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Existence of solutions to a nonlinear fractional diffusion equation with exponential growth*

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Abstract. In this paper, we study a Cauchy problem for a space–time fractional diffusion equation with exponential nonlinearity. Based on the standard L^p-L^q estimates of strongly continuous semigroup generated by fractional Laplace operator, we investigate the existence of global solutions for initial data with small norm in Orlicz space $\exp L^p(\mathbb{R}^d)$ and a time weighted $L^r(\mathbb{R}^d)$ space. In the framework of the Hölder interpolation inequality, we also discuss the existence of local solutions without the Orlicz space.

Keywords: diffusion equation, Caputo's fractional derivative, exponential growth, Orlicz spaces.

1 Introduction

Fractional calculus serves as a useful tool for describing nontrivial physical phenomena, such as anomalous diffusion phenomenon, and the memory effect of the complex or viscoelastic media; see, for example, [1, 6, 16, 17, 19, 20, 23, 26, 30, 31]. The nonlocality of fractional derivatives can also capture many interesting phenomena like the thermoelectric MHD non-Newtonian fluid in heat transfer [7], the Rayleigh–Stokes problem in a heated generalized second grade fluid [27], the viscoelasticity medium for wave equations [9, 11, 22], the jumps and long-distance interactions in Lévy processes [4], etc.

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In this paper, we consider a Cauchy problem for space–time fractional diffusion equation

$$\partial_t^\alpha u + (-\Delta)^\gamma u = f(u), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{1}$$

associated with an initial condition $u(0,x) = u_0(x)$, $x \in \mathbb{R}^d$, $d \geqslant 1$, where f is the exponential growth function, like asymptotic growth $f(u) \sim \mathrm{e}^{4\pi|u|^2}$ and with a vanishing behavior at zero and ∂_t^α standing for the Caputo fractional partial derivative of order $\alpha \in (0,1)$ defined by

$$\partial_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(s,x)\,\mathrm{d}s, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d;$$

see, e.g., [15]. In (1), $(-\Delta)^\gamma$ ($\gamma \in (0,1)$) stands for the fractional Laplacian operator defined by

$$(-\Delta)^\gamma u(x) = C_{\gamma,d} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2\gamma}}\,\mathrm{d}y$$

with $C_{\gamma,d} = \gamma 2^{2\gamma}\Gamma((d+2\gamma)/2)/(\pi^{d/2}\Gamma(1-\gamma))$.

Now we dwell on the literature dealing with the nonlinearity of exponential growth problems of diffusion and wave equations. The Cauchy problem for heat equation with exponential nonlinearity was studied by Ioku [13]. Inspired by this paper, Furioli et al. [10] discussed the asymptotic behavior and decay estimates of solutions for a nonlinear parabolic equation with exponential growth. Ioku et al. [14] obtained the existence and nonexistence results for a heat equation in Orlicz space $\exp L(\mathbb{R}^2)$. Fino and Kirane [8] investigated the global solutions for heat equation with fractional Laplacian and exponential nonlinearity with small initial data, also the local solution in Orlicz space. As for the wave equation, the global well-posed solutions with exponential growth-type nonlinearity was studied in the critical Sobolev space in [25]. In order to overcome a problem of invalidity of the embedding $H^1(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, Ibrahim et al. [12] discussed the existence and asymptotic behavior of finite energy solutions for large time for the subcritical case of the wave operators via Trudinger–Moser-type inequality. On the supercritical regime of large energies for smooth and radially symmetric initial data, Struwe [28] established the global well-posedness of solutions for a nonlinear wave equation with nonlinearity $f(u) \sim u\mathrm{e}^{u^2}$. Mahouachi and Saanouni [21] derived the well-posed and ill-posed results for a wave equation with exponential growth. In terms of the fractional derivatives, Bekkai et al. [3] discussed the local existence and blow-up of solution for a space–time fractional diffusion equation with nonlocal nonlinearity of the form $f(u) \sim J_t^{1-\alpha}(\mathrm{e}^u)$, where $J_t^{1-\alpha}$ represents the Riemann–Liouville fractional integral operator. Alsaedi et al. [2] proved the existence and uniqueness of the local mild solution for a system of space–time fractional evolution equations with nonlocal nonlinearities of exponential growth. They also established a blow-up result by applying Pokhozhaev capacity method and presented an estimate for the life span of blowing-up solutions under suitable conditions.

We note that there is no work concerning the existence of global solutions for space–time fractional diffusion equations with exponential growth data in $\mathbb{R}^d$. An appropriate space needs to be considered for this purpose, contrary to the technique of energy functional used in the previous works, this paper will consider the global solutions to the fractional Cauchy problem on Orlicz spaces $\exp L^p(\mathbb{R}^d)$ via the subordinate principle and the semigroup theory. It is worthwhile to notice that $C_0^\infty(\mathbb{R}^d)$ is not dense in $\exp L^p(\mathbb{R}^d)$, but it works in $L^p(\mathbb{R}^d)$ for $1 \leqslant p < \infty$. It is difficult to consider a generic global space $C([0,\infty), \exp L^p(\mathbb{R}^d))$ for solving the Cauchy problem (1). So, we shall deal with this issue in the sense of a weak topology. Furthermore, since $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \hookrightarrow \exp L^p(\mathbb{R}^d)$ and in order to solve Cauchy problem (1) under the minimum required conditions, we will establish the local solutions in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Our main goal in this paper is to investigate the existence of solutions for Cauchy problem (1). In the next section, we introduce the definition of Orlicz spaces, and the relevant solution operators of (1), and then we study the space–time estimates in the frameworks of L^p-L^q and L^p-$\exp L^q$. In Section 3, we establish the existence of global solution for initial data with small norm in space $\exp L^p(\mathbb{R}^d)$ and the decay estimate of solution. In Section 4, we prove the existence of local solutions in subspace $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ of the Orlicz spaces.

2 Orlicz spaces and space–time estimates

Throughout this paper, the notation $a \lesssim b$ stands for $a \leqslant Cb$, and $\sim$ stands for $a \leqslant Cb$ and $b \leqslant Ca$ for a positive generic constant C that does not depend on a, b. Symbols $\vee$ and $\wedge$ are expressed by $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, respectively. It is well known that the fractional Laplace operator $(-\Delta)^\gamma$ can generate a strongly continuous semigroup $T_\gamma(t) = \exp(-t(-\Delta)^\gamma))$ on $L^p(\mathbb{R}^d)$ for $p \geqslant 1$, $d \geqslant 1$ with its Fourier transformation $(-\Delta)^\gamma u = \mathcal{F}^{-1}(|\xi|^{2\gamma}\mathcal{F}(u))$. Moreover, a space–time estimate of this semigroup is given by

$$\left\|T_\gamma(t)\phi\right\|_{L^p(\mathbb{R}^d)} \lesssim t^{d(1/p-1/q)/(2\gamma)}\|\phi\|_{L^q(\mathbb{R}^d)}$$

for $t > 0$ and for all $\phi \in L^q(\mathbb{R}^d)$, $q \geqslant 1$; see [24].

