-\widehat{\alpha}}(0,T)$. As it was mentioned in Section 3.1, we have $C^{1-\widehat{\alpha}}(0,T) \subset W^{\infty,\widehat{\alpha}}_0(0,T)$. From Lemma 6 it follows that $y^n, F^{(f,\tau^n)}\big(y^n\big), G^{(g,\tau^n)}\big(y^n\big) \in C^{1-\widehat{\alpha}}(0,T)$. Thus, $x - y^n \in W^{\infty,\widehat{\alpha}}_0(0,T)$. Recall that elements of the space $W^{\infty,\widehat{\alpha}}_0(0,T)$ have finite norm $\|\cdot\|_{\widehat{\alpha},\lambda}$ with $\lambda \geqslant 0$.

Our proof starts with the observation that from Propositions 1 and 3 we obtain

$$\begin{aligned}\big\|x - y^n\big\|_{\widehat{\alpha},\lambda} &\leqslant \frac{1}{d}\big(\big\|F^{(f)}(x) - F^{(f)}(y^n)\big\|_{\widehat{\alpha},\lambda} + \big\|F^{(f)}\big(y^n\big) - F^{(f,\tau^n)}\big(y^n\big)\big\|_{\widehat{\alpha},\lambda}\\ &\quad + \big\|G^{(g)}(x) - G^{(g)}(y^n)\big\|_{\widehat{\alpha},\lambda} + \big\|G^{(g)}\big(y^n\big) - G^{(g,\tau^n)}\big(y^n\big)\big\|_{\widehat{\alpha},\lambda}\big)\\ &\leqslant \frac{1}{d}\bigg(\frac{C_{\widehat{\alpha},T} L\Gamma(1-\widehat{\alpha})}{\lambda^{1-\widehat{\alpha}}} + \frac{\Lambda_{\widehat{\alpha}}(h)C^{(3)}(1 + C^{(4)})}{\lambda^{1-2\widehat{\alpha}}}\bigg)\|x - y^n\|_{\widehat{\alpha},\lambda}\\ &\quad + \frac{1}{d}\big(\big\|F^{(f)}\big(y^n\big) - F^{(f,\tau^n)}\big(y^n\big)\big\|_{\widehat{\alpha},\lambda}\\ &\quad + \big\|G^{(g)}\big(y^n\big) - G^{(g,\tau^n)}\big(y^n\big)\big\|_{\widehat{\alpha},\lambda}\big),\end{aligned}$$

where the constant $C^{(4)}$ is defined in (19).

Assume that $\lambda = \lambda_1$ is sufficiently large such that

$$\frac{C_{\widehat{\alpha},T} L\Gamma(1-\widehat{\alpha})}{\lambda_1^{1-\widehat{\alpha}} d} + \frac{\Lambda_{\widehat{\alpha}}(h)C^{(3)}(1 + C^{(4)})}{\lambda_1^{1-2\widehat{\alpha}} d} \leqslant \frac{1}{2}.$$

Then

$$\begin{aligned}\|x - y^n\|_{\widehat{\alpha},\lambda_1} &\leqslant \frac{2}{d}\big(\big\|F^{(f)}\big(y^n\big) - F^{(f,\tau^n)}\big(y^n\big)\big\|_{\widehat{\alpha},\lambda_1}\\ &\quad + \big\|G^{(g)}\big(y^n\big) - G^{(g,\tau^n)}\big(y^n\big)\big\|_{\widehat{\alpha},\lambda_1}\big).\end{aligned}$$

From inequality (4) it follows that

$$\|x-y^n\|_{\widehat{\alpha},\infty} \leqslant \frac{2}{d}\left(1+\frac{T^{1-2\widehat{\alpha}}}{1-2\widehat{\alpha}}\right)\big[\big\|F^{(f)}\big(y^n\big)-F^{(f,\tau^n)}\big(y^n\big)\big\|_{1-\widehat{\alpha}} + \big\|G^{(g)}\big(y^n\big)-G^{(g,\tau^n)}\big(y^n\big)\big\|_{1-\widehat{\alpha}}\big]. \tag{27}$$

Now let us evaluate the right side of inequality (27). We first estimate the norm $\|F^{(f)}(y^n)-F^{(f,\tau^n)}(y^n)\|_{1-\widehat{\alpha}}$.

