Nonlinear Analysis: Modelling and Control, Vol. 29, No. 1, 111–123
https://doi.org/10.15388/namc.2024.29.33766

Chaotic single neuron model with periodic coefficients with period two

Inese Bula[a,b,1], **Michael A. Radin**[c]

[a]Department of Mathematics,
Faculty of Physics, Mathematics and Optometry,
University of Latvia,
Jelgavas str. 3, Riga, Latvia
inese.bula@lu.lv

[b]Institute of Mathematics and Computer Science
of University of Latvia,
Raina bulv. 29, Riga, Latvia

[c]Rochester Institute of Technology,
School of Mathematical Sciences,
Rochester, New York 14623, USA
michael.radin@rit.edu

Received: January 25, 2023 / **Revised:** October 16, 2023 / **Published online:** December 13, 2023

Abstract. Our goal is to investigate the piecewise linear difference equation $x_{n+1} = \beta_n x_n - g(x_n)$. This piecewise linear difference equation is a prototype of one neuron model with the internal decay rate β and the signal function g. The authors investigated this model with periodic internal decay rate β_n as a period-two sequence. Our aim is to show that for certain values of coefficients β_n, there exists an attracting interval for which the model is chaotic. On the other hand, if the initial value is chosen outside the mentioned attracting interval, then the solution of the difference equation either increases to positive infinity or decreases to negative infinity.

Keywords: neuron model, difference equation, periodic solution, unbounded solution, chaotic atractor.

1 Introduction

We study the following nonautonomous piecewise linear difference equation:

$$x_{n+1} = \beta_n x_n - g(x_n), \tag{1}$$

where $(\beta_n)_{n=0}^{\infty}$ is a periodic sequence with period two, where

$$\beta_n = \begin{cases} \beta_0 & \text{if } n \text{ is even}, \\ \beta_1 & \text{if } n \text{ is odd}, \end{cases} \qquad \beta_0 \neq \beta_1,\ \beta_0 > 0,\ \beta_1 > 0,$$

[1]Corresponding author.

and the function g presents the properties in the following form:

$$g(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases} \tag{2}$$

The studies on Eq. (1) commenced in [15], where J. Wu investigated the delayed differential equation

$$x'(t) = -g\bigl(x(t-\tau)\bigr) \tag{3}$$

that is used to model a single neuron with no internal decay. In this case, $g : \mathbb{R} \to \mathbb{R}$ is either a sigmoid function or a piecewise linear signal function, and $\tau \leqslant 0$ is a synaptic transmission delay. From (3) the corresponding piecewise difference equation was obtained as a discrete-time network of a single neuron model [7]

$$x_{n+1} = \beta x_n - g(x_n), \quad n = 0, 1, 2, \dots, \tag{4}$$

where $\beta > 0$ is an internal decay rate, and g is a signal function. Several authors investigated Eq. (4) (e.g., [3, 7, 14, 16–22]). In addition, Eq. (4) has been investigated as a single neuron model, where the signal function g is the piecewise constant function with McCulloch–Pitts nonlinearity (2).

In [1, 2], the authors studied the models by applying a different signal function (with more than one threshold). In [4–6], the authors investigated the periodic solutions of a discrete neuron model when $(\beta_n)_{n=0}^{\infty}$ is periodic with periods two and three.

If we consider the right side of difference equation (1) as a function $h : \mathbb{R} \to \mathbb{R}$ and let $x_n = h^n(x_0)$, $x_0 \in \mathbb{R}$, $n = 1, 2, \dots$, then we obtain the first-order difference equation $x_{n+1} = h(x_n)$ with initial condition $x_0 \in \mathbb{R}$. From the definition of difference equation (1) it follows that first iteration of function h is in the form

$$h(x) = \begin{cases} \beta_0 x - 1, & x \geqslant 0, \\ \beta_0 x + 1, & x < 0. \end{cases}$$

On the other hand, the second iteration emerges in the corresponding form

$$\begin{aligned} h^2(x) &= \begin{cases} \beta_1 h(x) - 1, & h(x) \geqslant 0, \\ \beta_1 h(x) + 1, & h(x) < 0, \end{cases} \\ &= \begin{cases} \beta_0\beta_1 x - \beta_1 - 1, & x \geqslant \frac{1}{\beta_0}, \\ \beta_0\beta_1 x - \beta_1 + 1, & 0 \leqslant x < \frac{1}{\beta_0}, \\ \beta_0\beta_1 x + \beta_1 - 1, & -\frac{1}{\beta_0} \leqslant x < 0, \\ \beta_0\beta_1 x + \beta_1 + 1, & x < -\frac{1}{\beta_0}. \end{cases} \end{aligned}$$

Observe that replacing the difference equation (1) with a function h does not do much. However, depending on the circumstance, sometimes it is more convenient to describe the dynamics more easily with the behavior of a function, and at other times – with a difference equation.

Our aim is to perform a qualitative analysis of difference equation (1) or function h. This includes the possibilities of chaotic behavior of the dynamical system. The chaotic behavior of Eq. (4) has not been studied so far and is of paramount interest.

The paper outlines the following results. Section 2 will present results about difference equation (1). Section 3 will analyze the Lyapunov exponent and find this exponent for dynamical system with (1). We will show that for certain values of coefficients β_0 and β_1, there exists a chaotic attractor. Finally, we present conclusions about our model and future work.