For any $\alpha \in (0,1)$, $\beta \in [0,\infty)$, $\gamma \in (0,1)$, we introduce a subordinate operator

$$\mathcal{A}_{\alpha,\beta}^\gamma(t) = \int_0^\infty \alpha^\beta \theta^\beta \zeta_\alpha(\theta) T_\gamma\left(t^\alpha\theta\right) \mathrm{d}\theta, \quad t \geqslant 0,$$

where $\zeta_\alpha(\cdot)$ is the Wright-type function

$$\zeta_\alpha(\theta) = \sum_{k=0}^\infty \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \quad z \in \mathbb{C}.$$

Obviously, the operator $\mathcal{A}_{\alpha,\beta}^\gamma(\cdot)$ is well defined due to the estimate

$$\begin{aligned}&\left\|\mathcal{A}_{\alpha,\beta}^\gamma(t)\phi\right\|_{L^p(\mathbb{R}^d)}\\ &\quad\leqslant \int_0^\infty \alpha^\beta \theta^\beta \zeta_\alpha(\theta)\left\|T_\gamma\left(t^\alpha\theta\right)\phi\right\|_{L^p(\mathbb{R}^d)} \mathrm{d}\theta \lesssim \int_0^\infty \alpha^\beta \theta^\beta \zeta_\alpha(\theta)\, \mathrm{d}\theta\, \|\phi\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

and the properties of $\zeta_\alpha(\cdot)$ (see [29])

$$\zeta_\alpha(\cdot) \geqslant 0, \quad \int_0^\infty \theta^\delta \zeta_\alpha(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \quad \delta \in (-1, \infty).$$

Specially, if u is a solution of (1), by using the strategy employed in [29], we have the following integral representation of Cauchy problem (1):

$$u(t) = \mathcal{A}^\gamma_{\alpha,0}(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{A}^\gamma_{\alpha,1}(t-s) f\big(u(s)\big)\,\mathrm{d}s. \tag{2}$$

Recall that a space is Orlicz type if it can be expressed as

$$\exp L^p\big(\mathbb{R}^d\big) = \big\{u \in L^1_{\mathrm{loc}}\big(\mathbb{R}^d\big)\colon \|u\|_{\exp L^p(\mathbb{R}^d)} < +\infty\big\},$$

endowed with the Luxemburg norm

$$\|u\|_{\exp L^p(\mathbb{R}^d)} := \inf\left\{\tau > 0\colon \int_{\mathbb{R}^d} \left(\exp\frac{|u(x)|^p}{\tau^p} - 1\right)\mathrm{d}x < 1\right\}.$$

Clearly, an Orlicz space is a Banach space.

Lemma 1. (*See* [13].) *For every* $1 \leqslant p \leqslant q < +\infty$, *the embedding* $\exp L^p(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *holds, moreover*

$$\|u\|_{L^q(\mathbb{R}^d)} \leqslant \left(\Gamma\left(\frac{q}{p}+1\right)\right)^{1/q}\|u\|_{\exp L^p(\mathbb{R}^d)}.$$

Lemma 2. (*See* [8].) *For every* $1 \leqslant q \leqslant p < +\infty$, *the embedding* $L^q(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \hookrightarrow \exp L^p(\mathbb{R}^d)$ *holds, moreover*

$$\|u\|_{\exp L^p(\mathbb{R}^d)} \leqslant \ln^{-1/p} 2\,\big(\|u\|_{L^q(\mathbb{R}^d)} + \|u\|_{L^\infty(\mathbb{R}^d)}\big).$$

Lemma 3. *Let* $\alpha \in (0,1)$, $\beta \in [0,\infty)$, $\gamma \in (0,1)$, *and let* $d(1/p - 1/q)/(2\gamma) < 1$ *for* $1 \leqslant p \leqslant q \leqslant +\infty$. *Then for* $t > 0$,

(i) $\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\alpha d/(2\gamma)(1/p-1/q)}\|f\|_{L^p(\mathbb{R}^d)}$;
(ii) $\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\|_{\exp L^p(\mathbb{R}^d)} \lesssim \|f\|_{\exp L^p(\mathbb{R}^d)}$ *for* $f \in \exp L^p(\mathbb{R}^d)$;
(iii) $\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\|_{\exp L^q(\mathbb{R}^d)} \lesssim t^{-\alpha d/(2\gamma p)} \ln^{-1/q}(t^{-\alpha d/(2\gamma)} + 1)\|f\|_{L^p(\mathbb{R}^d)}$ *for* $f \in L^p(\mathbb{R}^d)$;
(iv) $\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\|_{\exp L^p(\mathbb{R}^d)} \lesssim t^{-\alpha d/(2\gamma q)}\|f\|_{L^q(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}$ *for* $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$.

Proof. By the L^p-L^q estimates of semigroup $T_\gamma(t)$, for the operator $\mathcal{A}^\gamma_{\alpha,\beta}(\cdot)$, we obtain

$$\big\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\big\|_{L^q(\mathbb{R}^d)} \leqslant \int_0^\infty \alpha^\beta\theta^\beta\zeta_\alpha(\theta)\big\|T_\gamma\big(t^\alpha\theta\big)f\big\|_{L^q(\mathbb{R}^d)}\,\mathrm{d}\theta \lesssim t^{-\alpha d/(2\gamma r)}\|f\|_{L^p(\mathbb{R}^d)}$$

for $t>0$, where $1/r := 1/p - 1/q$, and then (i) follows. Additionally, for any $\tau>0$, $t>0$, it follows by Taylor expansion combined with (i) that

$$\begin{aligned}&\int_{\mathbb{R}^d}\bigg(\exp\bigg(\frac{\mathcal{A}^\gamma_{\alpha,\beta}(t)f}{\tau}\bigg)^p - 1\bigg)\,\mathrm{d}x \\ &\quad= \sum_{k=1}^\infty \frac{\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\|^{pk}_{L^{pk}(\mathbb{R}^d)}}{k!\tau^{pk}} \lesssim \sum_{k=1}^\infty \frac{\|f\|^{pk}_{L^{pk}(\mathbb{R}^d)}}{k!\tau^{pk}} \lesssim \int_{\mathbb{R}^d}\bigg(\exp\bigg(\frac{f}{\tau}\bigg)^p - 1\bigg)\,\mathrm{d}x,\end{aligned}$$

which implies that for $f\in\exp L^p(\mathbb{R}^d)$,

$$\big\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\big\|_{\exp L^p(\mathbb{R}^d)} \lesssim \|f\|_{\exp L^p(\mathbb{R}^d)}.$$

Next, by virtue of (i), for $t>0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\bigg(\exp\bigg(\frac{\mathcal{A}^\gamma_{\alpha,\beta}(t)f}{\tau}\bigg)^q - 1\bigg)\,\mathrm{d}x \\ &\quad\leqslant \sum_{k=1}^\infty \frac{\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\|^{qk}_{L^{qk}(\mathbb{R}^d)}}{k!\tau^{qk}} \lesssim \sum_{k=1}^\infty \frac{t^{-\alpha d/(2\gamma)(1/p-1/(qk))qk}\|f\|^{qk}_{L^p(\mathbb{R}^d)}}{k!\tau^{qk}},\end{aligned}$$

which implies that there exists a constant $C>0$ such that

$$\int_{\mathbb{R}^d}\bigg(\exp\bigg(\frac{\mathcal{A}^\gamma_{\alpha,\beta}(t)f}{\tau}\bigg)^q - 1\bigg)\,\mathrm{d}x \leqslant t^{\alpha d/(2\gamma)}\bigg(\exp\bigg(\frac{Ct^{-\alpha d/(2\gamma p)}\|f\|_{L^p(\mathbb{R}^d)}}{\tau}\bigg)^q - 1\bigg),$$

and then

$$\big\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\big\|_{\exp L^q(\mathbb{R}^d)} \lesssim t^{-\alpha d/(2\gamma p)}\ln^{-1/q}\big(t^{-\alpha d/(2\gamma)}+1\big)\|f\|_{L^p(\mathbb{R}^d)}.$$

