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Convergence results based on graph-Reich contraction in fuzzy metric spaces with application

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Abstract. This article introduces a novel class of Reich-type contractions that meet the graph preservation criteria in the context of complete fuzzy metric spaces. The provided contraction condition is satisfied through various forms of contractive mappings via directed graphs in the literature. Our key result is the natural extension of fuzzy metric spaces to fuzzy metrics enriched with a graph, which adds the understanding of fixed points in metric spaces within the realm of graph structure. The findings are further supported by examples and applications.

Keywords: fixed point, fuzzy metric, graph, Reich contraction.

1 Introduction

Fixed point theory is an efficient mathematical approach with numerous applications both within and outside of the field. The classical principle of contraction mapping states "if (C, d) is a complete metric space and $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ such that $d(\mathfrak{I}(\varpi), \mathfrak{I}(\varsigma)) \leqslant \alpha d(\varpi, \varsigma)$ for all $\varpi\varsigma \in \mathsf{C}$, where $\alpha \in [0, 1)$, then $\mathfrak{I}$ admits a unique fixed point". Banach's [1] fixed point hypothesis helps to compute mathematics for examining the existence of solutions to nonlinear integral equations, systems of linear equations, and nonlinear differential equations, as well as manifesting algorithms related to convergence. Many variations on Banach's fixed point theorem have been proposed; see [24].

In 1971, Ćirić [8], Reich [20], and Rus [19] established a fixed point theorem for mappings $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$, which has been around for decades and meets the following requirement:

$$d(\mathfrak{I}\varpi, \mathfrak{I}\varsigma) \leqslant a d(\varpi, \varsigma) + b\big(d(\varpi, \mathfrak{I}\varpi) + d(\varsigma, \mathfrak{I}\varsigma)\big) \quad \text{for all } \varpi\varsigma \in \mathsf{C}, \tag{1}$$

where $a, b \geqslant 0$, and $a + 2b < 1$. If $b = 0$, condition 1 reduces to Banach contraction:

$$d(\Im\varpi, \Im\varsigma) \leqslant a \cdot d(\varpi, \varsigma) \quad \text{for all } \varpi\varsigma \in \mathsf{C},$$

where $a \in [0, 1)$. If $a = 0$, condition 1 reduces to Kannan's contraction:

$$d(\Im\varpi, \Im\varsigma) \leqslant b\big(d(\varpi, \Im\varpi) + d(\varsigma, \Im\varsigma)\big) \quad \text{for all } \varpi\varsigma \in \mathsf{C},$$

where $b \in [0, 1/2)$.

Therefore, the fixed point results established in [8, 20] in slightly different forms are true generalizations of Banach's contraction principle [1] and Kannan's [16] fixed point theorem.

Kramosil and Michalek have presented the concept of fuzzy metric spaces in a variety of ways [17]. The Hausdorff topology of metric spaces, which was subsequently described by George and Veeramani [10], who extended an idea of fuzzy metric spaces. They also demonstrated how any metric produces fuzzy metrics, and there is now a wealth of literature on the subject. See [6, 9, 10, 13] for a few examples. It was with Grabiec's [11] paper that researchers first began looking into fixed point theory in fuzzy metric spaces. Then, within the framework of fuzzy metric spaces, Gregori and Sapena [12] proposed the concept of fuzzy contractive mappings and reported some fixed point results.

The study of fixed point theory with a graph plays a significant role in many recent investigations. In 2006, Rus et al. [19] discussed fixed point theorems in ordered L-spaces. However, in 2008, Jachymski [15] used the graphical structure to prove several fixed point findings in fixed point theory. Obtaining a fixed point theorem for G-Kannan maps was first proposed in 2012 by Bojor [3]. In 2014, Shukla et al. [22] generalized some fixed point theorems in graphical metric space that have applications to integral equations. In 2020, Chen et al. [4] presented fixed point solutions for set-valued graphical contraction mappings, and they also introduced a graphical convex metric space. In 2022, Younis et al. [26] discussed the graphical structure of extended b-metric spaces.

The existing approaches in literature in the setting of fuzzy metric spaces are only applicable to satisfy certain conditions. This limits their applicability in some cases. For example, some types of contractions do not guarantee the uniqueness of the fixed point. This can be a problem in some applications requiring a unique solution, whereas Reich contractions apply to a wide range of fuzzy metric spaces. These contractions guarantee the uniqueness of the fixed point, which is important in various applications. Reich contractions have a faster convergence rate than other types of contractions, resulting in quicker convergence to the fixed point.

We extend the definition of $f_{\mathbb{M}}$-space in the context of graph-preserving criteria. The following are primary aims of this manuscript:

(a) to develop a novel concept of $f_{\mathbb{M}}$-spaces utilizing Reich-type contraction;
(b) to introduce the concept of a generalization of multiple contraction mappings and fixed point results;
(c) to determine the presence of integral equation solutions using fuzzy graphical fixed point theory.

This paper is two-fold. First, we provide fixed point and common fixed point theorems for single-valued mappings in the realm of fuzzy metric spaces using Reich-type contractions. On the other hand, to prove fixed points, we introduce the $\mathfrak{J}$-Reich contractions as an expansion of fuzzy $\mathfrak{J}$-contraction. Our findings represent an expansion of Jachymski's [15] and a generalization of Gregori and Sapna's findings [12] in fuzzy metric spaces.

The work is divided into six sections. Section 2 is the summary of earlier literature that acts as inspiration for this research. The section includes some basic definitions helping to understand our findings more easily. In Section 3, we used fuzzy metric spaces to construct fixed points and common fixed point theorems for single-valued Reich contractions. Section at discusses the fixed point of $\mathfrak{J}$-Reich contraction in fuzzy metric spaces with a graph. The relevance of our findings is discussed in Section 5. Finally, Section 6 concludes with discussion of results and future prospects.

2 Preliminary concepts

To facilitate the reader, we will now review key concepts and properties that are relevant to this study. Throughout the paper, we will write fuzzy metric spaces as $f_{\mathbb{M}}$-spaces, and the sets of natural numbers be denoted by $\mathbb{N}$. The set of integers, i.e., $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$, $\mathbb{Z}^+ := \mathbb{N}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbols $\mathbb{R}_0^+$ denote the set of nonnegative real numbers. The notation $\mathbb{E}(\mathscr{L})$ stands for directed graph. Throughout the paper, all sets are assumed to be nonempty.

Definition 1. (See [21].) The binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm, written as $\diamond$-norm, if $([0,1], *)$ is topological monoid with unit 1 and satisfies the condition $a * b \leqslant c * d$ whenever $a \leqslant c$, $b \leqslant d$, and $a, b, c, d \in [0,1]$. Examples of $\diamond$-norm:

(i) Minimum $\diamond$-norm $\mathrm{M}(\varpi, \varsigma) = \min\{\varpi, \varsigma\}$;
(ii) Product $\diamond$-norm $\mathrm{P}(\varpi, \varsigma) = \varpi \cdot \varsigma$;
(iii) Lukasiewicz $\diamond$-norm $\mathrm{L}(\varpi, \varsigma) = \max\{\varpi + \varsigma - 1, 0\}$.

Definition 2. (See [10].) The 3-tuple $(\mathsf{C}, f_{\mathbb{M}}, *)$ is called $f_{\mathbb{M}}$-space if C is an nonempty set, $*$ is triangular norm denoted by $\diamond$-norm, and $\mathbb{M}$ is fuzzy set on $\mathsf{C} \times \mathsf{C} \times [0, \infty)$ satisfying the following conditions for all $\varpi, \varsigma, z \in \mathsf{C}$, and $t, s > 0$:

(i) $f_{\mathbb{M}}(\varpi, \varsigma, t) > 0$;
(ii) $f_{\mathbb{M}}(\varpi, \varsigma, t) = 1 \iff \varpi = \varsigma$;
(iii) $f_{\mathbb{M}}(\varpi, \varsigma, t) = f_{\mathbb{M}}(\varsigma, \varpi, t)$;
(iv) $f_{\mathbb{M}}(\varpi, \varsigma, t) * f_{\mathbb{M}}(\varsigma, z, s) \leqslant f_{\mathbb{M}}(\varpi, z, t + s)$;
(v) $f_{\mathbb{M}}(\varpi, \varsigma, t) : [0, \infty) \to [0,1]$ is left-continuous.

Definition 3. (See [10].) Let (ϖ_n) be a $f_{\mathbb{M}}$-space. The sequence (ϖ_n) is called convergent in $\mathsf{C} \Leftrightarrow f_{\mathbb{M}}(\varpi_n, \varpi, t) \to 1$ whenever $n \to \infty$.

For further synthesis and basic concepts like the Cauchy sequence, the convergence of the Cauchy sequence, and completeness in fuzzy setup, we refer the readers to [10].

Definition 4. (See [10].) Let (C, d) is a metric space. The triplet $(\mathsf{C}, d_{\mathbb{M}}, *)$ is called standard fuzzy metric space generated by metric d, where

$$d_{\mathbb{M}}(\varpi, \varsigma, t) = \frac{t}{t + d(\varpi, \varsigma)}.$$

It is worth noting that the topologies generated by the standard fuzzy metric and its corresponding measure are similar to one another.

Lemma 1. (*See* [10].) *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a* $f_{\mathbb{M}}$*-space. Then* $f_{\mathbb{M}}$ *is continuous function on* $\mathsf{C} \times \mathsf{C} \times (0, +\infty)$.

Lemma 2. (*See* [5].) *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is a* $f_{\mathbb{M}}$*-space. Then* $f_{\mathbb{M}}(\varpi, \varsigma, \cdot)$ *is nondecreasing for all* $\varpi, \varsigma \in \mathsf{C}$.

Definition 5. (See [12].) A self-mapping $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ within the context of a $f_{\mathbb{M}}$-space $(\mathsf{C}, f_{\mathbb{M}}, *)$ is called $\mathfrak{I}$-uniformly continuous if for all $q \in (0, 1)$, there exist $p \in (0, 1)$ such that for all $\varpi, \varsigma \in \mathsf{C}$,

$$f_{\mathbb{M}}(\varpi, \varsigma, t) \geqslant 1 - p \quad \Longrightarrow \quad f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t) \geqslant 1 - q.$$

Remark 1. (See [12].) If $\mathfrak{I}$ is $\mathfrak{I}$-uniformly continuous, then it is called uniformly continuous for the uniformity produced by $\mathbb{M}$ and continuous for the topology inferred from $\mathbb{M}$. We recommend [24] for more information on a uniform structure in a fuzzy metric environment.

