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On a (k,χ)-Hilfer fractional system with coupled nonlocal boundary conditions including various fractional derivatives and Riemann–Stieltjes integrals

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Abstract. In the present research, we investigate the existence and uniqueness of solutions for a system of (k,χ)-Hilfer fractional differential equations, subject to coupled nonlocal boundary conditions, which contain various fractional derivatives and Riemann–Stieltjes integrals. The uniqueness result relies on the Banach contraction mapping principle, while the existence results depend on the Leray–Schauder alternative and Krasnosel'skiĭ fixed point theorem. Examples are also constructed to illustrate the obtained results.

Keywords: systems of Hilfer fractional differential equations, fractional integrals, coupled nonlocal boundary conditions, existence of solutions, Riemann–Stieltjes integrals.

1 Introduction

Fractional differential equations (FDE) are applied to investigate the generalization of integer-order differential equations. They have been appeared in disparate scientific fields, such as chemistry, physics, biology, control theory etc., as they have a greater degree of freedom and are rather precise than the integer order. For the basic theory of fractional calculus and fractional differential equations, we refer to the monographs by Diethelm [14], Kilbas et al. [19], Miller and Ross [25], Podlubny [26] and Ahmed et al. [3]. Boundary value problems for fractional differential equations with different kinds of boundary conditions have been studied by many researchers. For example, fractional boundary value problems with integral and multipoint boundary conditions were studied in [30], with antiperiodic boundary conditions in [2], with integral boundary conditions with sequential Riemman–Liouville and Caputo fractional derivatives in [18], with multipoint strip boundary conditions in [6], for sequential hybrid ψ-Hilfer-type fractional differential equations in [9], for nonlinear impulsive (ρ_k, ϕ_k)-Hilfer fractional integro-differential equations with nonlocal multipoint fractional integral boundary conditions in [17] and separated boundary conditions in [29]. For a variety of results on boundary value problems for fractional differential equations and inclusions, we refer to the recent monograph [4].

In the literature, there are several kinds of fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, Hilfer, Katugampola etc. One of them, Hilfer fractional derivative [16], generalizes both Riemann–Liouville and Caputo fractional derivatives and especially interpolate between them. For applications of Hilfer fractional differential equations in many fields of mathematics, physics etc., see [28] and references cited therein.

Coupled systems of FDEs with different kinds of boundary conditions is one of the subjects in applied mathematics, which have been the interest of many researchers; see, for instance, [5, 10] and references therein. Hilfer coupled systems two-point boundary conditions were studied in [7], involving multipoint nonlocal boundary conditions in [13], involving Riemann–Stieltjes integral multistrip boundary conditions in [27], with nonlocal integral boundary conditions in [11], with nonlocal integro-multistrip-multipoint boundary conditions in [1] and coupled systems of nonlinear ψ-Hilfer hybrid fractional differential equations in [24] and [8]. Luca [23] studied a system of Riemann–Liouville fractional differential equations supplemented with coupled boundary conditions, which involve diverse fractional derivatives and Riemann–Stieltjes integrals.

In this paper, inspired by the above problems on Hilfer fractional coupled systems, a new coupled system of (k, χ)-Hilfer FDEs

$$\begin{aligned} {}^{k,H}\mathcal{D}^{\breve{\alpha}_1,\breve{\beta}_1;\chi} l(w) &= \Pi\big(w, l(w), k(w)\big), \quad w \in [0,1], \\ {}^{k,H}\mathcal{D}^{\breve{\alpha}_2,\breve{\beta}_2;\chi} k(w) &= \bar{\Pi}\big(w, l(w), k(w)\big), \quad w \in [0,1], \end{aligned} \tag{1}$$

supplemented with coupled nonlocal boundary conditions

$$\begin{aligned} l(0) &= 0, \qquad {}^{k,H}\mathcal{D}^{a_1,b_1,\chi} l(1) = \sum_{i=1}^{m} \int_0^1 {}^{k,H}\mathcal{D}^{\check{r}_i,\check{s}_i,\chi} k(s)\, \mathrm{d}H_i(s), \\ k(0) &= 0, \qquad {}^{k,H}\mathcal{D}^{a_2,b_2,\chi} k(1) = \sum_{i=1}^{n} \int_0^1 {}^{k,H}\mathcal{D}^{\check{u}_i,\check{v}_i,\chi} l(s)\, \mathrm{d}K_i(s) \end{aligned} \tag{2}$$

is introduced, where ${}^{k,H}\mathcal{D}^{\eta,\delta;\chi}$ denotes the (k, χ)-Hilfer derivative of order η, δ is a Hilfer parameter, and a constant $k > 0$, where constants $\eta \in \{\breve{\alpha}_1, \breve{\alpha}_2, a_1, a_2, \check{r}_i, \check{u}_i\}$ with $\breve{\alpha}_1, \breve{\alpha}_2 \in (1, 2]$ when $a_1, \check{u}_i < \breve{\alpha}_1$ and $a_2, \check{r}_i < \breve{\alpha}_2$ and $\delta \in \{\breve{\beta}_1, \breve{\beta}_2, b_1, b_2, \check{s}_i, \check{v}_i\}$,

$0 < \delta < 1$, two nonlinear functions $\mathit{\Pi}, \bar{\mathit{\Pi}} : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions, and the existent integrals in the boundary conditions represent Riemann–Stieltjes integral with H_i for $i = 1, 2, \dots, m$ and K_i for $i = 1, 2, \dots, n$ functions of bounded variations and the differentiable function χ with $\chi'(w) > 0$ for all $w \in [0,1]$.

The aim of the present work is to extend the existence theory for a class of coupled BVPs consisting (k, χ)-Hilfer FDEs of various orders. The presented study is substantial and advantageous due to the wider realm of the (k, χ)-Hilfer fractional operators. We will use standard tools of the fixed point theory to establish our results.

Here we emphasize that the novelty of system (1)–(2) lies in the fact that we study a (k, χ)-Hilfer fractional system with coupled nonlocal boundary conditions including various different-order fractional derivatives combined with Riemann–Stieltjes integrals. As far as we know, this is new in the literature. Our results are new and contribute significantly to the existing results on coupled systems for Hilfer fractional derivative. The main results not only enriches the existing literature but also specializes to several interesting cases by fixing the parameters involved in the problem at hand. The used method is standard, but its configuration to system (1)–(2) is new.

The structure of the remainder work will be arranged as follows. Some basic concepts are recalled in Section 2. An ancillary result is also presented, which will be applied to convert the mentioned problem into a fixed point problem. The main results, based on some classical fixed point theorems, are given in Section 3. Section 4 consists the numerical examples illustrating the obtained results.

2 Preliminaries

First, some definitions and lemmas related to this work are recalled.

Definition 1. (See [22].) Let $L \in L^1([0,1], \mathbb{R})$, $k > 0$ and χ is an increasing function with $\chi'(s) \neq 0$ for all $s \in [0,1]$. Then the (k, χ)-Riemann–Liouville fractional integral (or the k-Riemann–Liouville fractional integral with respect to a function χ) of order $\breve{\alpha} > 0$ ($\breve{\alpha} \in \mathbb{R}$) of the function L is given by

$$ {}^{k}\mathcal{I}^{\breve{\alpha};\chi} \mathit{\Pi}(w) = \frac{1}{k\Gamma_k(\breve{\alpha})} \int_0^w \chi'(s) \big(\chi(w) - \chi(s)\big)^{\breve{\alpha}/k - 1} \mathit{\Pi}(s) \, \mathrm{d}s. $$

Definition 2. (See [21].) Let $a, k \in \mathbb{R}^+$, $b \in [0,1]$, χ is an increasing function such that $\chi \in C^n([0,1], \mathbb{R})$, $\chi'(s) \neq 0, s \in [0,1]$, and $\mathit{\Pi} \in C^n([0,1], \mathbb{R})$. Then the (k, χ)-Hilfer fractional derivative of $\mathit{\Pi}$ of order a and type b is defined by

$$ {}^{k,H}\mathcal{D}^{a,b;\chi} \mathit{\Pi}(s) = {}^{k}\mathcal{I}^{b(nk-a);\chi} \left(\frac{k}{\chi'(s)} \frac{\mathrm{d}}{\mathrm{d}s} \right)^n {}^{k}\mathcal{I}^{(1-b)(nk-a);\chi} \mathit{\Pi}(s), \quad n = \left\lceil \frac{a}{k} \right\rceil. $$

Lemma 1. (*See* [21].) *Let $\breve{\mu}$, $k \in \mathbb{R}^+$ and $n = \lceil \breve{\mu}/k \rceil$. Assume that $L \in C^n([0,1],\mathbb{R})$ and ${}^k\mathcal{I}^{nk-\breve{\mu};\chi} L \in C^n([0,1],\mathbb{R})$. Then*

$$\begin{aligned} &{}^k\mathcal{I}^{\breve{\mu};\chi}\big({}^{k,RL}\mathcal{D}^{\breve{\mu};\chi}\Pi(w)\big) \\ &\quad = \Pi(w) - \sum_{j=1}^{n} \frac{(\chi(w)-\chi(0))^{\breve{\mu}/k-j}}{\Gamma_k(\breve{\mu}-jk+k)} \left[\left(\frac{k}{\chi'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^{n-j} {}^k\mathcal{I}^{nk-\breve{\mu};\chi}\Pi(w)\right]_{w=0}. \end{aligned}$$

Lemma 2. (*See* [21].) *Let $a, k \in \mathbb{R}^+$ with $a < k$, $b \in [0,1]$ and $\theta_k = a + b(k-a)$. Then*

$${}^k\mathcal{I}^{\theta_k;\chi}\big({}^{k,RL}\mathcal{D}^{\theta_k;\chi}\Pi\big)(w) = {}^k\mathcal{I}^{a;\chi}\big({}^{k,H}\mathcal{D}^{a,b;\chi}\Pi\big)(w), \quad \Pi \in C^n([0,1],\ \mathbb{R}),$$

where ${}^{k,RL}\mathcal{D}^{\theta_k;\chi}$ is the k-Riemann–Liouville fractional derivative with respect to a function χ of order $\theta_k > 0$ defined by

$${}^{k,RL}\mathcal{D}^{\theta_k;\chi}\Pi(w) = \left(\frac{k}{\chi'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^{n} {}^k\mathcal{I}^{(nk-b_k);\chi}\Pi(s), \quad n = \left\lceil \frac{\theta_k}{k} \right\rceil.$$

