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Fractal perspective on dynamics of dark matter and dark energy interactions

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Abstract. In this paper, the dynamics of intricate chaotic attractors of the nonlinear system modeling the dark matter and dark energy interactions is studied indulging fractal–fractional operator in the Caputo sense. The constructed strange attractors witness that the dynamics of the universe components is dominated by the fractal properties. The fractional entropies stemmed from the classical entropy are estimated with fractal parameter and graphically portrayed to measure the randomness of the dynamic variables associated with the proposed dynamical system.

Keywords: dark matter, dark energy, dynamics, fractal–fractional operator, entropy.

1 Introduction

All the atoms and light in the universe together constitute less than 5% of the total mass. The rest is composed of dark matter (26.8%) and dark energy (68.3%), which are invisible yet highly influence the structure, dynamics and evolution of the universe, according to Planck data [12]. Dark matter, which makes up the mass of most galaxies and galaxy clusters, is responsible for their organization on grand scales. Meanwhile, the unexplained force responsible for the universe's accelerating expansion is known as dark energy. In [14], authors proclaimed that gravity exposes both attractive and repulsive behavior, and the dark matter and dark energy are merely the properties of gravity. Following to a potential phenomenon referred to as the "dark matter and dark energy interaction," any two matter fields in particle physics or other more theoretical contexts can interact. The cosmology community is interested in this particular qualitative theory in light of its many possible consequences. As in the interaction model dark energy decay into dark matter, the interaction models of dark matter and dark energy provide an analogous, well-researched

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description of the dark sector of the Universe. They are driven by a workable solution to the so-called coincidence and cosmological constant concerns; refer [8, 11, 18].

In the cosmological settings of dark matter and dark energy, the application of dynamical systems has been revealing several surprising facts in connection with the theory of fractals and chaos. The dynamics of dark energy models is studied applying centre manifold theory to the dynamical system in [9]. The applications of dynamical systems in dark energy models ranging from scalar fields to modified gravity are discussed in detail in the thesis [22]. In [4], interactions among matter, dark matter and dark energy are investigated based on Lotka–Volterra-like equations. The study reported that interaction dynamics of these components are chaotic in nature, and the universe is evolving chaotically within the cosmic time-line. Later, in order to improve the accuracy of these results under special conditions, a new interaction concept is proposed to consider dark matter and dark energy as an open and interacting thermodynamic system in [5]. A nonlinear interaction function is generally a demanding tool to discuss the dynamics of the interacting model. Nonlinear interaction models, however, are always an intriguing way to investigate the dynamics of the Universe and see the extent to which we can extract any more insights from it. In this work, we investigate a nonlinear dynamical system using a fractal–fractional technique, motivated by the interacting scenarios. The nonlinear coupled equations modeling the Interaction between Dark Matter and Dark Energy (IDMDE) is presented as follows:

$$\begin{aligned} x'(t) &= y(t)z(t) - x(t), \\ y'(t) &= \big(z(t) - p\big)x(t) - y(t), \\ z'(t) &= 1 - x(t)y(t), \end{aligned} \tag{1}$$

where $x(t)$, $y(t)$ and $z(t)$ are variables governing the dynamics of dark matter and dark energy interactions, and $p \in \mathbb{R}^+$ is the control parameter of the system. In [5], it is shown that the dynamics of the system has concretely evidence that the interactions between dark matter and dark energy exhibit a chaotic behavior when $p = 3.46$, making a significant contribution in the cosmology.

Being instigated from the chaotic dynamics of the integer-order system (1), this work aims to construct a fractal–fractional-order system to potentially capture more fascinating dynamical behaviors such as self-similarity and complexity. The fractal–fractional-order systems generalize both integer-order and fractional-order systems; see [3]. An advancement in the field of differentiation and integration has been witnessed when the classical differentiation has been extended to the theme of nonlocal operators. The classical derivative is blended with the power-law kernel, and ultimately, this leads to the development of new calculus, popularly called as the fractional calculus. Among the various fractional calculus methods, the Riemann–Liouville and Caputo fractional derivative, Caputo–Fabrizio fractional derivative and Atangana–Baleanu fractional derivative are the three notable methods successfully applied in predicting chaotic behaviors of linear, nonlinear, autonomous and nonautonomous dynamical systems; see, for instance, [1, 10, 15, 19]. Since dynamical systems and fractals are opposite sides of the same coin, to explore the fractal properties while dealing the fractional-order systems, the concept of fractal differentiation is introduced; see [13, 20]. For detailed information on fractal dimension and its applications on nonlinear dynamics, we refer to [6, 7]. In case of fractal differentiation, if the fractal order tends to 1, then the classical derivative is recovered. In [2], Atangana has introduced three fractal–fractional differential and integral operators to attract more nonlocal problems displaying fractal behaviors.