Definition 6. (See [12].) A self-mapping $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ defined on $(\mathsf{C}, f_{\mathbb{M}}, *)$ is called fuzzy contractive mapping if there exists $l \in]0, 1[$ such that

$$\frac{1}{f_{\mathbb{M}}(f(\varpi), f(\varsigma), t)} - 1 \leqslant k\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right)$$

for every $\varsigma \neq \varpi \in \mathsf{C}$.

Proposition 1. (*See* [12].) *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a* $f_{\mathbb{M}}$*-space. If* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *is fuzzy contractive mapping, then* $\mathfrak{I}$ *is uniformly continuous.*

The following graph's concepts are similar to those in [14, 15, 25, 26].

Let C be a nonempty set in the sense of Jachymski [15], and let $\varDelta$ be the diagonal of $\mathsf{C} \times \mathsf{C}$. Also, consider the directed graph $\pounds = (\mathcal{V}(\pounds), \mathbb{E}(\pounds))$, where $\mathcal{V}(\pounds)$ is the vertex set of $\pounds$ so that it coincides with the set C, and $\mathbb{E}(\pounds)$ is the edge set of $\pounds$ that contains all the loops of $\pounds$, that is, $\mathbb{E}(\pounds) \supseteq \varDelta$. We designate the graph that results from reversing the orientation of $\mathbb{E}(\pounds)$ as $\pounds^{-1}$. Assuming the graph $\pounds$ has symmetrical edges, the notation $\breve{\pounds}$ is used to indicate this, i.e.,

$$\mathbb{E}(\breve{\pounds}) = \mathbb{E}(\pounds^{-1}) \cup \mathbb{E}(\pounds).$$

Assume that $\pounds$ is a directed graph with ϖ, and ς as its vertices. A path in $\pounds$ is described as a series of vertices $\{\varpi_j\}_{j=0}^{\mathbb{M}}$ with $m + 1$ vertices such that $v_0 = \varpi$, $v_m = \varsigma$, and

(ϖ_{j-1}, ϖ_j) are all present in $\mathbb{E}(\pounds)$, where $j = 1, 2, \dots, m$. If there is a path connecting each pair of its vertices, a graph $\pounds$ is called linked or connected. We refer to a graph as being weak connected when there is a path connecting every pair of its vertices and the graph $\pounds$ is undirected.

Shukla et al. [22] are to be credited with the following annotations [22]:

(a) $[\varpi]^l_\pounds = \{\varsigma \in \mathsf{C}\colon$ in the graph $\pounds,$ there is a directed path with length l that connects ϖ and $\varsigma\}$.

(b) $(vPw)_\pounds$: if there is a path leading from ϖ to ς in $\pounds$ and $\varsigma \in (vPw)_\pounds$, then ς is on the path $(vPw)_\pounds$.

Unless otherwise stated, we consider all the graphs to be direct.

We review some fundamental ideas related to the connectedness of graphs. They may all be found, for instance, in [14, 23, 25].

A path of length N ($N \in \mathbb{N}$) in a graph $\pounds$ from vertex ϖ directing to another vertex ς is the sequence $(\varpi i)_{i=0}^N$ of $N+1$ vertices such that $\varpi_0 = \varpi$, $\varpi_n = \varsigma$, and $(\varpi_{n-1}, \varpi_n) \in \mathbb{E}(\pounds)$, $1 \leqslant i \leqslant N$.

Definition 7. (See [14].) A mapping $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ is called a $\pounds$-fuzzy contraction in the context of $f_\mathbb{M}$-space $(\mathsf{C}, f_\mathbb{M}, *)$ if the following assertions are contented:

(i) $\mathfrak{I}$ is edge-preserving, i.e., for all $\varpi, \varsigma \in \mathsf{C}$, $(\varpi, \varsigma) \in \mathbb{E}(\pounds) \Rightarrow (\mathfrak{I}\varpi, \mathfrak{I}\varsigma) \in \mathbb{E}(\pounds)$;

(ii) There exists $\lambda \in (0, 1)$ such that for all $\varpi, \varsigma \in \mathsf{C}$ and $t > 0$,

$$(\varpi, \varsigma) \in \mathbb{E}(\pounds) \quad \Longrightarrow \quad \frac{1}{f_\mathbb{M}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant \lambda\left(\frac{1}{f_\mathbb{M}(\varpi, \varsigma, t)} - 1\right).$$

Example 1. (See [22].) Any constant function $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ (that is, $\mathfrak{I}\varpi = \varsigma$, $\varpi \in \mathsf{C}$, where $\varsigma \in \mathsf{C}$ fixed) is a $\pounds$-fuzzy contraction with arbitrary value of $\lambda \in]0, 1[$.

Definition 8. (See [22].) Let $(\mathsf{C}, f_\mathbb{M}, *)$ be a $f_\mathbb{M}$-space, and let $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ be a self mapping. We denote the nth iterate of $\mathfrak{I}$ on $\varpi \in \mathsf{C}$ by $\mathfrak{I}^n\varpi$, and $\mathfrak{I}^n\varpi = \mathfrak{I}\mathfrak{I}^{n-1}\varpi$ for all $n \in \mathbb{N}$ with $\mathfrak{I}^0\varpi = \varpi$.

The mapping $\mathfrak{I}$ is called Picard operator if $\mathfrak{I}$ has a unique fixed point u and $\lim_{n\to\infty} f_\mathbb{M}(\mathfrak{I}^n\varpi, u_\varpi, t) = 1$ for all $\varpi \in \mathsf{C}$, $\varsigma > 0$. The mapping $\mathfrak{I}$ is called a weak Picard operator if for all $\varpi \in \mathsf{C}$, there exists a fixed point $u_\varpi \in \mathsf{C}$ (which may depend on ϖ) of $\mathfrak{I}$ such that $\lim_{n\to\infty} f_\mathbb{M}(\mathfrak{I}^n\varpi, u_\varpi, t) = 1$ for all $t > 0$.

It is worth noting that each Picard operator is also a weak Picard operator. The fixed point of a weak Picard operator cannot be truely unique.

Definition 9. (See [22].) A subset A is called $\mathfrak{I}$-invariant if $\mathfrak{I}(A) \subset A$.

The next lemma will help with what comes next.

Lemma 3. (*See* [14].) *Let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a* $\pounds$*-fuzzy contraction. Then given* $\varpi \in \mathsf{C}$ *and* $\varsigma \in [\varpi]_{\tilde{\pounds}}$, *we have* $\lim_{n\to\infty} f_\mathbb{M}(\mathfrak{I}^n\varpi, \mathfrak{I}^n\varsigma, t) = 1$ *for all* $t > 0$.

The component of $\pounds$ that contains ϖ is made up of edges, vertices, and a path that starts at ϖ_0. In this instance, the relation R is defined on $\mathcal{V}(\pounds)$ by the rule:

- $\varsigma R z$ if there is a path in $\pounds$ from ς to z. In this scenario, $\mathcal{V}(\pounds_\varpi) = [\varpi]_\pounds$, where the equivalence class of this relation is $[\varpi]_\pounds$. Thus, the relationship between $\pounds_\varpi$ is self-evident.

We will now talk about different sorts of mapping continuity. The first is well known in metric fixed point theory and is frequently employed.

Definition 10. (See [7].) A mapping $\mathfrak{J} : \mathsf{C} \to \mathsf{C}$ is called orbitally continuous if for all $\varpi \in \mathsf{C}$ and the sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$\mathfrak{J}^{k_n}\varpi \to \varsigma \in \mathsf{C} \quad \Longrightarrow \quad \mathfrak{J}\bigl(\mathfrak{J}^{k_n}\varpi\bigr) \to \mathfrak{J}\varsigma \quad \text{as } n \to \infty.$$

Definition 11. (See [7].) A mapping $\mathfrak{J} : \mathsf{C} \to \mathsf{C}$ is called orbitally $\pounds$-continuous if given $\varpi \in \mathsf{C}$ and the sequence $(\varpi_n)_{n\in\mathbb{N}}$ such that

$$\varpi_n \to \varpi \quad \text{and} \quad (\varpi_n, \varpi_{n+1}) \in \mathbb{E}(\pounds),\ \ n \in \mathbb{N} \quad \Longrightarrow \quad \mathfrak{J}\varpi_n \to \mathfrak{J}\varpi.$$

Recent findings provide the necessary criteria for mapping to a Picard operator (PO) if (C, d) is graph-endowed. It was first discovered by Jachymski [15], who then explained its connection to the Kelisky–Rivlin theorem on iterates of the Bernstein operators in space $C[0, 1]$.

Definition 12. (See [15].) A self map $f : \mathsf{C} \to \mathsf{C}$ is called a Banach $\pounds$-contraction ($\pounds$-contraction) if f preserves edges, i.e., for all $\varpi, \varsigma \in \mathsf{C}$,

$$(\varpi, \varsigma) \in \mathbb{E}(\pounds) \quad \Longrightarrow \quad \bigl(f(\varpi), f(\varsigma)\bigr) \in \mathbb{E}(\pounds),$$

and edge weights decrease as follows in $\pounds$: there exist $\alpha \in (0, 1)$ such that for all $\varpi, \varsigma \in \mathsf{C}$,

$$(\varpi, \varsigma) \in \mathbb{E}(\pounds) \quad \Longrightarrow \quad d\bigl(f(\varpi), f(\varsigma)\bigr) \leqslant \alpha d(\varpi, \varsigma).$$

Definition 13. (See [22].) A mapping $\mathfrak{J} : \mathsf{C} \to \mathsf{C}$ is called a Ćirić–Rus–Reich mapping within the context of metric space (C, d) if for all $\varpi, \varsigma \in \mathsf{C}$,

$$d(\mathfrak{J}\varpi, \mathfrak{J}\varsigma) \leqslant \alpha d(\varpi, \varsigma) + \beta d(\varpi, \mathfrak{J}\varpi) + \gamma d(\varsigma, \mathfrak{J}\varsigma),$$

$\alpha + \beta + \gamma < 1$, $\alpha, \beta, \gamma > 0$.

Remark 2. This aforementioned condition is equivalent to the following statement by utilizing the symmetry of distance: there exist nonnegative numbers α and β with $\alpha + 2\beta < 1$ such that for all $\varpi, \varsigma \in \mathsf{C}$,

$$d(\mathfrak{J}\varpi, \mathfrak{J}\varsigma) \leqslant \alpha d(\varpi, \varsigma) + \beta\bigl[d(\varpi, \mathfrak{J}\varpi) + d(\varsigma, \mathfrak{J}\varsigma)\bigr].$$

As per the results proved by Reich [20], Ćirić [8], and Rus [19], we say that Ćirić–Reich–Rus mapping possesses a unique fixed point if (C, d) is complete.