Lemma 3. (*See* [21].) *Let $\breve{\zeta}$, $k \in \mathbb{R}^+$ and $\breve{\eta} \in \mathbb{R}$ such that $\breve{\eta}/k > -1$. Then*

$$\begin{aligned} {}^k\mathcal{I}^{\breve{\zeta},\chi}\big(\chi(s)-\chi(0)\big)^{\breve{\eta}/k} &= \frac{\Gamma_k(\breve{\eta}+k)}{\Gamma_k(\breve{\eta}+k+\breve{\zeta})}\big(\chi(s)-\chi(0)\big)^{(\breve{\eta}+\breve{\zeta})/k}, \\ {}^k\mathcal{D}^{\breve{\zeta},\chi}\big(\chi(s)-\chi(0)\big)^{\breve{\eta}/k} &= \frac{\Gamma_k(\breve{\eta}+k)}{\Gamma_k(\breve{\eta}+k-\breve{\zeta})}\big(\chi(s)-\chi(0)\big)^{(\breve{\eta}-\breve{\zeta})/k}. \end{aligned}$$

Lemma 4. (*See* [19].) *Let $\breve{\alpha}_1$, $\breve{\alpha}_2$, $\breve{\beta}$, $k \in (0,\infty)$ with $\breve{\alpha}_2 > \breve{\alpha}_1$ and $\breve{\beta} \in [0,1]$. Then*

$${}^{k,H}\mathcal{D}^{\breve{\alpha}_1,\beta;\varphi}\big({}^k\mathcal{I}_{0^+}^{\breve{\alpha}_2;\varphi}\big)\Pi(w) = {}^k\mathcal{I}_{0^+}^{\breve{\alpha}_2-\breve{\alpha}_1;\varphi}\Pi(w), \quad \Pi \in C\big([0,1],\mathbb{R}\big).$$

Remark 1. If $\theta_k = a + b(nk-a)$, then $b(nk-a) = \theta_k - a$ and $(1-b)(nk-a) = nk - \theta_k$, and hence, the (k,χ)-Hilfer fractional derivative has been defined in the form of (k,χ)-Riemann–Liouville fractional derivative as follows:

$$\begin{aligned} {}^{k,H}\mathcal{D}^{a,b;\chi}\Pi(w) &= {}^k\mathcal{I}_{l_1+}^{\theta_k-a;\chi}\left(\frac{k}{\chi'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^{n} {}^k\mathfrak{I}_{l_1+}^{nk-\theta_k;\chi}\Pi(w) \\ &= {}^k\mathfrak{I}_{l_1+}^{\theta_k-a;\chi}\big({}^{k,RL}\mathfrak{D}^{\theta_k;\chi}\Pi\big)(w). \end{aligned}$$

Note that for $b \in [0,1]$ and $n-1 < a/k \leqslant n$, we have $n-1 < \theta_k/k \leqslant n$.

3 An ancillary result

Lemma 5. *Let $k > 0$, $1 < \breve{\alpha}_1, \breve{\alpha}_2, a_1, a_2, \check{r}_i, \breve{u}_i \leqslant 2$, $\breve{\beta}_1, \breve{\beta}_2, b_1, b_2, \check{s}_i, \breve{v}_i \in [0,1]$, $\breve{\gamma}_1 = \breve{\alpha}_1 + \breve{\beta}_1(2k-\breve{\alpha}_1)$, $\breve{\gamma}_2 = \breve{\alpha}_2 + \breve{\beta}_2(2k-\breve{\alpha}_2)$, $\breve{\gamma}_3 = a_1 + b_1(2k-a_1)$, $\breve{\gamma}_4 = a_2 + b_2(2k-a_2)$, $\breve{\eta}_i = \check{r}_i + \check{s}_i(2k-\check{r}_i)$, $\breve{\delta}_i = \breve{u}_i + \breve{v}_i(2k-\breve{u}_i)$, $\Pi_1, \bar{\Pi}_1 \in C^2([0,1],\mathbb{R})$ and $\mathsf{A} \neq 0$. Then* *the unique solution of system*

$$\begin{aligned} &{}^{k,H}\mathcal{D}^{\breve{\alpha}_1,\breve{\beta}_1,\chi} l(w) = \varPi_1(w), \quad {}^{k,H}\mathcal{D}^{\breve{\alpha}_2,\breve{\beta}_2,\chi} k(w) = \bar{\varPi}_1(w), \quad w \in [0,1], \\ &l(0) = 0, \qquad {}^{k,H}\mathcal{D}^{a_1,b_1,\chi} l(1) = \sum_{i=1}^{m} \int_0^1 {}^{k,H}\mathcal{D}^{\breve{r}_i,\breve{s}_i,\chi} k(s)\,\mathrm{d}H_i(s), \\ &k(0) = 0, \qquad {}^{k,H}\mathcal{D}^{a_2,b_2,\chi} k(1) = \sum_{i=1}^{n} \int_0^1 {}^{k,H}\mathcal{D}^{\breve{u}_i,\breve{v}_i,\chi} l(s)\,\mathrm{d}K_i(s) \end{aligned} \tag{3}$$

is given by

$$\begin{aligned} l(w) = {}&{}^{k}I^{\breve{\alpha}_1;\chi}\varPi_1(w) + \frac{(\chi(w)-\chi(0))^{\breve{\gamma}_1/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_1)} \\ &\times \Biggl[\mathsf{A}_2\Biggl(\sum_{i=1}^{n}\int_0^1 {}^{k}I^{\breve{\alpha}_1-\breve{u}_i,\chi}\varPi_1(s)\,\mathrm{d}K_i(s) - {}^{k}\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\varPi}_1(1)\Biggr) \\ &+ \mathsf{A}_4\Biggl(\sum_{i=1}^{m}\int_0^1 {}^{k}I^{\alpha_2-\breve{r}_i,\chi}\bar{\varPi}_1(s)\,\mathrm{d}H_i(s) - {}^{k}\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\varPi_1(1)\Biggr)\Biggr] \end{aligned} \tag{4}$$

and

$$\begin{aligned} k(w) = {}&{}^{k}I^{\breve{\alpha}_2;\chi}\bar{\varPi}_1(w) + \frac{(\chi(t)-\chi(0))^{\breve{\gamma}_2/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_2)} \\ &\times \Biggl[\mathsf{A}_1\Biggl(\sum_{i=1}^{n}\int_0^1 {}^{k}I^{\breve{\alpha}_1-\breve{u}_i,\chi}\varPi_1(s)\,\mathrm{d}K_i(s) - {}^{k}\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\varPi}_1(1)\Biggr) \\ &+ \mathsf{A}_3\Biggl(\sum_{i=1}^{m}\int_0^1 {}^{k}I^{\alpha_2-\breve{r}_i,\chi}\bar{\varPi}_1(s)\,\mathrm{d}H_i(s) - {}^{k}\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\varPi_1(1)\Biggr)\Biggr], \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathsf{A}_1 &= \frac{(\chi(1)-\chi(0))^{(\breve{\gamma}_1-a_1)/k-1}}{\varGamma_k(\breve{\gamma}_1-a_1)}, \\ \mathsf{A}_2 &= \sum_{i=1}^{n}\frac{1}{\varGamma_k(\breve{\gamma}_2-\breve{r}_i)}\int_0^1 \bigl(\chi(s)-\chi(0)\bigr)^{(\breve{\gamma}_2-\breve{r}_i)/k-1}\,\mathrm{d}H_i(s), \\ \mathsf{A}_3 &= \sum_{i=1}^{m}\frac{1}{\varGamma_k(\breve{\gamma}_1-\breve{u}_i)}\int_0^1 \bigl(\chi(s)-\chi(0)\bigr)^{(\breve{\gamma}_1-\breve{u}_i)/k-1}\,\mathrm{d}K_i(s), \\ \mathsf{A}_4 &= \frac{(\chi(1)-\chi(0))^{(\breve{\gamma}_2-a_2)/k-1}}{\varGamma_k(\breve{\gamma}_2-a_2)} \end{aligned} \tag{6}$$

with

$$\mathsf{A} = \mathsf{A}_1\mathsf{A}_4 - \mathsf{A}_2\mathsf{A}_3.$$

Proof. Let (l,k) be a solution of (3). Taking the operators ${}^k\mathcal{I}^{\breve{\alpha}_1,\chi}$ and ${}^k\mathcal{I}^{\breve{\alpha}_2,\chi}$ on both sides of (3) and applying Lemmas 1 and 2, we get

$$l(w) = {}^kI^{\breve{\alpha}_1;\chi}\Pi_1(w) + c_0\frac{(\chi(w)-\chi(0))^{\breve{\gamma}_1/k-1}}{\Gamma_k(\breve{\gamma}_1)} + c_1\frac{(\chi(w)-\varphi(0))^{\breve{\gamma}_1/k-2}}{\Gamma_k(\breve{\gamma}_1-k)} \tag{7}$$

and

$$k(w) = {}^kI^{\breve{\alpha}_2;\chi}\bar{\Pi}_1(w) + d_0\frac{(\chi(w)-\chi(0))^{\breve{\gamma}_2/k-1}}{\Gamma_k(\breve{\gamma}_2)} + d_1\frac{(\chi(w)-\chi(0))^{\breve{\gamma}_2/k-2}}{\Gamma_k(\breve{\gamma}_2-k)}, \tag{8}$$

where

$$d_0 = \left[\left(\frac{k}{\chi'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right){}^kI^{2k-\breve{\gamma}_2:\chi}k(w)\right]_{w=0}, \qquad d_1 = \left[{}^kI^{2k-\breve{\gamma}_2:\chi}k(w)\right]_{w=0}.$$

