Nonlinear Analysis: Modelling and Control, Vol. 29, No. 3, 528–542
https://doi.org/10.15388/namc.2024.29.34969

Vilnius University Press

Abstract random differential equations with state-dependent delay using measures of noncompactness

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Received: September 10, 2022 / **Revised:** January 19, 2024 / **Published online:** April 5, 2024

Abstract. This paper is devoted to the existence of random mild solutions for a general class of second-order abstract random differential equations with state-dependent delay. The technique used is a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measures of noncompactness. An application related to partial random differential equations with state-dependent delay is presented.

Keywords: random differential equations, random mild solution, state-dependent delay, measure of noncompactness, Fréchet spaces.

1 Introduction

Many researchers have expressed interest in the study of differential equations with state-dependent delay since they are fundamental in applications and qualitative theory, and they describe many physical, chemical, and biological problems. For more details, see the papers of Büger and Martin [10], Si and Wang [27]. The literature on this topic is mostly concerned with first-order ordinary differential equations on finite dimensional spaces, we mention the works [1, 11, 12]. While there are several papers in which authors discuss differential problems with various forms of delays (see [5, 8, 18]), there are few studies on abstract second-order ordinary differential equations with state-dependent delay applied to partial differential equations, we cite [4, 19, 21, 22]. The authors of [2, 3, 6, 9, 16, 28] investigated multiple differential problems using various tools and approaches, one of which was the fixed point theory.

The authors of [13] presented the existence of mild solutions for the following partial integro–differential equations with state-dependent nonlocal local conditions:

$$\begin{aligned} y'(t) &= \Upsilon y(t) + \int_0^t Y(t-\varrho)y(\varrho)\,\mathrm{d}\varrho + ht(t, y_{\rho(t,y_t)}), \quad t \in \mathbb{R}^+, \\ y_0 &= K\bigl(\zeta(y), y\bigr) \in C\bigl([-p,0], E\bigr), \end{aligned}$$

where $\Upsilon : D(\Upsilon) \subset E \to E$ is the infinitesimal generator of C_0-semigroup $(\mathrm{T}(t))_{t\geqslant 0}$ on a Banach space E, $Y(t)$ is a closed linear operator with domain $D(Y) \supset D(\Upsilon)$, $h : \mathbb{R}_+ \times C([-p,0],E) \to E$, $\zeta : C([-p,+\infty),E) \to \mathbb{R}_+$, $\rho : \mathbb{R}_+ \times C([-p,0],E) \to \mathbb{R}_+$, and $K : \mathbb{R}_+ \times C([-p,+\infty),E) \to C([-p,0],E)$ are suitable functions (here and bellow $\mathbb{R}^+ = [0,+\infty)$).

In [20], Hernandez studied the global existence and uniqueness of solutions and well posedness of the following general class of abstract second-order differential equations with state dependent delay:

$$\begin{aligned} y''(t) &= \Upsilon y(t) + g\bigl(t, y(t-\sigma_1(t,y_t)), y'\bigl(t-\sigma_2(t,y_t)\bigr)\bigr), \quad t \in [0,a], \\ y_0 &= \chi \in \mathcal{B}, \quad (y')_0 = \chi' \in \mathcal{B}, \quad \mathcal{B} = C\bigl([-\varpi,0];E\bigr), \end{aligned}$$

where $\Upsilon : D(\Upsilon) \subset E \to E$ is the generator of a strongly continuous cosine family of bounded linear operators $(\mathcal{H}(t))_{t\in\mathbb{R}}$ defined on a Banach space $(E, \|\cdot\|)$, $\sigma_i \in C([0,a] \times \mathcal{B}; [0,\varpi])$ for $i = 1,2$, and $g \in C([0,a] \times E \times E; E)$.

Motivated by the above-mentioned papers and the works [24, 25], where the authors presented some existence results for a random fractional equation, in this paper, we discuss the existence of mild solutions defined on unbounded interval for general class of abstract second-order differential equations with state dependent delay of the form

$$y''(t,\mu) = \Phi y(t,\mu) + \hbar(t, y_{\sigma_1(t,y_t,\mu)}, y'_{\sigma_2(t,y_t,\mu)}, \mu), \quad t \in \mathbb{R}_+,\ \mu \in \Upsilon, \tag{1}$$

$$y(t,\mu) = \varphi(t,\mu), \quad t \in [-p,0], \tag{2}$$

$$y'(0,\mu) = \varphi'(0,\mu), \tag{3}$$

where $\Phi : D(\Phi) \subset E \to E$ is generator of strongly continuous cosine family of bounded linear operator $(\mathcal{H}(t))_{t\in\mathbb{R}})$ on a Banach space $(E, \|\cdot\|)$, and $\hbar : \mathbb{R}_+ \times C([-p,0],E) \times C([-p,0],E) \times \Upsilon \to E$, $\sigma_i : \mathbb{R}_+ \times C([-p,0],E) \times \Upsilon \to E$, and $\varphi \in C([-p,0],E) \times \Upsilon$ are suitable functions.