2 Some results about difference equation with period-two coefficients

First, notice that the difference equation (1) has no equilibrium points when $(\beta_n)_{n=0}^{\infty}$ is a periodic two sequence. In [4], we proved that Eq. (1) with (2) has no periodic orbits of odd period and has periodic orbits only with an even period. More precisely, we showed that if the coefficients $0 < \beta_0 \leqslant 1$ and $0 < \beta_1 \leqslant 1$, that is, coefficients are in region I (see Fig. 1), then only period-two solutions exist. If coefficients belong to region II, then period-four solutions exist. If the coefficients belong to region III, then period-two solutions exist. However, in this case, solutions with an arbitrary even period may also exist. The surprising situation emerges in the case when $\beta_1 = 1/\beta_0$ (except for $\beta_1 = \beta_0 = 1$). In this situation, there exist segments of initial conditions from which period-four solutions arise.

Theorem 1. (*See* [4].) *If* $\beta_0\beta_1 > 1$, *then the difference equation* (1) *has the corresponding two unstable periodic cycles with period two:*

$$\left\{-\frac{1+\beta_1}{\beta_0\beta_1-1}, -\frac{1+\beta_0}{\beta_0\beta_1-1}\right\} \quad \textit{and} \quad \left\{\frac{1+\beta_1}{\beta_0\beta_1-1}, \frac{1+\beta_0}{\beta_0\beta_1-1}\right\}.$$

Theorem 2. (*See* [4].) *Suppose* $\beta_0 > 1$ *and* $\beta_1 \geqslant 1$. *If there exists a positive integer* $n \geqslant 2$ *such that*

$$\beta_0^n\beta_1^n - 2\beta_0^{n-1}\beta_1^{n-1} + 1 > 0,$$

then the difference equation (1) *has an unstable periodic orbit of period* $2n$.

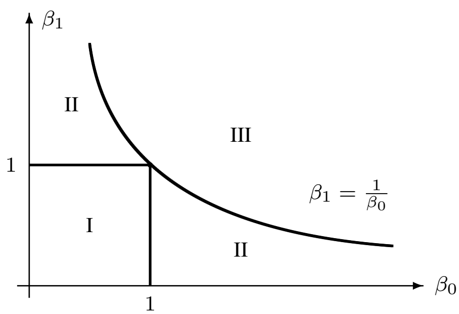

Figure 1. Existence of cycles depending on coefficients $\beta_0 > 0$ and $\beta_1 > 0$.

Note that by assumption $\beta_0 > 1$ and $\beta_1 \geqslant 1$ the inequality

$$\beta_0^{n-1}\beta_1^{n-1}(\beta_0\beta_1 - 2) + 1 > 0, \quad n \geqslant 2,$$

will hold only if $\beta_0\beta_1 - 2 \geqslant 0$. If $n = 2$, then

$$\beta_0^2\beta_1^2 - 2\beta_0\beta_1 + 1 = (\beta_0\beta_1 - 1)^2 > 0.$$

The above inequality holds always if $\beta_0\beta_1 > 1$. Indeed, the condition $\beta_1 \geqslant 1$ as in Theorem 2 does not need to hold for a cycle with period four to exist. For period four to exist, it is sufficient that conditions $\beta_0 > 1$ is fulfilled. For example, the cycle with a period four is formed by the following points:

$$x_0 = \frac{1+\beta_1}{\beta_0\beta_1 + 1} > 0, \qquad x_2 = -\frac{1+\beta_1}{\beta_0\beta_1 + 1} = -x_0 < 0,$$
$$x_1 = \frac{\beta_0 - 1}{\beta_0\beta_1 + 1} > 0, \qquad x_3 = -\frac{\beta_0 - 1}{\beta_0\beta_1 + 1} = -x_1 < 0.$$

Nothing can be said precisely about other cycles.

Next, we will prove that every solution with an initial condition that does not belong to an interval what boundaries are points of periodic cycle in Theorem 1 is unbounded.

Theorem 3. *Let $\beta_0\beta_1 > 1$. Then for any initial condition*

$$x_0 \notin \left[-\frac{1+\beta_1}{\beta_0\beta_1 - 1}, \frac{1+\beta_1}{\beta_0\beta_1 - 1}\right]$$

of difference equation (1)*, the solution is unbounded. More precisely,*

$$\lim_{n\to\infty} x_n = \begin{cases} -\infty & \text{if } x_0 < -\frac{1+\beta_1}{\beta_0\beta_1-1}, \\ +\infty & \text{if } x_0 > \frac{1+\beta_1}{\beta_0\beta_1-1}. \end{cases}$$

Proof. We will consider the following two cases: $x_0 < -(1+\beta_1)/(\beta_0\beta_1 - 1)$ or $x_0 > (1+\beta_1)/(\beta_0\beta_1 - 1)$.

First, we assume that $x_0 > (1+\beta_1)/(\beta_0\beta_1 - 1)$. The proof for other case is similar and will be omitted.