Due to the conditions $l(0)=0$ and $k(0)=0$ with (7) and (8), we get $c_1=0$ and $d_1=0$ since by Remark 1 $\breve{\gamma}_1/k-2<0$ and $\breve{\gamma}_2/k-2<0$. Now, applying the conditions

$${}^{k,H}\mathcal{D}^{a_1,b_1,\chi}l(1) = \sum_{i=1}^{m}\int_0^1 {}^{k,H}\mathcal{D}^{\breve{r}_i,\breve{s}_i,\chi}k(s)\,\mathrm{d}H_i(s)$$

and

$${}^{k,H}\mathcal{D}^{a_2,b_2,\chi}k(1) = \sum_{i=1}^{n}\int_0^1 {}^{k,H}\mathcal{D}^{\breve{u}_i,\breve{v}_i,\chi}l(s)\,\mathrm{d}K_i(s)$$

in (7) and (8) and using Lemmas 3 and 4, after inserting $c_1=0$ and $d_1=0$, we get

$$\begin{aligned}
&{}^k\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\Pi_1(1) + c_0\frac{(\chi(1)-\chi(0))^{(\breve{\gamma}_1-a_1)/k-1}}{\Gamma_k(\breve{\gamma}_1-a_1)} \\
&\quad = \sum_{i=1}^{m}\int_0^1 \left({}^kI^{\breve{\alpha}_2-\breve{r}_i,\chi}\bar{\Pi}_1(s)\right)\mathrm{d}H_i(s) \\
&\qquad + d_0\sum_{i=1}^{m}\frac{1}{\Gamma_k(\breve{\gamma}_2-\breve{r}_i)}\int_0^1 \left(\chi(s)-\chi(0)\right)^{(\breve{\gamma}_2-\breve{r}_i)/k-1}\mathrm{d}H_i(s)
\end{aligned} \tag{9}$$

and

$$
\begin{aligned}
&{}^{k}\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\Pi}_1(1)+d_0\frac{(\chi(1)-\chi(0))^{(\breve{\gamma}_2-a_2)/k-1}}{\Gamma_k(\breve{\gamma}_2-a_2)}\\
&\quad=\sum_{i=1}^{n}\int_0^1\left({}^{k}I^{\breve{\alpha}_1-\breve{u}_i,\chi}\Pi_1(s)\right)\mathrm{d}K_i(s)\\
&\qquad+c_0\sum_{i=1}^{n}\frac{1}{\Gamma_k(\breve{\gamma}_1-\breve{u}_i)}\int_0^1\left(\chi(s)-\chi(0)\right)^{(\breve{\gamma}_1-\breve{u}_i)/k-1}\mathrm{d}K_i(s). \qquad (10)
\end{aligned}
$$

In view of (6), (9) and (10), we conclude that

$$
\begin{aligned}
\mathsf{A}_1c_0-\mathsf{A}_2d_0&=\sum_{i=1}^{m}\int_0^1{}^{k}I^{\alpha_2-\check{r}_i,\chi}\bar{\Pi}_1(s)\,\mathrm{d}H_i(s)-{}^{k}\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\Pi_1(1),\\
-\mathsf{A}_3c_0+\mathsf{A}_4d_0&=\sum_{i=1}^{n}\int_0^1{}^{k}I^{\breve{\alpha}_1-\breve{u}_i,\chi}\Pi_1(s)\,\mathrm{d}K_i(s)-{}^{k}\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\Pi}_1(1).
\end{aligned} \qquad (11)
$$

By solving system (11) we conclude that

$$
\begin{aligned}
c_0=\frac{1}{\mathsf{A}}\Bigg[&\mathsf{A}_2\left(\sum_{i=1}^{n}\int_0^1{}^{k}I^{\breve{\alpha}_1-\breve{u}_i,\chi}\Pi_1(s)\,\mathrm{d}K_i(s)-{}^{k}\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\Pi}_1(1)\right)\\
&+\mathsf{A}_4\left(\sum_{i=1}^{m}\int_0^1{}^{k}I^{\alpha_2-\check{r}_i,\chi}\bar{\Pi}_1(s)\,\mathrm{d}H_i(s)-{}^{k}\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\Pi_1(1)\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
d_0=\frac{1}{\mathsf{A}}\Bigg[&\mathsf{A}_1\left(\sum_{i=1}^{n}\int_0^1{}^{k}I^{\breve{\alpha}_1-\breve{u}_i,\chi}\Pi_1(s)\,\mathrm{d}K_i(s)-{}^{k}\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\Pi}_1(1)\right)\\
&+\mathsf{A}_3\left(\sum_{i=1}^{m}\int_0^1{}^{k}I^{\alpha_2-\check{r}_i,\chi}\bar{\Pi}_1(s)\,\mathrm{d}H_i(s)-{}^{k}\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\Pi_1(1)\right)\Bigg].
\end{aligned}
$$

Replacing c_0, d_0, c_1, d_1 in (7) and (8), we get the solutions (4) and (5). It is easy to prove the converse by direct computation. The proof is completed. □

4 Existence and uniqueness results

Let $\mathbb{D}=C([0,1],\mathbb{R})=\{l:[0,1]\to\mathbb{R}\text{ is continuous}\}$. The space $\mathbb{D}$ is a Banach space with the norm $\|l\|=\max\{|l(w)|,\ w\in[0,1]\}$. Obviously, the product space $(\mathbb{D}\times\mathbb{D}, \|(l,k)\|)$ is also a Banach space with the norm $\|(l,k)\|=\|l\|+\|k\|$.

In view of Lemma 5, we define an operator $\mathcal{T}:\mathbb{D}\times\mathbb{D}\to\mathbb{D}\times\mathbb{D}$ by

$$\mathcal{T}(l,k)(w)=\begin{pmatrix}\mathcal{T}_1(l,k)(w)\\ \mathcal{T}_2(l,k)(w)\end{pmatrix}, \tag{12}$$

where for $w\in[0,1]$,

$$\begin{aligned}\mathcal{T}_1(l,k)(w) &= {}^kI^{\breve{\alpha}_1;\chi}\varPi_{l,k}(w)+\frac{(\chi(w)-\chi(0))^{\breve{\gamma}_1/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_1)}\\ &\quad\times\Biggl[\mathsf{A}_2\Biggl(\sum_{i=1}^{n}\int_0^1 {}^kI^{\breve{\alpha}_1-\breve{u}_i,\chi}\varPi_{l,k}(s)\,\mathrm{d}K_i(s)-{}^k\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\varPi}_{l,k}(1)\Biggr)\\ &\quad+\mathsf{A}_4\Biggl(\sum_{i=1}^{m}\int_0^1 {}^kI^{\alpha_2-\breve{r}_i,\chi}\bar{\varPi}_{l,k}(s)\,\mathrm{d}H_i(s)-{}^k\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\varPi_{l,k}(1)\Biggr)\Biggr]\end{aligned} \tag{13}$$

and

$$\begin{aligned}\mathcal{T}_2(l,k)(w) &= {}^kI^{\breve{\alpha}_2;\chi}\bar{\varPi}_{l,k}(w)+\frac{(\chi(w)-\chi(0))^{\breve{\gamma}_2/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_2)}\\ &\quad\times\Biggl[\mathsf{A}_1\Biggl(\sum_{i=1}^{n}\int_0^1 {}^kI^{\breve{\alpha}_1-\breve{u}_i,\chi}\varPi_{l,k}(s)\,\mathrm{d}K_i(s)-{}^k\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\varPi}_{l,k}(1)\Biggr)\\ &\quad+\mathsf{A}_3\Biggl(\sum_{i=1}^{m}\int_0^1 {}^kI^{\alpha_2-\breve{r}_i,\chi}\bar{\varPi}_{l,k}(s)\,\mathrm{d}H_i(s)-{}^k\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\varPi_{l,k}(1)\Biggr)\Biggr],\end{aligned} \tag{14}$$

where

$$\varPi_{l,k}(w)=\varPi\bigl(w,l(w),k(w)\bigr)\quad\text{and}\quad\bar{\varPi}_{l,k}(w)=\bar{\varPi}\bigl(w,l(w),k(w)\bigr).$$

For convenience, we set

$$\begin{aligned}\chi_1 &= \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\varGamma_k(\breve{\alpha}_1+k)}+\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_1)}\\ &\quad\times\Biggl(\mathsf{A}_2\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\varGamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s)+\mathsf{A}_4\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\varGamma_k(\breve{\alpha}_1-a_1+k)}\Biggr),\end{aligned} \tag{15$_1$}$$

$$\begin{aligned}\chi_2 &= \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_1)}\\ &\quad\times\Biggl(\mathsf{A}_2\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\varGamma_k(\breve{\alpha}_2-a_2+k)}+\mathsf{A}_4\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\varGamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr),\end{aligned} \tag{15$_2$}$$

$$\chi_3 = \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_2/k}}{\Gamma_k(\breve{\alpha}_2+k)} + \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_2/k-1}}{\mathsf{A}\Gamma_k(\breve{\gamma}_2)} \times \left(\mathsf{A}_1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2-a_2+k)} + \mathsf{A}_3\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\right), \quad (15_3)$$

$$\chi_1^* = \chi_1 - \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\Gamma_k(\breve{\alpha}_1+k)}, \qquad \chi_3^* = \chi_3 - \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_2/k}}{\Gamma_k(\breve{\alpha}_2+k)}, \quad (15_4)$$

$$\chi_4 = \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_2/k-1}}{\mathsf{A}\Gamma_k(\breve{\gamma}_2)} \times \left(\mathsf{A}_1\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s) + \mathsf{A}_3\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\right). \quad (15_5)$$

First, the existence of a unique solution to system (1) is proved via Banach's contraction mapping principle [12].

Theorem 1. *Suppose that*

(H1) *For all* $w \in [0,1]$ *and* $l_i, k_i \in \mathbb{R}$, $i = 1,2$, *we have*

$$\left|\Pi(w,l_1,l_2) - \Pi(w,k_1,k_2)\right| \leqslant p_1|l_1-k_1| + p_2|l_2-k_2|,$$
$$\left|\bar{\Pi}(w,l_1,l_2) - \bar{\Pi}(t,k_1,k_2)\right| \leqslant q_1|l_1-k_1| + q_2|l_2-k_2|$$

in which p_i, q_i, $i = 1,2$, *are real constants.*

Then system (1) *has a unique solution on* $[0,1]$, *provided that*

$$(\chi_1+\chi_4)(p_1+p_2) + (\chi_2+\chi_3)(q_1+q_2) < 1, \quad (16)$$

where χ_i, $i = 1,2,3,4$, *are given in* (15).

Proof. The assumptions of Banach's contraction mapping principle will be considered in the following steps:

(i) $\mathcal{T}(\mathbb{B}_x) \subseteq \mathbb{B}_x$ with $\mathcal{T}$ defined by (12) and $\mathbb{B}_x = \{(l,k) \in \mathbb{D}\times\mathbb{D}\colon \|(l,k)\| \leqslant x\}$ with
$$x \geqslant \frac{(\chi_1+\chi_4)\mathcal{M}_1 + (\chi_2+\chi_3)\mathcal{M}_2}{1 - [(\chi_1+\chi_4)(p_1+p_2) + (\chi_2+\chi_3)(q_1+q_2)]},$$
$\mathcal{M}_1 = \sup_{w\in[l_1,l_2]} \Pi(w,0,0) < \infty$, $\mathcal{M}_2 = \sup_{w\in[l_1,l_2]} \bar{\Pi}(w,0,0) < \infty$.