Assume that $s\in[t_k^n,t_{k+1}^n)$ for some $0\leqslant k\leqslant n-1$. Recall that $h\in W_T^{1-\alpha,\infty}(0,T)\subset W_T^{1-\widehat{\alpha},\infty}(0,T)\subset C^{1-\widehat{\alpha}}(0,T)$ for $\widehat{\alpha}>\alpha$. From assumptions (A1)–(A2) it follows that

$$\begin{aligned}
&\big|y_s^n-y^n\big(\tau_s^n\big)\big| \\
&\quad\leqslant d^{-1}\big|D\big(y_s^n\big)-D\big(y^n\big(\tau_s^n\big)\big)\big| \\
&\quad= d^{-1}\big|f\big(t_k^n,y^n\big(t_k^n\big)\big)\big(s-\tau_s^n\big)+g\big(t_k^n,y^n\big(t_k^n\big)\big)\big(h(s)-h\big(t_k^n\big)\big)\big| \\
&\quad\leqslant d^{-1}\big[L(s-t_k^n)+M\big|h(s)-h\big(t_k^n\big)\big|\big]\leqslant\lambda(\widehat{\alpha})\big(s-t_k^n\big)^{1-\widehat{\alpha}}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\big|f\big(s,y_s^n\big)-f\big(\tau_s^n,y^n\big(\tau_s^n\big)\big)\big| \\
&\quad\leqslant\big|f\big(s,y_s^n\big)-f\big(\tau_s^n,y_s^n\big)\big|+\big|f\big(\tau_s^n,y_s^n\big)-f\big(\tau_s^n,y^n\big(\tau_s^n\big)\big)\big| \\
&\qquad\leqslant L\big(s-\tau_s^n\big)+L\big|y_s^n-y^n\big(\tau_s^n\big)\big|\leqslant L\big(1+\lambda(\widehat{\alpha})\big)\big(s-t_k^n\big)^{1-\widehat{\alpha}},
\end{aligned} \tag{29}$$

where $\lambda(\widehat{\alpha})=d^{-1}[L+M|h|_{1-\widehat{\alpha}}]$. Recall that $\Delta_n<1$.

First, observe that if $t\leqslant t_{k+1}^n$,

$$\begin{aligned}
&\big|\big(F_t^{(f)}\big(y^n\big)-F_t^{(f,\tau^n)}\big(y^n\big)\big)-\big(F_s^{(f)}\big(y^n\big)-F_s^{(f,\tau^n)}\big(y^n\big)\big)\big| \\
&\quad\leqslant\int_s^t\big|f\big(u,y_u^n\big)-f\big(\tau_u^n,y^n\big(\tau_u^n\big)\big)\big|\,\mathrm{d}u\leqslant L\big(1+\lambda(\widehat{\alpha})\big)\int_s^t\big(u-t_k^n\big)^{1-\widehat{\alpha}}\,\mathrm{d}u \\
&\quad\leqslant L\big(1+\lambda(\widehat{\alpha})\big)\Delta_n^{1-\widehat{\alpha}}(t-s).
\end{aligned} \tag{30}$$

Assume that $t>t_{k+1}^n$. Then

$$\begin{aligned}
&\big|\big(F_t^{(f)}\big(y^n\big)-F_t^{(f,\tau^n)}\big(y^n\big)\big)-\big(F_s^{(f)}\big(y^n\big)-F_s^{(f,\tau^n)}\big(y^n\big)\big)\big| \\
&\quad\leqslant\int_{t_{k+1}^n}^t\big|f\big(u,y_u^n\big)-f\big(\tau_u^n,y^n\big(\tau_u^n\big)\big)\big|\,\mathrm{d}u \\
&\qquad+\int_s^{t_{k+1}^n}\big|f\big(u,y_u^n\big)-f\big(t_k^n,y^n\big(t_k^n\big)\big)\big|\,\mathrm{d}u,
\end{aligned} \tag{31}$$