This paper proposes the idea of $\pounds$-Reich operators in order to analyse the fixed points for Reich operators in $f_{\mathbb{M}}$-spaces given a graph $\pounds$. We assume that (C, d) is a metric space and that $\pounds$ is a directed graph with the criteria that $\mathcal{V}(\pounds) = \mathsf{C}, \mathbb{E}(\pounds) \supseteq \Delta$ and that $\pounds$ has no parallel edges. An instance of $Fix_{\Im}$ is a collection of all fixed points for a mapping $\Im$. The following concept serves as the foundation for the suggested outcomes throughout the work.

Definition 14. (See [2].) A mapping $\Im : \mathsf{C} \to \mathsf{C}$ in the framework of metric space (C, d) is called $\pounds$-Reich operator if the following assumptions hold true:

(i) $(\varpi, \varsigma) \in \mathbb{E}(\pounds) \Rightarrow (\Im\varpi, \Im\varsigma) \in \mathbb{E}(\pounds)$ for all $\varpi, \varsigma \in \mathsf{C}$;

(ii) For each $(\varpi, \varsigma) \in \mathbb{E}(\pounds)$,

$$d(\Im\varpi, \Im\varsigma) \leqslant \alpha d(\varpi, \varsigma) + \beta d(\varpi, \Im\varpi) + \gamma d(\varsigma, \Im\varsigma),$$

where $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma > 0$.

Lemma 4. (*See* [2].) *Let $\pounds$ be a graph endowing metric space (C, d), and let $\Im : \mathsf{C} \to \mathsf{C}$ be a $\pounds$-Ćirić–Rus–Reich operator. If the assertion $(\varpi, \Im\varpi) \in \mathbb{E}(\pounds)$ is satisfied by $\varpi \in \mathsf{C}$, then for all $n \in \mathbb{N}$,*

$$d\big(\Im^n\varpi, \Im^{n+1}\varpi\big) \leqslant \alpha^n d(\varpi, \Im\varpi),$$

where $\alpha = (a + b)/(1 - c)$.

Proof. Let $\varpi \in \mathsf{C}$ with $(\varpi, \Im\varpi) \in \mathbb{E}(\pounds)$. An easy induction shows $(\Im^n\varpi, \Im^{n+1}\varpi) \in \mathbb{E}(\pounds)$ for all $n \in \mathbb{N}$,

$$\begin{aligned} d\big(\Im^n\varpi, \Im^{n+1}\varpi\big) \leqslant{} & ad\big(\Im^{n-1}\varpi, \Im^n\varpi\big) + bd\big(\Im^{n-1}\varpi + \Im^n\varpi\big) \\ & + cd\big(\Im^n\varpi, \Im^{n+1}\varpi\big), \end{aligned}$$

which implies

$$d\big(\Im^n\varpi, \Im^{n+1}\varpi\big) \leqslant \alpha d\big(\Im^{n-1}\varpi, \Im^n\varpi\big),$$

where $\alpha = (a + b)/(1 - c)$. So, for all $n \in \mathbb{N}$, we get

$$d\big(\Im^n\varpi, \Im^{n+1}\varpi\big) \leqslant \alpha^n d(\varpi, \Im\varpi). \qquad \square$$

3 Fuzzy-Reich-type contraction and related convergence results

This section is devoted to produce fixed point and common fixed results for single-valued Reich-type contractions in the setting of $f_{\mathbb{M}}$-space.

Definition 15. Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be a complete $f_{\mathbb{M}}$-space. A self-mapping $\Im : \mathsf{C} \to \mathsf{C}$ is called a fuzzy Reich contraction if there is $\alpha \in (0, 1)$ such that for all $\varpi, \varsigma \in \mathsf{C}$ and $\Im > 0$,

$$\begin{aligned} \frac{1}{f_{\mathbb{M}}(\Im\varpi, \Im\varsigma, t)} - 1 \leqslant{} & \bigg\{a\bigg(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\bigg) + b\bigg(\frac{1}{f_{\mathbb{M}}(\varpi, T(\varpi), t)} - 1\bigg) \\ & + c\bigg(\frac{1}{f_{\mathbb{M}}(\varsigma, T(\varsigma), t)} - 1\bigg)\bigg\} \end{aligned}$$

with $a + b + c < 1$.

Theorem 1. *Suppose* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is complete* $f_{\mathbb{M}}$*-space. Let* $\mathcal{S}, \mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be self-mappings such that for all* $(\varpi, \varsigma) \in \mathsf{C} \times \mathsf{C}$*,*

$$\begin{aligned}\frac{1}{f_{\mathbb{M}}(\mathcal{S}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant \biggl\{ & a\biggl(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\biggr) + b\biggl(\frac{1}{f_{\mathbb{M}}(\varpi, \mathcal{S}(\varpi), t)} - 1\biggr) \\ & + c\biggl(\frac{1}{f_{\mathbb{M}}(\varsigma, \mathfrak{I}(\varsigma), t)} - 1\biggr)\biggr\}\end{aligned} \tag{2}$$

with $a, b, c \in [0, \infty)$*,* $a + b + c < 1$*. Then* $\mathcal{S}$ *and* $\mathfrak{I}$ *have unique common fixed point in* C*.*

Proof. Suppose ϖ_0 is an arbitrary point and define the sequence (ϖ_n) by

$$\mathcal{S}\varpi_{2j} = \varpi_{2j+1} \quad \text{and} \quad \mathfrak{I}\varpi_{2j+1} = \varpi_{2j+2}, \quad j = 0, 1, 2, \dots . \tag{3}$$

Using (2) and (3), we can write

$$\begin{aligned}&\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1 \\ &\quad = \frac{1}{f_{\mathbb{M}}(\mathcal{S}\varpi_{2j}, \mathfrak{I}\varpi_{2j+1}, t)} - 1 \\ &\quad \leqslant \biggl\{ a\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\biggr) + b\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \mathcal{S}\varpi_{2j}, t)} - 1\biggr) \\ &\qquad + c\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \mathfrak{I}\varpi_{2j+1}, t)} - 1\biggr)\biggr\} \\ &\quad = \biggl\{ a\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\biggr) + b\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\biggr) \\ &\qquad + c\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\biggr)\biggr\}.\end{aligned}$$

This implies

$$\begin{aligned}&(1 - c)\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\biggr) \\ &\quad \leqslant (a + b)\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\biggr) = \frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1 \\ &\quad < \frac{a + b}{1 - c}\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\biggr) = \lambda\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\biggr),\end{aligned}$$

where $(a + b)/(1 - c) = \lambda$. Similarly,

$$\begin{aligned}\frac{1}{f_{\mathbb{M}}(\varpi_{2j+2}, \varpi_{2j+3}, t)} - 1 &< \frac{a + b}{1 - c}\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\biggr) \\ &= \lambda\biggl(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\biggr).\end{aligned}$$

Continuing in this way, we get

$$\frac{1}{f_{\mathbb{M}}(\varpi_n, \varpi_{n+1}, t)} - 1 < \lambda\left(\frac{1}{f_{\mathbb{M}}(\varpi_{n-1}, \varpi_n, t)} - 1\right), \quad n \in \mathbb{N},$$

which yields

$$\begin{aligned}\frac{1}{f_{\mathbb{M}}(\varpi_n, \varpi_{n+1}, t)} - 1 &< \lambda\left(\frac{1}{f_{\mathbb{M}}(\varpi_{n-1}, \varpi_n, t)} - 1\right)\\ &< \lambda^2\left(\frac{1}{f_{\mathbb{M}}(\varpi_{n-2}, \varpi_{n-1}, t)} - 1\right) < \cdots\\ &< \lambda^n\left(\frac{1}{f_{\mathbb{M}}(\varpi_0, \varpi_1, t)} - 1\right), \quad n \in \mathbb{N}. \end{aligned}\tag{4}$$

Using (4), we can write

$$\begin{aligned}&\frac{1}{\sum_{k=n}^{M-1} f_{\mathbb{M}}(\varpi_k, \varpi_{k+1}, t)} - 1\\ &\quad= \left(\frac{1}{f_{\mathbb{M}}(\varpi_n, \varpi_{n+1}, t)} - 1\right) + \cdots + \left(\frac{1}{f_{\mathbb{M}}(\varpi_{m-1}, \varpi_{\mathbb{M}}, t)} - 1\right)\\ &\quad< \lambda^n\left(\frac{1}{f_{\mathbb{M}}(\varpi_0, \varpi_1, t)} - 1\right) + \cdots + \lambda^{M-1}\left(\frac{1}{f_{\mathbb{M}}(\varpi_0, \varpi_1, t)} - 1\right)\\ &\quad< \lambda^n\left[1 + \lambda + \lambda^2 + \cdots + \lambda^{M-n-1}\right]\left(\frac{1}{f_{\mathbb{M}}(\varpi_0, \varpi_1, t)} - 1\right)\\ &\quad\leqslant \frac{\lambda^n}{1-\lambda}\left(\frac{1}{f_{\mathbb{M}}(\varpi_0, \varpi_1, t)} - 1\right), \quad m > n.\end{aligned}$$

Since $\lim_{n\to\infty}(\lambda^n/(1-\lambda))f_{\mathbb{M}}(\varpi_0, \varpi_1, t) = 0$, for any $\delta > 0$, there exists some $n' \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda}\left(\frac{1}{f_{\mathbb{M}}(\varpi_0, \varpi_1, t)} - 1\right) < \delta, \quad n \geqslant n',$$

which yields that (ϖ_n) is a Cauchy sequence in C. Since $(\mathsf{C}, f_{\mathbb{M}}, *)$ is complete, there exists $v \in \mathsf{C}$ such that

$$\lim_{n\to\infty} f_{\mathbb{M}}(\varpi_n, v, t) = 1, \quad t \gg \theta.$$

To prove that v is a fixed point of S, assume that $f_{\mathbb{M}}(\mathcal{S}v, v, t) > 0$. Then

$$\begin{aligned}&\frac{1}{f_{\mathbb{M}}(\mathcal{S}v, \varpi_{2j+2}, t)} - 1\\ &\quad= \frac{1}{f_{\mathbb{M}}(\mathcal{S}v, \mathfrak{I}\varpi_{2j+1}, t)} - 1\\ &\quad\leqslant a\left(\frac{1}{f_{\mathbb{M}}(v, \varpi_{2j+1}, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(v, \mathcal{S}v)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \mathfrak{I}\varpi_{2j+1}, t)} - 1\right)\end{aligned}$$

$$
\begin{aligned}
&= a\left(\frac{1}{f_{\mathbb{M}}(v, \varpi_{2j+1}, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(v, \mathcal{S}v, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\right) \\
&\to b\left(\frac{1}{f_{\mathbb{M}}(v, \mathcal{S}v, t)} - 1\right) \quad \text{as } n \to \infty,
\end{aligned}
$$

and

$$
\limsup_{n\to\infty}\left(\frac{1}{f_{\mathbb{M}}(\mathcal{S}v, \varpi_{2j+2}, t)} - 1\right) \leqslant b\left(\frac{1}{f_{\mathbb{M}}(\mathcal{S}v, v, t)} - 1\right), \quad t \geqslant \theta,
$$

which implies

$$
(1-b)\left(\frac{1}{f_{\mathbb{M}}(v, \mathcal{S}v, t)} - 1\right) < 0, \quad t \geqslant \theta.
$$

