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Discrete fractional calculus with exponential memory: Propositions, numerical schemes and asymptotic stability*

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Abstract. A new fractional difference with an exponential kernel function is proposed in this study. First, a difference operator is defined by the exponential function. From the Cauchy problem of the nth-order difference equation, new fractional-order sum and differences are presented. The propositions between each other and numerical schemes are derived. Finally, fractional linear difference equations are presented, and exact solutions are given by using a new discrete Mittag-Leffler function.

Keywords: exponential functions, numerical scheme, fractional dynamic equations.

1 Introduction

The fractional derivative has extensive applications in various fields [12, 15]. Some new fractional derivatives were proposed. Particularly, the definitions of exponential type have been paid much attention. Very recently, a new fractional calculus with exponential memory was suggested in [10, 11]:

$$ {}^{C}D_{a}^{\alpha}x(t) := \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \left(\mathrm{e}^{\lambda t} - \mathrm{e}^{\lambda s}\right)^{-\alpha} x'(s)\,\mathrm{d}s, \quad 0<\lambda,\ 0<\alpha\leqslant 1,\ t\leqslant a. $$

The physical meaning was provided by the CTRW theory [9], and the kernel function played an important role in the waiting time probability density. The fractional calculus

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was also successfully used in fractional modeling of Dodson diffusion [11]. The function space [7] was analyzed in which the boundedness theorem can hold. The numerical method is investigated [16].

It is popular to study the discrete-time version from the time-scale theory [6]. In fact, some applications and theories were considered in the power-law definition [1, 2, 4], discrete-time fractional variational problems [5, 8], fractional chaotic maps [17, 21, 24], fractional recurrent neural network [14], Hadamard difference [20, 23] and finite-time stability [18, 22]. In view of these results, the new features and main reasons why we investigate discrete fractional calculus are the following:

- It exhibits a discrete memory effect, which is particularly suitable for data processing or discrete-time systems with memory.
- The propositions, for example, semigroup proposition, Leibniz integral law and composition law et al. hold, and they are convenient for qualitative analysis.
- The time scale unifies continuous and discrete-time systems.

Due to these new features, it is meaningful to find an exact discretization of the exponential fractional calculus, which is the purpose of this paper.

2 Preliminaries

Let us set the time scale

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1. (See [6].) For $t \in \mathbb{T}^{\kappa}$, $p : \mathbb{T} \to \mathbb{R}$ is a regressive function if it satisfies

$$1 + \mu(t)p(t) \neq 0.$$

Definition 2. (See [6].) The transformation $\xi_{\mu(\tau)} : \mathbb{C}_{\mu(\tau)} \to \mathbb{Z}_{\mu(\tau)}$ is defined by

$$\xi_{\mu(\tau)}(p(\tau)) = \begin{cases} \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)p(\tau)), & \mu(\tau) > 0, \\ p(\tau), & \mu(\tau) = 0, \end{cases}$$

where

$$\mathbb{C}_h := \left\{ z \in \mathbb{C}\colon\ z \neq -\frac{1}{h} \right\} \quad \text{and} \quad \mathbb{Z}_h := \left\{ z \in \mathbb{C}\colon\ -\frac{\pi}{h} < \operatorname{Im}(z) \leqslant \frac{\pi}{h} \right\}$$

for $h > 0$. Let $\mathbb{C}_0 := \mathbb{C}$ and $\mathbb{Z}_0 := \mathbb{C}$ for $h = 0$.

Definition 3 [Exponential function]. (See [6].) Suppose $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ is the set of regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$. If $p \in \mathcal{R}$, t and $a \in \mathbb{T}$, the exponential function is given as

$$e_p(t, a) := \exp\left(\int_a^t \xi_{\mu(\tau)}\bigl(p(\tau)\bigr)\, \Delta\tau \right).$$

Example 1. Suppose $\mathbb{T}^0 = (h\mathbb{N}_0)_a$ and $p(t) = \lambda$.

$$e_\lambda(t,a) = \exp\left(\int_a^t \xi_h(\lambda)\,\Delta\tau\right) = \exp\left(\sum_{\tau\in[a,t)} \log(1+h\lambda)\right) = \prod_{\tau\in[a,t)} (1+h\lambda)$$
$$= (1+h\lambda)^{(t-a)/h}$$

holds, and its inverse function reads

$$e_\lambda^{-1}(t,a) = a + \frac{\ln t}{\ln(1+h\lambda)}h.$$

Remark 1. The nonequidistant partition was proposed in general fractional differential equations [16], the step size is $h = (e_\lambda(b,a) - e_\lambda(a,a))/N$, and $e_\lambda(a,a) = 1$. We define

$$t_j = e_\lambda^{-1}(1+jh,\, a), \quad j = 0,1,\dots,N,$$
$$t_j \in \left\{a,\ a + \frac{\ln(1+h)}{\ln(1+h\lambda)}h,\ \dots,\ a + \frac{\ln(1+Nh)}{\ln(1+h\lambda)}h = b\right\},$$

and $e_\lambda(t_j,a) = 1 + jh$. Therefore, the following time scale is used in this paper:

$$\mathbb{T}_{\mathbb{N}_0} = a + \frac{\ln(1+h\mathbb{N}_0)}{\ln(1+h\lambda)}h.$$

If $e_\lambda(t_j,a) = 1 + (j+\alpha)h$, $j = 0,1,\dots,$ then

$$t_j \in \left\{a + \frac{\ln(1+\alpha h)}{\ln(1+h\lambda)}h,\ \dots,\ a + \frac{\ln(1+(k+\alpha)h)}{\ln(1+h\lambda)}h,\ \dots\right\}, \tag{1}$$

and we call $t_j \in \mathbb{T}_{\mathbb{N}_0+\alpha}$.

In addition, we denote

$$g_\beta(t) := a + \frac{\ln(e_\lambda(t,a) + \beta h)}{\ln(1+h\lambda)}h, \quad \beta \in \mathbb{R}. \tag{2}$$

Then we have

$$g_\beta(a) = a + \frac{\ln(1+\beta h)}{\ln(1+h\lambda)}h.$$

Theorem 1. *The following semigroup property holds*:

$$g_\gamma \circ g_\beta(t) = g_{\gamma+\beta}(t), \quad \beta,\gamma \in \mathbb{R},\ t \in \mathbb{T}_{\mathbb{N}_0}.$$

Proof. Let $t = a + \ln(1+kh)/\ln(1+h\lambda)h,\ k \in \mathbb{N}_0$. Then we have

$$e_\lambda\big(g_\beta(t),a\big) = 1 + (k+\beta)h.$$

According to the definition of $g_\nu(t)$, we give

$$g_\gamma \circ g_\beta(t) = a + \frac{\ln(e_\lambda(g_\beta(t), a) + \gamma h)}{\ln(1+h\lambda)}h = a + \frac{\ln(1+(k+\beta)h + \gamma h)}{\ln(1+h\lambda)}h$$
$$= a + \frac{\ln(e_\lambda(t,a) + (\beta+\gamma)h)}{\ln(1+h\lambda)}h = g_{\gamma+\beta}(t). \qquad \square$$

By Theorem 1, the propositions

$$g_{-\beta} \circ g_\beta(t) = t, \quad t \in \mathbb{T}_{\mathbb{N}_0},\ \beta \in \mathbb{R} \quad \text{and} \quad g_{-\beta} \circ g_\beta(a) = a$$

hold, respectively. They are useful in the sequel of the study.