and applying assumptions (A2), (28), we obtain

$$\begin{aligned}
&\int_{t_{k+1}^n}^{t} \left|f\left(u, y_u^n\right) - f\left(\tau_u^n, y^n\left(\tau_u^n\right)\right)\right| \mathrm{d}u \\
&\quad\leqslant L \int_{t_{k+1}^n}^{t} \left|y_u^n - y^n\left(\tau_u^n\right)\right| \mathrm{d}u + L \int_{t_{k+1}^n}^{t} \left|u - \tau_u^n\right| \mathrm{d}u \\
&\quad\leqslant L\Big(\sup_{t_{k+1}^n \leqslant s \leqslant t} \left|y_s^n - y^n\left(\tau_s^n\right)\right| + \varDelta_n\Big)\left(t - t_{k+1}^n\right) \\
&\quad\leqslant L\left(\lambda(\widehat{\alpha})\varDelta_n^{1-\widehat{\alpha}} + \varDelta_n\right)\left(t - t_{k+1}^n\right) \leqslant L\left(\lambda(\widehat{\alpha}) + 1\right)\varDelta_n^{1-\widehat{\alpha}}\left(t - t_{k+1}^n\right).
\end{aligned} \tag{32}$$

Consequently, we get

$$\begin{aligned}
&\left|\left(F_t^{(f)}\left(y^n\right) - F_t^{(f,\tau^n)}\left(y^n\right)\right) - \left(F_s^{(f)}\left(y^n\right) - F_s^{(f,\tau^n)}\left(y^n\right)\right)\right| \\
&\quad\leqslant L\left(1 + \lambda(\widehat{\alpha})\right)\varDelta_n^{1-\widehat{\alpha}}(t - s)
\end{aligned}$$

and

$$\left\|F^{(f)}\left(y^n\right) - F^{(f,\tau^n)}\left(y^n\right)\right\|_{1-\widehat{\alpha}} = O\left(\varDelta_n^{1-\widehat{\alpha}}\right). \tag{33}$$

Let us now estimate the norm $\|G^{(g)}(y^n) - G^{(g,\tau^n)}(y^n)\|_{1-\widehat{\alpha}}$. Recall that $y^n \in C^{1-\alpha}(0,T)$ for all $n \geqslant 1$. We first observe that

$$\left|g\left(t, y_t^n\right) - g\left(s, y_s^n\right)\right| \leqslant M|t-s| + M\left|y_t^n - y_s^n\right| \leqslant M|t-s| + M\widehat{C}|t-s|^{1-\alpha}$$

for a certain constant $\widehat{C}$ btained in Lemma 6. It is easy to see that

$$V_p\left(g\left(\cdot, y^n\right); [s,t]\right) \leqslant M(t-s) + M\widehat{C}(t-s)^{1-\alpha}. \tag{34}$$

Assume that $s \in [t_k^n, t_{k+1}^n)$ for some $0 \leqslant k \leqslant n-1$. From assumptions (A1) and (28) it follows that

$$\left|g\left(s, y_s^n\right) - g\left(\tau_s^n, y^n\left(\tau_s^n\right)\right)\right| \leqslant M\left(1 + \lambda(\widehat{\alpha})\right)\left(s - t_k^n\right)^{1-\widehat{\alpha}}. \tag{35}$$

Now let us go back to the estimate of the norm $\|G^{(g)}(y^n) - G^{(g,\tau^n)}(y^n)\|_{1-\widehat{\alpha}}$. Assume that $s \in [t_k^n, t_{k+1}^n)$ for some $0 \leqslant k \leqslant n-1$ and $t \leqslant t_{k+1}^n$. Recall that $\varDelta_n < 1$. Then from (34) and (35) it follows that