Noticing that $b < 1$, since $a + b + c < 1$, then $f_{\mathbb{M}}(v, \mathcal{S}v, t) = 1$, i.e., $\mathcal{S}v = v$. Similarly, Suppose $1/f_{\mathbb{M}(\gtrapprox, \Im\gtrapprox, \approx)} - 1 > 0$.

$$
\begin{aligned}
&\frac{1}{f_{\mathbb{M}}(\Im v, \varpi_{2j+1}, t)} - 1 \\
&\quad = \frac{1}{f_{\mathbb{M}}(\Im v, \mathcal{S}\varpi_{2j}, t)} - 1 \\
&\quad \leqslant a\left(\frac{1}{f_{\mathbb{M}}(v, \varpi_{2j}, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \mathcal{S}\varpi_{2j}, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(v, \Im v, t)} - 1\right) \\
&\quad = a\left(\frac{1}{f_{\mathbb{M}}(v, \varpi_{2j}, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(v, \mathcal{S}v, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\right) \\
&\quad \to c\left(\frac{1}{f_{\mathbb{M}}(v, \Im v, t)} - 1\right) \quad \text{as } n \to \infty,
\end{aligned}
$$

i.e.,

$$
(1-c)\left(\frac{1}{f_{\mathbb{M}}(v, \Im v, t)} - 1\right) < 0.
$$

Then

$$
\limsup_{n\to\infty}\left(\frac{1}{f_{\mathbb{M}}(\Im v, \varpi_{2j+1}, t)} - 1\right) \leqslant c\left(\frac{1}{f_{\mathbb{M}}(\Im v, v, t)} - 1\right), \quad t \geqslant \theta,
$$

which implies

$$
(1-c)\left(\frac{1}{f_{\mathbb{M}}(\Im v, v, t)} - 1\right) < 0, \quad t \geqslant \theta.
$$

Noticing that $c < 1$, since $a + b + c < 1$, then $f_{\mathbb{M}}(v, \Im v, t) = 1$, i.e., $\Im v = v$.

Let $u \in \mathsf{C}$ be any fixed point of $\Im$, and let $\mathcal{S} \in \mathsf{C}$, i.e., $u \neq v$. Then

$$
\begin{aligned}
\frac{1}{f_{\mathbb{M}}(u, v, t)} - 1 &= \frac{1}{f_{\mathbb{M}}(\mathcal{S}u, \Im v, t)} - 1 \\
&\leqslant a\left(\frac{1}{f_{\mathbb{M}}(u, v, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(u, \mathcal{S}u, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(v, \Im v, t)} - 1\right)
\end{aligned}
$$

$$= a\left(\frac{1}{f_{\mathbb{M}}(u,v,t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(u,u,t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(v,v,t)} - 1\right)$$
$$= (1-a)\left(\frac{1}{f_{\mathbb{M}}(u,v,t)} - 1\right).$$

Since $(1-a) < 1$, we get $f_{\mathbb{M}}(u,v,t) = 1$ and $u = v$. That is, $\mathfrak{I}$ has a unique fixed point. $\square$

Corollary 1. *Suppose* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is a complete* $f_{\mathbb{M}}$*-space. Let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a self-mapping such that for all* $(\varpi, \varsigma) \in \mathsf{C} \times \mathsf{C}$*,*

$$\frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi,\varsigma,t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi,\mathfrak{I}\varpi,\varsigma)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varsigma,\mathfrak{I}\varsigma,t)} - 1\right)$$

with $a+b+c > 1$*,* $a,b,c > 0$*. Then there is only one fixed point in* C *for* $\mathfrak{I}$*.*

The following corollary is derived from the assumption that $b = 0 = c$ in Corollary 1.

Corollary 2. *Suppose* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is a complete* $f_{\mathbb{M}}$*-space. Let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a self-mapping such that for* $(\varpi, \varsigma) \in \mathsf{C} \times \mathsf{C}$*,*

$$\frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi,\varsigma,t)} - 1\right)$$

with $a \in (0, \infty)$*. Then* $\mathfrak{I}$ *have at most one fixed point in* C*.*

In addition to the above significant findings, Theorem 1 leads us to the following fixed point result based on Kannan-type mappings.

Corollary 3. *Suppose* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is complete* $f_{\mathbb{M}}$*-space. Let* $\mathcal{S}, \mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be self-mappings. Suppose that for* $k \in [0,1)$*,*

$$\frac{1}{f_{\mathbb{M}}(\mathcal{S}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant \frac{k}{2}\left(\frac{1}{(f_{\mathbb{M}}(\varpi,\mathcal{S}\varpi,t)} - 1 + \frac{1}{f_{\mathbb{M}}(\varsigma,\mathfrak{I}\varsigma,t)} - 1\right)$$

where $(\varpi, \varsigma) \in \mathsf{C} \times \mathsf{C}$*. Then* $\mathcal{S}$ *and* $\mathfrak{I}$ *have at most one common fixed point in* C*.*

Proof. Assuming that ϖ_0 is an arbitrary point, define the sequence (ϖ_n) as follows:

$$\mathcal{S}\varpi_{2j} = \varpi_{2j+1} \quad \text{and} \quad \mathfrak{I}\varpi_{2j+1} = \varpi_{2j+2}, \quad j = 0,1,2,\dots .$$

Using (2) and (3), we can write

$$\begin{aligned}
&\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1 \\
&\quad = \frac{1}{f_{\mathbb{M}}(\mathcal{S}\varpi_{2j}, \Im\varpi_{2j+1}, t)} - 1 \\
&\quad \leqslant \frac{k}{2}\left\{\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \mathcal{S}\varpi_{2j}, t)} - 1\right) + \left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \Im\varpi_{2j+1}, t)} - 1\right)\right\} \\
&\quad = \frac{k}{2}\left\{\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\right) + \left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\right)\right\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left(1 - \frac{k}{2}\right)\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\right) \\
&\quad \leqslant \frac{k}{2}\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\right) = \frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1 \\
&\quad < \frac{k}{2-k}\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\right) = \lambda\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j}, \varpi_{2j+1}, t)} - 1\right),
\end{aligned}$$

where $k/(2-k) = \lambda$. Similarly.

$$\begin{aligned}
\frac{1}{f_{\mathbb{M}}(\varpi_{2j+2}, \varpi_{2j+3}, t)} - 1 &< \frac{k}{2-k}\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\right) \\
&= \lambda\left(\frac{1}{f_{\mathbb{M}}(\varpi_{2j+1}, \varpi_{2j+2}, t)} - 1\right).
\end{aligned}$$

The common fixed point of $\mathcal{S}$ and $\Im$ obtained by proceeding in the same manner as in Theorem 1. □

The following is the outcome of a single mapping in which $\mathcal{S}$ with $\Im$.

Corollary 4. *Suppose* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is a complete* $f_{\mathbb{M}}$*-space. Let* $\Im : \mathsf{C} \to \mathsf{C}$ *be self-mapping. Suppose that for* $k \in [0, 1)$ *and* $(\varpi, \varsigma) \in \mathsf{C} \times \mathsf{C}$*,*

$$\frac{1}{f_{\mathbb{M}}(\Im\varpi, \Im\varsigma, t)} - 1 \leqslant \frac{k}{2}\left(\frac{1}{f_{\mathbb{M}}(\varpi, \Im\varpi, t)} - 1 + \frac{1}{f_{\mathbb{M}}(\varsigma, \Im\varsigma, t)} - 1\right).$$

Then mapping $\Im$ *has at most one common fixed point in* C*.*

4 Fixed points of graph-Reich contractions

Definition 16. Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be a $f_{\mathbb{M}}$-space endowed with graph $\pounds$. The operator $\Im : \mathsf{C} \to \mathsf{C}$ is called fuzzy $\pounds$-Reich operator if:

(i) $(\varpi, \varsigma) \in \mathbb{E}(\pounds) \Rightarrow (\Im\varpi, \Im\varsigma) \in \mathbb{E}(\pounds)$ for all $\varpi, \varsigma \in \mathsf{C}$;

(ii) The values a, b, c are all positive, and for every pair $(\varpi, \varsigma) \in \mathbb{E}(\pounds)$, we have

$$\frac{1}{f_{\mathbb{M}}(\pounds\varpi, \pounds\varsigma, t)} - 1 \leqslant \left\{ a\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi, \pounds(\varpi), t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varsigma, \pounds(\varsigma), t)} - 1\right)\right\}.$$

Remark 3. Any Reich contraction is a Reich $\pounds_0$-contraction with $\pounds_0$ defined by $E(\pounds_0) = \mathsf{C} \times \mathsf{C}$. If a mapping $\mathfrak{I}$ is Reich $\pounds$-contraction with parameters a, b, c, and $b = c$, then it is a Reich $\tilde{\pounds}$-contraction.

Definition 17. Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be a metric space endowed with a directed graph $\pounds$. A mapping $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ is called $\pounds$-Reich contraction. For all $(\varpi, \varsigma) \in \mathbb{E}(\pounds)$, the following conditions are true if and only if $\alpha, \beta, \gamma \in [0, 1]$ and $\alpha + \beta + \gamma < 1$:

(i) $(\mathfrak{I}\varpi, \mathfrak{I}\varsigma) \in \mathbb{E}(\pounds)$;

(ii) $$\frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi, \mathfrak{I}\varpi, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varsigma, \mathfrak{I}\varsigma, t)} - 1\right).$$

When a, b, and c are given, we say that $\mathfrak{I}$ is a $\pounds$-Reich contraction.

Theorem 2. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a complete metric space endowed with a directed graph* $\pounds$. $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *is a* $\pounds$*-Reich contraction under the following assumptions*:

(i) *For any sequence* $\{\varpi_n\}$ *in* C*, if* $\lim_{n\to\infty} \varpi_n = \varpi \in \mathsf{C}$ *and* $(\varpi_n, \varpi_{n+1}) \in \mathbb{E}(\pounds)$ *for all* $n \in \mathbb{N}$*, then there is a subsequence* $\{\varpi n_k\}$ *of* $\{\varpi_n\}$ *such that* $(\varpi n_k, \varpi) \in \mathbb{E}(\pounds)$ *for all* $k \in \mathbb{N}$.

(ii) *If* $\pounds$ *is* $\mathfrak{I}$*-connected, then* $\mathfrak{I}$ *is a* PO.