(ii) $\mathcal{T}$ is a contraction.

To consider (i), let $(l,k) \in \mathbb{B}_x$ and $w \in [0,1]$. Then we have

$$\left|\mathcal{T}_1(l,k)(w)\right| \leqslant {}^k\mathcal{I}^{\breve{\alpha}_1;\chi}\left(\left|\Pi_{l,k}(w) - \Pi_{0,0}(w)\right| + \left|\Pi_{0,0}(w)\right|\right) + \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)}$$

$$
\begin{aligned}
&\times\Biggl[|\mathsf{A}_2|\Biggl(\sum_{i=1}^{n}\int_0^1 \mathcal{I}^{\breve{\alpha}_1-\breve{u}_i;\chi}\bigl(\bigl|\varPi_{l,k}(s)-\varPi_{0,0}(s)\bigr|+\bigl|\varPi_{0,0}(s)\bigr|\bigr)\,\mathrm{d}K_i(s)\\
&+{}^k\mathcal{I}^{\breve{\alpha}_2-a_2;\chi}\bigl(\bigl|\bar{\varPi}_{l,k}(1)-\bar{\varPi}_{0,0}(1)\bigr|+\bigl|\bar{\varPi}_{0,0}(1)\bigr|\bigr)\Biggr)\\
&+|\mathsf{A}_4|\Biggl(\sum_{i=1}^{m}\int_0^1 \mathcal{I}^{\breve{\alpha}_2-\breve{r}_i;\chi}\bigl(\bigl|\bar{\varPi}_{l,k}(s)-\bar{\varPi}_{0,0}(s)\bigr|+\bigl|\bar{\varPi}_{0,0}(s)\bigr|\bigr)\,\mathrm{d}H_i(s)\\
&+{}^k\mathcal{I}^{\breve{\alpha}_1-a_1;\chi}\bigl(\bigl|\varPi_{l,k}(s)-\varPi_{0,0}(s)\bigr|+\bigl|\varPi_{0,0}(s)\bigr|\bigr)\Biggr)\Biggr]\\
\leqslant{}&\frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\varGamma_k(\breve{\alpha}_1+k)}\bigl(p_1\|l\|+p_2\|k\|+\mathcal{M}_1\bigr)+\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\varGamma_k(\breve{\gamma}_1)}\\
&\times\Biggl[|\mathsf{A}_2|\Biggl(\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\varGamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\bigl(p_1\|l\|+p_2\|k\|+\mathcal{M}_1\bigr)\,\mathrm{d}K_i(s)\\
&+\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\varGamma_k(\breve{\alpha}_2-a_2+k)}\bigl(q_1\|l\|+q_2\|k\|+\mathcal{M}_2\bigr)\Biggr)\\
&+|\mathsf{A}_4|\Biggl(\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\varGamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\bigl(q_1\|l\|+q_2\|k\|+\mathcal{M}_2\bigr)\,\mathrm{d}H_i(s)\\
&+\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\varGamma_k(\breve{\alpha}_1-a_1+k)}\bigl(p_1\|l\|+p_2\|k\|+\mathcal{M}_1\bigr)\Biggr)\Biggr]\\
\leqslant{}&\bigl(p_1\|l\|+p_2\|k\|+\mathcal{M}_1\bigr)\Biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\varGamma_k(\breve{\alpha}_1+k)}+\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\varGamma_k(\breve{\gamma}_1)}\\
&\times\Biggl[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\varGamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s)+|\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\varGamma_k(\breve{\alpha}_1-a_1+k)}\Biggr]\Biggr\}\\
&+\bigl(q_1\|l\|+q_2\|k\|+\mathcal{M}_2\bigr)\Biggl[\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\varGamma_k(\breve{\gamma}_1)}\\
&\times\Biggl(|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\varGamma_k(\breve{\alpha}_2+a_2+k)}+|\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\varGamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr)\Biggr]\\
={}&\bigl(p_1\|l\|+p_2\|k\|+\mathcal{M}_1\bigr)\chi_1+\bigl(q_1\|l\|+q_2\|k\|+\mathcal{M}_2\bigr)\chi_2\\
={}&(p_1\chi_1+q_1\chi_2)\|l\|+(p_2\chi_1+q_2\chi_2)\|k\|+\chi_1\mathcal{M}_1+\chi_2\mathcal{M}_2\\
\leqslant{}&(p_1\chi_1+q_1\chi_2+p_2\chi_1+q_2\chi_2)r+\chi_1\mathcal{M}_1+\chi_2\mathcal{M}_2.
\end{aligned}
$$

Similarly, one can get that

$$\left|\mathcal{T}_2(l,k)(t)\right| \leqslant (p_1\chi_4 + q_1\chi_3 + p_2\chi_4 + q_2\chi_3)x + \chi_4\mathcal{M}_1 + \chi_3\mathcal{M}_2.$$

Consequently, adding the preceding inequalities, we get

$$\begin{aligned}\left\|\mathcal{T}(l,k)\right\| &= \left\|\mathcal{T}_1(l,k)\right\| + \left\|\mathcal{T}_2(l,k)\right\| \\ &\leqslant \left[(\chi_1+\chi_4)(p_1+p_2) + (\chi_2+\chi_3)(q_1+q_2)\right]x \\ &\quad + (\chi_1+\chi_4)\mathcal{M}_1 + (\chi_2+\chi_3)\mathcal{M}_2 \\ &\leqslant r.\end{aligned}$$

Hence, we get $\mathcal{T}(\mathbb{B}_x) \subseteq \mathbb{B}_x$.

Now we indicate that $\mathcal{T}$ is a contraction. For $(l_1,k_1),(l_2,k_2) \in \mathbb{D}\times\mathbb{D}$ and $w \in [0,1]$, we have

$$\begin{aligned}&\left|\mathcal{T}_1(l_2,k_2)(w) - \mathcal{T}_1(l_1,k_1)(w)\right| \\ &\quad\leqslant {}^k\mathcal{I}^{\breve{\alpha}_1;\chi}\left(\left|\Pi_{l_2,k_2}(w) - \Pi_{l_1,k_1}(w)\right|\right) + \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\ &\quad\quad\times\left[|\mathsf{A}_2|\left(\sum_{i=1}^{n}\int_0^1 \mathcal{I}^{\breve{\alpha}_1-\breve{u}_i;\chi}\left(\left|\Pi_{l_2,k_2}(s) - \Pi_{l_1,k_1}(s)\right|\right)\mathrm{d}K_i(s)\right.\right. \\ &\quad\quad\left. + {}^k\mathcal{I}^{\breve{\alpha}_2-a_2;\chi}\left(\left|\bar{\Pi}_{l_2,k_2}(1) - \bar{\Pi}_{l_1,k_1}(1)\right|\right)\right) \\ &\quad\quad + |\mathsf{A}_4|\left(\sum_{i=1}^{m}\int_0^1 \mathcal{I}^{\breve{\alpha}_2-\breve{r}_i;\chi}\left(\left|\bar{\Pi}_{l_2,k_2}(s) - \bar{\Pi}_{l_1,k_1}(s)\right|\right)\mathrm{d}H_i(s)\right. \\ &\quad\quad\left.\left. + {}^k\mathcal{I}^{\breve{\alpha}_1-a_1;\chi}\left(\left|\Pi_{l_2,k_2}(1) - \Pi_{l_1,k_1}(1)\right|\right)\right)\right] \\ &\quad\leqslant \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\Gamma_k(\breve{\alpha}_1+k)}\left(p_1\|l_2-l_1\| + p_2\|k_2-k_1\|\right) + \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\ &\quad\quad\times\left[|\mathsf{A}_2|\left(\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\left(p_1\|l_2-l_1\| + p_2\|k_2-k_1\|\right)\mathrm{d}K_i(s)\right.\right. \\ &\quad\quad\left. + \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2-a_2+k)}\left(q_1\|l_2-l_1\| + q_2\|k_2-k_1\|\right)\right) \\ &\quad\quad + |\mathsf{A}_4|\left(\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\left(q_1\|l_2-l_1\| + q_2\|k_2-k_1\|\right)\mathrm{d}H_i(s)\right. \\ &\quad\quad\left.\left. + \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\left(p_1\|l_2-l_1\| + p_2\|k_2-k_1\|\right)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \big(p_1\|l_2-l_1\|+p_2\|k_2-k_1\|\big)\left\{\frac{(\chi(1)-\chi(0))^{\check{\alpha}_1/k}}{\Gamma_k(\check{\alpha}_1+k)}+\frac{(\chi(1)-\chi(0))^{\check{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\check{\gamma}_1)}\right.\\
&\quad\left.\times\left[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\check{\alpha}_1-\check{u}_i)/k}}{\Gamma_k(\check{\alpha}_1-\check{u}_i+k)}\,\mathrm{d}K_i(s)+|\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\check{\alpha}_1-a_1)/k}}{\Gamma_k(\check{\alpha}_1-a_1+k)}\right]\right\}\\
&\quad+\big(q_1\|l_2-l_1\|+q_2\|k_2-k_1\|\big)\left\{\frac{(\chi(1)-\chi(0))^{\check{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\check{\gamma}_1)}\right.\\
&\quad\left.\times\left[|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\check{\alpha}_2-a_2)/k}}{\Gamma_k(\check{\alpha}_2+a_2+k)}+|\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\check{\alpha}_2-\check{r}_i)/k}}{\Gamma_k(\check{\alpha}_2-\check{r}_i+k)}\,\mathrm{d}H_i(s)\right]\right\}\\
&= \big(p_1\|l_2-l_1\|+p_2\|k_2-k_1\|\big)\chi_1+\big(q_1\|l_2-l_1\|+q_2\|k_2-k_1\|\big)\chi_2\\
&= (p_1\chi_1+q_1\chi_2)\|l_2-l_1\|+(p_2\chi_1+q_2\chi_2)\|k_2-k_1\|.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\big\|\mathcal{T}_1(l_2,k_2)-\mathcal{T}_1(l_1,k_1)\big\|\\
&\quad\leqslant (p_1\chi_1+q_1\chi_2+p_2\chi_1+q_2\chi_2)\big(\|l_2-l_1\|+\|k_2-k_1\|\big).
\end{aligned}\tag{17}
$$

Similarly, we can obtain that

$$
\begin{aligned}
&\big\|\mathcal{T}_2(l_2,k_2)-\mathcal{T}_2(l_1,k_1)\big\|\\
&\quad\leqslant (p_1\chi_4+q_1\chi_3+p_2\chi_4+q_2\chi_3)\big(\|l_2-l_1\|+\|k_2-k_1\|\big).
\end{aligned}\tag{18}
$$

Due to (17) and (18), we conclude that

$$
\begin{aligned}
&\big\|\mathcal{T}(l_2,k_2)(w)-\mathcal{T}(l_1,k_1)(w)\big\|\\
&\quad\leqslant \big((\chi_1+\chi_4)(p_1+p_2)+(\chi_2+\chi_3)(q_1+q_2)\big)\\
&\qquad\times\big(\|l_2-l_1\|+\|k_2-k_1\|\big).
\end{aligned}
$$

Thus, in view of condition (16), we conclude that the operator $\mathcal{T}$ is contraction and the Banach's contraction mapping principle implies that system (1) has a unique solution on $[0,1]$. The proof is completed. □

Now the existence results are established for system (1) by applying Leray–Schauder alternative [15] and Krasnosel'skiĭ's fixed point theorem [20].