Indeed, fractional derivative helps to address complex chaotic problems, and the fractal derivative reveals the hidden self-similar nature, thus bringing the two concepts of fractional order and fractal dimension into one (fractal–fractional) operator significantly contributes to uncover strange dynamics of complex systems. Based on the through literature study, we identified that the dynamics of IDMDE is discussed only with the integer-order system, and no work is reported with fractional or fractal derivative for dark matter and dark energy interactions. Our paper addresses this short fall by utilizing the numerical scheme of Caputo fractal–fractional derivative proposed in [2] to explore the dynamics of dark matter and dark energy interactions. For different values of fractional order and fractal dimension, the chaotic characteristics of IDMDE are investigated. Further, the dynamics of integer-order, fractional-order and fractal–fractional-order systems of IDMDE are compared using numerical simulations. The main purpose of this work is to enlighten that the dynamics of dark matter and dark energy interactions follows fractal patterns with self-similar nature in addition to chaotic behavior.

Besides discussing the dynamics of system (1) under fractal–fractional operator, its fractional entropies are studied in the fractal phase space. Entropy measures the rate of increase in dynamical complexity as the system evolves with time; for more information on the entropy of dynamical systems, see [23]. A variety of entropies with one, two and three parameters are reviewed for different probability distributions in [16]. In this work, entropy formulations are computed for the probability distributions of dynamic variables associated with the fractal–fractional IDMDE system. The entropies are taken with two parameters, namely, fractional order and fractal dimension, and upon particular conditions, they converge to the classical Shannon entropy. The computed entropies are simulated versus fractional order and fractal dimension to compare the randomness of each dynamic variable.

The rest of this paper is structured as follows. Section 2 proposes a fractal–fractional-order system in the Caputo sense to explore the dynamics of dark matter and dark energy interactions. The numerical scheme discussed in this section is employed in graphical simulations for the visual flavor. To further illustrate the dynamical complexity of the fractal–fractional IDMDE system, fractional-order entropies are computed with fractal dimension in Section 3. The results presented in Sections 2 and 3 are concluded in Section 4.

2 Modeling dark energy and dark matter interaction under fractal–fractional derivative

In this section, we introduce the dynamical system modeling the interaction between dark energy and dark matter using the definition of fractal–fractional derivative in the Caputo sense. Consider the IDMDE system modeled using the fractal–fractional derivative under the Riemann–Liouville sense and given by

$$\begin{aligned}
{}^{RL}D_{0,t}^{\alpha,\tau}x(t) &= y(t)z(t) - x(t),\\
{}^{RL}D_{0,t}^{\alpha,\tau}y(t) &= (z(t) - p)x(t) - y(t),\\
{}^{RL}D_{0,t}^{\alpha,\tau}x(t) &= 1 - x(t)y(t),
\end{aligned}$$

where ${}^{RL}D_{0,t}^{\alpha,\tau}x(t)$ is the fractal–fractional derivative of a fractal differentiable function $x(t)$ with order τ and the fractional order α in the Riemann–Liouville sense with power-law kernel [2] expressed by

$${}^{RL}D_{0,t}^{\alpha,\tau}x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t^\tau}\int_0^t (t-s)^{m-\alpha-1}x(s)\,\mathrm{d}s$$

with $m-1 < \alpha, \tau \leqslant m \in \mathbb{N}$ and $\mathrm{d}x(s)/\mathrm{d}s^\tau = \lim_{t\to s}(x(t)-x(s))/t^\tau - s^\tau)$. Since the fractional integral is differentiable, the fractal–fractional derivative can be converted into the following system:

$$\begin{aligned}
{}^{RL}D_{0,t}^{\alpha,\tau}x(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{\alpha}f(x,y,z,s)\,\mathrm{d}s\frac{1}{\tau t^{\tau-1}},\\
{}^{RL}D_{0,t}^{\alpha,\tau}y(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{\alpha}g(x,y,z,s)\,\mathrm{d}s\frac{1}{\tau t^{\tau-1}},\\
{}^{RL}D_{0,t}^{\alpha,\tau}z(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{\alpha}h(x,y,z,s)\,\mathrm{d}s\frac{1}{\tau t^{\tau-1}},
\end{aligned}\tag{2}$$

where

$$\begin{aligned}
f(x,y,z,s) &= y(s)z(s) - x(s),\\
g(x,y,z,s) &= (z(s) - p)x(s) - y(s),\\
h(x,y,z,s) &= 1 - x(s)y(s).
\end{aligned}$$

Then system (2) can be rewritten as

$$\begin{aligned}
{}^{RL}D_{0,t}^{\alpha}x(t) &= \tau t^{\tau-1}f(x,y,z,s)\,\mathrm{d}s,\\
{}^{RL}D_{0,t}^{\alpha}y(t) &= \tau t^{\tau-1}g(x,y,z,s)\,\mathrm{d}s,\\
{}^{RL}D_{0,t}^{\alpha}x(t) &= \tau t^{\tau-1}h(x,y,z,s)\,\mathrm{d}s.
\end{aligned}$$