It is important to highlight that our paper serves as a logical extension of the previously referenced studies. Specifically, our investigation focuses on second-order differential equations involving a random variable, as opposed to considering first-order differential equations in [13]. Furthermore, our research builds upon the work done in [20] by exploring differential equations on an unbounded domain in conjunction with random variables.

The following is how this work is organized. Section 2 presents notations, definitions, and lemmas that will be used throughout the work. Section 3 shows the existence of random mild solutions for abstract differential equations with state dependent delay. We will also provide an example to demonstrate the abstract consequence of our effort.

2 Preliminaries

We denote by $(\mathcal{G}(t))_{t\in\mathbb{R}}$ the associated sine family, where

$$\mathcal{G}(t) := \int_0^t \mathcal{H}(\varrho)\,\mathrm{d}\varrho.$$

Consider now the second-order abstract Cauchy problem

$$y''(t) = \Phi y(t) + \psi(t), \quad t \in \mathbb{R}^+, \tag{4}$$

$$y(0) = y, \qquad y'(0) = z, \tag{5}$$

where $\psi : \mathbb{R}^+ \to E$ is an integrable function, and $y, z \in E$. The function

$$y(t) = \mathcal{H}(t)y + \mathcal{G}(t)z + \int_0^t \mathcal{G}(t-\varrho)\psi(\varrho)\,\mathrm{d}\varrho, \quad t \in \mathbb{R}^+,$$

is a mild solution of (4)–(5), and when $y \in \Omega$, $y(\cdot)$ is a C^1 function on $\mathbb{R}^+$, and

$$y'(t) = \Phi\mathcal{G}(t)y + \mathcal{H}(t)z + \int_0^t \mathcal{H}(t-\varrho)\psi(\varrho)\,\mathrm{d}\varrho, \quad t \in \mathbb{R}^+.$$

We refer the reader to the papers of Fattorini [17], Vasil'ev and Piskarev [29] for additional details.

Let $\Theta := [0,\kappa]$, $\kappa > 0$. $L^1(\Theta, E)$ denotes the Banach space of Bochner-integrable functions $y : \Theta \to E$ with

$$\|y\|_{L^1} = \int_0^\kappa \big\|y(t)\big\|\,\mathrm{d}t.$$

By $\mathfrak{F}(E)$ we denote the Banach space of bounded linear operators from E into E with

$$\|\aleph\|_{\mathfrak{F}(E)} = \sup_{\|y\|=1} \big\|\aleph(y)\big\|.$$

Let $C(\Theta, E)$ be the Banach space of continuous functions from Θ into E with the norm

$$\|y\|_\infty = \sup\big\{\big\|y(t)\big\|,\, t \in \Theta\big\}.$$

Let $\mathfrak{D}_E$ be the σ-algebra of Borel subsets of E. The map $\bar{y} : \Upsilon \to E$ is measurable if for any $\mathfrak{B} \in \mathfrak{D}_E$, we have

$$\bar{y}^{-1}(\mathfrak{B}) = \{\mu \in \Upsilon\colon \bar{y}(\mu) \in \mathfrak{B}\} \subset \mathcal{Z}.$$

Definition 1. (See [23].) A mapping $U : \Upsilon \times E \to E$ is jointly measurable if for any $\mathfrak{B} \in \mathfrak{D}_E$, we have

$$U^{-1}(\mathfrak{B}) = \{(\mu, \bar{y}) \in \Upsilon \times E\colon U(\mu, \bar{y}) \in \mathfrak{B}\} \subset \mathcal{Z} \times \mathfrak{D}_E,$$

where $\mathcal{Z} \times \mathfrak{D}_E$ is the direct product of the σ-algebras $\mathcal{Z}$ and $\mathfrak{D}_E$ defined in Υ and E, respectively.

Lemma 1. (*See* [23].) *Let $U : \Upsilon \times E \to E$ be a mapping such that $U(\cdot, \bar{y})$ is measurable for all $\bar{y} \in E$, and let $U(\mu, \cdot)$ be continuous for all $\mu \in \Upsilon$. Then the map $(\mu, \bar{y}) \to U(\mu, \bar{y})$ is jointly measurable.*

Definition 2. (See [23].) A function $\lambda : \Theta \times E \times \Upsilon \to E$ is called random Carathéodory if the assumptions that follow are verified:

- The map $(t, \psi, \mu) \to \lambda(t, \psi, y, \mu)$ is jointly measurable for all $y \in E$, and
- $y \to \lambda(t, \psi, y, \mu)$ is continuous for almost all $(t, \psi) \in \Theta$ and $\mu \in \Upsilon$.