The last inequality can easily be proved by using the standard L^p-L^q estimates of $\mathcal{A}^\gamma_{\alpha,\beta}(\cdot)$ and the embedding $L^p(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\hookrightarrow \exp L^p(\mathbb{R}^d)$ in Lemma 2. In consequence, we get

$$\begin{aligned}\big\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\big\|_{\exp L^p(\mathbb{R}^d)} &\lesssim \big\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\big\|_{L^p(\mathbb{R}^d)} + \big\|\mathcal{A}^\gamma_{\alpha,\beta}(t)f\big\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim t^{-\alpha d/(2\gamma q)}\|f\|_{L^q(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Hence, this completes the proof. □

Observe that $C_0^\infty(\mathbb{R}^d)$ is not dense in $\exp L^p(\mathbb{R}^d)$ for $p \geqslant 1$; for more details, see [13, 14]. For the fractional version of $\mathcal{A}^\gamma_{\alpha,0}(\cdot)$, we have the following conclusion.

Remark 1. The operator $\mathcal{A}^\gamma_{\alpha,0}(t)$ is not strongly continuous at $t = 0$ in $\exp L^p(\mathbb{R}^d)$ for $p \geqslant 1$, that is, for any $\phi \in \exp L^p(\mathbb{R}^d)$, the following inequality holds:

$$\lim_{t\to 0} \left\| \mathcal{A}^\gamma_{\alpha,0}(t)\phi - \phi \right\|_{\exp L^p(\mathbb{R}^d)} \geqslant 1. \tag{3}$$

Proof. In fact, for any $\lambda > 0$, let $\mu_v(\lambda) := |\{x \in \mathbb{R}^d\colon |v(x)| > \lambda\}|$ be a distribution function of v, and let v^* be a nonincreasing rearrangement of v given by

$$v^*(r) := \inf\bigl\{\lambda > 0\colon \mu_v(\lambda) \leqslant r\bigr\},$$

and the maximal function of v^* is denoted by v^{**} as follows:

$$v^{**}(r) = \frac{1}{r} \int_0^r v^*(s)\,\mathrm{d}s.$$

By using the rearrangement technique, we have

$$\sup_{0<r<1} \frac{(\mathcal{A}^\gamma_{\alpha,0}\phi - \phi)^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)} \lesssim \left\| \mathcal{A}^\gamma_{\alpha,0}(t)\phi - \phi \right\|_{\exp L^p(\mathbb{R}^d)}$$

for any $t \geqslant 0$, where the following holds:

$$\|v\|_{\exp L^p(\mathbb{R}^d)} \lesssim \sup_{0<r<1} \frac{v^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)} + \|v\|_{L^p(\mathbb{R}^d)} \lesssim \|v\|_{\exp L^p(\mathbb{R}^d)}.$$

The first inequality can be established as in [18, Thm. 3.4], and function $\psi(t) = (1+\ln t)/\ln(1+t)$ has a maximum value for $t \geqslant 1$, while the second inequality can be shown by the method employed in [13, Lemma 5.2].

Therefore, due to the triangle inequality for v^{**}, that is, $(f+g)^{**} \leqslant f^{**} + g^{**}$, we have

$$\frac{\phi^{**}(r) - (\mathcal{A}^\gamma_{\alpha,0}(t)\phi)^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)} \leqslant \frac{(\phi - \mathcal{A}^\gamma_{\alpha,0}(t)\phi)^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)},$$

which implies from the nonnegative property of v^* ($v^*(r) \geqslant 0$ for any $r > 0$) that

$$\begin{aligned}\lim_{r\to 0} \frac{\phi^{**}(r) - (\mathcal{A}^\gamma_{\alpha,0}\phi)^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)} &\lesssim \sup_{0<r<1} \frac{(\mathcal{A}^\gamma_{\alpha,0}(t)\phi - \phi)^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)^{1/p}} \\ &\lesssim \left\| \mathcal{A}^\gamma_{\alpha,0}(t)\phi - \phi \right\|_{\exp L^p(\mathbb{R}^d)}.\end{aligned}$$

According to Lemmas 1 and 3, we get $\mathcal{A}^\gamma_{\alpha,0}(t)\phi \in L^\infty(\mathbb{R}^d)$ for all $t \geqslant \varepsilon$ with any $\varepsilon > 0$. This means that $(\mathcal{A}^\gamma_{\alpha,0}(t)\phi)^{**}(r) \in L^\infty(0,\infty)$ for all $t \geqslant \varepsilon$. Hence we have

$$\lim_{r\to 0} \frac{(\mathcal{A}^\gamma_{\alpha,0}(t)\phi)^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)} = 0.$$

Let

$$\phi(x)=\begin{cases}\frac{p-1-p\ln(\omega_d|x|^d)}{p(1-\ln(\omega_d|x|^d))^{(p-1)/p}}, & 0<|x|<1,\\ 0 & \text{otherwise,}\end{cases}$$

where ω_d is the measure of the unit ball in $\mathbb{R}^d$. For $\psi(t)=\mathrm{e}^{t^p}-1$ with $p\geqslant 1$, it is known that

$$\int\limits_{\mathbb{R}^d}\psi\big(\big|\phi(x)\big|\big)\,\mathrm{d}x=\int\limits_0^1\psi\big(\big|\phi^*(r)\big|\big)\,\mathrm{d}r.$$

Further, it is easy to check that $\phi^{**}(r)=\ln^{1/p}(\mathrm{e}/r)$ for $0<r<\omega_d$. Consequently, we get

$$1\lesssim\lim_{r\to 0}\frac{\phi^{**}(r)}{\ln^{1/p}(\mathrm{e}/r)}\lesssim\big\|\mathcal{A}^{\gamma}_{\alpha,0}(t)\phi-\phi\big\|_{\exp L^p(\mathbb{R}^d)},$$

which establishes (3). The proof is completed. □

In view of Remark 1, we consider a solution concept of Cauchy problem (1) in the following weak sense.

Definition 1. A function $u\in C((0,\infty);\,\exp L^p(\mathbb{R}^d))$ is a weak mild solution of Cauchy problem (1) if it satisfies the integral equation (2) in $\exp L^p(\mathbb{R}^d)$ for almost all $t>0$ and the initial data

$$\lim_{t\to 0}u(t)\stackrel{*}{=}u_0\quad\text{in }\exp L^p\big(\mathbb{R}^d\big).$$

Observe that $u(t)\to u_0$ as $t\to 0$ in weak-$*$ topology if and only if

$$\lim_{t\to 0}\int\limits_{\mathbb{R}^d}\big(u(t,x)\phi(x)-u_0(x)\phi(x)\big)\,\mathrm{d}x=0$$

for every $\phi\in L^1\ln^{1/p}L\,(\mathbb{R}^d)$, the predual space of $\exp L^p(\mathbb{R}^d)$.