Theorem 2. (*See* [6].) $e_p(\cdot, t_0)$ *is a solution of*

$$y^\Delta = p(t)y, \quad t \in \mathbb{T}, \qquad y(t_0) = 1. \tag{3}$$

Remark 2. Let $t \in \mathbb{T}_{\mathbb{N}_0}$, $p(t) = \lambda$ and $t_0 = a$. The solution $e_p(t, t_0)$ is reduced to $e_\lambda(t, a)$. It also holds that

$$\mathrm{e}_\lambda^\Delta(t,a) = \begin{cases} \lambda(1+h\lambda)^{(t-a)/h}, & h \neq 0, \\ \lambda \mathrm{e}^{\lambda(t-a)}, & h = 0. \end{cases}$$

3 Exponential fractional sum

Consider the initial value problem of the dynamic equation

$$\begin{aligned} &{}_{\mathbb{T}}\Delta^{n,e_\lambda} x(t) = f(t), \quad t \in \mathbb{T}_a, \\ &{}_{\mathbb{T}}\Delta^{i,e_\lambda} x(a) = 0, \quad i = 0,1,\dots,n-1, \end{aligned} \tag{4}$$

where

$${}_{\mathbb{T}}\Delta^{e_\lambda} x(t) := {}_{\mathbb{T}}\Delta^{1,e_\lambda} x(t) = \frac{1}{e_\lambda^\Delta(t,a)} x^\Delta(t)$$

and

$${}_{\mathbb{T}}\Delta^{n,e_\lambda} x(t) := \underbrace{{}_{\mathbb{T}}\Delta^{e_\lambda} \cdots {}_{\mathbb{T}}\Delta^{e_\lambda}}_{n} x(t).$$

On the time scale $\mathbb{T}_{\mathbb{N}_0}$, the unique solution of (4) is

$$\begin{aligned} x(t) &= \int_a^t \Delta t_1 \int_a^{t_1} \Delta t_2 \cdots \int_a^{t_{n-2}} \Delta t_{n-1} \\ &\quad \cdots \int_a^{t_{n-1}} e_\lambda^\Delta(t_1,a) e_\lambda^\Delta(t_2,a) e_\lambda^\Delta(t_{n-1},a) e_\lambda^\Delta(s,a) f(s)\,\Delta s \\ &= \frac{1}{(n-1)!} \int_a^{\rho^{n-1}(t)} \big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)_h^{\underline{n-1}} e_\lambda^\Delta(s,a) f(s)\,\Delta s, \end{aligned}$$

where the kernel function reads

$$\big(e_\lambda(t,a)-e_\lambda(\sigma(s),\,a)\big)^{\underline{n-1}}_h=\prod_{i=0}^{n-2}\big(e_\lambda(t,a)-e_\lambda(\sigma(s),\,a)-ih\big).$$

Definition 4 [h-factorial function]. For t and $\alpha\in\mathbb{R}$, the factorial function is given by

$$t^{\underline{\alpha}}_h:=h^\alpha\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)},\quad t\in\{\alpha h,\ (1+\alpha)h,\ \dots\}. \tag{5}$$

Here Γ is Euler gamma function.

Definition 5. Let $f:\mathbb{T}_{\mathbb{N}_0}\to\mathbb{R}$ and $0<\alpha$. The exponential fractional sum is given by

$$\begin{aligned}&{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}f(t)\\&:=\frac{1}{\Gamma(\alpha)}\int_a^{\rho^{\alpha-1}(t)}\big(e_\lambda(t,a)-e_\lambda(\sigma(s),a)\big)^{\underline{\alpha-1}}_h e_\lambda^\Delta(s,a)f(s)\,\Delta s,\quad t\in\mathbb{T}_{\mathbb{N}_0+\alpha}.\end{aligned} \tag{6}$$

Remark 3.

(i) By (5), the exponential fractional discrete kernel function can be rewritten as

$$\big(e_\lambda(t,a)-e_\lambda(\sigma(s),a)\big)^{\underline{\alpha-1}}_h=h^{\alpha-1}\frac{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(\sigma(s),a)}{h}+1)}{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(\sigma(s),a)}{h}+2-\alpha)}.$$

(ii) For $p(t)=\lambda>0$ in Eq. (3), $e_\lambda^\Delta(t,a)>0$, that is, $e_\lambda(t,a)$ is a strictly monotonically increasing function such that (6) is well defined.

(iii) By use of the function g_ν in (2), the integer-order sum is rewritten as

$${}_{\mathbb{T}}\Delta_a^{-n,e_\lambda}f(t)=\frac{h}{\Gamma(n)}\sum_{s=a}^{g_{-n}(t)}\big(e_\lambda(t,a)-e_\lambda(\sigma(s),a)\big)^{\underline{n-1}}_h f(s),\quad t\in\mathbb{T}_{\mathbb{N}_0+n}.$$

Generally, the exponential fractional-order sum reads

$${}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}f(t)=\frac{h}{\Gamma(\alpha)}\sum_{s=a}^{g_{-\alpha}(t)}\big(e_\lambda(t,a)-e_\lambda(\sigma(s),a)\big)^{\underline{\alpha-1}}_h f(s),\quad t\in\mathbb{T}_{\mathbb{N}_0+\alpha}.$$

4 Exponential fractional differences

Next, the exponential fractional difference is defined in terms of exponential fractional-order sum.

Definition 6. Let $f:\mathbb{T}_{\mathbb{N}_0}\to\mathbb{R}$, $0<\alpha$, $M=\lceil\alpha\rceil$, and let $\lceil\cdot\rceil$ is the ceiling function. The fractional difference of the exponential type is defined by

$${}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda}f(t):={}_{\mathbb{T}}\Delta^{M,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-(M-\alpha),e_\lambda}f(t),\quad t\in\mathbb{T}_{\mathbb{N}_0+M-\alpha}. \tag{7}$$

Remark 4. For arbitrary $M \in \mathbb{N}_1$, the integer-order difference's domain is

$$D\{{}_{\mathbb{T}}\Delta^{M,e_\lambda} f\} = D\{f\},$$

whereas the fractional-order difference is

$$D\{{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f\} = \mathbb{T}_{\mathbb{N}_0+M-\alpha}.$$

Theorem 3. *Suppose* $f : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$, $0 < \alpha$ *and* $M = \lceil \alpha \rceil$. *The fractional-order difference* (7) *equals to*

$${}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) = \frac{1}{\Gamma(-\alpha)} \int\limits_a^{\rho^{-\alpha-1}(t)} \bigl(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{-\alpha-1}} e_\lambda^\Delta(s,a) f(s)\,\Delta s,$$

and ${}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta^{M,e_\lambda} f(t)$ *if* $\alpha = M$, *where* $t \in \mathbb{T}_{\mathbb{N}_0+M-\alpha}$.