As we assumed that $\beta_0\beta_1 > 1$, then $x_0 > (1+\beta_1)/(\beta_0\beta_1 - 1) > 0$. Therefore

$$x_1 = \beta_0 x_0 - 1 > \frac{\beta_0(1+\beta_1)}{\beta_0\beta_1 - 1} - 1 = \frac{1+\beta_0}{\beta_0\beta_1 - 1} > 0.$$

Consequently, we obtain

$$x_2 = \beta_1 x_1 - 1 > \frac{\beta_1(1+\beta_0)}{\beta_0\beta_1 - 1} - 1 = \frac{1+\beta_1}{\beta_0\beta_1 - 1} > 0.$$

By iterations and induction we procure

$$x_{2k} > \frac{1+\beta_1}{\beta_0\beta_1 - 1} > 0 \quad \text{and} \quad x_{2k+1} > \frac{1+\beta_0}{\beta_0\beta_1 - 1} > 0, \quad k = 0, 1, 2, \ldots, .$$

On the other hand,

$$
\begin{aligned}
&x_0 > 0,\\
&x_1 = \beta_0 x_0 - 1 > 0,\\
&x_2 = \beta_0\beta_1 x_0 - \beta_1 - 1 > 0,\\
&x_3 = \beta_0^2\beta_1 x_0 - \beta_0\beta_1 - \beta_0 - 1 > 0,\\
&x_4 = \beta_0^2\beta_1^2 x_0 - \beta_0\beta_1^2 - \beta_0\beta_1 - \beta_1 - 1 > 0,\\
&\quad\dots,\\
x_{2k} &= \beta_0^k\beta_1^k x_0 - \beta_0^{k-1}\beta_1^k - \beta_0^{k-1}\beta_1^{k-1} - \cdots - \beta_0\beta_1 - \beta_1 - 1 > 0,\\
x_{2k+1} &= \beta_0^{k+1}\beta_1^k x_0 - \beta_0^k\beta_1^k - \beta_0^k\beta_1^{k-1} - \cdots - \beta_0\beta_1 - \beta_0 - 1 > 0,\\
x_{2k+2} &= \beta_0^{k+1}\beta_1^{k+1} x_0 - \beta_0^k\beta_1^{k+1} - \beta_0^k\beta_1^k - \cdots - \beta_0\beta_1 - \beta_1 - 1 > 0,\\
x_{2k+3} &= \beta_0^{k+2}\beta_1^{k+1} x_0 - \beta_0^{k+1}\beta_1^{k+1} - \beta_0^{k+1}\beta_1^k - \cdots - \beta_0\beta_1 - \beta_0 - 1 > 0,\\
&\quad\dots .
\end{aligned}
$$

Since $x_0 > (1+\beta_1)/(\beta_0\beta_1 - 1) > 0$, then there exists $\varepsilon > 0$ such that $x_0 = ((1+\beta_1)+\varepsilon)/(\beta_0\beta_1 - 1)$. We then obtain the corresponding difference between the neighboring even-ordered terms x_{2k} and x_{2k+2}:

$$
\begin{aligned}
x_{2k+2} - x_{2k} &= \beta_0^{k+1}\beta_1^{k+1}x_0 - \beta_0^k\beta_1^{k+1} - \beta_0^k\beta_1^k - \beta_0^k\beta_1^k x_0\\
&= \beta_0^k\beta_1^k\big((\beta_0\beta_1 - 1)x_0 - \beta_1 - 1\big)\\
&= \beta_0^k\beta_1^k\left(\frac{(\beta_0\beta_1 - 1)((1+\beta_1)+\varepsilon)}{\beta_0\beta_1 - 1} - \beta_1 - 1\right)\\
&= \beta_0^k\beta_1^k \cdot \varepsilon > 0, \quad k = 0, 1, 2, \dots .
\end{aligned}
$$

Since $\beta_0\beta_1 > 1$, then we see that $\lim_{k\to\infty}(\beta_0\beta_1)^k = +\infty$ and therefore

$$
x_0 < x_2 < x_4 < \cdots < x_{2k} < x_{2k+2} < \cdots \quad \text{and} \quad \lim_{k\to\infty} x_{2k} = +\infty.
$$

Similarly, we procure the difference between the neighboring odd-ordered terms x_{2k+1} and x_{2k+3}

$$
\begin{aligned}
x_{2k+3} - x_{2k+1} &= \beta_0^{k+2}\beta_1^{k+1}x_0 - \beta_0^{k+1}\beta_1^{k+1} - \beta_0^{k+1}\beta_1^k - \beta_0^{k+1}\beta_1^k x_0\\
&= \beta_0^{k+1}\beta_1^k\big((\beta_0\beta_1 - 1)x_0 - \beta_1 - 1\big)\\
&= \beta_0^{k+1}\beta_1^k\left(\frac{(\beta_0\beta_1 - 1)((1+\beta_1)+\varepsilon)}{\beta_0\beta_1 - 1} - \beta_1 - 1\right)\\
&= \beta_0^k\beta_1^k \cdot \beta_0 \cdot \varepsilon > 0, \quad k = 0, 1, 2, \dots .
\end{aligned}
$$

In addition, we get

$$
x_1 < x_3 < x_5 < \cdots < x_{2k+1} < x_{2k+3} < \cdots \quad \text{and} \quad \lim_{k\to\infty} x_{2k+1} = +\infty.
$$

We then conclude that if $(1+\beta_1)/(\beta_0\beta_1 - 1) < x_0$, then $\lim_{n\to\infty} x_n = +\infty$. □

Our goal is to determine the largest possible invariant set.

Definition 1. Let $f : X \to X$ be a map. A set $A \subset X$ is said to be invariant under the map f if $f(A) = A$.