3 The global existence

In this section, we show the global existence of solutions to the Cauchy problem (1). In order to achieve this aim, we assume that the exponential growth function f associated with $f(u)\sim|u|^\sigma$ near zero is given by

$$f(0)=0,\qquad\big|f(u)-f(v)\big|\lesssim|u-v|\big(|u|^{\sigma-1}\mathrm{e}^{\lambda|u|^\varsigma}+|u|^{\sigma-1}\mathrm{e}^{\lambda|v|^\varsigma}\big)$$

for some constants $\varsigma,\lambda>0$ and $\sigma\geqslant 1$. A typical example satisfying previous nonlinearity is $f(u)=u(\mathrm{e}^{4\pi u^2}-1)$, it is considered in [5].

Theorem 1. *Let* $1 \leqslant d < 2\gamma p$, $p \geqslant 1$, $\gamma \in (0,1)$, *and* $\sigma \geqslant 1 + 2\gamma p/d$. *If there exists* $\chi > 0$ *such that, for all* $u_0 \in \exp L^p(\mathbb{R}^d)$, $\|u_0\|_{\exp L^p(\mathbb{R}^d)} \leqslant \chi$, *then there exists a unique weak mild solution* u *of the Cauchy problem* (1) *satisfying*

$$\lim_{t\to 0} \big\|u(t) - \mathcal{A}^{\gamma}_{\alpha,0}(t)u_0\big\|_{\exp L^p(\mathbb{R}^d)} = 0.$$

Moreover, for some $r > 2\gamma p^2/d + p$, *the decay estimate holds:*

$$\big\|u(t)\big\|_{L^r(\mathbb{R}^d)} \lesssim t^{-\alpha d/(2\gamma)(1/p-1/r)} \|u_0\|_{\exp L^p(\mathbb{R}^d)}.$$

Proof. Let $\varrho = \alpha d/(2\gamma)(1/p - 1/r)$. For any $\varepsilon > 0$, define a complete metric space X_ε by

$$X_\varepsilon := \Big\{u \in C\big((0,\infty); \exp L^p\big(\mathbb{R}^d\big)\big): \\ \sup_{t>0} t^{\varrho}\big\|u(t)\big\|_{L^r(\mathbb{R}^d)} + \|u\|_{L^\infty(0,\infty;\exp L^p(\mathbb{R}^d))} \leqslant \varepsilon\Big\},$$

endowed with the distance $d(u,v) = \sup_{t>0} t^{\varrho}\|u(t) - v(t)\|_{L^r(\mathbb{R}^d)}$.

In the sequel, we set an operator Q by

$$Q(u)(t) := \mathcal{A}^{\gamma}_{\alpha,0}(t)\phi + \int_0^t (t-s)^{\alpha-1}\mathcal{A}^{\gamma}_{\alpha,1}(t-s) f\big(u(s)\big)\,\mathrm{d}s.$$

We solve the current problem with the exponential growth term by a contraction mapping argument by splitting the proof into four steps.

Step 1. Q is a contraction on X_ε. For any $u, v \in X_\varepsilon$, Lemma 3(i) implies that

$$\big\|Q(u)(t) - Q(v)(t)\big\|_{L^r(\mathbb{R}^d)} \lesssim \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)}\big\|f(u) - f(v)\big\|_{L^l(\mathbb{R}^d)}\,\mathrm{d}s,$$

where $d(1/l - 1/r) < 2\gamma$. From the assumption of f it follows by Taylor expansion that

$$\int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)}\big\|f(u) - f(v)\big\|_{L^l(\mathbb{R}^d)}\,\mathrm{d}s \\ \lesssim \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)}\big\||u-v|\big(|u|^{\sigma-1+\varsigma k} + |v|^{\sigma-1+\varsigma k}\big)\big\|_{L^l(\mathbb{R}^d)}\,\mathrm{d}s.$$

By Hölder and Minkowski inequalities, for $1/l = 1/r + 1/p$, we have

$$\big\||u-v|\big(|u|^{\sigma-1+\varsigma k} + |v|^{\sigma-1+\varsigma k}\big)\big\|_{L^l(\mathbb{R}^d)} \\ \lesssim \|u-v\|_{L^r(\mathbb{R}^d)}\big\|\big(|u|^{\sigma-1+\varsigma k} + |v|^{\sigma-1+\varsigma k}\big)\big\|_{L^p(\mathbb{R}^d)} \\ \lesssim \|u-v\|_{L^r(\mathbb{R}^d)}\big(\|u\|^{\sigma-1+\varsigma k}_{L^{p(\sigma-1+\varsigma k)}(\mathbb{R}^d)} + \|v\|^{\sigma-1+\varsigma k}_{L^{p(\sigma-1+\varsigma k)}(\mathbb{R}^d)}\big).$$

For some $0 < c < (\sigma-1)(r-p)/(\gamma pr) \wedge 2(\sigma-2)/(d(\sigma-1))$, let

$$\zeta = \frac{(1-\vartheta)\gamma p^2 rc}{\vartheta(r-p-\gamma p^2 c)}, \qquad \vartheta = \frac{\gamma prc}{(\sigma-1+\varsigma k)(r-p)}, \qquad \frac{1}{\gamma p} = \frac{2}{d} - c.$$

Then $\zeta \geqslant p$, $0 < \vartheta \leqslant 1$ for each $k \in \mathbb{N} \cup \{0\}$ and $\alpha - \alpha d/(2\gamma p) - \varrho(\sigma-1+\varsigma k)\vartheta = 0$. Hence the Hölder interpolation inequality implies that

$$\|u\|_{L^{p(\sigma-1+\varsigma k)}(\mathbb{R}^d)} \leqslant \|u\|_{L^r(\mathbb{R}^d)}^{\vartheta} \|u\|_{L^\zeta(\mathbb{R}^d)}^{1-\vartheta},$$

where

$$\frac{1}{p(\sigma-1+\varsigma k)} = \frac{\vartheta}{r} + \frac{1-\vartheta}{\zeta}.$$

Additionally, for any $y \in \exp L^p(\mathbb{R}^d)$, Lemma 2 shows that

$$\|y\|_{L^\zeta(\mathbb{R}^d)}^{(\sigma-1+\varsigma k)(1-\vartheta)} \lesssim \left(\Gamma\left(\frac{\zeta}{p}+1\right)\right)^{(\sigma-1+\varsigma k)(1-\vartheta)/\zeta} \|y\|_{\exp L^p(\mathbb{R}^d)}^{(\sigma-1+\varsigma k)(1-\vartheta)}.$$

By virtue of $\Gamma(x+1) \leqslant Cx^{x+1/2}$ for all $x \geqslant 1$ and for some constant $C > 0$, from Stirling's formula and the inequality $(\sigma-1+\varsigma k)(1-\vartheta) \leqslant \zeta$ it follows that

$$\left(\Gamma\left(\frac{\zeta}{p}+1\right)\right)^{(\sigma-1+\varsigma k)(1-\vartheta)/\zeta} \leqslant C^k \Gamma(k+1).$$