$$\begin{aligned}
&\left|\left(G_t^{(g)}\left(y^n\right) - G_t^{(g,\tau^n)}\left(y^n\right)\right) - \left(G_s^{(g)}\left(y^n\right) - G_s^{(g,\tau^n)}\left(y^n\right)\right)\right| \\
&\quad\leqslant \left|\int_s^t \left[g\left(u, y_u^n\right) - g\left(s, y_s^n\right)\right] \mathrm{d}h_u\right| + \left|\left[g\left(s, y_s^n\right) - g\left(t_k^n, y^n\left(t_k^n\right)\right)\right]\left(h(t) - h(s)\right)\right| \\
&\quad\leqslant C_{p,p} V_p\left(g\left(\cdot, y^n\right); [s,t]\right) V_p\left(h; [s,t]\right) + M\left(1 + \lambda(\widehat{\alpha})\right)\varDelta_n^{1-\widehat{\alpha}}|h|_{1-\widehat{\alpha}}(t-s)^{1-\widehat{\alpha}} \\
&\quad\leqslant M\left[C_{p,p}(1 + \widehat{C}) + \left(1 + \lambda(\widehat{\alpha})\right)\right]\varDelta_n^{1-\widehat{\alpha}}|h|_{1-\widehat{\alpha}}(t-s)^{1-\widehat{\alpha}}.
\end{aligned} \tag{36}$$

Assume that $t > t^n_{k+1}$. Using the restriction of the theorem to the exponent of the Hölder condition θ, we can apply inequality (7) and obtain

$$\begin{aligned}
&\left|\int\limits_{t^n_{k+1}}^{t} \left[g\left(u, y^n_u\right) - g\left(\tau^n_u, y^n\left(\tau^n_u\right)\right)\right] \mathrm{d}h_u\right| \\
&\quad \leqslant C_{p,p/\theta} V_{p/\theta}\left(g\left(\cdot, y^n\right) - g\left(\cdot, y^n\left(\tau^n\right); \left[t^n_{k+1}, t\right]\right) V_p\left(h; \left[t^n_{k+1}, t\right]\right) \\
&\qquad + C_{p,p/\theta} V_{p/\theta}\left(g\left(\cdot, y^n\left(\tau^n\right)\right) - g\left(\tau^n, y^n\left(\tau^n\right); \left[t^n_{k+1}, t\right]\right) V_p\left(h; \left[t^n_{k+1}, t\right]\right). \qquad (37)
\end{aligned}$$

From Lemma 1 it follows that

$$\begin{aligned}
&V_{p/\theta}\left(g\left(\cdot, y^n\right) - g\left(\cdot, y^n\left(\tau^n\right); \left[t^n_{k+1}, t\right]\right) \\
&\quad \leqslant M\left[V_{p/\theta}\left(y^n - y^n\left(\tau^n\right); \left[t^n_{k+1}, t\right]\right) \right.\\
&\qquad \left. + 2 \sup_{t^n_{k+1} \leqslant s \leqslant t} \left|y^n_s - y^n\left(\tau^n_s\right)\right| \left[V^\theta_p\left(y^n; [0, T]\right) + T\right]\right].
\end{aligned}$$

Next, applying inequality (10) and $V_p(y^n(\tau^n); [0, T]) \leqslant V_p(y^n; [0, T])$, we get

$$\begin{aligned}
&V_{p/\theta}\left(y^n - y^n\left(\tau^n\right); \left[t^n_{k+1}, t\right]\right) \leqslant V_{p/\theta}\left(y^n - y^n\left(\tau^n\right); [0, T]\right) \\
&\quad \leqslant \left(\operatorname{Osc}\left(y^n - y^n\left(\tau^n\right); [0, T]\right)\right)^{1-\theta} V^\theta_p\left(y^n - y^n\left(\tau^n\right); [0, T]\right) \\
&\quad \leqslant 2\left(\operatorname{Osc}\left(y^n - y^n\left(\tau^n\right); [0, T]\right)\right)^{1-\theta} V^\theta_p\left(y^n; [0, T]\right).
\end{aligned}$$

From (28) it follows that

$$\left|\left(y^n_s - y^n\left(\tau^n_s\right)\right) - \left(y^n_t - y^n\left(\tau^n_t\right)\right)\right| \leqslant 2\lambda(\widehat{\alpha}) \Delta_n^{1-\widehat{\alpha}}.$$