Although we are interested in all such coefficients β_0 and β_1 for which $\beta_0\beta_1 > 1$, it is obvious that there is no invariant interval for all such coefficients. For example, let $\beta_0 = 2$ and $\beta_1 = 3$, then the significant period-two cycles are $\{-0.8, -0.6\}$ and $\{0.8, 0.6\}$. However, the solutions with an initial value x_0 from the interval $I = [-0.8, 0.8]$ may not belong to the interval I. For example, if $x_0 = 0.7$, then we procure

$$\begin{aligned}
x_1 &= 2 \cdot 0.7 - 1 = 0.4,\\
x_2 &= 3 \cdot 0.4 - 1 = 0.2,\\
x_3 &= 2 \cdot 0.2 - 1 = -0,6,\\
x_4 &= 3 \cdot (-0.6) + 1 = -\mathbf{0.8},\\
x_5 &= \mathbf{2} \cdot (-0.8) + 1 = -\mathbf{0.6},\\
x_6 &= \mathbf{3} \cdot (-0,6) + 1 = -0.8, \ \dots \ .
\end{aligned}$$

Observe that this solution is eventually periodic with the corresponding period-two cycle $\{-0.8, -0.6\}$. On the other hand, if we start with $x_0 = 0$, then

$$\begin{aligned}
x_1 &= 2 \cdot 0 - 1 = -1,\\
x_2 &= 3 \cdot (-1) + 1 = -2,\\
x_3 &= 2 \cdot (-2) + 1 = -3,\\
x_4 &= 3 \cdot (-3) + 1 = -8, \ \dots \ .
\end{aligned}$$

Notice that $x_{n+1} < x_n$ for all $n \geqslant 1$ and $\lim_{n\to\infty} x_n = -\infty$.

The last example shows that invariant interval always contains 0 and -1. The invariant interval must contain the entire interval $[-1, 1]$.

It is easy to prove. In fact, if $0 < \beta_0 \leqslant 2$, $0 < \beta_1 \leqslant 2$, then the function h is invariant in $[-1, 1]$ (that is, $h : [-1, 1] \to [-1, 1]$).

Now suppose that $x_0 \in [-1, 1]$. Then the following statements hold true:

(i) if $0 \leqslant x_0 \leqslant 1$, then $-1 = 0 - 1 \leqslant h(x_0) = \beta_i x_0 - 1 \leqslant 2 \cdot 1 - 1 = 1$, $i = 0, 1$;
(ii) if $-1 \leqslant x_0 < 0$, then $-1 = 2 \cdot (-1) + 1 \leqslant h(x_0) = \beta_i x_0 + 1 < 0 + 1 = 1$, $i = 0, 1$.

Is it possible to extend the invariant interval and the set of coefficients β_0 and β_1?

By Theorem 3 it follows that

$$[-1, 1] \subset \left[-\frac{1+\beta_1}{\beta_0\beta_1 - 1}, \frac{1+\beta_1}{\beta_0\beta_1 - 1}\right] \quad \text{and} \quad [-1, 1] \subset \left[-\frac{1+\beta_0}{\beta_0\beta_1 - 1}, \frac{1+\beta_0}{\beta_0\beta_1 - 1}\right].$$

Then we see that

$$\frac{1+\beta_1}{\beta_0\beta_1 - 1} \geqslant 1, \qquad \frac{1+\beta_0}{\beta_0\beta_1 - 1} \geqslant 1.$$

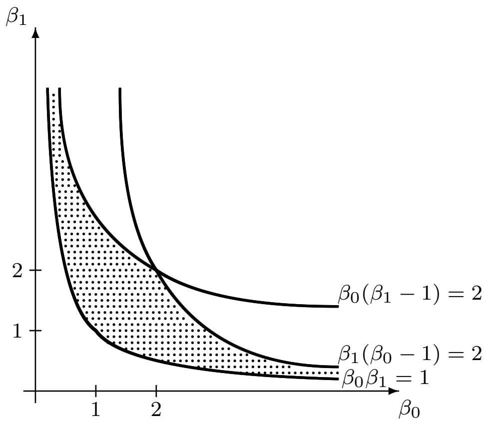

Figure 2. Area of coefficients β_0 and β_1 that satisfies the inequalities $\beta_0\beta_1 > 1$, $2 + \beta_1 - \beta_0\beta_1 \geqslant 0$ and $2 + \beta_0 - \beta_0\beta_1 \geqslant 0$.

Since $\beta_0\beta_1 > 1$, then

$$\begin{aligned} 1 + \beta_1 &\geqslant \beta_0\beta_1 - 1, \\ 1 + \beta_0 &\geqslant \beta_0\beta_1 - 1 \end{aligned} \quad \text{or} \quad \begin{aligned} 2 + \beta_1 - \beta_0\beta_1 &\geqslant 0, \\ 2 + \beta_0 - \beta_0\beta_1 &\geqslant 0. \end{aligned}$$

The area satisfying the system of inequalities is shown in Fig. 2.

Theorem 4. *Let* $\beta_0\beta_1 > 1$. *Let* $(1 + \beta_1)/(\beta_0\beta_1 - 1) \geqslant 1$ *and* $(1 + \beta_0)/(\beta_0\beta_1 - 1) \geqslant 1$. *Then for any initial condition*

$$x_0 \in I = \left[-\frac{1 + \beta_1}{\beta_0\beta_1 - 1}, \frac{1 + \beta_1}{\beta_0\beta_1 - 1}\right]$$

of difference equation (1)*, the solution is bounded. More precisely,*

$$x_n \in \begin{cases} I, & n = 0, 2, 4, \dots, \\ I_1, & n = 1, 3, 5, \dots, \end{cases} \qquad I_1 = \left[-\frac{1 + \beta_0}{\beta_0\beta_1 - 1}, \frac{1 + \beta_0}{\beta_0\beta_1 - 1}\right].$$

Proof. If $x_0 = -(1 + \beta_1)/(\beta_0\beta_1 - 1)$ or $(1 + \beta_1)/(\beta_0\beta_1 - 1)$, then $x_1 = -(1 + \beta_0)/(\beta_0\beta_1 - 1)$ or $(1 + \beta_0)/(\beta_0\beta_1 - 1)$, respectively, where the endpoints of interval I are initial points of a period-two cycle, and hence $x_1 \in I_1$.