Moreover, by the fact

$$\int_0^t (t-s)^{a-1} s^{b-1}\,\mathrm{d}s = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} t^{a+b-1}, \quad a, b > 0,\ t > 0,$$

combined with $d < 2p\gamma$, $0 < \alpha < 1$, and $\varrho + \varrho(\sigma-1+\varsigma k)\vartheta < 1$, we obtain

$$\begin{aligned}
&\big\|Q(u)(t) - Q(v)(t)\big\|_{L^r(\mathbb{R}^d)} \\
&\quad\lesssim \sum_{k=0}^{\infty} C^k \lambda^k \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \|u-v\|_{L^r(\mathbb{R}^d)} \big(\|u\|_{L^r(\mathbb{R}^d)}^{(\sigma-1+\varsigma k)\vartheta} \|u\|_{\exp L^p(\mathbb{R}^d)}^{(\sigma-1+\varsigma k)(1-\vartheta)} \\
&\qquad + \|v\|_{L^r(\mathbb{R}^d)}^{(\sigma-1+\varsigma k)\vartheta} \|v\|_{\exp L^p(\mathbb{R}^d)}^{(\sigma-1+\varsigma k)(1-\vartheta)}\big)\,\mathrm{d}s \\
&\quad\lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\sigma-1+\varsigma k} \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} s^{-\varrho-\varrho(\sigma-1+\varsigma k)\vartheta}\,\mathrm{d}s\, d(u,v) \\
&\quad\lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\sigma-1+\varsigma k} t^{-\varrho} d(u,v),
\end{aligned}$$

which means that for any $u \in X_\varepsilon$, there exists a constant $C > 0$ such that

$$d\big(Q(u), Q(v)\big) \leqslant C \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\sigma-1+\varsigma k} d(u,v).$$

For some small enough $\varepsilon > 0$ satisfying

$$C \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\sigma-1+\varsigma k} \leqslant \frac{1}{4},$$

we deduce that Q is a contraction on X_ε.

Step 2. Q maps X_ε into itself. Continuity of Q follows from that of the operator $\mathcal{A}^{\gamma}_{\alpha,\beta}(t)$ and the strongly continuous behavior of semigroup $T_\gamma(t)$ for all $t \geqslant 0$. Since $g(x) = \ln(x+1) - x/2 = 0$ has two zero points, it follows by letting $a \in (2,4)$ with $a/2 = \ln(a+1)$ that $\ln^{-1/p}((t-s)^{-\alpha d/(2\gamma)} + 1) \leqslant 2^{1/p}(t-s)^{\alpha d/(2\gamma p)}$ for $0 \leqslant s \leqslant t - a^{-2\gamma/(\alpha d)}$, and $\ln^{-1/p}((t-s)^{-\alpha d/(2\gamma)} + 1) \leqslant 1$ for $t - a^{-2\gamma/(\alpha d)} \leqslant s \leqslant t$. Therefore, by Lemma 3(iii), for some $1 \leqslant d < 2p\gamma$, we have

$$\begin{aligned}
&\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{A}^{\gamma}_{\alpha,1}(t-s) f(u)\,\mathrm{d}s \right\|_{\exp L^p(\mathbb{R}^d)} \\
&\quad \lesssim \int_0^t (t-s)^{\alpha-1-\alpha d/(2p\gamma)} \ln^{-1/p}(t-s)^{-\alpha d/(2\gamma)} + 1 \,\big\|f(u)\big\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad \lesssim \int_0^{t-a^{-2\gamma/(\alpha d)}} (t-s)^{\alpha-1} \big\|f(u)\big\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \\
&\qquad + \int_{t-a^{-2\gamma/(\alpha d)}}^{t} (t-s)^{\alpha-1-\alpha d/(2p\gamma)} \big\|f(u)\big\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad \lesssim \int_0^t (t-s)^{\alpha-1} \big\|f\big(u(s)\big)\big\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s + \sup_{s>0} \big\|f\big(u(s)\big)\big\|_{L^p(\mathbb{R}^d)} \\
&\quad =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By Taylor expansion, we have

$$\big|f(u)\big| \leqslant \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |u|^{\varsigma k+\sigma}.$$

For $r > 2\gamma p^2/d + p$ and $\sigma \geqslant 1 + 2\gamma p/d$, let

$$\theta = \frac{2\gamma p r}{d(r-p)(\varsigma k+\sigma)}, \qquad \varpi = \frac{2\gamma p^2 r(1-\theta)}{\theta(d(r-p) - 2\gamma p^2)}.$$

Clearly, for each $k \in \mathbb{N} \cup \{0\}$, we have $\theta \in (0,1]$ and $\alpha = (\varsigma k + \sigma)\theta\varrho$. Next, by virtue of Hölder interpolation inequality, we get

$$\|u\|_{L^{(\varsigma k+\sigma)p}(\mathbb{R}^d)} \lesssim \|u\|^{\theta}_{L^r(\mathbb{R}^d)} \|u\|^{1-\theta}_{L^{\varpi}(\mathbb{R}^d)},$$

where

$$\frac{1}{(\varsigma k + \sigma)p} = \frac{\theta}{r} + \frac{1-\theta}{\varpi}.$$

Similar to the proof to Step 1, it follows by Lemma 1 that

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1} \|u(s)\|^{\varsigma k+\sigma}_{L^{(\varsigma k+\sigma)p}(\mathbb{R}^d)}\, \mathrm{d}s \\
&\quad\lesssim \left(\Gamma\left(\frac{\varpi}{p}+1\right)\right)^{(\varsigma k+\sigma)(1-\theta)/\varpi} \int_0^t (t-s)^{\alpha-1} \|u(s)\|^{(\varsigma k+\sigma)\theta}_{L^r(\mathbb{R}^d)} \|u(s)\|^{(\varsigma k+\sigma)(1-\theta)}_{\exp L^p(\mathbb{R}^d)}\, \mathrm{d}s \\
&\quad\lesssim C^k \Gamma(k+1) \varepsilon^{\varsigma k+\sigma}
\end{aligned}$$

for $u \in X_\varepsilon$ and $\Gamma(\varpi/p+1) \leqslant C^k \Gamma(k+1)$ since $(\varsigma k+\sigma)(1-\theta) \leqslant \varpi$. Therefore, by $\||u|^{\varsigma k+\sigma}\|_{L^p(\mathbb{R}^d)} = \|u\|^{\varsigma k+\sigma}_{L^{(\varsigma k+\sigma)p}(\mathbb{R}^d)}$, we obtain

$$\mathrm{I} \leqslant \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{\alpha-1} \big\| |u(s)|^{\varsigma k+\sigma} \big\|_{L^p(\mathbb{R}^d)}\, \mathrm{d}s \lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k+\sigma}. \tag{4}$$

Let us prove the second term. In fact, by the assumption of f and Hölder inequality with $1/p = 1/(a_1 p) + 1/(a_2 p)$ for some constants $a_1 \geqslant 1 \vee 1/\varsigma$, $a_2 \geqslant 1$, we have

$$\big\|f(u)\big\|_{L^p(\mathbb{R}^d)} \lesssim \big\|\mathrm{e}^{\lambda|u|^\varsigma} - 1\big\|_{L^{a_1 p}(\mathbb{R}^d)} \|u\|^{\sigma}_{L^{\sigma a_2 p}(\mathbb{R}^d)} + \|u\|^{\sigma}_{L^{\sigma p}(\mathbb{R}^d)}. \tag{5}$$