Specifically, we assume that $(\mathsf{C}, f_{\mathbb{M}}, *)$ is a $f_{\mathbb{M}}$-space endowed with a graph $\pounds$ and $\pounds$ is directed graph such that the set $V(\pounds) = \mathsf{C}$, $\mathbb{E}(\pounds) \supseteq \Delta$. Let $\pounds$ has no parallel edges. We assign to each edge a unique element $(\mathsf{C}, f_{\mathbb{M}}, *)$.

In order to demonstrate the forthcoming fixed point theorems, the following results are useful.

Definition 18. Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be a $f_{\mathbb{M}}$-space endowed with a graph $\pounds$, and let $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ be a mapping. We say that the graph $\pounds$ is $\mathfrak{I}$-connected if for all vertices ϖ_i, ς of $\pounds$ with $(\varpi_i, \varsigma) \notin \mathbb{E}(\pounds)$, there exists a path in $\pounds$, $(\varpi_i)_{i=0}^N$ from ϖ to ς such that $\varpi_0 = \varpi$, $\varpi_n = \varsigma$ and $(\varpi_i, \mathfrak{I}\varpi_i) \in \mathbb{E}(\pounds)$ for all $i = 1, \dots, N-1$. A graph $\pounds$ is weak $\mathfrak{I}$-connected if $\tilde{\pounds}$ is $\mathfrak{I}$-connected.

Lemma 5. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a* $f_{\mathbb{M}}$*-space endowed with a graph* $\pounds$*, and let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a* $\pounds$*-Ćirić–Reich operator such that the graph* $\pounds$ *is* $\mathfrak{I}$*-connected. For all* $\varpi \in \mathsf{C}$*, the sequence* $(\mathfrak{I}^n\varpi)_{n\in\mathbb{N}}$ *is a Cauchy sequence.*

Proof. Put $\varpi \in \mathsf{C}$ as a fixed value. Two cases are discussed:

Case 1. If $(\varpi, \Im\varpi) \in \mathbb{E}(\pounds)$, then by Lemma 4 we have

$$\frac{1}{f_{\mathbb{M}}(\Im^n\varpi, \Im^{n+1}\varpi, t)} - 1 \leqslant \alpha^n \left(\frac{1}{f_{\mathbb{M}}(\varpi, \Im\varpi, t)} - 1 \right)$$

for all $n \in \mathbb{N}^*$, where $\alpha = (a+b)/(1-c)$. Because $\alpha < 1$, we get

$$\sum_{n=0}^{\infty} \frac{1}{f_{\mathbb{M}}(\Im^n\varpi, \Im^{n+1}\varpi, t)} - 1 = \frac{1}{1-\alpha} \left(\frac{1}{f_{\mathbb{M}}(\varpi, \Im\varpi, t)} - 1 \right) < \infty,$$

and the usual reasoning demonstrates that Cauchy sequences have the form $(\Im^n\varpi)_{n\in\mathbb{N}}$.

Case 2. If $(\varpi, \Im\varpi) \notin \mathbb{E}(\pounds)$, there is a path from ϖ to $\Im\varpi$ in $\pounds$, $(\varpi_i)_{i=0}^N$, where $\varpi_0 = \varpi$, $\varpi_n = \Im\varpi$ with $(\varpi_{i-1}, \varpi_i) \in \mathbb{E}(\pounds)$ for all $i = 1, \dots, N$, and $(\varpi_i, \Im\varpi_i) \in \mathbb{E}(\pounds)$ for all $i = 1, \dots, N-1$. Consequently, by using the inequality of the fuzzy metric triangle, we have

$$\begin{aligned}
&\frac{1}{f_{\mathbb{M}}(\Im^n\varpi, \Im^{n+1}\varpi, t)} - 1 \\
&\quad\leqslant \sum_{i=1}^{N} \left(\frac{1}{f_{\mathbb{M}}(\Im^n\varpi_{i-1}, \Im^n\varpi_i, t)} - 1 \right) \\
&\quad\leqslant a \sum_{i=1}^{N} \left(\frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_{i-1}, \Im^{n-1}\varpi_i, t)} - 1 \right) + b \sum_{i=1}^{N} \left(\frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_{i-1}, \Im^n\varpi_{i-1}, t)} - 1 \right) \\
&\qquad + c \sum_{i=1}^{N} \left(\frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_i, \Im^n\varpi_i, t)} - 1 \right) \\
&\quad\leqslant a \sum_{i=1}^{N} \left(\frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_{i-1}, \Im^{n-1}\varpi_i, t)} - 1 \right) + b \left(\frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi, \Im^n\varpi, t)} - 1 \right) \\
&\qquad + b\alpha^{n-1} \sum_{i=2}^{N} \left(\frac{1}{f_{\mathbb{M}}(\varpi_{i-1}, \Im\varpi_{i-1}, t)} - 1 \right) + c \left(\frac{1}{f_{\mathbb{M}}(\Im^n\varpi, \Im^{n+1}\varpi, t)} - 1 \right) \\
&\qquad + c\alpha^{n-1} \sum_{i=1}^{N-1} \left(\frac{1}{f_{\mathbb{M}}(\varpi_i, \Im\varpi_i, t)} - 1 \right).
\end{aligned}$$

So, let us define

$$\varpi_n := \sum_{i=1}^{N} \frac{1}{f_{\mathbb{M}}(\Im^n\varpi_{i-1}, \Im^n\varpi_i, t)} - 1, \quad n \in \mathbb{N},$$

and

$$r(\varpi) := (b+c) \sum_{i=2}^{N} \frac{1}{f_{\mathbb{M}}(\varpi_{i-1}, \Im\varpi_{i-1}, t)} - 1.$$

After that,

$$\varpi_n \leqslant (a+b)\varpi_{n-1} + (b+c)\alpha^{n-1} r(\varpi) + c\varpi_n,$$

hence,

$$\varpi_n \leqslant \alpha\varpi_{n-1} + \frac{b+c}{1-c}\,\alpha^{n-1} r(\varpi), \quad \alpha := \frac{a+b}{1-c}.$$

Via elementary computations, we get

$$\varpi_n \leqslant n\,\frac{b+c}{1-c}\,\alpha^{n-1} r(\varpi)$$

for all $n \in \mathbb{N}$. Since $\alpha \in [0, 1]$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{f_{\mathbb{M}}(\mathfrak{I}^n\varpi, \mathfrak{I}^{n+1}\varpi, t)} - 1 \leqslant \sum_{n=0}^{\infty} \varpi_n \leqslant \frac{b+c}{1-c}\,r(\varpi) \sum_{n=0}^{\infty} n\alpha^{n-1} \\ = \frac{b+c}{(1-c)(1-\alpha)^2}\,r(\varpi) < \infty.\end{aligned}$$

A common proof technique reveals that $(\mathfrak{I}^n\varpi)_{n\geqslant 0}$ is a Cauchy sequence. □

This article's main result is stated in the following theorem.

Theorem 3. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a complete* $f_{\mathbb{M}}$*-space endowed with a graph* $\pounds$*, and let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a* $\pounds$*-Reich operator. Assume that*:

(i) $\pounds$ *is* $\mathfrak{I}$*-connected*;
(ii) *For any* $(\varpi_n)_{n\in\mathbb{N}}$ *in* C*, if* $\varpi_n \to \varpi$ *and* $(\varpi_n, \varpi_{n+1}, t) \in \mathbb{E}(\pounds)$ *for* $n \in \mathbb{N}$*, then a subsequence* $(\varpi k_n)_{n\in\mathbb{N}}$ *with* $(\varpi k_n, \varpi, t) \in \mathbb{E}(\pounds)$ *for* $n \in \mathbb{N}$.

Then $\mathfrak{I}$ *is a PO.*

Proof. By Lemma 5, $(\mathfrak{I}^n\varpi)_{n\geqslant 0}$ is a Cauchy sequence for all $\varpi \in \mathsf{C}$, and from hypothesis we have that $(\mathfrak{I}^n\varpi)_{n\geqslant 0}$ is convergent. Let $\varpi, \varsigma \in \mathsf{C}$, then $(\mathfrak{I}^n\varpi)_{n\geqslant 0} \to \varpi^*$, and $(\mathfrak{I}^n t)_{n\geqslant 0} \to \varsigma^*$ as $n \to \infty$.

Case 1. If $(\varpi, \varsigma) \in \mathbb{E}(\pounds)$, we have $(\mathfrak{I}^n\varpi, \mathfrak{I}^n\varsigma, t) \in \mathbb{E}(\pounds)$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\frac{1}{f_{\mathbb{M}}(\mathfrak{I}^n\varpi, \mathfrak{I}^n\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\mathfrak{I}^{n-1}\varpi, \mathfrak{I}^{n-1}\varsigma, t)} - 1\right) \\ + b\left(\frac{1}{f_{\mathbb{M}}(\mathfrak{I}^{n-1}\varpi, \mathfrak{I}^{n}\varpi, t)} - 1\right) \\ + c\left(\frac{1}{f_{\mathbb{M}}(\mathfrak{I}^{n-1}\varsigma, \mathfrak{I}^{n}\varsigma, t)} - 1\right)\end{aligned}$$

for all $n \in \mathbb{N}^*$. Letting $n \to \infty$, $1/f_{\mathbb{M}}(\varpi^*, \varsigma^*, t) - 1 \leqslant a(1/f_{\mathbb{M}}(\varpi^*, \varsigma^*, t) - 1)$, because $a \in [0, 1)$, we obtain $\varpi^* = \varsigma^*$.