To indulge the initial conditions, the Riemann–Liouville derivative is replaced with the Caputo derivative. Thus,

$$\begin{aligned}
{}^{C}D_{0,t}^{\alpha}x(t) &= \tau t^{\tau-1}f(x,y,z,s)\,\mathrm{d}s,\\
{}^{C}D_{0,t}^{\alpha}y(t) &= \tau t^{\tau-1}g(x,y,z,s)\,\mathrm{d}s,\\
{}^{C}D_{0,t}^{\alpha}x(t) &= \tau t^{\tau-1}h(x,y,z,s)\,\mathrm{d}s,
\end{aligned}\tag{3}$$

where ${}^{C}D_{0,t}^{\alpha,\tau}x(t)$ is the fractal–fractional derivative of a fractal-differentiable function $x(t)$ with order τ and the fractional order α in the Caputo sense [2] expressed by

$$ {}^{C}D_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\mathrm{d}x(s)}{\mathrm{d}s} (t-s)^{m-\alpha-1}\,\mathrm{d}s $$

with $0 < m-1 < \alpha \leqslant m \in \mathbb{N}$. On applying the Riemann–Liouville fractional integral on both sides, we obtain

$$ \begin{aligned} x(t) &= x(0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau-1}(t-s)^{\alpha-1} f(x,y,z,s)\,\mathrm{d}s, \\ y(t) &= y(0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau-1}(t-s)^{\alpha-1} g(x,y,z,s)\,\mathrm{d}s, \\ z(t) &= z(0) + \frac{\tau}{\Gamma(\alpha)} \int_0^t s^{\tau-1}(t-s)^{\alpha-1} h(x,y,z,s)\,\mathrm{d}s. \end{aligned} \tag{4} $$

For $n \in \mathbb{N}$, the solution of system (4) at $t = t_{n+1}$ is expressed as

$$ \begin{aligned} x(t_{n+1}) &= x(0) + \frac{\tau}{\Gamma(\alpha)} \int_0^{t_{n+1}} s^{\tau-1}(t_{n+1}-s)^{\alpha-1} f(x,y,z,s)\,\mathrm{d}s, \\ y(t_{n+1}) &= y(0) + \frac{\tau}{\Gamma(\alpha)} \int_0^{t_{n+1}} s^{\tau-1}(t_{n+1}-s)^{\alpha-1} g(x,y,z,s)\,\mathrm{d}s, \\ z(t_{n+1}) &= z(0) + \frac{\tau}{\Gamma(\alpha)} \int_0^{t_{n+1}} s^{\tau-1}(t_{n+1}-s)^{\alpha-1} h(x,y,z,s)\,\mathrm{d}s. \end{aligned} $$

Approximating the above integrals in the finite interval $[t_j, t_{j+1}]$, it is seen that

$$ \begin{aligned} x(t_{n+1}) &= x(0) + \frac{\tau}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} s^{\tau-1}(t_{n+1}-s)^{\alpha-1} f(x,y,z,s)\,\mathrm{d}s, \\ y(t_{n+1}) &= y(0) + \frac{\tau}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} s^{\tau-1}(t_{n+1}-s)^{\alpha-1} g(x,y,z,s)\,\mathrm{d}s, \\ z(t_{n+1}) &= z(0) + \frac{\tau}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} s^{\tau-1}(t_{n+1}-s)^{\alpha-1} h(x,y,z,s)\,\mathrm{d}s. \end{aligned} \tag{5} $$

The piece-wise Lagrangian interpolation is employed to approximate the functions $s^{\tau-1}f(x,y,z,s)$, $s^{\tau-1}g(x,y,z,s)$ and $s^{\tau-1}h(x,y,z,s)$ such that

$$\begin{aligned}
P_j(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\tau-1} f\bigl(x(j),y(j),z(j),t(j)\bigr) \\
&\quad - \frac{s-t_j}{t_j-t_{j-1}} t_{j-1}^{\tau-1} f\bigl(x(j-1),y(j-1),z(j-1),t(j-1)\bigr), \\
Q_j(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\tau-1} g\bigl(x(j),y(j),z(j),t(j)\bigr) \\
&\quad - \frac{s-t_j}{t_j-t_{j-1}} t_{j-1}^{\tau-1} g\bigl(x(j-1),y(j-1),z(j-1),t(j-1)\bigr), \\
R_j(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\tau-1} h\bigl(x(j),y(j),z(j),t(j)\bigr) \\
&\quad - \frac{s-t_j}{t_j-t_{j-1}} t_{j-1}^{\tau-1} h\bigl(x(j-1),y(j-1),z(j-1),t(j-1)\bigr).
\end{aligned} \tag{6}$$

Substituting Eq. (6) into Eq. (5) and solving the integrals, we obtain the numerical scheme of system (2)