For $a_1 p \geqslant 1$, $\Gamma(x+1) \leqslant C x^{x+1/2}$, we obtain by Stirling's formula that

$$\begin{aligned}
\big\|\mathrm{e}^{\lambda|u|^\varsigma} - 1\big\|_{L^{a_1 p}(\mathbb{R}^d)} &\leqslant \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \|u\|^{\varsigma k}_{L^{\varsigma k a_1 p}(\mathbb{R}^d)} \\
&\leqslant \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \big(\Gamma(\varsigma k a_1 + 1)\big)^{1/(a_1 p)} \|u\|^{\varsigma k}_{\exp L^p(\mathbb{R}^d)} \\
&\lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k}.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\sup_{t>0} \big\|f(u)\big\|_{L^p(\mathbb{R}^d)} &\lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k} \|u\|^{\sigma}_{\exp L^p(\mathbb{R}^d)} + \|u\|^{\sigma}_{\exp L^p(\mathbb{R}^d)} \\
&\lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k+\sigma} + \varepsilon^{\sigma}.
\end{aligned} \tag{6}$$

By Lemma 3(ii), we find that

$$\left\| \mathcal{A}_{\alpha,0}^{\gamma}(t)u_0 \right\|_{\exp L^p(\mathbb{R}^d)} \lesssim \|u_0\|_{\exp L^p(\mathbb{R}^d)}. \tag{7}$$

Then it follows by (4), (6), and (7) that

$$\left\| Q(u)(t) \right\|_{\exp L^p(\mathbb{R}^d)} \lesssim \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k+\sigma} + \varepsilon^{\sigma} + \|u_0\|_{\exp L^p(\mathbb{R}^d)}. \tag{8}$$

Furthermore, by using Lemmas 3(i) and 1 and letting $f(v) = 0$ for $v = 0$ as in Step 1, we get

$$\begin{aligned} \left\| Q(u)(t) \right\|_{L^r(\mathbb{R}^d)} &\leqslant \left\| \mathcal{A}_{\alpha,0}^{\gamma}(t)u_0 \right\|_{L^r(\mathbb{R}^d)} + \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{A}_{\alpha,1}^{\gamma}(t-s) f(u)\,\mathrm{d}s \right\|_{L^r(\mathbb{R}^d)} \\ &\lesssim t^{-\varrho} \|u_0\|_{\exp L^p(\mathbb{R}^d)} + t^{-\varrho} \sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k+\sigma}. \end{aligned} \tag{9}$$

Consequently, by virtue of (8) and (9), there exists a constant $C > 0$ such that

$$\left\| Q(u) \right\|_{X_\varepsilon} \leqslant C \left(\|u_0\|_{\exp L^p(\mathbb{R}^d)} + \sum_{k=0}^{\infty} \lambda^k \varepsilon^{\varsigma k+\sigma} + \varepsilon^{\sigma} \right).$$

If we take $\varepsilon = 4C\chi$ with χ small enough such that $C\varepsilon^{\sigma} < \varepsilon/4$ and $C\sum_{k=0}^{\infty} C^k \lambda^k \varepsilon^{\varsigma k} < 1$, then Q is a contraction from X_ε into itself. Thus, by the contraction mapping principle, there exists a unique solution to the Cauchy problem (1).

Step 3. Next, we prove the continuity of solution at zero. According to Lemma 3(iv), we have

$$\begin{aligned} &\left\| u(t) - \mathcal{A}_{\alpha,0}^{\gamma}(t)u_0 \right\|_{\exp L^p(\mathbb{R}^d)} \\ &\quad \lesssim \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \left\| f(u) \right\|_{L^p(\mathbb{R}^d)} \mathrm{d}s + \int_0^t \left\| f(u) \right\|_{L^p(\mathbb{R}^d)} \mathrm{d}s. \end{aligned}$$

For the estimate of $\|f(u)\|_{L^p(\mathbb{R}^d)}$ given by (5) and for any $u \in X_\varepsilon$ with small $\varepsilon > 0$, we have

$$\begin{aligned} &\left\| u(t) - \mathcal{A}_{\alpha,0}^{\gamma}(t)u_0 \right\|_{\exp L^p(\mathbb{R}^d)} \\ &\quad \lesssim \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \left\| u(s) \right\|_{\exp L^p(\mathbb{R}^d)} \mathrm{d}s + \int_0^t \left\| u(s) \right\|_{\exp L^p(\mathbb{R}^d)} \mathrm{d}s \\ &\quad \lesssim t^{\alpha(1-d/(2\gamma p))} \|u\|_{L^\infty(0,\infty;\, \exp L^p(\mathbb{R}^d))} + t\|u\|_{L^\infty(0,\infty;\, \exp L^p(\mathbb{R}^d))} \\ &\quad \to 0 \quad \text{as } t \to 0. \end{aligned}$$

Step 4. Finally we check the weak-$*$ convergence at $t = 0$. Let $X = L^1 \ln^{1/p} L(\mathbb{R}^d)$ be the predual space of $\exp L^p(\mathbb{R}^d)$ (that is, $X^* = \exp L^p(\mathbb{R}^d)$). Since X is a Banach space and $C_0^\infty(\mathbb{R}^d)$ is dense in X, it follows by the properties of $\zeta_\alpha(\cdot)$ that

$$\begin{aligned}
&\langle \mathcal{A}^\gamma_{\alpha,0}(t)u_0 - u_0, \phi\rangle_{X^*,X} \\
&\quad= \int_{\mathbb{R}^d} \big(\mathcal{A}^\gamma_{\alpha,0}(t)u_0(x) - u_0(x)\big)\phi(x)\,\mathrm{d}x \\
&\quad= \int_{\mathbb{R}^d}\int_0^\infty \zeta_\alpha(\theta)\big(\exp\big((-\Delta)^\gamma t^\alpha\theta\big)u_0(x) - u_0(x)\big)\phi(x)\,\mathrm{d}\theta\,\mathrm{d}x \\
&\quad= \int_0^\infty \zeta_\alpha(\theta)\int_{\mathbb{R}^d}\big(\exp((-\Delta)^\gamma t^\alpha\theta)u_0(x) - u_0(x)\big)\phi(x)\,\mathrm{d}x\,\mathrm{d}\theta \\
&\quad= \int_0^\infty \zeta_\alpha(\theta)\int_{\mathbb{R}^d}\big(\exp\big((-\Delta)^\gamma t^\alpha\theta\big)\phi(x) - \phi(x)\big)u_0(x)\,\mathrm{d}x\,\mathrm{d}\theta \\
&\quad= \int_{\mathbb{R}^d} \big(\mathcal{A}^\gamma_{\alpha,0}(t)\phi(x) - \phi(x)\big)u_0(x)\,\mathrm{d}x,
\end{aligned}$$

which, by the Hölder inequality for Orlicz space, implies that

$$\big|\langle \mathcal{A}^\gamma_{\alpha,0}(t)u_0 - u_0, \phi\rangle_{X^*,X}\big| \lesssim \|u_0\|_{X^*}\big\|\mathcal{A}^\gamma_{\alpha,0}(t)\phi - \phi\big\|_X.$$

By virtue of the density of $C_0^\infty(\mathbb{R}^d)$ in X, we have $\|\mathcal{A}^\gamma_{\alpha,0}(t)\phi - \phi\|_X \to 0$ as $t \to 0$. Consequently, the conclusions are achieved, and the proof is complete. □

Remark 2. Notice that the solution of Cauchy problem (1) is in $\exp L^2(\mathbb{R})$ for $\gamma \in (1/4, 1/2]$, while the global solution may not exist for $\gamma \in (0, 1/4]$. If $\gamma \to 1$ in (1), then one can establish the global existence result for $\gamma \in (0, 1)$ by the same method by replacing the operator $(-\Delta)^\gamma$ with the Laplace operator.