If $0 \leqslant x_0 < (1 + \beta_1)/(\beta_0\beta_1 - 1)$, then

$$-1 = 0 - 1 \leqslant x_1 = \beta_0 x_0 - 1 < \frac{\beta_0(1 + \beta_1)}{\beta_0\beta_1 - 1} - 1 = \frac{1 + \beta_0}{\beta_0\beta_1 - 1}.$$

Similarly, if $-(1 + \beta_1)/(\beta_0\beta_1 - 1) < x_0 < 0$, then

$$-\frac{1 + \beta_0}{\beta_0\beta_1 - 1} = -\frac{\beta_0(1 + \beta_1)}{\beta_0\beta_1 - 1} + 1 < x_1 = \beta_0 x_0 + 1 < 0 + 1 = 1.$$

Hence $x_1 \in I_1$.

If $0 \leqslant x_0 \leqslant (1+\beta_0)/(\beta_0\beta_1 - 1)$, then

$$-1 = 0 - 1 \leqslant x_2 = \beta_1 x_1 - 1 \leqslant \frac{\beta_1(1+\beta_0)}{\beta_0\beta_1 - 1} - 1 = \frac{1+\beta_1}{\beta_0\beta_1 - 1}.$$

Similarly, if $-(1+\beta_0)/(\beta_0\beta_1 - 1) \leqslant x_1 < 0$, then

$$-\frac{1+\beta_1}{\beta_0\beta_1 - 1} = -\frac{\beta_1(1+\beta_0)}{\beta_0\beta_1 - 1} + 1 \leqslant x_2 = \beta_1 x_1 + 1 < 0 + 1 = 1.$$

Hence $x_2 \in I$. For all other x_n, $n > 2$, the proof is similar. □

Remark. If $\beta_1 > \beta_0$, then $I_1 \subset I$. If $\beta_1 < \beta_0$, then $I \subset I_1$. This means that if $\beta_1 > \beta_0$, then $h: I \to I$, but in second case, $h: I \to I_1$. We remark that if we choose $x_0 \in I_1 \setminus I$ in second case, then this solution by Theorem 3 will tend to infinity.

3 Lyapunov exponent and chaotic attractor

Let f be function with domain I. The orbit of point $x_0 \in I$ is a set $\{x_0,\ x_1 = f(x_0),\ x_2 = f(x_1), \dots\}$.

Definition 2. The *Lyapunov exponent* $\lambda(x_0)$ of the orbit $\{x_0, x_1, x_2, \dots\}$ is defined as

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\left|f'(x_k)\right|,$$

provided that the limit exists.

In [8], the authors showed that if the Lyapunov exponent $\lambda > 0$, then the sensitivity dependence on initial conditions exists. The Lyapunov exponent at a point x measures the growth in error per iteration. As the Lyapunov exponent becomes larger, the magnification of error becomes greater.

Theorem 5. *If $\beta_0\beta_1 > 1$, then function h have a positive Lyapunov exponent for all $x_0 \notin C = \{0\} \cup \{x\colon \exists j \in \mathbb{N},\ x_j = h^j(x) = 0\}$.*

Proof. If x_0 is 0 or such that $x_j = h^j(x_0) = 0$ for some j, then $\lambda(x_0)$ is not defined because the derivative is not defined. Such points make up a countable set C. For every $\beta_0 > 0$ and $\beta_1 > 0$ and arbitrary initial point $x_0 \notin C$, the Lyapunov exponent is

$$\begin{aligned}
\lambda(x_0) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\left|h'(x_k)\right| \\
&= \lim_{n\to\infty} \frac{1}{n}(\ln\beta_0 + \ln\beta_1 + \cdots + \ln\beta_0 + \ln\beta_1 + j\cdot\ln\beta_0) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\frac{\ln(\beta_0\beta_1)\cdot(n-j)}{2} + j\cdot\ln\beta_0\right) = \frac{\ln(\beta_0\beta_1)}{2},
\end{aligned}$$

where

$$j = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

□

The authors in [9] show that the first nonequilibrium chaotic system has been introduced by Sprott [13] in 1994. We will show that for certain values of the coefficients β_1 and β_2, Eq. (1) forms a chaotic system.

A discrete dynamical system, denoted by DDS for short, is the description of an evolutive phenomenon in terms of a map f whose image is contained in its domain I. Then the pair $\{I, f\}$ is called DDS.

Definition 3. (See [12] and [10, 11].) A set $A \subset I$ is called an attractor for a DDS $\{I, f\}$ if the following conditions hold:

(i) A is closed;
(ii) A is invariant;
(iii) there exists $\eta > 0$ such that, for any $x \in I$ fulfilling $\operatorname{dist}(x, A) < \eta$, we have $\lim_{k\to\infty} \operatorname{dist}(f^k(x), A) = 0$;
(iv) A is a minimal, that is, there are no proper subsets of A fulfilling (i), (ii) and (iii).

In previous definition, the distance from a point $x \in \mathbb{R}$ to a closed set $K \subset \mathbb{R}$ is defined as

$$\operatorname{dist}(x, K) = \min\{|x - y|,\, y \in K\}.$$

Definition 4. (See [12].) If A is an attractor of function f, then the set

$$\Big\{x \in \mathbb{R}\colon \lim_{k\to\infty} f^k(x) \in A\Big\}$$

is called an *attraction basin* of attractor A.