$$\begin{aligned}
&x(t_{n+1}) \\
&\quad = x(0) + \frac{\tau(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \bigl[t_j^{\tau-1} f\bigl(x(j),y(j),z(j),t(j)\bigr)\bigl((n+1-j)^\alpha(n-j+2+\alpha) \\
&\qquad - (n-j)^\alpha(n-j+2+2\alpha)\bigr) - t_{j-1}^{\tau-1} f\bigl(x(j-1),y(j-1),z(j-1),t(j-1)\bigr) \\
&\qquad \times \bigl((n+1-j)^{\alpha+1} - (n-j)^\alpha(n-j+1+\alpha)\bigr)\bigr], \\
&y(t_{n+1}) \\
&\quad = y(0) + \frac{\tau(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \bigl[t_j^{\tau-1} g\bigl(x(j),y(j),z(j),t(j)\bigr)\bigl((n+1-j)^\alpha(n-j+2+\alpha) \\
&\qquad - (n-j)^\alpha(n-j+2+2\alpha)\bigr) - t_{j-1}^{\tau-1} g\bigl(x(j-1),y(j-1),z(j-1),t(j-1)\bigr) \\
&\qquad \times \bigl((n+1-j)^{\alpha+1} - (n-j)^\alpha(n-j+1+\alpha)\bigr)\bigr], \\
&z(t_{n+1}) \\
&\quad = z(0) + \frac{\tau(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \bigl[t_j^{\tau-1} h\bigl(x(j),y(j),z(j),t(j)\bigr)\bigl((n+1-j)^\alpha(n-j+2+\alpha) \\
&\qquad - (n-j)^\alpha(n-j+2+2\alpha)\bigr) - t_{j-1}^{\tau-1} h\bigl(x(j-1),y(j-1),z(j-1),t(j-1)\bigr) \\
&\qquad \times \bigl((n+1-j)^{\alpha+1} - (n-j)^\alpha(n-j+1+\alpha)\bigr)\bigr].
\end{aligned} \tag{7}$$

The numerical scheme presented in Eq. (7) is employed to approximate the graphical solutions of the following IDMDE system modeled using the Caputo fractal–fractional derivative operator. Consider the following IDMDE system:

$$\begin{aligned}
{}^{C}D_{0,t}^{\alpha,\tau}x(t) &= \tau t^{\tau-1}f(x,y,z,s)\,\mathrm{d}s,\\
{}^{C}D_{0,t}^{\alpha,\tau}y(t) &= \tau t^{\tau-1}g(x,y,z,s)\,\mathrm{d}s,\\
{}^{C}D_{0,t}^{\alpha,\tau}x(t) &= \tau t^{\tau-1}h(x,y,z,s)\,\mathrm{d}s,
\end{aligned} \tag{8}$$

where $f(x,y,z,s) = y(s)z(s) - x(s)$, $g(x,y,z,s) = (z(s) - p)x(s) - y(s)$ and $h(x,y,z,s) = 1 - x(s)y(s)$ with control parameter $p = 3.46$. The dynamics of the proposed system is studied by maintaining a fixed fractional order α, while varying the fractal dimension τ and, conversely, by keeping the fractal dimension τ fixed and altering the fractional order α. The system's behavior is graphically portrayed with two set of initial values $x(0) = y(0) = z(0) = 1$ and $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$ in Figs. 1–3 and Figs. 4–6, respectively. In Fig. 1, both the fractional order and fractal dimension are set to 1 (i.e., $\alpha = \tau = 1$), it is and observed that the results coincide with the dynamics of integer-order systems. Figure 1(a) represents the chaotic attractor of the three-dimensional dynamical system (8), Figs. 1(b), 1(c) and 1(d) illustrate the dynamics of two variables x vs y, x vs z and y vs z, respectively. In Fig. 2, blue (2(a)), pink (2(b)), orange (2(c)) and green (2(d)) curves represent the chaotic attractors obtained by fixing $\alpha = 1$ and $\tau = 0.73$, whereas red (2(a)), black (2(b)), blue (2(c)) and violet (2(d)) are the chaotic attractors obtained by fixing $\alpha = 0.92$ and $\tau = 1$. Specifically, decreasing fractal dimension preserves the self-similar pattern while manifesting a decrease in the oscillation count within the system. The tendency toward a limit cycle attractor is also significant. On the other side, a reduction in fractional order generates one portion of attractor with less oscillation count. By varying both τ and α, the dynamical behaviour of the system is simulated in Fig. 3. With the values $\tau = 0.92$ and $\alpha = 0.92$, Fig. 3 depicts strange attractors with lacking perfect symmetry, which is caused by the power-law effect as this function not possesses statistical setting. Such complex attractors could not be captured neither by fractional differential operator nor by fractal differential operator at individual basis. Thus, a combination of fractional and fractal operator enables the system for depicting additional complexities.

Figure 4(a) depicts the chaotic attractor of system (8) with the initial conditions $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$ fixing both τ and α to be 1. Figures 4(b), 4(c) and 4(d) demonstrate attractors in the phase planes x vs y, x vs z and y vs z, respectively. As both fractional order and fractal dimension are one, Fig. 4 resembles the attractors corresponding to the integer-order system. Figure 5 presents the dynamical evolution of the system with varying τ and α. The curves in blue (5(a)), pink (5(b)), orange (5(c)) and green (5(d)) represent the simulations with $\alpha = 1$ and $\tau = 0.73$, whereas red (5(a)), black (5(b)), blue (5(c)) and violet (5(d)) are the simulations with $\alpha = 0.92$ and $\tau = 1$. While varying the fractal dimension, it is observed that the system evolves towards a butterfly-type attractors, which seems to be symmetric, on the other hand, decreasing the fractional order produces the attractor with less oscillations. Figure 6 portrays strange attractors with $\tau = 0.72$ and $\alpha = 0.92$. Comparison of varying fractal dimension τ with fixed fractional order α and, similarly, fixed fractal dimension τ with varying fractional order α of system (1) are graphically presented in Figs. 7 and 8 corresponding to two different set of initial conditions. Ultimately, the dynamics of the nonlinear chaotic systems are significantly influenced by the fractal and fractional orders.