Theorem 2. *Let $\Pi,\bar{\Pi}:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be continuous functions satisfying the following condition*:

(H2) *There exist real constants $\upsilon_i,\bar{\upsilon}_i\geqslant 0$, $i=1,2$, and $\upsilon_0,\bar{\upsilon}_0>0$ such that for all $l_i\in\mathbb{R}$, $i=1,2$,*

$$
\begin{aligned}
\big|\Pi(w,l_1,l_2)\big| &\leqslant \upsilon_0+\upsilon_1|l_1|+\upsilon_2|l_2|,\\
\big|\bar{\Pi}(w,l_1,l_2)\big| &\leqslant \bar{\upsilon}_0+\bar{\upsilon}_1|l_1|+\bar{\upsilon}_2|l_2|.
\end{aligned}
$$

Then at least one solution is obtained for system (1) *on* $[0,1]$ *if*

$$(\chi_1+\chi_3)\upsilon_1+(\chi_2+\chi_4)\bar{\upsilon}_1<1 \quad \textit{and} \quad (\chi_1+\chi_3)\upsilon_2+(\chi_2+\chi_4)\bar{\upsilon}_2<1,$$

where χ_i, $i=1,2,3,4$, *are given in* (15).

Proof. Due to the continuity property of Π and $\bar{\Pi}$, we conclude that the operator $\mathcal{T}$ is also continuous. Now the uniformly bounded property of the operator $\mathcal{T}$ is considered. Let $\mathbb{B}_r=\{(l,k)\in\mathbb{D}\times\mathbb{D}\colon \|(l,k)\|\leqslant r\}$. For all $l,k\in B_r$, by (H3) we have

$$\begin{aligned}\bigl|\Pi(w,l(w),k(w)\bigr| &\leqslant \upsilon_0+\upsilon_1|l|+\upsilon_2|k| \leqslant \upsilon_0+\upsilon_1\|l\|+\upsilon_2\|k\| \\ &\leqslant \upsilon_0+(\upsilon_1+\upsilon_2)r:=\vartheta_1,\end{aligned}$$

and similarly,

$$\bigl|\bar{\Pi}(w,l(w),k(w)\bigr| \leqslant \bar{\upsilon}_0+(\bar{\upsilon}_1+\bar{\upsilon}_2)r:=\vartheta_2.$$

Then, for any $(l,k)\in B_r$, we have

$$\begin{aligned}&\bigl|\mathcal{T}_1(l,k)(w)\bigr| \\ &\quad\leqslant \vartheta_1\Biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\Gamma_k(\breve{\alpha}_1+k)}+\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\ &\qquad\times\Biggl[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s)+|\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\Biggr]\Biggr\} \\ &\qquad+\vartheta_2\Biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\ &\qquad\times\Biggl[|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2+a_2+k)}+|\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr]\Biggr\},\end{aligned}$$

which yields

$$\bigl\|\mathcal{T}_1(l,k)\bigr\| \leqslant \chi_1\vartheta_1+\chi_2\vartheta_2.$$

Likewise, one can get

$$\bigl\|\mathcal{T}_2(l,k)\bigr\| \leqslant \chi_3\vartheta_1+\chi_4\vartheta_2.$$

Therefore, we have

$$\bigl\|\mathcal{T}(l,k)\bigr\|=\bigl\|\mathcal{T}_1(l,k)\bigr\|+\bigl\|\mathcal{T}_2(l,k)\bigr\| \leqslant (\chi_1+\chi_3)\vartheta_1+(\chi_2+\chi\chi_4)\vartheta_2,$$

which indicates that the operator $\mathcal{T}$ is uniformly bounded.

Now we indicate that the operator $\mathcal{T}$ is equicontinuous. Let $w_1, w_2 \in [0,1]$ with $w_1 < w_2$. Then we have

$$\begin{aligned}
&\bigl|\mathcal{T}_1\bigl(l(w_2),k(w_2)\bigr) - \mathcal{T}_1\bigl(l(w_1),k(w_1)\bigr)\bigr| \\
&\quad\leqslant \frac{\vartheta_1}{\Gamma_k(\breve{\alpha}_1)}\Biggl|\int_0^{w_1}\chi'(s)\bigl[\bigl(\chi(w_2)-\chi(s)\bigr)^{\breve{\alpha}_1/k-1} - \bigl(\chi(w_1)-\chi(s)\bigr)^{\breve{\alpha}_1/k-1}\bigr]\,\mathrm{d}s \\
&\qquad + \int_{w_1}^{w_2}\chi'(s)\bigl(\chi(w_2)-\chi(s)\bigr)^{\breve{\alpha}_1/k-1}\,\mathrm{d}s\Biggr| \\
&\qquad + \frac{(\chi(w_2)-\chi(0))^{\breve{\gamma}_1/k-1} - (\chi(w_1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\
&\qquad \times\Biggl\{\vartheta_1\Biggl[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s) + |\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\Biggr] \\
&\qquad + \breve{P}_2\Biggl[|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2+a_2+k)} + |\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr]\Biggr\} \\
&\quad\leqslant \frac{\vartheta_1}{\Gamma_k(\breve{\alpha}_1)}\bigl[2\bigl(\chi(w_2)-\chi(w_1)\bigr)^{\breve{\alpha}_1/k} + \bigl|\bigl(\chi(w_2)-\chi(0)\bigr)^{\breve{\alpha}_1/k} - \bigl(\chi(w_1)-\chi(0)\bigr)^{\breve{\alpha}_1/k}\bigr|\bigr] \\
&\qquad + \frac{(\chi(w_2)-\chi(0))^{\breve{\gamma}_1/k-1} - (\chi(w_1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\
&\qquad \times\Biggl\{\vartheta_1\Biggl[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s) + |\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\Biggr] \\
&\qquad + \vartheta_2\Biggl[|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2+a_2+k)} + |\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr]\Biggr\}.
\end{aligned}$$

The right hand of the above inequality is independent of (l,k) and tends to zero as $w_2 - w_1 \to 0$. Thus, $\mathcal{T}_1(l,k)$ is equicontinuous. Similarly, we can show that the operator $\mathcal{T}_2(l,k)$ is also equicontinuous. Thus, the operator $\mathcal{T}(l,k)$ is equicontinuous. Hence, the operator $\mathcal{T}$ is completely continuous.

Finally, we indicate that the set

$$\varXi = \bigl\{(l,k)\in\mathbb{D}\times\mathbb{D}\colon\ (l,k) = \lambda\mathcal{T}(l,k),\ 0\leqslant\lambda\leqslant 1\bigr\}$$

is bounded. Let $(l,k)\in\varXi$. Then $(\bar{l},k) = \lambda\mathcal{T}(l,k)$, and for all $w\in[0,1]$, we have

$$l(w) = \lambda\mathcal{T}_1(l,k)(w), \qquad k(w) = \lambda\mathcal{T}_2(l,k)(w).$$

Thus,

$$\begin{aligned}
&\bigl|l(w)\bigr| \\
&\quad\leqslant \bigl(\upsilon_0 + \upsilon_1\bigl|l(w)\bigr| + \upsilon_2\bigl|k(w)\bigr|\bigr)\biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\Gamma_k(\breve{\alpha}_1+k)} + \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\
&\qquad\times\Biggl[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s) + |\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\Biggr]\Biggr\} \\
&\qquad+ \bigl(\bar{\upsilon}_0 + \bar{\upsilon}_1\bigl|l(w)\bigr| + \bar{\upsilon}_2\bigl|k(w)\bigr|\bigr)\biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\
&\qquad\times\Biggl(|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2+a_2+k)} + |\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr)\Biggr\}
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|k(w)\bigr| \\
&\quad\leqslant \bigl(\bar{\upsilon}_0 + \bar{\upsilon}_1\bigl|l(w)\bigr| + \bar{\upsilon}_2\bigl|k(w)\bigr|\bigr)\biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\alpha}_2/k}}{\Gamma_k(\breve{\alpha}_2+k)} + \frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_2)} \\
&\qquad\times\Biggl[|\mathsf{A}_1|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-A_2)/k-1}}{\Gamma_k(\breve{\alpha}_2-A_2+k)} + |\mathsf{A}_3|\sum_{i=1}^{m}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k-1}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Biggr]\Biggr\} \\
&\qquad+ \bigl(\upsilon_0 + \upsilon_1\bigl|l(w)\bigr| + \upsilon_2\bigl|k(w)\bigr|\bigr)\biggl\{\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_2/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_2)} \\
&\qquad\times\Biggl[|\mathsf{A}_1|\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k-1}}{\Gamma_k(\breve{\alpha}_2-\breve{u}_i+k)}\,\mathrm{d}K_i(s) + |\mathsf{A}_3|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k-1}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\Biggr]\Biggr\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|l\| &\leqslant \bigl(\upsilon_0 + \upsilon_1\|l\| + \upsilon_2\|k\|\bigr)\chi_1 + \bigl(\bar{\upsilon}_0 + \bar{\upsilon}_1\|l\| + \bar{\upsilon}_2\|k\|\bigr)\chi_2, \\
\|k\| &\leqslant \bigl(\bar{\upsilon}_0 + \bar{\upsilon}_1\|l\| + \bar{\upsilon}_2\|k\|\bigr)\chi_4 + \bigl(\upsilon_0 + \upsilon_1\|l\| + \upsilon_2|\|k\|\bigr)\chi_3,
\end{aligned}$$

and hence,

$$\begin{aligned}
\|l\| + \|k\| \leqslant{}& (\chi_1+\chi_4)\upsilon_0 + \bigl((\chi_2+\chi_3)\bar{\upsilon}_0 + (\chi_1+\chi_4)\upsilon_1 + (\chi_2+\chi_3)\bar{\upsilon}_1\bigr)\|l\| \\
&+ (\chi_1+\chi_4)\upsilon_2 + (\chi_2+\chi_3)\bar{\upsilon}_2)\|k\|.
\end{aligned}$$