Definition 5. (See [10, 11].) An invariant set A is called a chaotic attractor, provided that it is an attractor and f has sensitive dependence on initial conditions on A (or f have a positive Lyapunov exponent on A).

So far, in our research, we have not identified all possible cycles for any parameter values $\beta_0 > 1$ and $\beta_1 > 1$. Before we prove the next theorem, we note that for all $\beta_0 > 1$ and $\beta_1 > 1$, there exists a cycle with period two $\{(\beta_1 - 1)/(\beta_0\beta_1 - 1),\, (1 - \beta_0)/(\beta_0\beta_1 - 1)\}$ and there exists a cycle with period four $\{(1 + \beta_1)/(\beta_0\beta_1 + 1),\, (\beta_0 - 1)/(\beta_0\beta_1 + 1),\, -(1 + \beta_1)/(\beta_0\beta_1 + 1),\, -(\beta_0 - 1)/(\beta_0\beta_1 + 1)\}$. Both cycles lie inside the interval $[-1, 1]$. Although the sets consisting of points of one cycle are invariant, they do not form attractors here because, due to the fact that $\beta_0 > 1$ and $\beta_1 > 1$, cycles are unstable, property (iii) of the definition of an attractor is not fulfilled.

Theorem 6. *Let* $1 < \beta_0 \leqslant 2$, $1 < \beta_1 \leqslant 2$ *and* $\beta_0 \neq \beta_1$. *Then* $[-1, 1]$ *is a chaotic attractor of function* h*, and attraction basin is*

$$\left]-\frac{1 + \beta_1}{\beta_0\beta_1 - 1}, \frac{1 + \beta_1}{\beta_0\beta_1 - 1}\right[.$$

Proof. If $1 < \beta_0 \leqslant 2$, $1 < \beta_1 \leqslant 2$ and $\beta_0 \neq \beta_1$, then $(1+\beta_1)/(\beta_0\beta_1 - 1) > 1$ as

$$1 + \beta_1 > \beta_0\beta_1 - 1 \quad \iff \quad 2 > \beta_1(\beta_0 - 1).$$

Our aim is to show that for all $x_0 \in]-(1+\beta_1)/(\beta_0\beta_1-1), (1+\beta_1)/(\beta_0\beta_1-1)[\setminus[-1,1]$, the orbit by the function h eventually falls in the interval $[-1,1]$. We will only consider the case when $1 < x_0 < (1+\beta_1)/(\beta_0\beta_1-1)$. The case when $-(1+\beta_1)/(\beta_0\beta_1-1) < x_0 < -1$ is similar and will be omitted.

If $1 < x_0 < (1+\beta_1)/(\beta_0\beta_1-1)$, then

$$0 < \beta_0 - 1 < x_1 = \beta_0 x_0 - 1 < \frac{\beta_0(1+\beta_1)}{\beta_0\beta_1 - 1} - 1 = \frac{1+\beta_0}{\beta_0\beta_1 - 1}.$$

If $0 < x_1 \leqslant 1$, then the proof is complete. If this is not the case, then $1 < x_1 < (1+\beta_0)/(\beta_0\beta_1 - 1)$ and therefore

$$0 < \beta_1 - 1 < x_2 = \beta_1 x_1 - 1 < \frac{\beta_1(1+\beta_0)}{\beta_0\beta_1 - 1} - 1 = \frac{1+\beta_1}{\beta_0\beta_1 - 1}.$$

Provided that $x_n \notin [-1,1]$, by induction we then conclude that

$$1 < x_{2k} < \frac{1+\beta_1}{\beta_0\beta_1 - 1} \quad \text{and} \quad 1 < x_{2k+1} < \frac{1+\beta_0}{\beta_0\beta_1 - 1}, \quad k = 0, 1, 2, \ldots.$$

Notice that all the points are in form

$$1 < x_0 < \frac{1+\beta_1}{\beta_0\beta_1 - 1},$$
$$1 < x_1 = \beta_0 x_0 - 1 < \frac{1+\beta_0}{\beta_0\beta_1 - 1},$$
$$1 < x_2 = \beta_0\beta_1 x_0 - \beta_1 - 1 < \frac{1+\beta_1}{\beta_0\beta_1 - 1},$$
$$1 < x_3 = \beta_0^2\beta_1 x_0 - \beta_0\beta_1 - \beta_0 - 1 < \frac{1+\beta_0}{\beta_0\beta_1 - 1},$$
$$1 < x_4 = \beta_0^2\beta_1^2 x_0 - \beta_0\beta_1^2 - \beta_0\beta_1 - \beta_1 - 1 < \frac{1+\beta_1}{\beta_0\beta_1 - 1},$$
$$\ldots,$$
$$1 < x_{2k} = \beta_0^k\beta_1^k x_0 - \beta_0^{k-1}\beta_1^k - \beta_0^{k-1}\beta_1^{k-1} - \cdots - \beta_0\beta_1 - \beta_1 - 1 < \frac{1+\beta_1}{\beta_0\beta_1 - 1},$$
$$1 < x_{2k+1} = \beta_0^{k+1}\beta_1^k x_0 - \beta_0^k\beta_1^k - \beta_0^k\beta_1^{k-1} - \cdots - \beta_0\beta_1 - \beta_0 - 1 < \frac{1+\beta_0}{\beta_0\beta_1 - 1},$$
$$1 < x_{2k+2} = \beta_0^{k+1}\beta_1^{k+1} x_0 - \beta_0^k\beta_1^{k+1} - \beta_0^k\beta_1^k - \cdots - \beta_0\beta_1 - \beta_1 - 1 < \frac{1+\beta_1}{\beta_0\beta_1 - 1},$$
$$1 < x_{2k+3} = \beta_0^{k+2}\beta_1^{k+1} x_0 - \beta_0^{k+1}\beta_1^{k+1} - \beta_0^{k+1}\beta_1^k - \cdots - \beta_0\beta_1 - \beta_0 - 1 < \frac{1+\beta_0}{\beta_0\beta_1 - 1},$$
$$\ldots.$$