Case 2. If $(\varpi, \varsigma, t) \notin \mathbb{E}(\pounds)$, there is a path in $\pounds, (\varpi i)_{i=0}^{N}$ from ϖ to ς such that $\varpi_0 = \varpi, \varpi_n = \varsigma$ with $(\varpi_{i-1}, \varpi_i) \in \mathbb{E}(\pounds)$ for all $i = 1, \dots, N$, and $(\varpi i, \Im\varpi i) \in \mathbb{E}(\pounds)$ for all $i = 1, \dots, N-1$. Then $(\Im^n\varpi_{i-1}, \Im^n\varpi i) \in \mathbb{E}(\pounds)$ for all $n \in \mathbb{N}$, and $i = 1, \dots, N$, and by the triangle inequality we get

$$\begin{aligned}
\frac{1}{f_{\mathbb{M}}(\Im^n\varpi, \Im^n\varsigma, t)} - 1 &\leqslant \sum_{i=1}^{N} \frac{1}{f_{\mathbb{M}}(\Im^n\varpi_{i-1}, \Im^n\varpi_i, t)} - 1 \\
&\leqslant a\biggl(\sum_{i=1}^{N} \frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_{i-1}, \Im^{n-1}\varpi_i, t)} - 1\biggr) \\
&\quad + b\biggl(\sum_{i=1}^{N} \frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_{i-1}, \Im^n\varpi_{i-1}, t)} - 1\biggr) \\
&\quad + c\biggl(\sum_{i=1}^{N} \frac{1}{f_{\mathbb{M}}(\Im^{n-1}\varpi_i, \Im^n\varpi_i, t)} - 1\biggr).
\end{aligned}$$

By Lemma 5 and the hypothesis we state that the series $(\Im^n\varpi i)_{n\geqslant 0}$ is convergent as does the continuity, which states that the sequence $(1/f_{\mathbb{M}}(\Im^n\varpi i - 1, \Im^n\varpi i, t) - 1)_{n\in\mathbb{N}}$ is convergent. Let $\lim_{n\to\infty} 1/f_{\mathbb{M}}(\Im^n\varpi i - 1, \Im^n\varpi i, t) - 1 := \ell_i$ for all $i = 1, \dots, N$. When $n \to \infty$, we obtain $\ell_i = 0$ for all $i = 1, \dots, N$, that is,

$$\frac{1}{f_{\mathbb{M}}(\varpi^*, \varsigma^*, t)} - 1 \leqslant 0.$$

Hence $\varpi^* = \varsigma^*$. Therefore, for all $\varpi \in \mathsf{C}$, there exists a unique $\varpi^* \in \mathsf{C}$ such that

$$\lim_{n\to\infty} \Im^n\varpi = \varpi^*.$$

We will explain that now $\varpi^* \in Fix_{\Im}$. Because the graph $\pounds$ is $\Im$-connected, there is at least $\varpi_0 \in \mathsf{C}$ such that $(\varpi_0, \Im\varpi_0) \in \mathbb{E}(\pounds)$, so $(\Im^n\varpi_0, \Im^{n+1}\varpi_0) \in \mathbb{E}(\pounds)$ for all $n \in \mathbb{N}$. But $\lim_{n\to\infty} \Im^n\varpi_0 = \varpi^*$, then there is a subsequence $(\Im^{k_n}\varpi_0)_{n\in\mathbb{N}}$ with $(\Im^{k_n}\varpi_0, \varpi^*) \in \mathbb{E}(\pounds)$ for all $n \in \mathbb{N}$. We get

$$\begin{aligned}
&\frac{1}{f_{\mathbb{M}}(\varpi^*, \Im\varpi^*, t)} - 1 \\
&\quad \leqslant \biggl(\frac{1}{f_{\mathbb{M}}(\varpi^*, \Im^{k_n+1}\varpi_0, t)} - 1\biggr) + \biggl(\frac{1}{f_{\mathbb{M}}(\Im^{k_n+1}\varpi_0, \Im\varpi^*, t)} - 1\biggr) \\
&\quad \leqslant \frac{1}{f_{\mathbb{M}}(\varpi^*, \Im^{k_n+1}\varpi_0, t)} - 1 + a\biggl(\frac{1}{f_{\mathbb{M}}(\Im^{k_n}\varpi_0, \varpi^*, t)} - 1\biggr) \\
&\qquad + b\biggl(\frac{1}{f_{\mathbb{M}}(\Im^{k_n+1}\varpi_0, \Im^{k_n}\varpi_0, t)} - 1\biggr) + c\biggl(\frac{1}{f_{\mathbb{M}}(\varpi^*, \Im\varpi^*, t)} - 1\biggr).
\end{aligned}$$

Now, letting $n \to \infty$, we obtain

$$\frac{1}{f_{\mathbb{M}}(\varpi^*, \Im\varpi^*, t)} - 1 \leqslant c\biggl(\frac{1}{f_{\mathbb{M}}(\varpi^*, \Im\varpi^*, t)} - 1\biggr) \quad \Longrightarrow \quad \varpi^* = \Im\varpi^*,$$

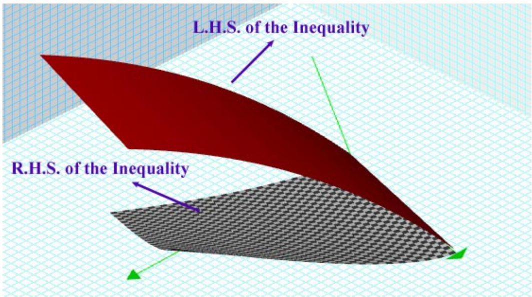

Figure 1. Validation of inequality (5).

that is, $\varpi^* \in Fix_{\mathfrak{I}}$. If we have $\mathfrak{I}\varsigma = \varsigma$ for some $\varsigma \in \mathsf{C}$, then from above we must have $\mathfrak{I}^n\varsigma \to \varpi^*$, so $\varsigma = \varpi^*$, and therefore, $\mathfrak{I}$ is a PO. $\square$

Example 2. Let $\mathsf{C} = Q \cup Q' = \mathbb{R}$, and let $f_{\mathbb{M}} : \mathsf{C} \times \mathsf{C} \times (0, \infty) \to (0, 1]$ be the fuzzy metric defined by $f_{\mathbb{M}}(\varpi, \varsigma, t) = \varsigma/(\varsigma + d(\varpi, \varsigma))$, where d is the usual metric on C. Define the transformation $\mathfrak{I} : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{I}\varpi = \begin{cases} 1 & \text{if } \varpi \in Q, \\ 0 & \text{if } \varpi \in Q'. \end{cases}$$

Suppose that $\eta(\varsigma) = \sqrt{\varsigma}$ for all $\varsigma \in (0, 1)$. If $\varpi, \varsigma \in Q$, then $\mathfrak{I}\varpi = \mathfrak{I}\varsigma = 1$ and $\mathbb{E}\varpi = \mathbb{E}\varsigma = [0, 4]$. Also,

$$f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, kt) = 1 \geqslant \eta\bigl(f_{\mathbb{M}}(\mathbb{E}\varpi, \mathbb{E}\varsigma, t)\bigr)$$

when $\varpi, \varsigma \in Q'$. Then $\mathfrak{I}\varpi = \mathfrak{I}\varsigma = 0$ and $\mathbb{E}\varpi = \mathbb{E}\varsigma = [7, 9]$. In this case, we have

$$f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, kt) = 1 \geqslant \eta\bigl(f_{\mathbb{M}}(\mathbb{E}\varpi, \mathbb{E}\varsigma, t)\bigr).$$

If $\varpi \in Q$ and $\varsigma \in Q'$ or $\varpi \in Q'$, and $\varsigma \in Q$, then $\mathbb{E}\varpi = [0, 4]$ and $\mathbb{E}\varsigma = [7, 9]$ or $\mathbb{E}\varpi = [7, 9]$ and $\mathbb{E}\varsigma = [0, 4]$. For $k \geqslant 3/4$, we have

$$f_{\mathbb{M}}\left(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, \frac{3}{4}t\right) = \frac{\frac{3}{4}\varsigma}{\frac{3}{4}\varsigma + 1} \geqslant \eta\bigl(f_{\mathbb{M}}(\mathbb{E}\varpi, \mathbb{E}\varsigma, t)\bigr). \tag{5}$$

Note that $\mathfrak{I}(\mathsf{C}) = \{0, 1\} \subset \mathbb{E}\varpi = [0, 4] \cup [7, 9]$. Thus all the conditions of Theorem 3 are fulfilled.

4.1 Well posedness in $f_{\mathbb{M}}$-spaces

Definition 19. Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be $f_{\mathbb{M}}$-space, and let $\mathfrak{I}$ be a set of self-mappings C. The common fixed point problem of $\mathfrak{I}$ is called well-posed if

(i) $\mathfrak{I}$ has a unique common fixed point ϖ in C, that is, ϖ is the unique point in C such that $A\varpi = \varpi$ for all $A \in \mathfrak{I}$;

(ii) For every sequence $\{\varpi_n\}$ in C,

$$\lim_{n\to\infty} f_{\mathbb{M}}(\varpi_n, A\varpi_n, t) = 1, \quad A \in \mathfrak{I} \quad \Longrightarrow \quad \lim_{n\to\infty} f_{\mathbb{M}}(\varpi_n, \varpi, t) = 1.$$

Example 3. Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be a $f_{\mathbb{M}}$-space, and let $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ be a Reich contraction. The operator $\mathfrak{I}$ provides a positive framework of the fixed point problem. Indeed, $Fix_{\mathfrak{I}} = \{u\}$, and let $\varpi_n \in \mathsf{C}$, $n \in \mathbb{N}$, be such that $1/f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t) - 1 \to 0$ as $n \to \infty$. We have

$$\begin{aligned}
\frac{1}{f_{\mathbb{M}}(\varpi_n, u, t)} - 1 &\leqslant \varsigma\left(\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1 + \frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi_n, \mathfrak{I}u, t)} - 1\right) \\
&\leqslant \varsigma\left\{\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1 + a\left(\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1\right)\right. \\
&\quad \left. + b\left(\frac{1}{f_{\mathbb{M}}(u, \mathfrak{I}u, \varsigma)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varpi_n, u, t)} - 1\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{f_{\mathbb{M}}(\varpi_n, u, t)} - (1 - c\varsigma)\frac{1}{f_{\mathbb{M}}(\varpi_n, u, t)} - 1 &\leqslant \varsigma\left(\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1\right) \\
&\quad + a\varsigma\left(\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1\right) \\
(1 - c\varsigma)\frac{1}{f_{\mathbb{M}}(\varpi_n, u, t)} - 1 &\leqslant (\varsigma + a\varsigma)\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1 \\
\frac{1}{f_{\mathbb{M}}(\varpi_n, u, t)} - 1 &\leqslant \frac{\varsigma + a\varsigma}{1 - c\varsigma}\,\frac{1}{f_{\mathbb{M}}(\varpi_n, \mathfrak{I}\varpi_n, t)} - 1 \to 0
\end{aligned}$$

as $n \to \infty$.

Corollary 5. *Let $(\mathsf{C}, f_{\mathbb{M}}, *)$ be a complete $f_{\mathbb{M}}$-space endowed with a graph $\pounds$, and let $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ performs the role of an operator. Suppose that*:

(i) *$\pounds$ is weak $\mathfrak{I}$-connected*;

(ii) *There exist nonnegative numbers a and b satisfying the condition $a + 2b < 1$ such that, for each $(\varpi, \varsigma, t) \in \mathbb{E}(\pounds)$, we have*

$$\begin{aligned}
\frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi, T\varsigma, t)} - 1 &\leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right) \\
&\quad + b\left(\frac{1}{f_{\mathbb{M}}(\varpi, \mathfrak{I}\varpi, t)} - 1 + \frac{1}{f_{\mathbb{M}}(\varsigma, \mathfrak{I}\varsigma, t)} - 1\right);
\end{aligned}$$

(iii) *For any $(\varpi_n)_{n\in\mathbb{N}}$ in C, if $\varpi_n \to \varpi$ and $(\varpi_n, \varpi_{n+1}, t) \in \mathbb{E}(\pounds)$ for $n \in \mathbb{N}$, there is a subsequence $(\varpi k_n)_{n\in\mathbb{N}}$ with $(\varpi k_n, \varpi, t) \in \mathbb{E}(\pounds)$ for $n \in \mathbb{N}$.*

Then $\mathfrak{I}$ is a PO.