Thus,

$$\operatorname{Osc}\left(y^n - y^n\left(\tau^n\right); [0, T]\right) \leqslant 2\lambda(\widehat{\alpha}) \Delta_n^{1-\widehat{\alpha}}$$

and

$$V_{p/\theta}\left(y^n - y^n\left(\tau^n\right); \left[t^n_{k+1}, t\right]\right) \leqslant 4\left(\lambda(\widehat{\alpha})\right)^{1-\theta} \Delta_n^{(1-\widehat{\alpha})(1-\theta)} V^\theta_p\left(y^n; [0, T]\right).$$

Furthermore,

$$\begin{aligned}
&V_{p/\theta}\left(g\left(\cdot, y^n\right) - g\left(\cdot, y^n\left(\tau^n\right)\right); \left[t^n_{k+1}, t\right]\right) V_p\left(h; \left[t^n_{k+1}, t\right]\right) \\
&\quad \leqslant 4M\left(\lambda(\widehat{\alpha})\right)^{1-\theta} V^\theta_p\left(y^n; [0, T]\right) h_{1-\widehat{\alpha}} \Delta_n^{(1-\widehat{\alpha})(1-\theta)} \left(t - t^n_{k+1}\right)^{1-\widehat{\alpha}}. \qquad (38)
\end{aligned}$$

Now we estimate the second term in (37). From (10) we get

$$\begin{aligned}
&V_{p/\theta}\left(g\left(\cdot, y^n\left(\tau^n\right)\right) - g\left(\tau^n, y^n\left(\tau^n\right)\right); \left[t^n_{k+1}, t\right]\right) V_p\left(h; \left[t^n_{k+1}, t\right]\right) \\
&\quad \leqslant \left(\operatorname{Osc}\left(g\left(\cdot, y^n\left(\tau^n\right)\right) - g\left(\tau^n, y^n\left(\tau^n\right)\right); [0, T]\right)\right)^{1-\theta} \\
&\qquad \times V^\theta_p\left(g\left(\cdot, y^n\left(\tau^n\right)\right) - g\left(\tau^n, y^n\left(\tau^n\right)\right); [0, T]\right).
\end{aligned}$$

Since

$$\begin{aligned}&\big|\big[g\big(t,y^n\big(\tau^n(t)\big)\big)-g\big(\tau^n(t),y^n\big(\tau^n(t)\big)\big)\big]\\&\quad-\big[g\big(s,y^n\big(\tau^n(s)\big)\big)-g\big(\tau^n(s),y^n\big(\tau^n(s)\big)\big)\big]\big|\\&\leqslant M\big|t-\tau^n(t)\big|+M\big|s-\tau^n(s)\big|\leqslant 2M\Delta_n,\end{aligned}$$

then

$$\operatorname{Osc}\big(g\big(\cdot,y^n\big(\tau^n\big)\big)-g\big(\tau^n,y^n\big(\tau^n\big)\big);[0,T]\big)\leqslant 2M\Delta_n.$$

Moreover, repeating the proof of (25), we obtain

$$\begin{aligned}V_p\big(g\big(\cdot,y^n\big(\tau^n\big)\big);[0,T]\big)&\leqslant M\big(T+V_p\big(y^n;[0,T]\big)\big),\\V_p\big(g\big(\tau^n,y^n\big(\tau^n\big)\big);[0,T]\big)&\leqslant M\big(T+V_p\big(y^n;[0,T]\big)\big).\end{aligned}$$

Thus,

$$\begin{aligned}&V_{p/\theta}\big(g\big(\cdot,y^n\big(\tau^n\big)\big)-g\big(\tau^n,y^n\big(\tau^n\big);\big[t^n_{k+1},t\big]\big)V_p\big(h;\big[t^n_{k+1},t\big]\big)\\&\quad\leqslant 2M\Delta_n^{1-\theta}\big(T+V_p\big(y^n;[0,T]\big)\big)^\theta V_p\big(h;\big[t^n_{k+1},t\big]\big)\\&\quad\leqslant 2M\big(T+V_p\big(y^n;[0,T]\big)\big)^\theta h_{1-\widehat{\alpha}}\Delta_n^{1-\theta}\big(t-t^n_{k+1}\big)^{1-\widehat{\alpha}}.\end{aligned}\tag{39}$$