Next, note that the difference between the even-ordered iterations x_{2k} and x_{2k+2} is

$$\begin{aligned} x_{2k} - x_{2k+2} &= \beta_0^k \beta_1^k x_0 - \left(\beta_0^{k+1}\beta_1^{k+1} x_0 - \beta_0^k \beta_1^{k+1} - \beta_0^k \beta_1^k\right) \\ &= \beta_0^k \beta_1^k \left((1 - \beta_0\beta_1)x_0 + \beta_1 + 1\right) \\ &= \beta_0^k \beta_1^k (\beta_0\beta_1 - 1)\left(\frac{\beta_1 + 1}{\beta_0\beta_1 - 1} - x_0\right), \quad k = 0, 1, 2, \dots . \end{aligned}$$

Since $\beta_0\beta_1 > 1$ and $1 < x_0 < (1 + \beta_1)/(\beta_0\beta_1 - 1)$, then we obtain $\lim_{k\to\infty}(\beta_0\beta_1)^k = +\infty$ and $(\beta_1 + 1)/(\beta_0\beta_1 - 1) - x_0 > 0$. Hence the difference between x_{2k} and x_{2k+2} increases, and we then get $x_0 > x_2 > x_4 > \dots > x_{2k} > x_{2k+2} > \cdots$. Thus we conclude that there exists $k \in \mathbb{N}$ such that $x_{2k} \leqslant 1$.

Similarly, the difference between odd-ordered iterations x_{2k+1} and x_{2k+3} is

$$x_{2k+1} - x_{2k+3} = \beta_0^k \beta_1^k (\beta_0\beta_1 - 1)\left(\frac{\beta_0 + 1}{\beta_0\beta_1 - 1} - x_0\right), \quad k = 0, 1, 2, \dots .$$

The difference between x_{2k+1} and x_{2k+3} increases, and we get $x_1 > x_3 > x_5 > \dots > x_{2k+1} > x_{2k+3} > \cdots$. Thus we conclude that there exists $k \in \mathbb{N}$ such that $x_{2k+1} \leqslant 1$.

From what has just been proved it follows that all periodic points (cycles) lie in the interval $[-1, 1]$. Since $\beta_0 > 1$ and $\beta_1 > 1$, then all cycles are unstable. It is impossible to choose an even smaller set than $[-1, 1]$ that satisfies the definition of an attractor. In case $1 < \beta_0 \leqslant 2$, $1 < \beta_1 \leqslant 2$ and $\beta_0 \neq \beta_1$, the interval $[-1, 1]$ is an invariant set for the function h, and the Lyapunov exponent is positive $\ln(\beta_0\beta_1)/2 > \ln 1 = 0$ for all $x_0 \notin C = \{0\} \cup \{x\colon \exists j \in \mathbb{N},\, x_j = h^j(x) = 0\}$. C is countable set (similar as for Tent map). $\eta = (1 + \beta_1)/(\beta_0\beta_1 - 1) - 1 > 0$ – this was proved above. The interval $[-1, 1]$ is a chaotic attractor of function h. □

Example. Suppose that $\beta_0 = 1.9$ and $\beta_1 = 1.35$. In this case, we obtain the period-two cycle $\{1.501597444, 1.8530035144\}$ and the basin of attraction $]-1.501597444, 1.501597444[$. If we start with initial condition $x_0 = 1.49$ (a point close to the boundary of the interval), then we observe the situation described in Theorem 6, where the first seven iterations of the solution are greater than 1. Then $x_8 = 0.99958814 < 1$, and all other points of the solution lie in the interval $[-1, 1]$. In Fig. 3, we see that $x_0 > x_2 > x_4 > x_6 > x_8$ and $x_1 > x_3 > x_5 > x_7$. The behavior of the other points cannot be clearly described, but all other points of the solution lie in the interval $[-1, 1]$.

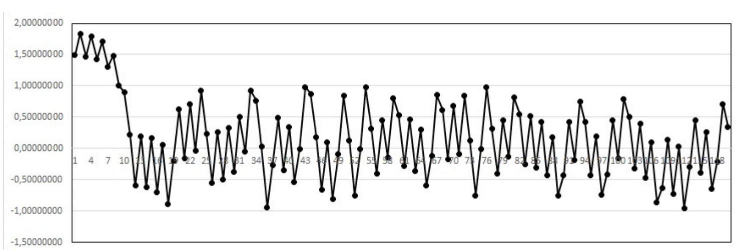

Figure 3. First 120 values of solution of difference equation (1) when $\beta_0 = 1.9$, $\beta_1 = 1.35$ and $x_0 = 1.49$.

4 Conclusion

In this article, we investigated the boundedness and the chaotic character of solutions of Eq. (1). First, we determined the necessary and sufficient conditions for every solution to either diverge to $+\infty$ or to diverge to $-\infty$ as two subsequences. This then led us to determining the existence of invariant and attracting intervals, where the chaotic behavior of solutions of Eq. (1) arise. The most important result of the article is the last Theorem 6, which shows the possibility of constructing a chaotic attractor with noncontinuous functions.