Proof. In the case $\alpha = M$, according to the definition, it is easy to give

$${}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta^{M,e_\lambda} f(t).$$

In the case $M - 1 < \alpha < M$, we rewrite Eq. (7) as

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) \\
&\quad= {}_{\mathbb{T}}\Delta^{M,e_\lambda}\left[\frac{h}{\Gamma(M-\alpha)} \sum_{s=a}^{g_{-(M-\alpha)}(t)} \bigl(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-1}} f(s)\right] \\
&\quad= {}_{\mathbb{T}}\Delta^{M-1,e_\lambda}\,{}_{\mathbb{T}}\Delta^{1,e_\lambda}\left[\frac{h}{\Gamma(M-\alpha)} \sum_{s=a}^{g_{-(M-\alpha)}(t)} \bigl(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-1}} f(s)\right] \\
&\quad= \frac{{}_{\mathbb{T}}\Delta^{M-1}}{\Gamma(M-\alpha)}\left[\sum_{s=a}^{g_{-(M-\alpha-1)}(t)} \bigl(e_\lambda(\sigma(t),a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-1}} f(s)\right. \\
&\qquad\left. - \sum_{s=a}^{g_{-(M-\alpha)}(t)} \bigl(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-1}} f(s)\right] \\
&\quad= \frac{{}_{\mathbb{T}}\Delta^{M-1}}{\Gamma(M-\alpha)}\left\{\sum_{s=a}^{g_{-(M-\alpha)}(t)} f(s)\bigl[\bigl(e_\lambda(\sigma(t),a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-1}}\right. \\
&\qquad\left. - \bigl(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-1}}\bigr] + h^{M-\alpha-1}\Gamma(M-\alpha) f\bigl(g_{-(M-\alpha-1)}(t)\bigr)\right\} \\
&\quad= \frac{{}_{\mathbb{T}}\Delta^{M-1}}{\Gamma(M-\alpha)}\left\{\sum_{s=a}^{g_{-(M-\alpha)}(t)} f(s) h\bigl(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\bigr)_h^{\underline{M-\alpha-2}}(M-\alpha-1)\right.
\end{aligned}$$

$$+ h^{M-\alpha-1}\Gamma(M-\alpha) f\big(g_{-(M-\alpha-1)}(t)\big)\Bigg\}$$

$$= {}_{\mathbb{T}}\Delta^{M-1}\left(\sum_{s=a}^{g_{-(M-\alpha-1)}(t)} \frac{h}{\Gamma(M-\alpha-1)} f(s)\big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)^{\frac{M-\alpha-2}{h}}\right).$$

Repeat $M-1$ times to obtain

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) \\
&= {}_{\mathbb{T}}\Delta^{M-2,e_\lambda}\left[\sum_{s=a}^{g_{-(M-\alpha-2)}(t)} \frac{h}{\Gamma(M-\alpha-2)} f(s)\big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)^{\frac{M-\alpha-3}{h}}\right] \\
&= {}_{\mathbb{T}}\Delta^{M-M,e_\lambda}\left[\sum_{s=a}^{g_\alpha(t)} \frac{h}{\Gamma(-\alpha)} f(s)\big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)^{\frac{-\alpha-1}{h}}\right] \\
&= \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{g_\alpha(t)} e_\lambda^\Delta(s,a) f(s)\big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)^{\frac{-\alpha-1}{h}} \mu(s).
\end{aligned}$$

The proof is completed. □

We now can directly give the Caputo-like difference, which can be considered as a discrete version of [10, 11].

Definition 7. Let $f : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$, $0 < \alpha$ and $M = \lceil \alpha \rceil$. The exponential Caputo difference of order α is defined by

$${}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} f(t) := {}_{\mathbb{T}}\Delta_a^{-(M-\alpha),e_\lambda}\, {}_{\mathbb{T}}\Delta^{M,e_\lambda} f(t), \quad t \in \mathbb{T}_{\mathbb{N}_0+M-\alpha}.$$

Theorem 4. *Let $f : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$, $0 < \alpha$ and $M = \lceil \alpha \rceil$. For $t = g_{\alpha+N-1}(a)$ and $s = g_j(a)$, the following numerical scheme of the fractional-order sum is obtained:*

$${}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda} f(t) = \frac{h^\alpha}{\Gamma(\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(N-j+1+\alpha)}{\Gamma(N-j)} f\big(g_j(a)\big).$$

For $t = g_{M+N-1-\alpha}(a)$ and $s = g_j(a)$, $j = 0, 1, \ldots, M+N-1$, numerical schemes of the fractional differences are

$${}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) = \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{M+N-1} \frac{\Gamma(M+N-j-1-\alpha)}{\Gamma(M+N-j)} f\big(g_j(a)\big)$$

and

$${}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} f(t) = \frac{h^{M-\alpha}}{\Gamma(M-\alpha)} \sum_{j=0}^{M+N-1} \frac{\Gamma(M+N-j-1-\alpha)}{\Gamma(M+N-j)}\, {}_{\mathbb{T}}\Delta^{M,e_\lambda} f\big(g_j(a)\big),$$

respectively.

5 Fractional composition rules

Next, consider several compositions of exponential fractional-order sum and difference.

Lemma 1. *Let* $\beta > 0$. *Then*

$$ {}_{\mathbb{T}}\Delta^{1,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} = \beta\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta-1}}, \quad t \in \mathbb{T}_{\mathbb{N}_0+\beta}. $$

In addition, for $\alpha > 0$,

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} \\ &\quad = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta+\alpha}}, \quad t \in \mathbb{T}_{\mathbb{N}_0+\alpha+\beta}, \end{aligned} \tag{8} $$

and

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{\alpha,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} \\ &\quad = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta-\alpha}}, \quad t \in \mathbb{T}_{\mathbb{N}_0+\beta+M-\alpha}. \end{aligned} $$

Proof. Suppose $t = g_{n+\beta}(a) \in \mathbb{T}_{\mathbb{N}_0+\beta}$, $n \in \mathbb{N}_0$. Then we obtain

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta^{1,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} \\ &\quad = \frac{1}{e_\lambda^\Delta(t,r)}\Big(\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}}\Big)^\Delta \\ &\quad = \frac{(e_\lambda(\sigma(t),a) - e_\lambda(a,a))_h^{\underline{\beta}} - \big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}}}{e_\lambda(\sigma(t),a) - e_\lambda(t,a)} \\ &\quad = \beta h^{\beta-1}\frac{\Gamma(n+\beta+1)}{\Gamma(n+2)} \end{aligned} $$

and

$$ \big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta-1}} = h^{\beta-1}\frac{\Gamma(\beta+n+1)}{\Gamma(n+2)}. $$

Thus, we arrive at

$$ {}_{\mathbb{T}}\Delta^{1,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} = \beta(e_\lambda(t,a) - e_\lambda(a,a))_h^{\underline{\beta-1}}. \tag{9} $$

Next, utilize mathematical induction to prove

$$ {}_{\mathbb{T}}\Delta^{M,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-M+1)}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta-M}}. \tag{10} $$

When $M = 1$, according to (9), the fractional difference of the power-law function (10) holds. Suppose that the following holds for $M = k$:

$$ {}_{\mathbb{T}}\Delta^{k,e_\lambda}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-k)}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_h^{\underline{\beta-k}}. $$

Then

$$
\begin{aligned}
&{}_{\mathbb{T}}\Delta^{k+1,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta}{h}}}\\
&\quad=\frac{1}{e_\lambda^\Delta(t,a)}\big[{}_{\mathbb{T}}\Delta^{k,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta}{h}}}\big]^\Delta\\
&\quad=\frac{1}{e_\lambda^\Delta(t,a)}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-k)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta-k}{h}}}\right]^\Delta\\
&\quad=h^{\beta-(k+1)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-k)}\frac{\Gamma(n+\beta+1)}{\Gamma(n+2+k)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-(k+1))}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta-(k+1)}{h}}}\\
&\quad=\frac{\Gamma(\beta+1)}{\Gamma(\beta-k)}h^{\beta-(k+1)}\frac{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(a,a)}{h}+1)}{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(a,a)}{h}+1-(\beta-(k+1)))}\\
&\quad=h^{\beta-(k+1)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-k)}\frac{\Gamma(n+\beta+1)}{\Gamma(n+k+2)}.
\end{aligned}
$$

Therefore,

$$
{}_{\mathbb{T}}\Delta^{k+1,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta}{h}}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-k)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta-(k+1)}{h}}}.
$$

As a result, for arbitrary $M\in\mathbb{N}_1$,

$$
{}_{\mathbb{T}}\Delta^{M,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta}{h}}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-M)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\frac{\beta-M}{h}}}
$$

holds.