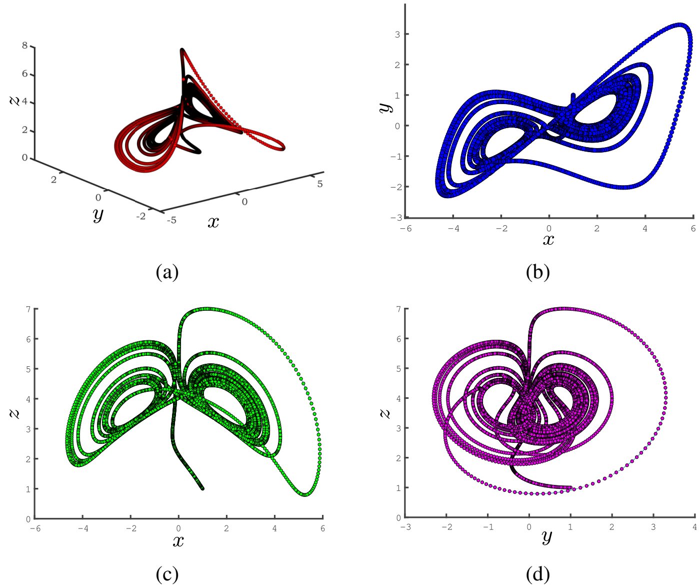


Figure 1. Dynamical behavior of fractal–fractional system (8) with $x(0) = y(0) = (0) = 1$ and $\alpha = \tau = 1$.

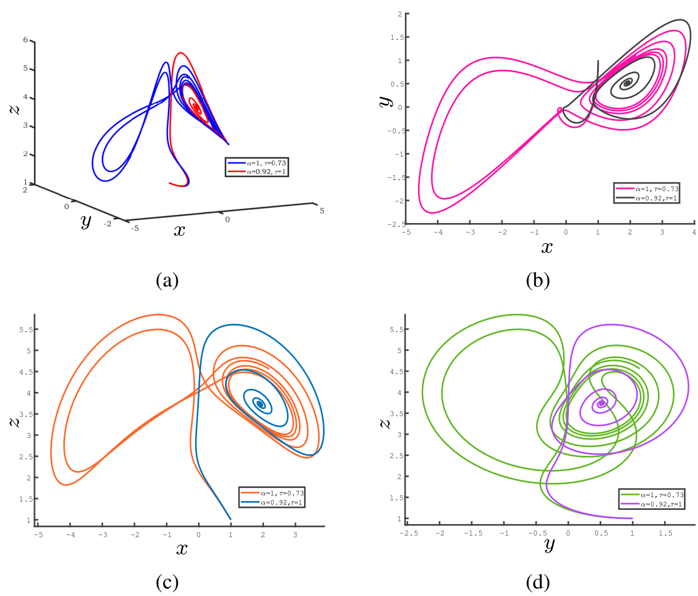


Figure 2. Dynamical behavior of fractal–fractional system (8) with $x(0) = y(0) = (0) = 1$, varying α and τ.

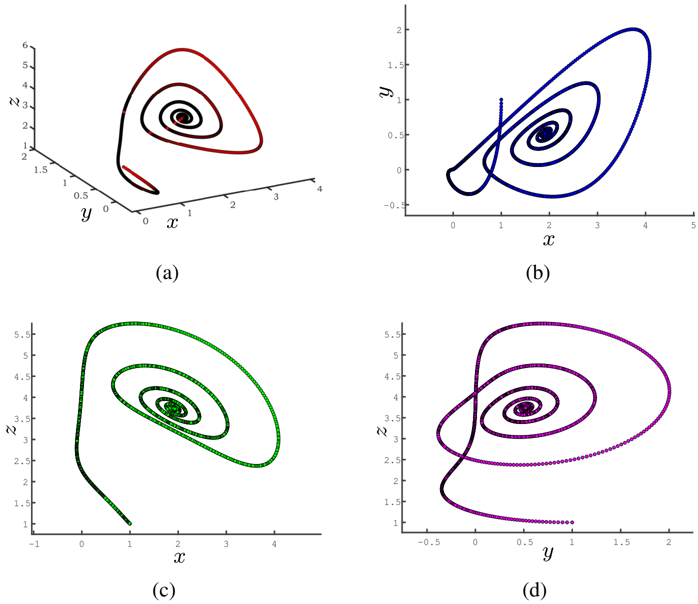


Figure 3. Dynamical behavior of fractal–fractional system (8) with $x(0) = y(0) = (0) = 1$, $\alpha = 0.92$ and $\tau = 0.92$.