Our aim is to proceed with the examination of the boundedness and the chaotic character of solutions of Eq. (1) when $(\beta_n)_{n=0}^{\infty}$ is a periodic sequence with period three and higher. In particular, it is of paramount interest to investigate the monotonic properties of Eq. (1), that is, into how many subsequences to decompose the solution of Eq. (1). Furthermore, our objectives are to determine the existence of invariant and the attracting intervals of Eq. (1).

References

1. A. Anisimova, M. Avotina, I. Bula, Periodic orbits of single neuron models with internal decay rate $0 < \beta \leqslant 1$, *Math. Model. Anal.*, **18**(3):325–345, 2013, https://doi.org/10.3846/13926292.2013.804462.
2. A. Anisimova, M. Avotina, I.Bula, Periodic and chaotic orbits of a neuron model, *Math. Model. Anal.*, **20**(1):30–52, 2015, https://doi.org/10.3846/13926292.2015.1000411.
3. H. Bin, L. Huang, G. Zhang, Convergence and periodicity of solutions for a class of difference systems, *Adv. Difference Equ.*, **2006**:70461, 2006, https://doi.org/10.1155/ADE/2006/70461.
4. I. Bula, M.A. Radin, Periodic orbits of a neuron model with periodic internal decay rate, *Appl. Math. Comput.*, **266**:293–303, 2015, https://doi.org/10.1016/j.amc.2015.05.097.
5. I. Bula, M.A. Radin, Eventually periodic solutions of single neuron model, *Nonlinear Anal. Model. Control*, **25**(6):903–918, 2020, https://doi.org/10.15388/namc.2020.25.20555.
6. I. Bula, M.A. Radin, N. Wilkins, Neuron model with a period three internal decay rate, *Electron. J. Qual. Theory Differ. Equ.*, **2017**:46, 2017, https://doi.org/10.14232/ejqtde.2017.1.46.
7. Y. Chen, All solutions of a class of difference equations are truncated periodic, *Appl. Math. Lett.*, **15**:975–979, 2002, https://doi.org/10.1016/S0893-9659(02)00072-1.
8. S.N. Elaydi, *Discrete Chaos: With Applications in Science and Engineering,* 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2007, https://doi.org/10.1201/9781420011043.
9. H. Jahanshahi, O. Orozco-López, J.M. Munoz-Pacheco, N.D. Alotaibi, C. Volos, Z. Wang, R. Sevilla-Escoboza, Y.M. Chu, Simulation and experimental validation of a non-equilibrium chaotic system, *Chaos Solitons Fractals*, **143**:110539, 2021, https://doi.org/10.1016/j.chaos.2020.110539.

10. C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRS Press, Boca Raton, FL, 1995.
11. R.C. Robinson, *An Introduction to Dynamical Systems: Continuous and Discrete*, Pearson Prentice Hall, Upper Saddle River, NJ, 2004.
12. E. Salinelli, F. Tomarelli, *Discrete Dynamical Models*, Unitext, Vol. 76, Springer, Cham, 2014.
13. J.C. Sprott, Some simple chaotic flows, *Phys. Rev. E*, **50**(2):647–650., 1994, https://doi.org/10.1103/PhysRevE.50.R647.
14. Z. Wei, L. Huang, Y. Meng, Unboundedness and periodicity of solutions for a discrete-time network model of three neurons, *Appl. Math. Model.*, **32**(7):1463–1474, 2008, https://doi.org/10.1016/j.apm.2007.06.016.
15. J. Wu, *Introduction to Neural Dynamics and Signal Transmission Delay*, De Gruyter Ser. Nonlinear Anal. Appl., Vol. 6, De Gruyter, Berlin, 2001, https://doi.org/10.1515/9783110879971.
16. T. Yi, Z. Zhou, Periodic solutions of difference equations, *J. Math. Anal. Appl.*, **286**(1):220–229, 2003, https://doi.org/10.1016/S0022-247X(03)00473-6.
17. Z. Yuan, L. Huang, All solutions of a class of discrete-time systems are eventually periodic, *Appl. Math. Comput.*, **158**(2):537–546, 2004, https://doi.org/10.1016/j.amc.2003.09.011.
18. Z. Yuan, L. Huang, Y. Chen, Convergence and periodicity of solutions for a discrete-time network model of two neurons, *Math. Comput. Modelling*, **35**(9–10):941–950, 2002, https://doi.org/10.1016/S0895-7177(02)00061-4.
19. Z. Zhou, Periodic orbits on discrete dynamical systems, *Comput. Math. Appl.*, **45**(6–9):1155–1161, 2003, https://doi.org/10.1016/S0898-1221(03)00075-0.
20. Z. Zhou, J. Wu, Stable periodic orbits in nonlinear discrete-time neural networks with delayed feedback, *Comput. Math. Appl.*, **45**(6–9):935–942, 2003, https://doi.org/10.1016/S0898-1221(03)00066-X.
21. Z. Zhou, J. Yu, L. Huang, Asymptotic behavior of delay difference systems, *Comput. Math. Appl.*, **42**(3–5):283–290, 2001, https://doi.org/10.1016/S0898-1221(01)00153-5.
22. H. Zhu, L. Huang, Dynamics of a class of nonlinear discrete-time neural networks, *Comput. Math. Appl.*, **48**(1–2):85–94, 2004, https://doi.org/10.1016/j.camwa.2004.01.006.