Next, we prove (8) in two cases. For $h=1$ and $t\in a+\ln(1+\mathbb{N}_0+\alpha+\beta)/\ln(1+\lambda)=\mathbb{T}^*_{\mathbb{N}_0+\alpha+\beta}$, we have $e_\lambda(t,a)\in 1+\alpha+\beta+\mathbb{N}_0$. According to the property [13]

$$
\Delta_{\alpha+\mu}^{-v}(t-a)^{\underline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}(t-a)^{\underline{\mu+v}},
$$

there is

$$
{}_{\mathbb{T}^*}\Delta_{g_\beta^\star(a)}^{-\alpha,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\beta}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)^{\underline{\alpha+\beta}},
$$

where $g_\beta^\star(a)=a+\ln(1+\beta)/\ln(1+\lambda)$.

When $h \neq 1$,

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_{h}^{\underline{\beta}} \\
&\quad= \frac{h}{\Gamma(\alpha)}\sum_{s=g_\beta(a)}^{g_{-\alpha}(t)}\big(e_\lambda(t,a)-e_\lambda(\sigma(s),a)\big)_{h}^{\underline{\alpha-1}}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_{h}^{\underline{\beta}} \\
&\quad= \frac{h^{\alpha+\beta}}{\Gamma(\alpha)}\sum_{s=g_\beta(a)}^{g_\alpha(t)}\frac{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(\sigma(s),a)}{h}+1)}{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(\sigma(s),a)}{h}-\alpha+2)}\frac{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(a,a)}{h}+1)}{\Gamma(\frac{e_\lambda(t,a)-e_\lambda(a,a)}{h}+1-\beta)}.
\end{aligned}$$

For $t = g_{k+\alpha+\beta}(a)$, $k \in \mathbb{N}_0$, and $s = g_{k+\beta}(a)$, one can obtain

$$\begin{aligned}
\frac{e_\lambda(t,a)-e_\lambda(\sigma(s),\,a)}{h} &= \frac{(1+h\lambda)^{(t-a)/h}-(1+h\lambda)^{(\sigma(s)-a)/h}}{h} \\
&= \alpha+\beta+k-(\beta+k+1).
\end{aligned}$$

On $\mathbb{T}^*_{\mathbb{N}_0+\alpha+\beta}$, suppose $t^* = g^\star_{k+\alpha+\beta}(a)$ and $s^* = g^\star_{k+\beta}(a)$ such that

$$e_\lambda\big(t^*,a\big)-e_\lambda\big(\sigma(s^*),a\big) = \alpha+\beta+k-(\beta+k+1).$$

Thus,

$$\frac{e_\lambda(t,a)-e_\lambda(\sigma(s),a)}{h} = e_\lambda(t^*,a)-e_\lambda\big(\sigma(s^*),a\big)$$

and

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_{h}^{\underline{\beta}} \\
&\quad= \frac{h^{\alpha+\beta}}{\Gamma(\alpha)}\sum_{s=g^\star_\beta(a)}^{g^\star_{-\alpha}(t^\star)}\frac{\Gamma(e_\lambda(t^*,a)-e_\lambda(\sigma(s^*),a)+1)\Gamma(e_\lambda(t^*,a)-e_\lambda(a^*,a)+1)}{\Gamma(e_\lambda(t^*,a)-e_\lambda(\sigma(s^*),a)-\alpha+2)\Gamma(e_\lambda(t^*,a)-e_\lambda(a^*,a)+1-\beta)} \\
&\quad= \frac{h^{\alpha+\beta}}{\Gamma(\alpha)}\sum_{s=g^\star_\beta(a)}^{g^\star_{-\alpha}(t^\star)}\big(e_\lambda(t^*,a)-e_\lambda\big(\sigma(s^*),a\big)\big)^{\underline{\alpha-1}}\big(e_\lambda(t^*,a)-e_\lambda(a^*,a)\big)^{\underline{\beta}} \\
&\quad= h^{\alpha+\beta}{}_{\mathbb{T}}\Delta_{s=g^\star_\beta(a)}^{-\alpha,e_\lambda}\big(e_\lambda(t^*,a)-e_\lambda(a^*,a)\big)^{\underline{\beta}} \\
&\quad= h^{\alpha+\beta}\big(e_\lambda(t^*,a)-e_\lambda(a^*,a)\big)^{\underline{\beta+\alpha}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \\
&\quad= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_{h}^{\underline{\beta+\alpha}}.
\end{aligned}$$

Finally, we give

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{\alpha,e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_h^{\underline{\beta}}\\
&\quad={}_{\mathbb{T}}\Delta^{M,e_\lambda}\Big[{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-(M-\alpha),e_\lambda}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_h^{\underline{\beta}}\Big]\\
&\quad={}_{\mathbb{T}}\Delta^{M,e_\lambda}\bigg[\frac{\Gamma(\beta+1)}{\Gamma(M-\alpha+\beta+1)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_h^{\underline{M-\alpha+\beta}}\bigg]\\
&\quad=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\big(e_\lambda(t,a)-e_\lambda(a,a)\big)_h^{\underline{\beta-\alpha}}.
\end{aligned}$$

□

Theorem 5. *Let $f:\mathbb{T}_{\mathbb{N}_0}\to\mathbb{R}$ and $\alpha,\beta>0$. The following semigroup property holds*:

$$\begin{aligned}
{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\beta,e_\lambda}f(t)&={}_{\mathbb{T}}\Delta_a^{-\alpha-\beta,e_\lambda}f(t)\\
&={}_{\mathbb{T}}\Delta_{g_\alpha(a)}^{-\beta,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}f(t),\quad t\in\mathbb{T}_{\mathbb{N}_0+\alpha+\beta}.
\end{aligned}$$