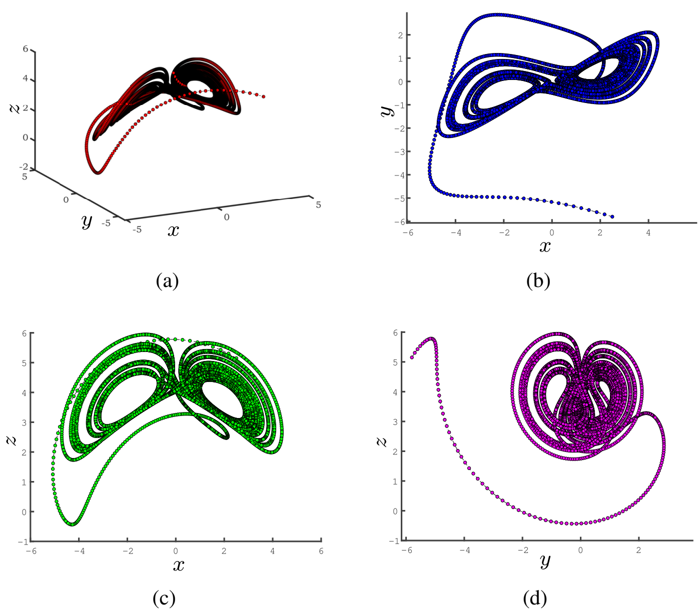


Figure 4. Dynamical behavior of fractal–fractional system (8) with $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$, $\alpha = \tau = 1$.

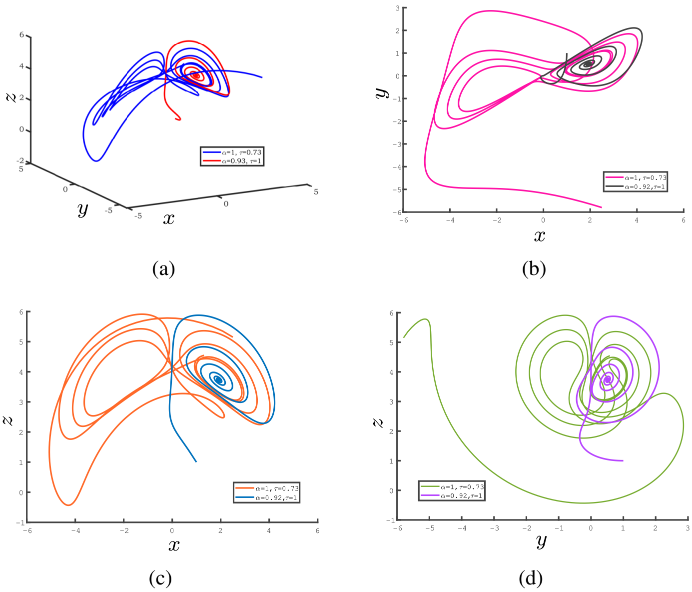


Figure 5. Dynamical behavior of fractal–fractional system (8) with $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$, varying α and τ.

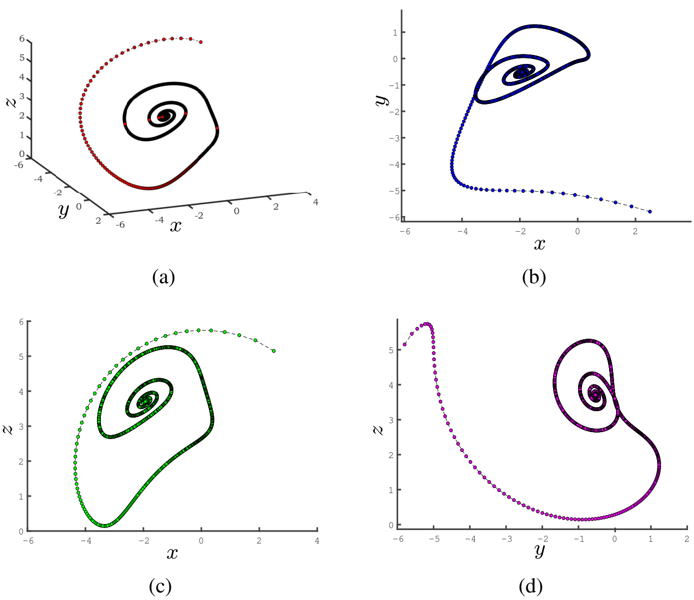
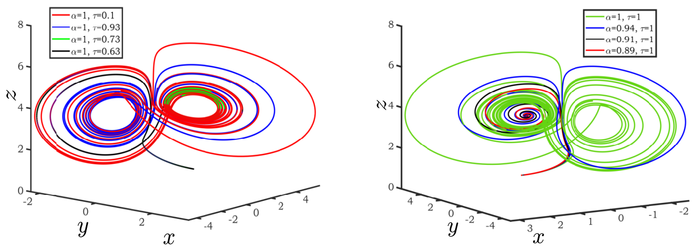
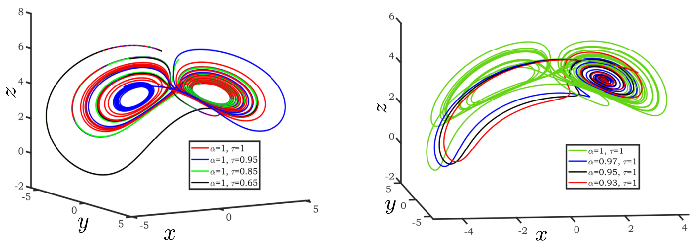


Figure 6. Dynamical behavior of fractal–fractional system (8) with $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$, $\alpha = 0.98$ and $\tau = 0.72$.