Remark 3. Consider the embedding $H^{s,q}(\mathbb{R}^d) \hookrightarrow \exp L^\Psi(\mathbb{R}^d)$ togethered with the Trudinger inequality (see, e.g., [10]), where $\exp L^\Psi(\mathbb{R}^d)$ is an Orlicz space defined by the convex function

$$\Psi(t) := \exp\big(t^{q/(q-1)}\big) - \sum_{j=0}^{k-1}\frac{t^{jq/(q-1)}}{j!},$$

k is the smallest integer satisfying $k \geqslant q - 1$, $H^{s,q}(\mathbb{R}^d)$ is the Sobolev spaces for any $s \in \mathbb{R}$ and $1 < q < \infty$ defined by

$$H^{s,q}\big(\mathbb{R}^d\big) = \big\{\psi \in \mathcal{S}'\big(\mathbb{R}^d\big)\colon (1 - \Delta)^{s/2} \in L^q\big(\mathbb{R}^d\big)\big\}.$$

Then the solution of the Cauchy problem (1) can be considered for initial data $u_0 \in H^{s,q}(\mathbb{R}^d)$. In particular, the growth of the nonlinearity at infinity is of the form $f(u) \sim \mathrm{e}^{u^{q/(q-1)}}$ for $u_0 \in H^{n/q,q}(\mathbb{R}^d)$.

4 The local existence

In this section, we set $X = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for each $d \geqslant 1$ and obtain the local solutions to Cauchy problem (1) for a small initial data $u_0 \in X$. We are concerned with the local existence and uniqueness of mild solution of (1). First, we give the definition of a mild solution of (1). Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, by Lemma 2, it is clear that $X \hookrightarrow \exp L_0^p(\mathbb{R}^d)$ for all $p \geqslant 1$, where $\exp L_0^p(\mathbb{R}^d)$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in $\exp L^p(\mathbb{R}^d)$ with respect to the same norm; see, for example, [14]. So, it is natural to consider the local solution in X without the Orlicz space.

Definition 2. Let $u_0 \in X$ and $T > 0$. We say that $u \in C([0,T]; X)$ is a mild solution to Cauchy problem (1) if (2) holds.

Theorem 2. *Let $u_0 \in X$ and $\sigma > 3/2$. Then the Cauchy problem* (1) *has a unique mild solution on $[0, T_*]$ for some $T_* > 0$.*

Proof. For given $T > 0$ and $R > 0$, we define a ball in Banach space $C([0,T]; X)$ by

$$B_R = \bigl\{u \in C\bigl([0,T]; X\bigr):\ \|u\|_* \leqslant R\bigr\},$$

where the norm $\|u\|_* = \|u\|_{L^\infty(0,T;X)}$. Considering the operator Q defined in Theorem 1, we shall show the existence of local solution by the fixed point argument. We first verify that $Q(B_R) \subset B_R$.

In fact, for $\sigma > 3/2$, let $\theta = 1/(\varsigma k + \sigma)$ for each $k \in \mathbb{N} \cup \{0\}$. Clearly, $\theta \in (0,1)$. The Hölder interpolation inequality implies that

$$\|u\|_{L^{\varsigma k+\sigma}(\mathbb{R}^d)} \lesssim \|u\|_{L^1(\mathbb{R}^d)}^{\theta} \|u\|_{L^\infty(\mathbb{R}^d)}^{1-\theta}.$$

For any $u \in B_R$, we obtain

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1} \bigl\|u(s)\bigr\|_{L^{\varsigma k+\sigma}(\mathbb{R}^d)}^{\varsigma k+\sigma} \,\mathrm{d}s &\lesssim \int_0^t (t-s)^{\alpha-1} \bigl\|u(s)\bigr\|_{L^1(\mathbb{R}^d)}^{(\varsigma k+\sigma)\theta} \bigl\|u(s)\bigr\|_{L^\infty(\mathbb{R}^d)}^{(\varsigma k+\sigma)(1-\theta)} \,\mathrm{d}s \\ &\lesssim T^\alpha R^{\varsigma k+\sigma},\end{aligned}$$

which yields by Lemma 3(i) that

$$\begin{aligned}\bigl\|Q(u)(t)\bigr\|_{L^1(\mathbb{R}^d)} &\lesssim \|u_0\|_{L^1(\mathbb{R}^d)} + \int_0^t (t-s)^{\alpha-1} \bigl\|f(u)(s)\bigr\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}s \\ &\lesssim \|u_0\|_X + \sum_{k=0}^\infty \int_0^t (t-s)^{\alpha-1} \bigl\||u|^{\varsigma k+\sigma}\bigr\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}s \\ &\lesssim \|u_0\|_X + \sum_{k=0}^\infty \frac{\lambda^k}{k!} \int_0^t (t-s)^{\alpha-1} \bigl\|u(s)\bigr\|_{L^{\varsigma k+\sigma}(\mathbb{R}^d)}^{\varsigma k+\sigma} \,\mathrm{d}s.\end{aligned}$$

Therefore, for any $u \in B_R$, we have

$$\|Q(u)(t)\|_{L^1(\mathbb{R}^d)} \lesssim \|u_0\|_X + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} T^\alpha R^{\varsigma k+\sigma} \lesssim \|u_0\|_X + T^\alpha R^\sigma \mathrm{e}^{\lambda R^\varsigma}.$$

In addition, for any $p \in (d/(2\gamma) \vee 1, \infty)$, we have

$$\begin{aligned}
\|Q(u)(t)\|_{L^\infty(\mathbb{R}^d)} &\lesssim \|u_0\|_{L^\infty(\mathbb{R}^d)} + \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \|f(u)(s)\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \\
&\lesssim \|u_0\|_X + \sum_{k=0}^{\infty} \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \big\||u|^{\varsigma k+\sigma}\big\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \\
&\lesssim \|u_0\|_X + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \|u(s)\|_{L^{(\varsigma k+\sigma)p}(\mathbb{R}^d)}^{\varsigma k+\sigma}\,\mathrm{d}s.
\end{aligned}$$