Proof. The operator $\mathfrak{I}$ is clearly a $\tilde{\pounds}$-Reich operator with $b = c$. As a result of Theorem 3, the conclusion is drawn. $\square$

Corollary 6. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a complete* $f_{\mathbb{M}}$*-space endowed with a graph* $\pounds$*, and let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a Banach* $\pounds$*-contraction. We suppose that*:

(i) $\pounds$ *is weak* $\mathfrak{I}$*-connected*;

(ii) *For any* $(\varpi_n)_{n\in\mathbb{N}}$ *in* C*, if* $\varpi_n \to \varpi$ *and* $(\varpi_n, \varpi_{n+1}) \in \mathbb{E}(\pounds)$ *for* $n \in \mathbb{N}$*, there is a subsequence* $(\varpi k_n)_{n\in\mathbb{N}}$ *with* $(\varpi k_n, \varpi) \in \mathbb{E}(\pounds)$ *for* $n \in \mathbb{N}$*.*

Then $\mathfrak{I}$ *is a PO.*

Proof. If $\mathfrak{I}$ is a Banach $\pounds$-contraction with constant $\alpha \in [0, 1)$, then $\mathfrak{I}$ is a $\tilde{\pounds}$-Reich operator with constants $a = \alpha$, $b = c = 0$. Then mapping $\mathfrak{I}$ is PO from Corollary 5. □

Corollary 7. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a complete metric space endowed with a graph* $\pounds$*, and let* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *be a* $\pounds$*-Kannan mapping. We suppose that*:

(i) $\pounds$ *is weak* $\mathfrak{I}$*-connected*;

(ii) *For any* $(\varpi_n)_{n\in\mathbb{N}}$ *in* C*, if* $\varpi_n \to \varpi$*, and* $(\varpi_n, \varpi_{n+1}) \in \mathbb{E}(\pounds)$ *for* $n \in \mathbb{N}$*, there is a subsequence* $(\varpi k_n)_{n\in\mathbb{N}}$ *with* $(\varpi k_n, \varpi) \in \mathbb{E}(\pounds)$ *for* $n \in \mathbb{N}$*.*

Then $\mathfrak{I}$ *is a PO.*

Proof. If $\mathfrak{I}$ is a $\pounds$-Kannan with constant α, then $\mathfrak{I}$ is a $\tilde{\pounds}$-Reich operator with the constants $a = 0$, $b = c = \alpha$, and the resulting mapping is PO from Corollary 5.

Using Theorem 3, we can determine that the fixed point of the Ćirić–Rus–Reich operator holds true in partially ordered metric spaces. □

Corollary 8. *Consider the case where the set* $(\mathsf{C}, \leqslant)$ *is only partially ordered, and the metric space* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *is complete. Just for argument, assume that the increasing operator* $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ *is valid. Three statements below hold true*:

(i) *There exist the real numbers* $a, b, c > 0$ *with* $a + b + c < 1$ *such that, for each* $\varpi, \varsigma \in \mathsf{C}$ *with* $\varpi \leqslant \varsigma$*, we have*

$$\frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi, \mathfrak{I}\varpi, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varsigma, \mathfrak{I}\varsigma, t)} - 1\right);$$

(ii) *For each* $\varpi, \varsigma \in \mathsf{C}$*, incomparable elements of* $(\mathsf{C}, \leqslant)$*, there exists* $z \in \mathsf{C}$ *such that* $\varpi \leqslant z$*,* $\varsigma \leqslant z$*, and* $z \leqslant \mathfrak{I}z$;

(iii) *If an increasing sequence* (ϖ_n) *converges to* ϖ *in* C*, then* $\varpi_n \leqslant \varpi$ *for all* n*.*

Then $\mathfrak{I}$ *is a PO.*

Proof. Consider the graph $\pounds$ with $\mathcal{V}(\pounds) = \mathsf{C}$ and

$$\mathbb{E}(\pounds) = \{(\varpi, \varsigma, t) \in \mathsf{C} \times \mathsf{C} \colon \varpi \leqslant \varsigma\}.$$

The mapping $\mathfrak{I}$ is a Ćirić–Rus–Reich operator since it is increasing and (i) holds. The graph $\pounds$ is $\mathfrak{I}$-connected via Theorem 3. This result is deduced from the premises in this theorem. □

In the ones that follow, we demonstrate that Theorem 3 is a corollary of [18]'s fixed point theorem for cyclic Reich operators.

Let $p \geqslant 2$, and let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space C. A mapping $\Im : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is called cyclical operator if

$$T(A_i) \subseteq A_{i+1}, \quad i \in \{1, 2, \dots, p\}, \quad \text{and} \quad A_{p+1} := A_1.$$

Theorem 4. *Let $A_1, A_2, \dots, A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space (C, d), and let $\Im$: $\cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a cyclical operator. There exist nonnegative numbers a, b, c, $a+b+c < 1$, such that for each pair $(\varpi, \varsigma, t) \in A_i \times A_{i+1}$, $i \in \{1, 2, \dots, p\}$,*

$$\frac{1}{f_{\mathbb{M}}(\Im\varpi, \Im\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi, \Im\varpi, t)} - 1\right) + c\left(\frac{1}{f_{\mathbb{M}}(\varsigma, \Im\varsigma, t)} - 1\right).$$

Then $\Im$ is a PO.

Proof. Denote $Y = \cup_{i=1}^p A_i$. Then (Y, d) is a complete metric space. Consider the graph $\pounds$ with $\mathcal{V}(\pounds) = Y$ and

$$\mathbb{E}(\pounds) = \big\{(\varpi, \varsigma, t) \in Y \times Y\colon \text{there exist } i \in \{1, 2, \dots, p\} \text{ such that } \varpi \in A_i \text{ and } \varsigma \in A_{i+1}\big\}.$$

Because $\Im$ is a cyclic operator, we get

$$(\Im\varpi, \Im\varsigma) \in \mathbb{E}(\pounds), \text{ for all } (\varpi, \varsigma, t) \in \mathbb{E}(\pounds).$$

Via hypothesis, the operator $\Im$ is a $\pounds$-Reich operator, and the graph $\pounds$ is $\Im$-connected.

Let $(\varpi_n)_{n\in\mathbb{N}}$ be in C such that $\varpi_n \to \varpi$, and let $(\varpi_n, \varpi_{n+1}, t) \in \mathbb{E}(\pounds)$ for $n \in \mathbb{N}$. Then there is $j \in \{1, 2, \dots, n\}$ such that $\varpi \in A_j$. However, the sequence $\{\varpi_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, \dots, p\}$. The subsequence of sequence $\{\varpi_n\}$ is formed by the terms, which are in A_{j-1} and satisfy the requirements (ii) of Theorem 3. □

5 Applications

Let C be any set, and let symbol $\varDelta$ represents the diagonal formed by the $\mathsf{C} \times \mathsf{C}$. $\pounds(\mathcal{V}, \mathbb{E})$ is an undirected graph, where the set of vertices $\mathcal{V}$ is a subset of C, and $\mathbb{E}$ is the set of graph's edges that contains all loops, i.e., $\varDelta \subseteq \mathbb{E}$. Assuming that the graph $\pounds$ does not have any parallel edges, it can be matched to the pair $(\mathcal{V}, \mathbb{E})$.

We present the procedure of a natural selection of transformation $\Im$, which is motivated by the aforementioned fact: $\mathcal{V} \to \mathcal{V}$.

If (ϖ, ς) are in the $\mathcal{V}$, then (ϖ, ς) represents an edge between ϖ and ς. We say there is a path between vertices ϖ and ς of a structure if certain edges connect ϖ and ς. In this instance, we define $[\varpi, \varsigma]$ as a path that originates at ϖ and ends at ς (we designate vertex ς as terminal vertex and vertex ϖ as a reference vertex). A vertex $w \in \mathcal{V}$ is referred as a natural selection of $\Im : \mathcal{V} \to \mathcal{V}$ if $\Im w$ is terminal vertex of $[w, \Im w]$.

Let ϖ and ς represent the two vertices of a $\pounds$. Giving each edge of a graph a specific weight is a common practice. The weight of an edge between ϖ and ς can be determined by multiplying the positive real value produced by the distance calculation between ϖ and ς.

In this illustration, we allot fuzzy weight $f_{\mathbb{M}}(\varpi, \varsigma, t)$ (a number between 0 and 1) to an edge (ϖ, ς) at $\Im$, where $\Im$ is translated as a time. In this case, we have bigger adaptability in choosing the weights particularly when one is questionable or confounded at a certain point of time in assigning a weight to an edge (ϖ, ς). We establish the existence of a graph vertex v such that its image, subjected to a graphical transformation satisfying specific contraction requirements, becomes the last vertex of a path starting at v.

$f_{\mathbb{M}}(\varpi, \varsigma, t) = 1$ for all $\Im > 0$ if and only if a path $[\varpi, \varsigma]$ defines a loop.

Define

$$D(\mathbb{E}\varpi, \mathbb{E}\varsigma, t) = \sup\bigl\{ f_{\mathbb{M}}(u, v, t),\ u \in \mathbb{E}\varpi, v \in \mathbb{E}\varsigma \bigr\}.$$

In this last section, we prove that our results built up within the previous section enable us to solve a few certain functional integral equations. Also, we provide few applications related to our results.

Definition 20. (See [2].) Let Ψ denote all functions $\Psi : [0, \infty) \to [0, \infty)$ satisfying:

(i) Ψ is strictly increasing and continuous;

(ii) $\Psi(t) = 0$ if and only if $t = 0$.

We define $\Psi(c_o) = \int_0^{c_o} \varpi(\varsigma)\,\mathrm{d}t$, where ϖ is strictly increasing and continuous function for all $\varsigma > 0$. Moreover, for each $\varsigma > 0$, $\varpi(\varsigma) > 0$. This implies that $\varpi(c_o) = 0$ if and only if $c_o = 0$.