(a) (b)

Figure 7. Dynamics of system (8) with $x(0) = y(0) = z(0) = 1$: (a) fixed α and varying τ, (b) fixed τ and varying α.



(a) (b)

Figure 8. Dynamics of system (8) with $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$: (a) fixed α and varying τ, (b) fixed τ and varying α.

3 Fractional entropy in fractal phase space

This section discusses fractional entropies for the proposed dynamical system in the fractal phase space, that is, entropies involving fractional order and fractal dimension. The Shannon entropy, denoted by S, of a discrete probability distribution $\mathbf{P} = \{p_1, p_2, \dots, p_N\}$ with $\sum_i p_i = 1$ and $p_i \geqslant 0$ is defined as

$$S = \sum_i p_i I(p_i) = -\sum_i p_i \ln p_i, \tag{9}$$

the expected value of the information content is represented by $I(p_i) = -\ln p_i$. In [21], authors have introduced a fractional entropy involving two-parameters expression and given by

$$S^1_{q,\beta} = \sum_i p_i^q (-\ln p_i)^\beta +, \quad q, \beta > 0. \tag{10}$$

The two parameters q and β are associated with fractality and fractionality, respectively. Therefore, entropy expressed in Eq. (10) is referred as a fractional entropy in the fractal phase space. In addition, if $\{q, \beta\} = \{1, 1\}$, then the expression in Eq. (10) reduces to the

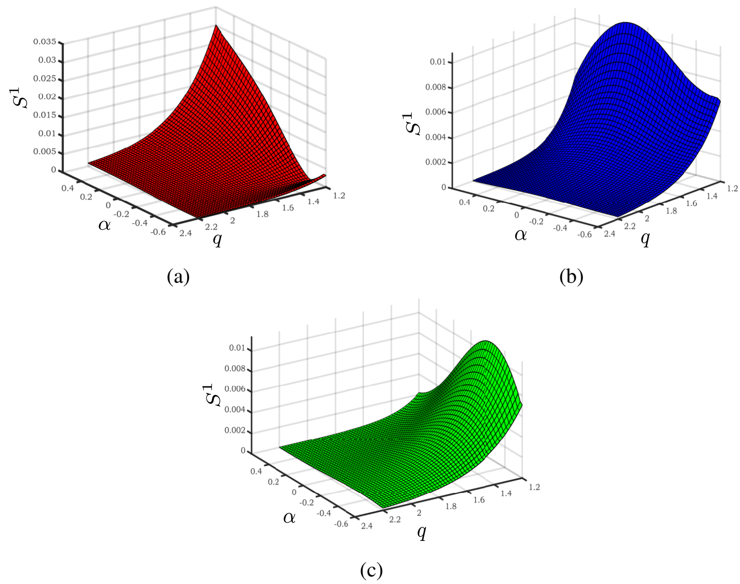


Figure 9. Values of $S^1_{q,\beta}$ versus $\beta \in [-0.6, 0.6]$ and $q \in [1.2, 2.2]$ for probability distributions of (a) $x(t)$, (b) $y(t)$ and (c) $z(t)$ associated with the dynamical system (8).

standard Shannon entropy expressed in Eq. (9). In the perspective of fractional calculus, the formulation of information of order $\beta \in \mathbb{R}$ is given by

$$I_\beta(p_i) = D^\beta I(p_i) = -\frac{p_i^{-\beta}}{\Gamma(\beta+1)}(\ln p_i + \tilde{\Psi}), \tag{11}$$

where D^β denotes a fractional derivative operator, $\tilde{\Psi} = \Psi(1) - \Psi(1-\beta)$, and $\Psi(\cdot)$ represents the digamma function. In [17], authors defined one more fractional entropy based on $I_\beta(p_i)$ expressed in Eq. (11), and its formula is provided by

$$\begin{aligned} S^2_{q,\beta} &= \frac{1}{1-q}\ln\biggl\{\sum_i p_i \exp\bigl[(1-q)I_\beta(p_i)\bigr]\biggr\} \\ &= \frac{1}{1-q}\ln\biggl\{\sum_i p_i \exp\biggl[(1-q)\frac{p_i^{-\beta}}{\Gamma(\beta+1)}\biggr]\biggr\}. \end{aligned} \tag{12}$$

For more details on fractional entropies with one and two parameters, readers are recommended to visit [16]. Using the above presented formulations in Eqs. (10) and (12), the fractional entropies are estimated for the probability distributions of dynamic variables $x(t)$, $y(t)$ and $z(t)$ associated with the IDMDE system (8). The estimated entropies are graphically illustrated individually for each variable in the specified domain. For the values of $q \in [1.2, 2.2]$ and $\beta \in [-0.6, 0.6]$, the two-parameter entropy $S^1_{q,\beta}$ is computed