Consequently, from inequality

$$\begin{aligned}&\big|\big(G_t^{(g)}\big(y^n\big)-G_t^{(g,\tau^n)}\big(y^n\big)\big)-\big(G_s^{(g)}\big(y^n\big)-G_s^{(g,\tau^n)}\big(y^n\big)\big)\big|\\&\quad\leqslant\left|\int_{t^n_{k+1}}^{t}\big[g\big(u,y^n_u\big)-g\big(\tau^n_u,y^n\big(\tau^n_u\big)\big)\big]\,\mathrm{d}h_u\right|+\left|\int_{s}^{t^n_{k+1}}\big[g\big(u,y^n_u\big)-g\big(s,y^n_s\big)\big]\,\mathrm{d}h_u\right|\end{aligned}$$

and (36)–(39) it follows that

$$\big\|G^{(g)}\big(y^n\big)-G^{(g,\tau^n)}\big(y^n\big)\big\|_{1-\widehat{\alpha}}=O\big(\Delta_n^{(1-\widehat{\alpha})(1-\theta)}\big).$$

The proof is now easily completed. □

Proof of Corollary 2. We verify the conditions of Theorem 4. We first observe that

$$\frac{1}{H}-1<\widetilde{\theta}<1,\qquad\frac{\widetilde{\theta}}{1+\widetilde{\theta}}=1-(\gamma-\varepsilon)>1-\gamma_0>1-H.$$

Let $\alpha=1-\gamma-\varepsilon$, $\widehat{\alpha}=1-\gamma$. Note that $\alpha<1-\gamma_0-\varepsilon=\alpha_0-\varepsilon<1/2$, $\widehat{\alpha}+\varepsilon=1-(\gamma-\varepsilon)<1-\gamma_0=\alpha_0\leqslant 1/2$, $\widetilde{\theta}=\theta<\delta$. Moreover, $1-H<\alpha<\widehat{\alpha}$. Thus, conditions of Theorem 4 are satisfied, and Eq. (13) has a unique solution $x\in C^{1-\widetilde{\alpha}}(0,T)=C^\gamma(0,T)$.

It is easy to check that

$$(1-\widehat{\alpha})(1-\widetilde{\theta})=\gamma(1-\widetilde{\theta})>2\gamma-1-\varepsilon.$$

From this we conclude that for any $0 < \varepsilon < (\gamma - \gamma_0) \wedge (H - \gamma)$,

$$\left\|x - y^n\right\|_{1-\gamma,\infty} = O\left(\Delta_n^{2\gamma-1-\varepsilon}\right).$$

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Appendix

Let

$$X_t = x_0 + \Phi(X_t) - \Phi(x_0) + \int_0^t (\beta - \alpha X_s)\,\mathrm{d}s + \sigma B_t^H, \quad t \geqslant 0, \tag{A.1}$$

be a fractional Vasicek model with a resisting force Φ and $\alpha, \beta \in \mathbb{R}$, $\sigma \geqslant 0$. To find out how a force affects the behavior of a process, consider two types of resistant forces:

$$\Phi(x) = \begin{cases} \Phi_1(x) = k_1 \mathrm{e}^{-\lambda(x-a_1)} & \text{if } x > a_1, \\ \Phi_2(x) = k_2(a_1 - x) + \Phi_1(a_1) & \text{if } x \leqslant a_1 \end{cases} \tag{A.2}$$

and

$$\Phi(x) = \begin{cases} \Phi_1(x) = k_1 \mathrm{e}^{-\lambda(x-a_1)} & \text{if } x > a_1, \\ \Phi_2(x) = k_2(a_1 - x) + \Phi_1(a_1) & \text{if } a_2 \leqslant x \leqslant a_1, \\ k_3(a_2 - x) + \Phi_2(a_2) & \text{if } x < a_2, \end{cases} \tag{A.3}$$

where $\lambda \in (0, 1)$; $a_1, a_2, k_1, k_2, k_3 > 0$. By these two resistant forces, we obtain single and double soft-wall models.

Note that the function $D(x) = x - \Phi(x) : \mathbb{R} \to \mathbb{R}$ satisfies assumptions (H3), where $\Phi(x)$ is defined in (A.3). For the forces (A.2), the proof is simpler than for (A.3). The following lemma will be useful.