Proof. Since $f:\mathbb{T}_{\mathbb{N}_0}\to\mathbb{R}$, $\alpha,\beta>0$ and $t\in\mathbb{T}_{\mathbb{N}_0+\alpha+\beta}$, we give

$$\begin{aligned}
{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\beta,e_\lambda}f(t)&=\frac{1}{\Gamma(\alpha)}\sum_{s=g_\beta(a)}^{g_{-\alpha}(t)}\big(e_\lambda(t,a)-e_\lambda(\sigma(s),a)\big)_h^{\underline{\alpha-1}}\\
&\quad\times h\sum_{\tau=a}^{g_{-\beta}(s)}\frac{1}{\Gamma(\beta)}\big(e_\lambda(s,a)-e_\lambda(\sigma(\tau),a)\big)_h^{\underline{\beta-1}}hf(\tau)\\
&=\frac{1}{\Gamma(\beta)}\sum_{\tau=a}^{g_{-(\alpha+\beta)}(t)}hf(\tau)\,{}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha}\big(e_\lambda(t,a)-e_\lambda(\sigma(\tau),a)\big)_h^{\underline{\beta-1}}\\
&=\frac{1}{\Gamma(\alpha+\beta)}\sum_{\tau=a}^{g_{-(\alpha+\beta)}(t)}\Big[\big(e_\lambda(t,a)-e_\lambda\big(\sigma(\tau),a\big)\big)_h^{\underline{\alpha+\beta-1}}f(\tau)h\Big]\\
&=\frac{1}{\Gamma(\alpha+\beta)}\,{}_{\mathbb{T}}\Delta_a^{-(\alpha+\beta)}f(t).
\end{aligned}$$

For arbitrary α and β,

$${}_{\mathbb{T}}\Delta_{g_\beta(a)}^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\beta,e_\lambda}f(t)={}_{\mathbb{T}}\Delta_a^{-\alpha-\beta,e_\lambda}f(t)={}_{\mathbb{T}}\Delta_{g_\alpha(a)}^{-\beta,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}f(t)$$

holds. So the proof is completed. □

Theorem 6. *Assume $f:\mathbb{T}_{\mathbb{N}_0}\to\mathbb{R}$, $M\in\mathbb{N}_1$, $M-1<\alpha\leqslant M$. When $k\in\mathbb{N}_1$, the propositions are as follows*:

$$\begin{aligned}
{}_{\mathbb{T}}\Delta^{k,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}f(t)&={}_{\mathbb{T}}\Delta_a^{k-\alpha,e_\lambda}f(t),\quad t\in\mathbb{T}_{\mathbb{N}_0+\alpha},\qquad(11)\\
{}_{\mathbb{T}}\Delta^{k,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda}f(t)&={}_{\mathbb{T}}\Delta_a^{k+\alpha,e_\lambda}f(t),\quad t\in\mathbb{T}_{\mathbb{N}_0+M-\alpha}.
\end{aligned}$$

Proof. For $\alpha = M$ and $t \in \mathbb{T}_{\mathbb{N}_0+1}$, we get

$$\begin{aligned}{}_{\mathbb{T}}\Delta^{1,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-1,e_\lambda} f(t) &= \frac{1}{e_\lambda^\Delta(t,a)}\left[\int_a^t e_\lambda^\Delta(s,a) f(s)\,\Delta s\right]^\Delta(t) \\ &= \frac{1}{e_\lambda^\Delta(t,a)}\frac{\sum_{s=t}^{\rho(\sigma(t))} e_\lambda^\Delta(s,a) f(s)\mu(s)}{\mu(t)} \\ &= f(t).\end{aligned}$$

In addition, for $t \in \mathbb{T}_{\mathbb{N}_0+k}$ and $q \in \mathbb{N}_1$, we give

$$\begin{aligned}{}_{\mathbb{T}}\Delta^{q,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-q,e_\lambda} f(t) &= {}_{\mathbb{T}}\Delta^{q-1,e_\lambda}\left[{}_{\mathbb{T}}\Delta^{1,e_\lambda}{}_{\mathbb{T}}\Delta_{g_{k-1}(a)}^{-1,e_\lambda}\left({}_{\mathbb{T}}\Delta_a^{-(q-1),e_\lambda} f(t)\right)\right] \\ &= {}_{\mathbb{T}}\Delta^{q-1,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-(q-1),e_\lambda} f(t) = \cdots = f(t).\end{aligned}$$

Thus, for arbitrary $t \in \mathbb{T}_{\mathbb{N}_0+M}$,

$${}_{\mathbb{T}}\Delta^{k,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-M,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta^{k-M,e_\lambda}\left[{}_{\mathbb{T}}\Delta^{M,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-M,e_\lambda} f(t)\right] = {}_{\mathbb{T}}\Delta^{k-M,e_\lambda} f(t)$$

holds for $k \geqslant M$. When $k < M$, we obtain

$${}_{\mathbb{T}}\Delta^{k,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-M,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta^{k,e_\lambda}{}_{\mathbb{T}}\Delta_{g_{M-k}(a)}^{-k,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-(M-k),e_\lambda} f(t) = {}_{\mathbb{T}}\Delta_a^{k-M,e_\lambda} f(t)$$

As a result, the proof of proposition (11) is completed.

For $M - k < \alpha < M$ and $t \in \mathbb{T}_{\mathbb{N}_0+M-\alpha}$,

$$\begin{aligned}{}_{\mathbb{T}}\Delta^{1,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) &= {}_{\mathbb{T}}\Delta^{1,e_\lambda}\left[\frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{g_\alpha(t)}\left(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\right)_h^{\underline{-\alpha-1}} f(s)h\right] \\ &= \frac{1}{\Gamma(-(1+\alpha))}\sum_{s=a}^{g_{\alpha+1}(t)}\left[\left(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\right)_h^{\underline{-\alpha-2}} f(s)h\right] \\ &= {}_{\mathbb{T}}\Delta_a^{\alpha+1,e_\lambda} f(t)\end{aligned}$$

holds.

As a result, for arbitrary $t \in \mathbb{T}_{\mathbb{N}_0+M-\alpha}$ and $q \in \mathbb{N}_1$, one can get

$$\begin{aligned}{}_{\mathbb{T}}\Delta^{q,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) &= {}_{\mathbb{T}}\Delta^{q-1,e_\lambda}{}_{\mathbb{T}}\Delta^{1,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta^{q-1,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\alpha+1,e_\lambda} f(t) \\ &= \cdots = {}_{\mathbb{T}}\Delta_a^{q+\alpha,e_\lambda} f(t).\end{aligned}$$ □

From Theorem 6 it is easy to derive the theorems.

Theorem 7. *Let $f : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$, $\beta > 0$ and $M - 1 < \alpha \leqslant M$. Then*

$${}_{\mathbb{T}}\Delta_{g_\beta(a)}^{\alpha,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-\beta,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta_a^{\alpha-\beta,e_\lambda} f(t), \quad t \in \mathbb{T}_{\mathbb{N}_0+\beta+M-\alpha}.$$

Theorem 8. *Let* $f : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$, $\alpha > 0$, $k \in \mathbb{N}_1$ *and* $t \in \mathbb{T}_{\mathbb{N}_0+\alpha}$. *Then*

$$ {}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta^{k,e_\lambda} f(t) = {}_{\mathbb{T}}\Delta_a^{k-\alpha,e_\lambda} f(t) - \sum_{j=0}^{k-1} \frac{(e_\lambda(t,a)-1)_{h}^{\underline{\alpha-k+j}}}{T(\alpha+j-k+1)}\,{}_{\mathbb{T}}\Delta^{j,e_\lambda} f(a). \qquad (12) $$

When $0 < \beta, M = \lceil \beta \rceil$ *and* $t \in \mathbb{T}_{\mathbb{N}_0+M-\beta+\alpha}$, *the following composition law holds*:

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta_{g_{M-\beta}(a)}^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{\beta,e_\lambda} f(t) \\ &\quad= {}_{\mathbb{T}}\Delta_a^{\beta-\alpha,e_\lambda} f(t) \\ &\qquad- \sum_{j=0}^{M-1} \frac{(e_\lambda(t,a)-e_\lambda(g_{M-\beta}(a)))_{h}^{\underline{\alpha-M+j}}}{\Gamma(\alpha+j-M+1)}\,{}_{\mathbb{T}}\Delta_a^{j-M+\beta,e_\lambda} f\big(g_{M-\beta}(a)\big). \end{aligned} $$

Proof. For $k = 1$, we obtain

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta^{1,e_\lambda} f(t) \\ &\quad= \frac{1}{\Gamma(\alpha)} \int_a^t \big(e_\lambda(t,a) - e_\lambda\big(\sigma(s),a\big)\big)_{h}^{\underline{\alpha-1}} e_\lambda^{\Delta}(s,a)\big({}_{\mathbb{T}}\Delta^{1,e_\lambda} f(s)\big)\,\Delta s \\ &\quad= \frac{1}{\Gamma(\alpha)} \int_a^t \big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)_{h}^{\underline{\alpha-1}} f^{\Delta}(s)\Delta s \\ &\quad= \frac{1}{\Gamma(\alpha-1)} \int_a^t \big(e_\lambda(t,a) - e_\lambda(\sigma(s),a)\big)_{h}^{\underline{\alpha-2}} e_\lambda^{\Delta}(s,a) f(s)\,\Delta s \\ &\qquad- \frac{(e_\lambda(t,a) - e_\lambda(a,a))_{h}^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(a) \\ &\quad= {}_{\mathbb{T}}\Delta_a^{1-\alpha,e_\lambda} f(t) - \frac{(e_\lambda(t,a) - e_\lambda(a,a))_{h}^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(a). \end{aligned} $$

Let (12) holds for the case $k = n - 1$. Then it follows that

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta^{n,e_\lambda} f(t) \\ &\quad= {}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta^{1,e_\lambda}\,{}_{\mathbb{T}}\Delta^{n-1,e_\lambda} f(t) \\ &\quad= {}_{\mathbb{T}}\Delta_a^{1-\alpha,e_\lambda}\,{}_{\mathbb{T}}\Delta^{n-1,e_\lambda} f(t) - \frac{1}{\Gamma(\alpha)}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_{h}^{\underline{\alpha-1}}\,{}_{\mathbb{T}}\Delta^{n-1,e_\lambda} f(a) \\ &\quad= {}_{\mathbb{T}}\Delta^{1,e_\lambda}\,{}_{\mathbb{T}}\Delta^{n-1,e_\lambda}\,{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda} f(t) \\ &\qquad- \sum_{j=0}^{n-2} \frac{(e_\lambda(t,a) - e_\lambda(a,a))_{h}^{\underline{\alpha-n+j+1}}}{\Gamma(\alpha+j-n+2)}\,{}_{\mathbb{T}}\Delta^{j,e_\lambda} f(a) \\ &\qquad- \frac{1}{\Gamma(\alpha)}\big(e_\lambda(t,a) - e_\lambda(a,a)\big)_{h}^{\underline{\alpha-1}}\,{}_{\mathbb{T}}\Delta^{n-1,e_\lambda} f(a) \end{aligned} $$

$$= {}_{\mathbb{T}}\Delta^{n,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-\alpha,e_\lambda} f(t) - \sum_{j=0}^{n-1} \frac{(e_\lambda(t,a) - e_\lambda(a,a))^{\frac{\alpha-n+j}{h}}}{\Gamma(\alpha+j-n+1)} {}_{\mathbb{T}}\Delta^{j,e_\lambda} f(a)$$
$$= {}_{\mathbb{T}}\Delta_a^{n-\alpha,e_\lambda} f(t) - \sum_{j=0}^{n-1} \frac{(e_\lambda(t,a) - e_\lambda(a,a))^{\frac{\alpha-n+j}{h}}}{\Gamma(\alpha+j-n+1)} {}_{\mathbb{T}}\Delta^{j,e_\lambda} f(a).$$

For arbitrary $k \in \mathbb{N}_1$, equality (12) holds.

For α and $M-1 < \beta \leqslant M$, we give

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_{g_{M-\beta}(a)}^{-\alpha,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\beta,e_\lambda} f(t)\\
&\quad= {}_{\mathbb{T}}\Delta_{g_{M-\beta}(a)}^{M-\alpha,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-(M-\beta),e_\lambda} f(t)\\
&\qquad - \sum_{j=0}^{M-1} \frac{(e_\lambda(t,a) - e_\lambda(g_{M-\beta}(a)))^{\frac{\alpha-M+j}{h}}}{\Gamma(\alpha+j-M+1)} {}_{\mathbb{T}}\Delta^{j,e_\lambda}{}_{\mathbb{T}}\Delta_a^{-(M-\beta),e_\lambda} f\big(g_{M-\beta}(a)\big)\\
&\quad= {}_{\mathbb{T}}\Delta_a^{\beta-\alpha,e_\lambda} f(t)\\
&\qquad - \sum_{j=0}^{M-1} \frac{(e_\lambda(t,a) - e_\lambda(g_{M-\beta}(a)))^{\frac{\alpha-M+j}{h}}}{\Gamma(\alpha+j-M+1)} {}_{\mathbb{T}}\Delta_a^{j-M+\beta,e_\lambda} f\big(g_{M-\beta}(a)\big).
\end{aligned}$$
□

Theorem 9. *Suppose $f : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$, $0 < \alpha, \beta$, $L = \lceil \alpha \rceil$ and $M = \lceil \beta \rceil$. The fractional difference*

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_{g_{M-\beta}(a)}^{\alpha,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\beta,e_\lambda} f(t)\\
&\quad= {}_{\mathbb{T}}\Delta_a^{\alpha+\beta,e_\lambda} f(t)\\
&\qquad - \sum_{j=0}^{M-1} \left[\frac{(e_\lambda(t,a) - e_\lambda(g_{M-\beta}(a)))^{\frac{-\alpha-M+j}{h}}}{\Gamma(-\alpha+j-M+1)} {}_{\mathbb{T}}\Delta^{j-M+\beta,e_\lambda} f\big(g_{M-\beta}(a)\big) \right]
\end{aligned}$$

holds for $t \in \mathbb{T}_{\mathbb{N}_0+L-\alpha+M-\beta}$.