Moreover, letting $r \in (1, \sigma p)$, we have that $\vartheta = r/((\varsigma k+\sigma)p) \in (0,1)$ for each $k \in \mathbb{N} \cup \{0\}$. Hence the Hölder interpolation inequality yields

$$\|u\|_{L^{(\varsigma k+\sigma)p}(\mathbb{R}^d)} \lesssim \|u\|_{L^r(\mathbb{R}^d)}^{\vartheta} \|u\|_{L^\infty(\mathbb{R}^d)}^{1-\vartheta} \lesssim \|u\|_{L^1(\mathbb{R}^d)}^{\varpi\vartheta} \|u\|_{L^\infty(\mathbb{R}^d)}^{1-\varpi\vartheta}$$

for $\varpi = 1/r \in (0,1)$. This shows that

$$\begin{aligned}
&\|Q(u)(t)\|_{L^\infty(\mathbb{R}^d)} \\
&\quad\lesssim \|u_0\|_X + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)} \|u(s)\|_X^{(\varsigma k+\sigma)\varpi\vartheta} \|u(s)\|_{L^\infty(\mathbb{R}^d)}^{(\varsigma k+\sigma)(1-\varpi\vartheta)}\,\mathrm{d}s \\
&\quad\lesssim \|u_0\|_X + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} T^{\alpha-\alpha d/(2\gamma p)} R^{\varsigma k+\sigma} \quad \lesssim \|u_0\|_X + T^{\alpha-\alpha d/(2\gamma p)} R^\sigma \mathrm{e}^{\lambda R^\varsigma}.
\end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\|Q(u)(t)\|_X \leqslant C\|u_0\|_X + CT^{\alpha-\alpha d/(2\gamma p)} R^\sigma \mathrm{e}^{\lambda R^\varsigma} + CT^\alpha R^\sigma \mathrm{e}^{\lambda R^\varsigma}.$$

Consequently, letting $R = 2C\|u_0\|_X$ and choosing T small enough, we get

$$C\big(T^\alpha + T^{\alpha-\alpha d/(2\gamma p)}\big) R^{\sigma-1} \mathrm{e}^{\lambda R^\varsigma} \leqslant \frac{1}{2}. \tag{10}$$

Thus, we deduce that $\|Q(u)\|_X \leqslant R$ and hence $Q(u) \in B_R$ for any $u \in B_R$.

Next, we verify that Q is a contraction map. Let $\mu = 1/(2(\varsigma k+\sigma-1))$ for $\sigma > 3/2$, $k \in \mathbb{N} \cup \{0\}$, and observe that $\mu \in (0,1)$. Then, by Hölder interpolation inequality, we have

$$\begin{aligned}\big\||u|^{\sigma-1}\mathrm{e}^{\lambda|u|^\varsigma}\big\|_{L^2(\mathbb{R}^d)} &= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\|u\|^{\varsigma k+\sigma-1}_{L^{2(\varsigma k+\sigma-1)}(\mathbb{R}^d)} \\ &\lesssim \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\|u\|^{\mu(\varsigma k+\sigma-1)}_{L^1(\mathbb{R}^d)}\|u\|^{(1-\mu)(\varsigma k+\sigma-1)}_{L^\infty(\mathbb{R}^d)} \\ &\lesssim R^{\sigma-1}\mathrm{e}^{\lambda R^\varsigma}\end{aligned}$$

for any $u \in B_R$. Letting $u, v \in B_R$, it follows by Lemma 3(i) and the Hölder inequality that

$$\begin{aligned}\big\|f(u)(s)-f(v)(s)\big\|_{L^1(\mathbb{R}^d)} &\lesssim \big\||u-v|\big(|u|^{\sigma-1}\mathrm{e}^{\lambda|u|^\varsigma}+|v|^{\sigma-1}\mathrm{e}^{\lambda|v|^\varsigma}\big)\big\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \|u-v\|_{L^2(\mathbb{R}^d)}\big\||u|^{\sigma-1}\mathrm{e}^{\lambda|u|^\varsigma}+|v|^{\sigma-1}\mathrm{e}^{\lambda|v|^\varsigma}\big\|_{L^2(\mathbb{R}^d)} \\ &\lesssim R^{\sigma-1}\mathrm{e}^{\lambda R^\varsigma}\|u-v\|^{1/2}_{L^1(\mathbb{R}^d)}\|u-v\|^{1/2}_{L^\infty(\mathbb{R}^d)} \\ &\lesssim R^{\sigma-1}\mathrm{e}^{\lambda R^\varsigma}\|u-v\|_X.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\big\|Q(u)(t)-Q(v)(t)\big\|_{L^1(\mathbb{R}^d)} &\lesssim \int_0^t (t-s)^{\alpha-1}\big\|f(u)(s)-f(v)(s)\big\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}s \\ &\lesssim T^\alpha R^{\sigma-1}\mathrm{e}^{\lambda R^\varsigma}\|u-v\|_X.\end{aligned}$$

On the other hand, for any fixed $p \in (d/(2\gamma)\vee 1, \infty)$, we have by the previous arguments that

$$\begin{aligned}&\big\|f(u)(s)-f(v)(s)\big\|_{L^p(\mathbb{R}^d)} \\ &\quad\lesssim \big\||u-v|\big(|u|^{\sigma-1}\mathrm{e}^{\lambda|u|^\varsigma}+|v|^{\sigma-1}\mathrm{e}^{\lambda|v|^\varsigma}\big)\big\|_{L^p(\mathbb{R}^d)} \\ &\quad\lesssim \|u-v\|_{L^{2p}(\mathbb{R}^d)}\big\||u|^{\sigma-1}\mathrm{e}^{\lambda|u|^\varsigma}+|v|^{\sigma-1}\mathrm{e}^{\lambda|v|^\varsigma}\big\|_{L^{2p}(\mathbb{R}^d)} \\ &\quad\lesssim \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\|u-v\|^{1/(2p)}_{L^1(\mathbb{R}^d)}\|u-v\|^{1-1/(2p)}_{L^\infty(\mathbb{R}^d)}\big(\|u\|^{\varsigma k+\sigma-1}_{L^{2(\varsigma k+\sigma-1)p}(\mathbb{R}^d)}+\|v\|^{\varsigma k+\sigma-1}_{L^{2(\varsigma k+\sigma-1)p}(\mathbb{R}^d)}\big) \\ &\quad\lesssim \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\|u-v\|_X R^{\varsigma k+\sigma-1} \lesssim R^{\sigma-1}\mathrm{e}^{\lambda R^\varsigma}\|u-v\|_X,\end{aligned}$$

which implies that

$$\begin{aligned}\big\|Q(u)(t)-Q(v)(t)\big\|_{L^\infty(\mathbb{R}^d)} &\lesssim \int_0^t (t-s)^{\alpha-1-\alpha d/(2\gamma p)}\big\|f(u)(s)-f(v)(s)\big\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s \\ &\lesssim T^{\alpha-\alpha d/(2\gamma p)}R^{\sigma-1}\mathrm{e}^{\lambda R^\varsigma}\|u-v\|_X.\end{aligned}$$

Thus, there exists a $C > 0$ (may be the same C given in (10)) such that

$$\left\|Q(u)(t) - Q(v)(t)\right\|_X \leqslant CT^{\alpha-\alpha d/(2\gamma p)} R^{\sigma-1} \mathrm{e}^{\lambda R^{\varsigma}} \|u - v\|_X.$$

Let T be small enough such that (10) holds, then Q is a contraction on B_R. Since $(-\Delta)^\gamma$ generates a strongly continuous semigroup $T_\gamma(t)$ on $L^1(\mathbb{R}^d)$, it is easy to check the continuity of Q. Hence, according to the Banach fixed point theorem, the Cauchy problem (1) admits a unique local mild solution $u \in B_R$. This finishes the proof. □

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