Therefore,

$$\bigl\|(l,k)\bigr\| \leqslant \frac{(\chi_1+\chi_4)\upsilon_0 + (\chi_2+\chi_3)\bar{\upsilon}_0}{M_0},$$

where M_0 is defined by

$$M_0 = \min\bigl\{1 - \bigl[(\chi_1+\chi_4)\upsilon_1 + (\chi_2+\chi_3)\bar{\upsilon}_1\bigr],\, 1 - \bigl[(\chi_1+\chi_4)\upsilon_2 + (\chi_2+\chi_3)\bar{\upsilon}_2\bigr]\bigr\},$$

which implies that $\varXi$ is bounded. Thus, using Leray–Schauder alternative [15], system (1) has at least one solution on $[0,1]$. The proof is complete. □

Now Krasnosel'skiĭ fixed point theorem [20] is applied to present next result.

Theorem 3. *Let $\varPi, \bar{\varPi} : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying* (H2). *In addition, we assume that*

(H3) *There exist continuous functions B_1 and $B_2 \in C([0,1],\mathbb{R}^+)$ satisfying*

$$\bigl|\varPi(w,l,k)\bigr| \leqslant B_1(w), \qquad \bigl|\bar{\varPi}(w,l,k)\bigr| \leqslant B_2(w)$$

for each $(w,l,k) \in [0,1]\times\mathbb{R}\times\mathbb{R}$.

Then system (1) *has at least one solution on $[0,1]$ if*

$$(\chi_1^* + \chi_4)(p_1+p_2) + (\chi_2+\chi_3^*)(q_1+q_2) < 1, \tag{19}$$

where χ_i, $i=2,4$, and χ_i^*, $i=1,3$, are given in (15).

Proof. We divide the operator $\mathcal{T}$ defined by (12) into four operators $\mathcal{T}_{1,1}$, $\mathcal{T}_{1,2}$, $\mathcal{T}_{2,1}$ and $\mathcal{T}_{2,2}$ as follows:

$$\mathcal{S}_{1,1}(l,k)(w) = {}^kI^{\breve{\alpha}_1;\chi}\varPi_{l,k}(w), \quad \mathcal{S}_{2,1}(l,k)(w) = {}^kI^{\breve{\alpha}_2;\chi}\bar{\varPi}_{l,k}(w), \quad w\in[0,1],$$

$$\begin{aligned}
&\mathcal{S}_{1,2}(l,k)(w)\\
&\quad= \frac{(\chi(w)-\chi(0))^{\breve{\gamma}_1/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_1)}\\
&\qquad\times\Biggl[\mathsf{A}_2\Biggl(\sum_{i=1}^{n}\int_0^1 {}^kI^{\breve{\alpha}_1-\breve{u}_i,\chi}\varPi_{l,k}(s)\,\mathrm{d}K_i(s) - {}^k\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\varPi}_{l,k}(1)\Biggr)\\
&\qquad+\mathsf{A}_4\Biggl(\sum_{i=1}^{m}\int_0^1 {}^kI^{\alpha_2-\breve{r}_i,\chi}\bar{\varPi}_{l,k}(s)\,\mathrm{d}H_i(s) - {}^k\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}f_{l,k}(1)\Biggr)\Biggr], \quad w\in[0,1],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{S}_{2,2}(l,k)(w)\\
&\quad= \frac{(\chi(w)-\chi(0))^{\breve{\gamma}_2/k-1}}{\mathsf{A}\varGamma_k(\breve{\gamma}_2)}\\
&\qquad\times\Biggl[\mathsf{A}_1\Biggl(\sum_{i=1}^{n}\int_0^1 {}^kI^{\breve{\alpha}_1-\breve{u}_i,\chi}\varPi_{l,k}(s)\,\mathrm{d}K_i(s) - {}^k\mathcal{I}^{\breve{\alpha}_2-a_2,\chi}\bar{\varPi}_{l,k}(1)\Biggr)\\
&\qquad+\mathsf{A}_3\Biggl(\sum_{i=1}^{m}\int_0^1 {}^kI^{\alpha_2-\breve{r}_i,\chi}\bar{\varPi}_{l,k}(s)\,\mathrm{d}H_i(s) - {}^k\mathcal{I}^{\breve{\alpha}_1-a_1,\chi}\varPi_{l,k}(1)\Biggr)\Biggr], \quad w\in[0,1].
\end{aligned}$$

Obviously, $\mathcal{T}_1 = \mathcal{S}_{1,1} + \mathcal{S}_{1,2}$, $\mathcal{T}_2 = \mathcal{S}_{2,1} + \mathcal{S}_{2,2}$. Now we introduce the ball: $\mathbb{B}_\delta = \{(l,k) \in \mathbb{D} \times \mathbb{D}\colon \|(l,k)\| \leqslant \delta\}$ with $\delta \geqslant (\chi_1 + \chi_3)\|B_1\| + (\chi_2 + \chi_4)\|B_2\|$. For any $(l_1,l_2),(k_1,k_2) \in B_\delta$, similar to Theorem 2, we get

$$\big|\mathcal{S}_{1,1}(l_1,k_2)(w) + \mathcal{S}_{1,2}(k_1,k_2)(w)\big| \leqslant \chi_1\|B_1\| + \chi_2\|B_2\|.$$

Similarly, we have

$$\big|\mathcal{S}_{2,1}(l_1,k_2)(w) + \mathcal{S}_{2,2}(k_1,k_2)(w)\big| \leqslant \chi_3\|B_1\| + \chi_4\|B_2\|.$$

Thus, we obtain

$$\big\|\mathcal{T}_1(l_1,l_2) + \mathcal{T}_2(k_1,k_2)\big\| \leqslant (\chi_1 + \chi_3)\|B_1\| + (\chi_2 + \chi_4)\|\|B_2\| < \delta,$$

which implies that $\mathcal{T}_1(l_1,l_2) + \mathcal{T}_2(k_1,k_2) \in \mathbb{B}_\delta$.

To indicate that the operator $(\mathcal{S}_{1,2}, \mathcal{S}_{2,2})$ is a contraction, let (l_1,l_2), $(k_1,k_2) \in \mathbb{B}_\delta$. Then, similar to the proof of Theorem 1, we get

$$\begin{aligned}
&\big|\mathcal{S}_{1,2}(l_2,k_2)(w) - \mathcal{S}_{1,2}(l_1,k_1)(w)\big| \\
&\quad\leqslant \big(p_1\|l_2 - l_1\| + p_2\|k_2 - k_1\|\big)\bigg\{\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\
&\qquad\times\Bigg[|\mathsf{A}_2|\sum_{i=1}^{n}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-\breve{u}_i)/k}}{\Gamma_k(\breve{\alpha}_1-\breve{u}_i+k)}\,\mathrm{d}K_i(s) + |\mathsf{A}_4|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_1-a_1)/k}}{\Gamma_k(\breve{\alpha}_1-a_1+k)}\Bigg]\bigg\} \\
&\qquad+ \big(q_1\|l_2 - l_1\| + q_2\|k_2 - k_1\|\big)\frac{(\chi(1)-\chi(0))^{\breve{\gamma}_1/k-1}}{|\mathsf{A}|\Gamma_k(\breve{\gamma}_1)} \\
&\qquad\times\Bigg[|\mathsf{A}_2|\frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-a_2)/k}}{\Gamma_k(\breve{\alpha}_2+a_2+k)} + |\mathsf{A}_4|\sum_{i=1}^{m}\int_0^1 \frac{(\chi(1)-\chi(0))^{(\breve{\alpha}_2-\breve{r}_i)/k}}{\Gamma_k(\breve{\alpha}_2-\breve{r}_i+k)}\,\mathrm{d}H_i(s)\Bigg] \\
&\quad= \big(p_1\|l_2 - l_1\| + p_2\|k_2 - k_1\|\big)\chi_1^* + \big(q_1\|l_2 - l_1\| + q_2\|k_2 - k_1\|\big)\chi_2 \\
&\quad= (p_1\chi_1^* + q_1\chi_2)\|l_2 - l_1\| + (p_2\chi_1^* + q_2\chi_2)\|k_1 - k_2\|, \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
&\big|\mathcal{S}_{2,2}(l_1,k_1)(w) - \mathcal{S}_{2,2}(l_2,k_2)(w)\big| \\
&\quad\leqslant (m_1\chi_4 + q_1\chi_3^*)\|l_2 - l_1\| + (p_2\chi_4 + q_2\chi_3^*)\|k_2 - k_1\|. \qquad (21)
\end{aligned}$$

From (20) and (21) we have

$$\begin{aligned}
&\big\|(\mathcal{S}_{1,2},\mathcal{S}_{2,2})(l_1,k_1) - (\mathcal{S}_{1,2},\mathcal{S}_{2,2})(l_2,k_2)\big\| \\
&\quad\leqslant \big\{(\chi_1^* + \chi_4)(p_1+p_2) + (\chi_2 + \chi_3^*)(q_1+q_2)\big\}\big(\|l_1 - l_2\| + \|k_1 - k_2\|\big),
\end{aligned}$$

which shows that $(\mathcal{S}_{1,2}, \mathcal{S}_{2,2})$ is a contraction by applying condition (19).