Proof. Since $t \in \mathbb{T}_{\mathbb{N}_0+L-\alpha+M-\beta}$, we get

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_{g_{M-\beta}(a)}^{\alpha,e_\lambda}{}_{\mathbb{T}}\Delta_a^{\beta,e_\lambda} f(t)\\
&\quad= {}_{\mathbb{T}}\Delta^{L,e_\lambda}{}_{\mathbb{T}}\Delta_{g_{M-\beta}(a)}^{-(L-\alpha),e_\lambda}{}_{\mathbb{T}}\Delta_a^{\beta,e_\lambda} f(t)\\
&\quad= {}_{\mathbb{T}}\Delta_a^{\alpha+\beta,e_\lambda} f(t)\\
&\qquad - \sum_{j=0}^{M-1} \left[\frac{{}_{\mathbb{T}}\Delta^{L,e_\lambda}(e_\lambda(t,a) - e_\lambda(g_{M-\beta}(a),a))^{\frac{L-M+j-\alpha}{h}}}{\Gamma(L+j-M+1-\alpha)} {}_{\mathbb{T}}\Delta_a^{j-(M-\beta),e_\lambda} f\big(g_{M-\beta}(a)\big) \right]\\
&\quad= {}_{\mathbb{T}}\Delta_a^{\alpha+\beta,e_\lambda} f(t)\\
&\qquad - \sum_{j=0}^{M-1} \frac{(e_\lambda(t,a) - e_\lambda(g_{M-\beta}(a)))^{\frac{-\alpha-M+j}{h}}}{\Gamma(-\alpha-M+j+1)} \Delta_a^{j-M+\beta,e_\lambda} f\big(g_{M-\beta}(a)\big).
\end{aligned}$$
□

Theorem 10. *Suppose $F(x(\cdot),\cdot) : \mathbb{T}_{\mathbb{N}_0} \to \mathbb{R}$ and $0 < \alpha \leqslant 1$. The Cauchy problem*

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} x(t) = F\big(x\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},\\
&{}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda} x\big(g_{1-\alpha}(a)\big) = C,
\end{aligned} \tag{13}$$

is equivalent to the fractional sum equation

$$\begin{aligned}
x(t) = {}& \frac{(e_\lambda(t,a) - e_\lambda(g_{1-\alpha}(a),a))_{\frac{}{h}}^{\alpha-1}}{\Gamma(\alpha)} C\\
&+ {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(x\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_1}.
\end{aligned} \tag{14}$$

Proof. Since $\phi(t)$ is the solution of (13), then we give

$$\begin{gathered}
{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} \phi(t) = F\big(\phi(g_{\alpha-1}(t)), g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},\\
{}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda} \phi\big(g_{1-\alpha}(a)\big) = C.
\end{gathered}$$

According to Theorem 8, we obtain

$$\begin{aligned}
\phi(t) = {}& \frac{(e_\lambda(t,a) - e_\lambda(g_{1-\alpha}(a),a))_{\frac{}{h}}^{\alpha-1}}{\Gamma(\alpha)} {}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda} \phi\big(g_{1-\alpha}(a)\big)\\
&+ {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi(g_{\alpha-1}(t)), g_{\alpha-1}(t)\big).
\end{aligned}$$

From the initial value condition we get

$$\begin{aligned}
\phi(t) = {}& \frac{(e_\lambda(t,a) - e_\lambda(g_{1-\alpha}(a)))_{\frac{}{h}}^{\alpha-1}}{\Gamma(\alpha)} C\\
&+ {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_1}.
\end{aligned}$$

Thus, $\phi(t)$ is a solution of (14).

If $\phi(t)$ is a solution of (14), we give

$$\begin{aligned}
{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} \phi(t) = {}& \frac{{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} (e_\lambda(t,a) - e_\lambda(g_{1-\alpha}(a),a))_{\frac{}{h}}^{\alpha-1}}{\Gamma(\alpha)} C\\
&+ {}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big).
\end{aligned}$$

Due to

$${}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} \big(e_\lambda(t,a) - e_\lambda\big(g_{1-\alpha}(a),a\big)\big)_{\frac{}{h}}^{\alpha-1} = 0,$$

we get

$$\begin{aligned}
&{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda} {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big)\\
&\quad = {}_{\mathbb{T}}\Delta_{g_1(a)}^{\alpha,e_\lambda} {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big)\\
&\quad = F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big)
\end{aligned}$$

and

$$ {}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda}\phi(t) = F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big). $$

In addition, we also can take the fractional sum of order $1-\alpha$ to check the initial condition

$$ \begin{aligned} {}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}\phi(t) &= \frac{{}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}\big(e_\lambda(t,a) - e_\lambda\big(g_{1-\alpha}(a), a\big)\big)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C \\ &\quad + {}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}{}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big) \end{aligned} $$

and

$$ \frac{1}{\Gamma(\alpha)}{}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}\big(e_\lambda(t,a) - e_\lambda\big(g_{1-\alpha}(a), a\big)\big)^{\frac{\alpha-1}{h}} = 1. $$

Thus,

$$ \begin{aligned} {}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}\phi(t) &= C + {}_{\mathbb{T}}\Delta_{g_1(a)}^{\alpha-1,e_\lambda}{}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big) \\ &= C + {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-1,e_\lambda} F\big(\phi\big(g_{\alpha-1}(t)\big), g_{\alpha-1}(t)\big). \end{aligned} $$

Due to

$$ {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-1,e_\lambda} F\big(\phi\big(g_{\alpha-1}(a)\big), g_{\alpha-1}(a)\big) = 0, $$

we get

$$ {}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}\phi\big(g_{\alpha-1}(a)\big) = C. $$

Thereby, $\phi(t)$ is the solution of (13) from which the proof is completed. □

We define a Mittag-Leffler function as

$$ \begin{aligned} &\epsilon_{\alpha,\alpha}\big(\eta, (t-\sigma(a))^{\underline{\alpha}}\big) \\ &\quad := \sum_{k=0}^{\infty} \eta^k \frac{(e_\lambda(t,a) - e_\lambda(g_{(k+1)(1-\alpha)}(a), a))^{\frac{(k+1)\alpha-1}{h}}}{\Gamma(k\alpha+\alpha)}, \quad t \in \mathbb{T}_{\mathbb{N}_1}. \end{aligned} $$

The uniform convergence can be proved by d'Alembert's criterion.

Example 2. Consider an exponential fractional discrete-time equation

$$ \begin{aligned} &{}_{\mathbb{T}}\Delta_a^{\alpha,e_\lambda}x(t) = \eta x\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},\ 0 < \alpha \leqslant 1, \\ &{}_{\mathbb{T}}\Delta_a^{\alpha-1,e_\lambda}x\big(g_{1-\alpha}(a)\big) = C. \end{aligned} \tag{15} $$

According to Theorem 10, we derive

$$ \begin{aligned} x(t) &= \frac{(e_\lambda(t,a) - e_\lambda(g_{1-\alpha}(a), a)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C \\ &\quad + \eta{}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} x\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_1}. \end{aligned} \tag{16} $$

We obtain that

$$x_{k+1}(t) = x_0(t) + \eta\, {}_{\mathbb{T}}\Delta_{g_{1-\alpha}(a)}^{-\alpha,e_\lambda} x_k\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_1},$$

where $k \in \mathbb{N}_0$, and the initial iteration $x_0(t)$ reads

$$x_0(t) = \frac{(e_\lambda(t,a) - e_\lambda(g_{1-\alpha}(a),a))^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C, \quad t \in \mathbb{T}_{\mathbb{N}_1}.$$

Then we give the iteration scheme as

$$x_{n+1}(t) = C \sum_{k=0}^{n+1} \eta^k \frac{(e_\lambda(t,a) - e_\lambda(g_{(k+1)(1-\alpha)}(a),a))^{\frac{(k+1)\alpha-1}{h}}}{\Gamma(k\alpha+\alpha)}, \quad t \in \mathbb{T}_{\mathbb{N}_1}.$$

Finally, we present the solution of (15) as

$$x(t) = C\epsilon_{\alpha,\alpha}\big(\eta, \big(t-\sigma(a)\big)^{\frac{\alpha}{}}\big), \quad t \in \mathbb{T}_{\mathbb{N}_1}.$$

We can give exact values of the solution or the discrete Mittag-Leffler function. Set $t_k = a + \ln(1+kh)/\ln(1+h\lambda)h$ and $s = a + (\ln(1+(j+1-\alpha)h)/\ln(1+h\lambda))h$, then $e_\lambda(t_k,a) = 1+kh$ and $e_\lambda(s,a) = 1+(j+1-\alpha)h$ such that the numerical scheme of (16) reads

$$x(t_k) = h^{\alpha-1} \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)} C + \eta h^\alpha \sum_{j=0}^{k-1} \frac{\Gamma(k-1-j+\alpha)}{\Gamma(\alpha)\Gamma(k-j)} x(t_j).$$

We need the following theorem to discuss the asymptotic stability.