Theorem 5. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a complete* $f_{\mathbb{M}}$*-space, and let* $f : \mathsf{C} \to \mathsf{C}$ *be a mapping satisfying*

$$f_{\mathbb{M}}(\varpi, \varsigma, t) = 1, \qquad \int_0^{f_{\mathbb{M}}(f\varpi, f\varsigma, k\varsigma)} \varpi(\varsigma)\,\mathrm{d}\varsigma \geqslant \int_0^{\lambda(\varpi, \varsigma, t)} \varpi(\varsigma)\,\mathrm{d}\varsigma,$$

where

$$\lambda(\varpi, \varsigma, t) = \min\left\{ \frac{f_{\mathbb{M}}(\varsigma, f\varsigma, t)[1 + f_{\mathbb{M}}(\varpi, f\varpi, t)]}{[1 + f_{\mathbb{M}}(\varpi, \varsigma, t)]},\ f_{\mathbb{M}}(\varpi, \varsigma, t) \right\}$$

for all $\varpi, \varsigma \in \mathsf{C}, \varpi \in \Psi$ *and* $k \in (0, 1)$. *Then* f *has a unique fixed point.*

Proof. By taking $\varpi(\varsigma) = 1$ and applying Theorem 3, we obtain the result. $\square$

Theorem 6. *Let* $(\mathsf{C}, f_{\mathbb{M}}, *)$ *be a complete* $f_{\mathbb{M}}$*-space, and let* $f : \mathsf{C} \to \mathsf{C}$ *be a mapping satisfying*

$$f_{\mathbb{M}}(\varpi, \varsigma, t) = 1, \qquad \int\limits_0^{f_{\mathbb{M}}(f\varpi, f\varsigma, k\varsigma)} \varpi(\varsigma)\,\mathrm{d}\varsigma \geqslant \phi\Biggl\{ \int\limits_0^{\lambda(\varpi,\varsigma,t)} \varpi(\varsigma)\,\mathrm{d}t \Biggr\},$$

where

$$\lambda(\varpi, \varsigma, t) = \min\biggl\{ \frac{f_{\mathbb{M}}(\varsigma, f\varsigma, t)[1 + f_{\mathbb{M}}(\varpi, f\varpi, t)]}{[1 + f_{\mathbb{M}}(\varpi, \varsigma, t)]},\ f_{\mathbb{M}}(\varpi, \varsigma, t) \biggr\}$$

for all $\varpi, \varsigma \in \mathsf{C}\varpi \in \Psi$*,* $k \in (0, 1)$*, and* $\phi \in \Phi$*. Then* f *has a unique fixed point.*

Proof. Since $\phi(a) > a$, $0 < a < 1$, this result is the consequence of Theorem 3. □

Theorem 7. *Consider the following implicit-type integral equation*:

$$\gamma\varpi(\varsigma) + \int\limits_{-1}^{\varsigma} \kappa\bigl(\varsigma, s, \varpi(s), \Im\varpi(s)\bigr)\,\mathrm{d}s, \quad 0 < \gamma < \frac{1}{3},\ \varsigma \in [-1, 1],$$

where $\kappa \in C([-1, 1])$ *in Banach space* $E = C([-1, 1], \mathbb{R})$ *is scalar continuous functions. The investigation is essentially based on the properties of kernel* $\kappa(\cdot, \cdot, \cdot, \cdot)$*, and* $\varpi(s) \in C[-1, 1]$ *is the unknown function.*

Also, suppose the following assumptions contend:

(i) $\kappa(\varsigma, s, \varpi(s), \Im\varpi(s)) \geqslant 0$ *for* $\Im, s \in [-1, 1]$ *such that* $\kappa(\cdot, \cdot, 0, \cdot) \neq 0$;

(ii) *The mapping* $\Im$ *defined by* $\Im\varpi(\varsigma) = \int_{-1}^{\varsigma} \kappa(\varsigma, s, \varpi(s), \Im\varpi(s))\,\mathrm{d}s$ *satisfies* $\Im\varpi \in \mathbb{R}$ *for all* $\varpi \in \mathbb{R}$ *and*

$$\|\Im\varpi - \Im\varsigma\| < (1 - \gamma)\|\varpi - \varsigma\|, \quad \varpi, \varsigma \in \mathbb{R}\ (\varpi \neq \varsigma);$$

(iii) *For a given* $\epsilon > 0$*, there exists* $\delta < (1 - 3\gamma)/2$ *such that if* $\varpi, \varsigma \in \mathbb{R}$ *and* $\|\varpi - \varsigma\| \geqslant \epsilon$*,*

$$\bigl|\Im\varpi(\varsigma) - \Im\varsigma(\varsigma)\bigr| \leqslant \delta\bigl(|\varpi - \Im\varpi| + \bigl|\varsigma(\varsigma) - \Im\varsigma(\varsigma)\bigr|\bigr), \quad \Im \in [-1, 1].$$

Then $\Im$ *has unique fixed point in* $\mathbb{R}$*.*

Proof. Consider the metric d defined on $C[-1, 1]$ by

$$d(\varpi, \psi) = \max_{\varsigma \in [-1,1]} \bigl|\varpi(\varsigma) - \psi(\varsigma)\bigr|\bigr) = \max_{\varsigma \in [0,1]} \bigl|\varpi(\varsigma) - \psi(\varsigma)\bigr|$$

for all $\varpi, \psi \in C[-1, 1]$. Then $(C[-1, 1], d)$ is a complete metric space. The binary operation $*$ is defined by product $\diamond$-norm $\mathrm{P}(\varpi, \varsigma) = \varpi \cdot \varsigma$.

A standard fuzzy metric $d_{\mathbb{M}} : U^2 \times (0, \infty) \to [0, 1]$ is given as

$$\frac{1}{d_{\mathbb{M}}(\varpi, \varsigma, t)} - 1 = \frac{d(\varpi, \varsigma)}{t} \quad \text{for } t > 0,\ \varpi, \varsigma \in \mathsf{C}.$$

Then, easily one can verify that $d_{\mathbb{M}}$ is triangular and $(\mathsf{C}, f_{\mathbb{M}}, *)$ is a complete fuzzy metric space.

Let $\mathsf{C} = C[-1,1]$, and define the operator $\mathfrak{I} : \mathsf{C} \to \mathsf{C}$ by

$$\mathfrak{I}\varpi(\varsigma) = \gamma\,\varpi(\varsigma) + \int_{-1}^{\varsigma} \kappa\bigl(\varsigma, s, \varpi(s)\bigr), \quad \varsigma \in \mathsf{C}.$$

We have

$$\varpi(\varsigma) - \mathfrak{I}\varpi(\varsigma) = (1-\gamma)\varpi(\varsigma) - \int_{-1}^{\varsigma} \kappa\bigl(\varsigma, s, \varpi(s), \mathfrak{I}\varpi(s)\bigr)\,\mathrm{d}s.$$

Let $\varpi, \varsigma \in \mathbb{R}$ with $\|\varpi - \varsigma\| \geqslant \epsilon$. Then by using our assumptions we get

$$\begin{aligned}
&\bigl|\mathfrak{I}\varpi(t) - \mathfrak{I}\varsigma(t)\bigr| \\
&\quad= \left|\gamma\bigl(\varpi(t) - \varsigma(t)\bigr) + \int_{-1}^{\varsigma} \kappa\bigl(\varsigma, s, \varpi(s), \mathfrak{I}\varpi(s)\bigr)\,\mathrm{d}s - \int_{-1}^{\varsigma} \kappa\bigl(\varsigma, s, \varsigma(s), \mathfrak{I}\varsigma(s)\bigr)\,\mathrm{d}s\right| \\
&\quad\leqslant \gamma\bigl|\varpi(\varsigma) + \mathfrak{I}\varpi(\varsigma) - \mathfrak{I}\varpi(\varsigma) + \mathfrak{I}\varsigma(\varsigma) - \mathfrak{I}\varsigma(\varsigma) - \varsigma(\varsigma)\bigr| \\
&\qquad+ \left|\int_{-1}^{\varsigma} \bigl[\kappa(\varsigma, s, \varpi(s), \mathfrak{I}\varpi(s)\bigr) - \kappa\bigl(\varsigma, s, \varsigma(s), \mathfrak{I}\varsigma(s)\bigr)\bigr]\,\mathrm{d}s\right| \\
&\quad\leqslant \gamma\bigl(\|\varpi - \mathfrak{I}\varpi\| + \|\varsigma - \mathfrak{I}\varsigma\| + \|\mathfrak{I}\varpi - \mathfrak{I}\varsigma\|\bigr) + \delta\bigl(\|\varpi - \mathfrak{I}\varpi\| + \|\varsigma - \mathfrak{I}\varsigma\|\bigr),
\end{aligned}$$

which gives that

$$\|\mathfrak{I}\varpi - \mathfrak{I}\varsigma\| \leqslant \gamma\|\mathfrak{I}\varpi - \mathfrak{I}\varsigma\| + (\gamma + \delta)\bigl(\|\varpi - \mathfrak{I}\varpi\| + \|\varsigma - \mathfrak{I}\varsigma\|\bigr).$$

Hence

$$\|\mathfrak{I}\varpi - \mathfrak{I}\varsigma\| \leqslant \frac{\gamma + \delta}{1 - \gamma}\bigl(\|\varpi - \mathfrak{I}\varpi\| + \|\varsigma - \mathfrak{I}\varsigma\|\bigr),$$

which satisfies

$$\begin{aligned}
\frac{1}{f_{\mathbb{M}}(\mathfrak{I}\varpi, \mathfrak{I}\varsigma, t)} - 1 \leqslant a\left(\frac{1}{f_{\mathbb{M}}(\varpi, \varsigma, t)} - 1\right) + b\left(\frac{1}{f_{\mathbb{M}}(\varpi, \mathfrak{I}\varpi, \varsigma)} - 1\right) \\
+ c\left(\frac{1}{f_{\mathbb{M}}(\varsigma, \mathfrak{I}\varsigma, t)} - 1\right)
\end{aligned}$$

for all $(\varpi, \varsigma) \in \mathsf{C} \times \mathsf{C}$ and $a + b + c > 1$, $a = 0$ and $b, c > 0$. Then there is only one fixed point in C for $\mathfrak{I}$.

Since $0 < \delta < (1 - 3\gamma)/2$, then $(\gamma + \delta)/(1 - \gamma) < 1/2$, and the result is an immediate consequence of Theorem 3. □

6 Conclusion and future scope

In the situation of $f_{\mathbb{M}}$-spaces, we developed new and suitable extensions of Reich contraction and discovered their applicability to the solution of integral equations. This is the first attempt to solve $\pounds$-graph-Reich contraction in $f_{\mathbb{M}}$-spaces that we are aware of. A few nontrivial examples and an application confirm the solution's uniqueness. Finally, we have used the manuscript's major contents to offer a new application in which we can derive the existence of a solution to the class of integral equations under very generic conditions.

On the one hand, additional research is needed to reformulate the contractivity condition so that fixed point theory can be developed in more general abstract metric spaces.

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