Lemma A1. *Assume that the function $D : \mathbb{R} \to \mathbb{R}$ is differentiable, invertable, and such that $|D'|_\infty \geqslant c > 0$. Then inequality* (H3)(ii) *is satisfied.*

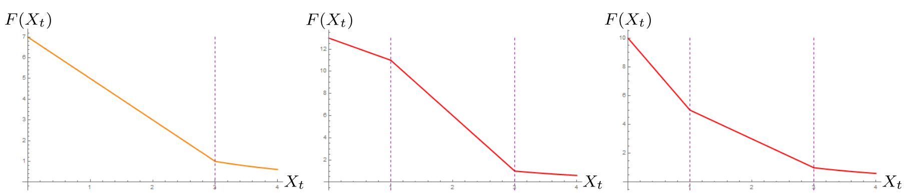

Figure A1. Force profiles: single soft-wall, double soft-wall.

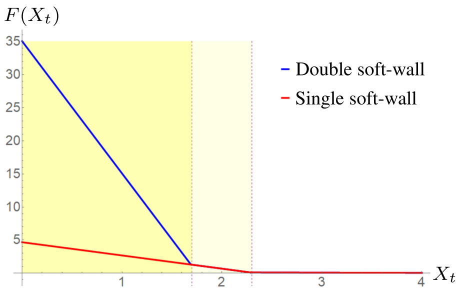

Figure A2. Single (A.2) and double (A.3) force $\varPhi$ profiles for $\lambda = 0.5$, $k_1 = 0.001$, $k_2 = 2$, $k_3 = 20$, $a_1 = 2.3$, $a_2 = 1.7$.

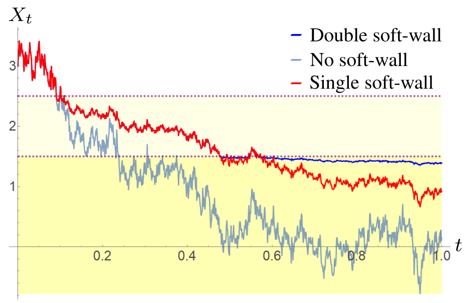

Figure A3. Trajectories of the fractional Vasicek process for $\alpha = 1$, $\beta = 3$, $H = 0.3$, $\lambda = 0.5$, $T = 1$, $n = 1000$.

The statement of the lemma follows from Remarks 6 and 7 in [11].

Suppose that $x, y > a_1$. Since $\varPhi_1'(x) = -k_1\lambda e^{-\lambda(x-a_1)} \leqslant 0$, $D(x)$ is differentiable, invertible, and such that $|D'|_\infty > 1$. From Lemma A1 it follows that $|D(x) - D(y)| \geqslant |x - y|$. Further,

$$\big|D(x)-D(y)\big| = \begin{cases} (k_2+1)(y-x) & \text{if } x < y \leqslant a_1, \\ (k_3+1)(y-x) & \text{if } x < y \leqslant a_2, \\ (\varPhi_1(a_1)-\varPhi_1(x)) + (x-y) + k_2(a_1-y) & \text{if } y \leqslant a_1 < x, \\ (\varPhi_2(a_1)-\varPhi_1(x)) + (x-y) + k_2(a_2-y) & \text{if } y \leqslant a_2,\ x > a_1, \\ (\varPhi_2(a_2)-\varPhi_2(x)) + (x-y) + k_3(a_2-y) & \text{if } y \leqslant a_2,\ a_2 \leqslant x \leqslant a_1. \end{cases}$$

Thus, $|D(x) - D(y)| \geqslant |x - y|$. Moreover, $D(x) : \mathbb{R} \to \mathbb{R}$ is strictly increasing and surjective. Thus, assumption (H3)(i) is satisfied.

For the fractional Vasicek process (A.1) and soft-wall resistant force profiles (A.2) and (A.3) illustrated in Fig. A2, we obtain the following trajectories of the fractional Vasicek process.

The simulation results show that when the process crosses the soft wall boundary, the force *pushes* the process trajectories to the boundary.

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