Theorem 11. (*See* [3].) *If the isolated zeros, off the nonnegative real axis, of*

$$Q(z) = 1 - \frac{\eta h^\alpha}{z}\left(\frac{z}{z-1}\right)^\alpha$$

strictly inside the unit circle, then

$$\begin{aligned} &{}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} x(t) = \eta x\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},\ 0<\alpha\leqslant 1, \\ &x(a) = C. \end{aligned}$$

is asymptotically stable.

Example 3. Consider the following fractional discrete relaxation equation:

$$\begin{aligned} &{}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} x(t) = -x\big(g_{\alpha-1}(t)\big) + \frac{\beta}{1+x(g_{\alpha-1}(t))}, \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},\ 0<\alpha\leqslant 1, \\ &x(a) = C. \end{aligned} \tag{17}$$

We get Eq. (17)'s equilibrium points $x_1^* = (-1+\sqrt{1+4\beta})/2$ and $x_2^* = (-1-\sqrt{1+4\beta})/2$, and its linearization equations read

$$
\begin{aligned}
{}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} x(t) = {} & \frac{4\beta\sqrt{1+4\beta}}{(1+\sqrt{1+4\beta})^2} \\
& - \frac{2(\sqrt{1+4\beta}+1+4\beta)}{(1+\sqrt{1+4\beta})^2} x\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
{}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} x(t) = {} & \frac{-4\beta\sqrt{1+4\beta}}{(1-\sqrt{1+4\beta})^2} \\
& - \frac{2(\sqrt{1+4\beta}+1+4\beta)}{(1-\sqrt{1+4\beta})^2} x\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha}.
\end{aligned}
$$

We only take Eq. (18) as an example. The other one is similar to analyze. Using the transform $y(t) = x(t) - x_1^*(t)$, the fractional difference equation (18) can be rewritten as

$$
{}_{\mathbb{T}}^{C}\Delta_a^{\alpha,e_\lambda} y(t) = \eta y\big(g_{\alpha-1}(t)\big), \quad t \in \mathbb{T}_{\mathbb{N}_0+1-\alpha},\ 0 < \alpha \leqslant 1,
$$

where $\eta = -2(\sqrt{1+4\beta}+1+4\beta)/(1+\sqrt{1+4\beta})^2$.

According to Theorem 11, the coefficient β should satisfy

$$
\frac{2(\sqrt{1+4\beta}+1+4\beta)}{(1+\sqrt{1+4\beta})^2} = \frac{1}{h^\alpha}|z|^{1-\alpha}\big(|z|+1\big)^\alpha, \quad 0 < |z| < 1.
$$

As a result, β can be determined as

$$
-\frac{1}{4} < \beta < \frac{1}{(4(\frac{h}{2})^\alpha - 2)^2} - \frac{1}{4} \quad \text{if } h > 2^{1-1/\alpha},
$$

or

$$
\beta > -\frac{1}{4} \quad \text{if } h \leqslant 2^{1-1/\alpha}.
$$

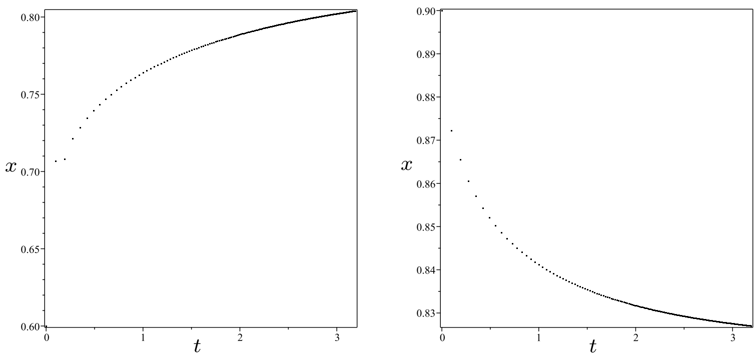
Equation (17) is local asymptotically stable.

Let $\alpha = 0.5$ and $\alpha = 0.6$, respectively. Set $\lambda = 1$, $h = 0.1$, $\beta = 1.5$ and $x(a) = 0.6,\ 0.9$ near $x_1^* = 0.823$. These parameters satisfy the asymptotic stability conditions. The numerical illustrations support the theoretical results in Fig. 1.

We reconsider the standard h-fractional difference equation with power-law memory effects

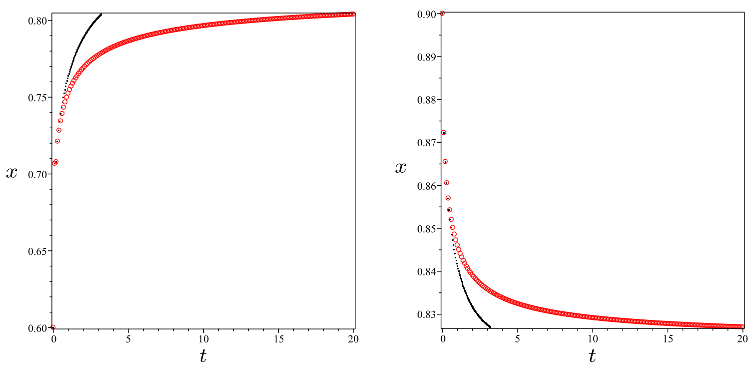
$$
\begin{aligned}
& {}_{h}^{C}\Delta_a^{\alpha} x(t) = -x(t+\alpha h - h) + \frac{\beta}{1 + x(t+\alpha h - h)}, \\
& t \in (h\mathbb{N})_{a+(1-\alpha)h},\ 0 < \alpha \leqslant 1, \\
& x(a) = C.
\end{aligned}
\tag{19}
$$

where ${}_{h}^{C}\Delta_a^{\alpha} x(t)$ is the Caputo h-difference [5, 19]. Within the same parameters, we can observe that the exponential fractional difference equation (17) exhibits a shorter tail in Fig. 2.



(a) $\alpha = 0.5$ (b) $\alpha = 0.6$

Figure 1. Numerical solutions of Example 3.



(a) $\alpha = 0.5$ (b) $\alpha = 0.6$

Figure 2. Exponential fractional difference versus the Caputo h-difference [19]: the black point – Eq. (17); the red circle – Eq. (19).

6 Conclusions

Since the time-scale theory provides an exact discretization method for the continuous fractional calculus, we turn to the classical way on the set (1), and we define a discrete exponential fractional calculus. We report some often-used propositions and theorems, which are useful for qualitative analysis of fractional difference equations. Compared with the classical discrete fractional calculus, our definitions show different memory effects, which provide an alternative tool in discrete modeling such as image processing, population dynamics et al. In addition, there are more problems needed to be addressed, for example, numerical analysis, stability theory and what is the relationship between the discrete and continuous-time systems. We also need a comparative study of the different fractional differences or equations to determine which fractional difference is the most efficient.

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