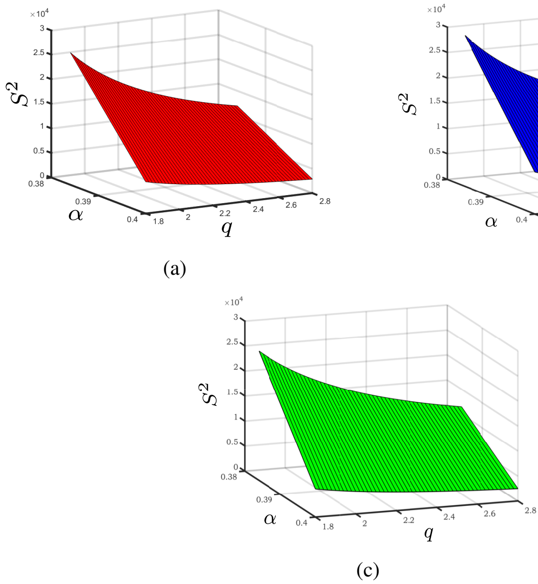


Figure 10. Values of $S^2_{q,\beta}$ versus $\beta \in [-0.4, 0.4]$ and $q \in [1.8, 2.8]$ for probability distributions of (a) $x(t)$, (b) $y(t)$ and (c) $z(t)$ associated with the dynamical system (8).

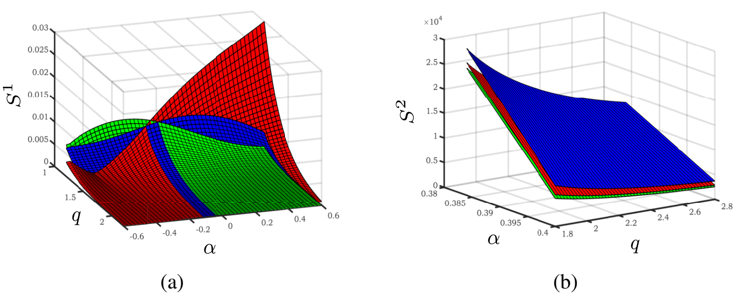


Figure 11. Overlapping mesh of (a) $S^1_{q,\beta}$ and (b) $S^2_{q,\beta}$ versus α and q for dynamic variables x, y and z associated with system (8).

for x, y and z and plotted in Figs. 9(a), 9(b) and 9(c), respectively. Similarly, the entropy $S^2_{q,\beta}$ is computed for x, y and z with the values of $q \in [1.8, 2.8]$ and $\beta \in [-0.4, 0.4]$, and its corresponding graphs is provided in Fig. 10. It is observed that in the considered domain of q and α, the entropies $S^1_{q,\beta}$ and $S^2_{q,\beta}$ do not diverge though varies slightly. To compare the dynamics of x, y and z, the fractional entropies mesh are plotted in a single graph in Fig. 11. In Fig. 11(a), three mesh of $S^1_{q,\beta}$ corresponding to red (x), blue (y) and green (z) are plotted. The meshes overlap with each other portraying that each one shows different dynamical behavior. On the other side, three mesh red (x), blue (y) and green (z) of $S^2_{q,\beta}$ sliding one above the other depicts that the rate of dynamics vary moderately.

4 Conclusion

The fractal dimension measures roughness and captures irregularities where the traditional notion of integer dimensionality fails. The interesting connection between fractal dimension and fractional order is aroused from the fact that fractals often show noninteger dimension. This attribute highlights the necessity of employing fractal–fractional derivative to better portray and understand complex systems. The application of fractal–fractional derivative facilitates a profound understanding of physical phenomena displaying intricate self-similar dynamics across scales. In this work, the dynamics of dark matter and dark energy interactions are studied under fractal–fractional derivative in the Caputo sense. As a result of indulging both fractional order α and fractal dimension τ, new strange attractors are obtained with higher complexities than the existing attractors associated with the integer-order dynamical system. The results obtained confirm that at the limiting case of $\alpha = 1$ and $\tau = 1$, the original attractors (of the integer-order system) are retrieved, indicating the diversity of the proposed system. The examples discussed with two different initial conditions $x(0) = y(0) = z(0) = 1$ and $x(0) = 2.5$, $y(0) = -5.8$, $z(0) = 5.15$ clarify that the chaos and the self-similarity highly influence the dynamics of IDMDE system, and its dynamics is sensitive to the initial conditions. In addition, decreasing fractal dimension preserves the self-similarity of the attractors, and decreasing fractional order produces complex asymmetric attractors. The fractional entropies for the probabilities of dynamic variables of fractal–fractional IDMDE system are discussed, the results aid to measure the statistical complexity of the system. The findings of this paper contribute to a deeper understanding of the dynamics of dark matter and dark energy in the universe. The fractal–fractional IDMDE system evidences the chaotic dynamics of universe along with fractal properties, benefiting as a tool to understand how nature functions.

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