Now, due to the continuity property of the functions f and g, we conclude that the operator $(\mathcal{S}_{1,1},\mathcal{S}_{2,1})$ is also continuous. Moreover, the operator $(\mathcal{S}_{1,2},\mathcal{S}_{2,1})$ is uniformly bounded on $\mathbb{B}_\delta$, since

$$\big\|\mathcal{S}_{1,1}(l,k)\big\| \leqslant \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\Gamma_k(\breve{\alpha}_1+k)}\|B_1\|,$$

and

$$\big\|\mathcal{S}_{2,1}(l,k)\big\| \leqslant \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_2/k}}{\Gamma_k(\breve{\alpha}_2+k)}\|B_2\|.$$

Hence, we obtain

$$\big\|(\mathcal{S}_{1,1},\mathcal{S}_{2,1})(l,k)\big\| \leqslant \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_1/k}}{\Gamma_k(\breve{\alpha}_1+k)}\|B_1\| + \frac{(\chi(1)-\chi(0))^{\breve{\alpha}_2/k}}{\Gamma_k(\breve{\alpha}_2+k)}\|B_2\|,$$

which implies that the set $(\mathcal{S}_{1,1},\mathcal{S}_{2,1})\mathbb{B}_\delta$ is also uniformly bounded. Now the equicontinuous property of the set $(\mathcal{S}_{1,1},\mathcal{S}_{2,1})\mathbb{B}_\delta$ is showed. For any $(l,k)\in\mathbb{B}_\delta$ and $w_1,w_2\in[0,1]$ with $w_1<w_2$, we have

$$\begin{aligned}
&\big|\mathcal{S}_{1,1}(l,k)(w_2)-\mathcal{S}_{1,1}(l,k)(w_1)\big| \\
&\quad\leqslant \frac{1}{\Gamma_k(\breve{\alpha}_1)}\Bigg|\int_0^{w_1}\chi'(s)\big[\big(\chi(w_2)-\chi(s)\big)^{\breve{\alpha}_1/k-1}-\big(\chi(w_1)-\chi(s)\big)^{\breve{\alpha}_1/k-1}\big] \\
&\qquad\qquad\times \Pi\big(s,l(s),k(s)\big)\,\mathrm{d}s \\
&\qquad+\int_{w_1}^{w_2}\chi'(s)\big(\chi(w_2)-\chi(s)\big)^{\breve{\alpha}_1/k-1}\Pi\big(s,l(s),k(s)\big)\,\mathrm{d}s\Bigg| \\
&\quad\leqslant \frac{\|B_1\|}{\Gamma_k(\breve{\alpha}_1+k)}\big[2\big(\chi(w_2)-\chi(w_1)\big)^{\breve{\alpha}_1/k} \\
&\qquad+\big|\big(\chi(w_2)-\chi(0)\big)^{\breve{\alpha}_1/k}-\big(\chi(w_1)-\chi(0)\big)^{\breve{\alpha}_1/k}\big|\big],
\end{aligned}$$

which tends to zero as $w_1\to w_2$, independently of $(l,k)\in\mathbb{B}_\delta$. Similarly, we can show that $|\mathcal{S}_{2,1}(l,k)(w_2)-\mathcal{S}_{2,1}(l,k)(w_1)|\to 0$ as $w_1\to w_2$ independently of $(l,k)\in\mathbb{B}_\delta$. Thus, $|(\mathcal{S}_{1,1},\mathcal{S}_{2,1})(l,k)(w_2)-(\mathcal{S}_{1,1},\mathcal{S}_{2,1})(l,k)(w_1)|$ tends to zero, as $w_1\to w_2$, and hence, $(\mathcal{S}_{1,1},\mathcal{S}_{2,1})$ is equicontinuous and by applying Arzelà–Ascoli theorem, we conclude that the operator $(\mathcal{S}_{1,1},\mathcal{S}_{2,1})$ is compact on $\mathbb{B}_\delta$.

Hence, by applying Krasnosel'skiĭ fixed point theorem system (1) has at least one solution on $[0,1]$. The proof is finished. □

5 Numerical examples

In this section, some illustrated examples will be shown presenting the advantage of the obtained theorems from the previous section by setting some parameters.

Consider the following (k,χ)-Hilfer fractional system with coupled nonlocal boundary conditions including various fractional derivatives and Riemann–Stieltjes integrals:

$$\begin{aligned} {}^{8/7,H}\mathcal{D}^{9/7,2/5;\sqrt{w+1}}l(w) &= \Pi\big(w,l(w),k(w)\big), \quad w\in[0,1],\\ {}^{8/7,H}\mathcal{D}^{11/7,35/;\sqrt{w+1}}k(w) &= \bar{\Pi}\big(w,l(w),k(w)\big), \quad w\in[0,1], \end{aligned} \tag{22}$$

supplemented with coupled nonlocal boundary conditions

$$\begin{aligned} \breve{l}(0) = 0, &\qquad k(0)=0,\\ {}^{8/7,H}\mathcal{D}^{1/9,4/5,\sqrt{w+1}}\breve{l}(1) &= \int_0^1 {}^{8/7,H}\mathcal{D}^{1/3,1/7,\sqrt{w+1}}k(s)d\big(s^3+2s\big)\\ &\quad + \int_0^1 {}^{8/7,H}\mathcal{D}^{5/9,2/7,\sqrt{w+1}}k(s)\,\mathrm{d}\big(s^3+3s\big)\\ &\quad + \int_0^1 {}^{8/7,H}\mathcal{D}^{7/9,3/7,\sqrt{t+1}}k(s)\,\mathrm{d}\big(s^3+4s\big),\\ {}^{8/7,H}\mathcal{D}^{8/9,1/5,\sqrt{w+1}}k(1) &= \int_0^1 {}^{8/7,H}\mathcal{D}^{2/9,4/7,\sqrt{w+1}}\bar{\Pi}(s)\,\mathrm{d}\big(s^2+5s\big)\\ &\quad + \int_0^1 {}^{8/7,H}\mathcal{D}^{/4/9,5/7,\sqrt{t+1}}\bar{\Pi}(s)\,\mathrm{d}\big(s^2+6s\big)\\ &\quad + \int_0^1 {}^{8/7,H}\mathcal{D}^{2/3,6/7,\sqrt{t+1}}\bar{\Pi}(s)\,\mathrm{d}\big(s^2+7s\big). \end{aligned} \tag{23}$$

Here $\breve{\alpha}_1 = 9/7$, $\breve{\alpha}_2 = 11/7$, $\breve{\beta}_1 = 2/5$, $\breve{\beta}_2 = 3/5$, $\chi(w) = \sqrt{w+1}$, $k = 8/7$, $a_1 = 1/9$, $a_2 = 8/9$, $b_1 = 4/5$, $b_2 = 1/5$, $m = 3$, $\check{r}_1 = 1/3$, $\check{r}_2 = 5/9$, $\check{r}_3 = 7/9$, $\check{s}_1 = 1/7$, $\check{s}_2 = 2/7$, $\check{s}_3 = 3/7$, $H_1(s) = s^3+2s$, $H_2(s) = s^3+3s$, $H_3(s) = s^3+4s$, $n = 3$, $\check{u}_1 = 2/9$, $\check{u}_2 = 4/9$, $\check{u}_3 = 2/3$, $\check{v}_1 = 4/7$, $\check{v}_2 = 5/7$, $\check{v}_3 = 6/7$, $K_1(s) = s^2+5s$, $K_2(s) = s^2+6s$, $K_3(s) = s^2+7s$. Observe that $\chi'(w) = 1/(2\sqrt{w+1}) > 0$ for $w\in[0,1]$, $\breve{\gamma}_1 = 59/35$ with $\breve{\gamma}_1 < 2k$ and $\breve{\gamma}_2 = 2$ with $\breve{\gamma}_2 < 2k$. To calculate the function k-Gamma, we use the well-known relation

$$\Gamma_k(x) = k^{x/k-1}\Gamma\left(\frac{x}{k}\right).$$

All detail can be used to compute constants as $\mathsf{A}_1 \approx 0.7668878267$, $\mathsf{A}_2 \approx 9.046930524$, $\mathsf{A}_3 \approx 19.47984730$, $\mathsf{A}_4 \approx 1.011581212$, $|\mathsf{A}| \approx 175.4570558$, $\chi_1 \approx 0.5622110429$, $\chi_2 \approx 0.02490732732$, $\chi_3 \approx 0.2454906089$, $\chi_1^* \approx 0.2608746201$, $\chi_3^* \approx 0.0428239670$, $\chi_4 \approx 0.03568021627$.

(i) Consider the functions Π and $\bar{\Pi}$ given by

$$\begin{aligned} \Pi(w,l,k) &= \frac{10}{w+38}\,\frac{l^2+2|l|}{1+|l|} + \frac{10}{(t+4)^2}\sin k + \frac{1}{2}w^2 + 3w + 6, \\ \bar{\Pi}(w,l,k) &= \frac{10}{(w+2)^4+1}\tan^{-1} l + \frac{5}{2w+9}\,\frac{|k|}{1+|k|} + \frac{1}{3}\mathrm{e}^w + 4. \end{aligned} \tag{24}$$

Note that Π and $\bar{\Pi}$ satisfy the Lipschitz condition since

$$\bigl|\Pi(w,l_1,k_1) - \Pi(w,l_2,k_2)\bigr| \leqslant \frac{10}{19}|l_1-l_2| + \frac{5}{8}|k_1-k_2|$$

and

$$\bigl|\bar{\Pi}(w,l_1,k_1) - \bar{\Pi}(w,l_2,k_2)\bigr| \leqslant \frac{10}{17}|l_1-l_2| + \frac{5}{9}|k_1-k_2|.$$

Setting constants as $p_1 = 10/19$, $p_2 = 5/8$, $q_1 = 10/17$ and $q_2 = 5/9$ leads to

$$(\chi_1+\chi_4)(p_1+p_2) + (\chi_2+\chi_3)(q_1+q_2) \approx 0.9976403327 < 1.$$

Thus, conditions (H2) and (16) in Theorem 1 are fulfilled. Then the unique solution of problem (22)–(23) with Π and $\bar{\Pi}$ given by (24) is guaranteed on $[0,1]$.

(ii) If the unbounded term

$$\frac{10}{w+38}\,\frac{l^2+2|l|}{1+|l|} \quad \text{is replaced by bounded term} \quad \frac{10}{w+18}\,\frac{|l|}{1+|l|}$$

in (24), then

$$\Pi(w,l,k) = \frac{10}{w+18}\,\frac{|l|}{1+|l|} + \frac{10}{(w+4)^2}\sin k + \frac{1}{2}w^2 + 3w + 6. \tag{25}$$

In this case, Π and $\bar{\Pi}$ are bounded as

$$\bigl|\Pi(w,l,k)\bigr| \leqslant \frac{10}{w+18} + \frac{10}{(w+4)^2} + \frac{1}{2}w^2 + 3w + 6 := B_1(w)$$

and

$$\bigl|\bar{\Pi}(w,l,k)\bigr| \leqslant \frac{10\pi}{2(w+2)^4+2} + \frac{5}{2w+9} + \frac{1}{3}e^w + 4 := B_2(w).$$

Then constant p_1 is changed to be $p_1 = 5/9$, and consequently,

$$(\chi_1+\chi_4)(p_1+p_2) + (\chi_2+\chi_3)(q_1+q_2) \approx 1.015122533 > 1,$$

which is a contradiction to (16) in Theorem 1. This means that we can not claim the uniqueness in this case. However, the inequality

$$(\chi_1^*+\chi_4)(p_1+p_2) + (\chi_2+\chi_3^*)(q_1+q_2) \approx 0.4275698945 < 1,$$

holds, and all assumptions in Theorem 3 are satisfied. Therefore, (k,χ)-Hilfer fractional system with coupled nonlocal boundary conditions including various fractional derivatives and Riemann–Stieltjes integrals (22)–(23) with $\varPi$ given in (25) and $\bar{\varPi}$ given in (24) has at least one solution on $[0,1]$.

(iii) Let the nonlinear functions be appeared as

$$\begin{aligned} \varPi(w,l,k) &= \frac{1}{5}w + \frac{|l|^{2023}}{2(1+l^{2022})}e^{-|k|} + \frac{1}{3}k\sin^{12} l, \\ \bar{\varPi}(w,l,k) &= \frac{1}{4}w + \frac{18}{\pi}l\tan^{-1}|k| + 10\frac{k^{2022}}{1+|k|^{2021}}\cos^{14} l. \end{aligned} \tag{26}$$

We remark that $\varPi$ and $\bar{\varPi}$ are bounded as

$$\bigl|\varPi(w,l,k)\bigr| \leqslant \frac{1}{5} + \frac{1}{2}|l| + \frac{1}{3}|k| \quad \text{and} \quad \bigl|\bar{\varPi}(w,l,k)\bigr| \leqslant \frac{1}{4} + 9l + 10|k|$$

for all $w \in [0,1]$ and $l,k \in \mathbb{R}$. Choosing constants as $\upsilon_0 = 1/5$, $\upsilon_1 = 1/2$, $\upsilon_2 = 1/3$, $\bar{\upsilon}_0 = 1/4$, $\bar{\upsilon}_1 = 9$, $\bar{\upsilon}_2 = 10$, we have

$$(\chi_1+\chi_3)\upsilon_1 + (\chi_2+\chi_4)\bar{\upsilon}_1 \approx 0.9491387182 < 1$$

and

$$(\chi_1+\chi_3)\upsilon_2 + (\chi_2+\chi_4)\bar{\upsilon}_2 \approx 0.8751093198 < 1.$$

Thus, all conditions in Theorem 2 are satisfied, and we can use it to conclude that the boundary value problem (22)–(23) with nonlinear functions $\varPi$ and $\bar{\varPi}$ given in (26) has at least one solution on $[0,1]$.

6 Conclusions

In this paper, we have derived the existence and uniqueness results for a system of (k,χ)-Hilfer fractional differential equations, subject to coupled nonlocal boundary conditions containing various fractional derivatives and Riemann–Stieltjes integrals. We have applied the standard tools of the fixed point theory to establish the desired results. The used methods are standard, but their configuration in the present problem is new. Our results are new and significantly contribute to the literature on nonlocal coupled systems of (k,χ)-Hilfer fractional differential equations.

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References

1. R.P. Agarwal, A. Assolami, A. Alsaedi, B. Ahmad, Existence results and Ulam–Hyers stability for a fully coupled system of nonlinear sequential Hilfer fractional differential equations and integro-multistrip-multipoint boundary conditions, *Qual. Theory Dyn. Syst.*, **21**:125, 2022, https://doi.org/10.1007/s12346-022-00650-6.
2. B. Ahmad, Y. Alruwaily, A. Alsaedi, J.J. Nieto, Fractional integro-differential equations with dual anti-periodic boundary conditions, *Differ. Integral Equ.*, **33**:181–206, 2020, https://doi.org/10.57262/die/1584756018.
3. B. Ahmad, A. Alsaedi, S.K. Ntouyas, J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, New York, 1999, https://doi.org/10.1007/978-3-319-52141-1.
4. B. Ahmad, S.K. Ntouyas, *Nonlocal Nonlinear Fractional-Order Boundary Value Problems*, World Scientific, Singapore, 2021, https://doi.org/10.1142/12102.
5. S. Aibout, S. Abbas, M. Benchohra, M. Bohner, A coupled Caputo–Hadamard fractional differential system with multipoint boundary conditions, *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, **29**:125–136, 2022.
6. M. Alam, A. Zada, I.-L. Popa, A. Kheiryan, Sh. Rezapour, M.K. A. Kaabar, A fractional differential equation with multi-point strip boundary condition involving the Caputo fractional derivative and its Hyers–Ulam stability, *Bound. Value Probl.*, **2021**(73), 2021, https://doi.org/10.1186/s13661-021-01549-y.
7. M.A. Almalahi, M.S. Abdo, S.K. Panchat, Existence and Ulam–Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations, *Res. Appl. Math.*, **10**: 100142, 2021, https://doi.org/10.1016/j.rinam.2021.100142.
8. M.A. Almalahi, O. Bazighifan, S.K. Panchal, S.S. Askar, G.I. Oros, Analytical study of two nonlinear coupled hybrid systems involving generalized Hilfer fractional operators, *Fractal Fract.*, **2022**(4):178, 2022, https://doi.org/10.3390/fractalfract5040178.
9. A. Boutiara, J. Alzabut, G.M.Selvan, D. Vignesh, Analysis and applications of sequential hybrid ψ-Hilfer fractional differential equations and inclusions in Banach algebra, *Qual. Theory Dyn. Syst.*, **22**:11–32, 2023, https://doi.org/10.1007/s12346-022-00710-x.
10. A. Boutiara, M.M. Matar, T. Abdeljawad, F. Jarad, Existence and stability analysis for Caputo generalized hybrid Langevin differential systems involving three-point boundary conditions, *Bound. Value Probl.*, **2023**(22), 2023, https://doi.org/10.1186/s13661-023-01710-9.
11. A. Cernea, Existence and stability analysis for Caputo generalized hybrid Langevin differential systems involving three-point boundary conditions, *Bound. Value Probl.*, **2023**(22), 2023, https://doi.org/10.3390/foundations2010020.
12. K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985, https://doi.org/10.1007/978-3-662-00547-7.
13. K. Dhawan, R.K. Vats, R.P. Agarwal, Qualitative analysis of coupled fractional differential equations involving Hilfer derivative, *An. Stiinţ. Univ. "Ovidius" Constanţa, Ser. Mat.*, **30**:191–217, 2022, https://doi.org/10.2478/auom-2022-0011.
14. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, New York, 2010, https://doi.org/10.1007/978-3-642-14574-2.

15. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, New York, 2005, https://doi.org/10.1007/978-0-387-21593-8.
16. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000, https://doi.org/10.1142/3779.
17. M. Kaewsuwan, R. Phuwapathanapun, W. Sudsutad, J. Alzabut, C. Thaiprayoon, J. Kongson, Nonlocal impulsive fractional integral boundary value problem for (ρ_k, ϕ_k)-Hilfer fractional integro-differential equations, *Mathematics*, **2022**(4):3874, 2022, https://doi.org/10.3390/math10203874.
18. Ch. Kiataramkul, W. Yukunthorn, S.K. Ntouyas, J. Tariboon, Sequential Riemann–Liouville and Hadamard–Caputo fractional differential systems with nonlocal coupled fractional integral boundary conditions, *Axioms*, **2021**(10):1714, 2021, https://doi.org/10.3390/axioms10030174.
19. A.A. Kilbas, H.M.Srivastava, J.J. Trujillo, *Theory and Applications of the Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006, https://doi.org/10.1016/s0304-0208(06)80001-0.
20. M.A. Krasnosel'skiǐ, Two remarks on the method of successive approximations, *Usp. Mat. Nauk*, **10**:123–127, 1955, https://doi.org/10.4213/rm10144.
21. K.D. Kucche, A.D. Mali, On the nonlinear (k, φ)-Hilfer fractional differential equations, *Chaos Solitons Fractals*, **152**:111335, 2021, https://doi.org/10.1016/j.chaos.2021.111335.
22. Y.C. Kwun, G. Farid, W. Nazeer, S. Ullah, S.M. Kang, Generalized Riemann–Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities, *IEEE Access*, **6**:64946–64953, 2018, https://doi.org/10.1109/access.2018.2878266.
23. R. Luca, On a system of Riemann–Liouville fractional differential equations with coupled nonlocal boundary conditions, *Adv. Difference Equ.*, **2021**:134, 2021, https://doi.org/10.1186/s13662-021-03303-1.
24. A.D. Mali, K.D. Kucche, J.V.C. Sousa, On coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations, *Int. J. Nonlinear Sci. Numer. Simul.*, **24**:1425–1445, 2023, https://doi.org/10.1515/ijnsns-2021-0012.
25. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
26. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
27. A. Samadi, S.K. Ntouyas, S. Asawasamrit, J. Tariboon, Existence and uniqueness for coupled systems of Hilfer type sequential fractional differential equations involving Riemann–Stieltjes integral multi-strip boundary conditions, *J. Math.*, **2022**:7454388, 2022, https://doi.org/10.1155/2022/7454388.
28. T.T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York, 1973, https://doi.org/10.1016/s0076-5392(08)x6011-4.
29. J. Tariboon, A. Cuntavepanit, S.K. Ntouyas, W. Nithiarayaphaks, Separated boundary value problems of sequential Caputo and Hadamard fractional differential equations, *J. Funct. Spaces*, **2018**(1):101–110, 2009, https://doi.org/10.1155/2018/6974046.
30. Y. Wang, S. Liang, Q. Wang, Existence results for fractional differential equations with integral and multi-point boundary conditions, *Bound. Value Probl.*, **2018**(4), 2018, https://doi.org/10.1186/s13661-017-0924-4.