The map $U : \Upsilon \times E \to E$ is a random operator if $U(\mu, y)$ is measurable in μ for all $y \in E$ and it is given as $U(\mu)y = U(\mu, y)$. $U(\mu)$ is a random operator on E. A random operator $U(\mu)$ on E is continuous if $U(\mu, y)$ is continuous in y for all $\mu \in \Upsilon$.

Definition 3. (See [15].) Let $\mathcal{P}(\mathcal{S})$ be the family of all nonempty subsets of $\mathcal{S}$, and let $\mathfrak{R}$ be a mapping from Υ into $\mathcal{P}(\mathcal{S})$. $U = \{(\mu, \psi)\colon \mu \in \Upsilon, \psi \in \mathfrak{R}(\mu)\} \to \mathcal{S}$ is a random operator with stochastic domain $\mathfrak{R}$ if $\mathfrak{R}$ is measurable (i.e., for all closed $\mathfrak{B} \subset \mathcal{S}$, $\{\mu \in \Upsilon\colon \mathfrak{R}(\mu) \cap \mathfrak{B} \neq \emptyset\}$ is measurable), and for all open $\tilde{\mathfrak{B}} \subset \mathcal{S}$ and all $\psi \in \mathcal{S}$, $\{\mu \in \Upsilon\colon \psi \in \mathfrak{R}(\mu), U(\mu, \psi) \in \tilde{\mathfrak{B}}\}$ is measurable. U is continuous if every $U(\mu)$ is continuous. A mapping $\psi : \Upsilon \to \mathcal{S}$ is a random fixed point of U if for almost all $\mu \in \Upsilon$, $\psi(\mu) \in \mathfrak{R}(\mu)$ and $U(\mu)\psi(\mu) = \psi(\mu)$, and for all open $\tilde{\mathfrak{B}} \subset \mathcal{S}$, $\{\mu \in \Upsilon\colon \psi(\mu) \in \tilde{\mathfrak{B}}\}$ is measurable.

Let $C(\mathbb{R}_+)$ be the Fréchet space of continuous functions y from $\mathbb{R}_+$ into E with the family seminorms

$$\|y\|_j = \sup_{t \in [0,j]} \|y(t)\|, \quad j \in \mathbb{N},$$

and the distance

$$d(y, \varkappa) = \sum_{j=1}^{\infty} \frac{2^{-j}\|y - \varkappa\|_j}{1 + \|y - \varkappa\|_j}, \quad y, \varkappa \in C(\mathbb{R}_+).$$

Definition 4. (See [14].) Let $\mathcal{M}_\Omega$ be the family of all nonempty and bounded subsets of a Fréchet space Ω. $\{\alpha_j\}_{j\in\mathbb{N}}$, where $\alpha_j : \mathcal{M}_\Omega \to \mathbb{R}^+$ is a family of measures of noncompactness in Ω if it verifies the following for all $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \in \mathcal{M}_\Omega$:

(i) $\{\alpha_j\}_{j\in\mathbb{N}}$ is full, i.e., $\alpha_j(\mathcal{K}) = 0$ for $j \in \mathbb{N}$ if and only $\mathcal{K}$ precompact;
(ii) $\alpha_j(\mathcal{K}_1) \leqslant \alpha_j(\mathcal{K}_2)$ for $\mathcal{K}_1 \subset \mathcal{K}_2$ and $j \in \mathbb{N}$;
(iii) $\alpha_j(\operatorname{conv}\mathcal{K}) = \alpha_j(\mathcal{K})$ for $j \in \mathbb{N}$;
(iv) If $\{\mathcal{K}_i\}_{i=1,\dots}$ is a sequence of closed sets from $\mathcal{M}_\Omega$ such that $\mathcal{K}_{i+1} \subset \mathcal{K}_i$, $i = 1,\dots,$ and if $\lim_{i\to\infty}\alpha_j(\mathcal{K}_i) = 0$ for each $j \in \mathbb{N}$, then the intersection set $\mathcal{K}_\infty := \cap_{i=1}^{\infty}\mathcal{K}_i$ is nonempty.

Example 1. For $\mathcal{K} \in \mathcal{M}_\Omega$, $y \in \mathcal{K}$, $j \in \mathbb{N}$, and $\varepsilon > 0$, let us denote by $\mu^j(y,\varepsilon)$, $j \in \mathbb{N}$, the modulus of continuity of the function y on the interval $[0,j]$, that is,

$$\mu^j(y,\varepsilon) = \sup\bigl\{\bigl|y(t) - y(\varrho)\bigr|t,\ \varrho \in [0,j],\ |t-\varrho| \leqslant \varepsilon\bigr\}.$$

Further, let us put

$$\mu^j(\mathcal{K},\varepsilon) = \sup\bigl\{\mu^j(y,\varepsilon),\ y \in \mathcal{K}\bigr\}, \qquad \mu_0^j(\mathcal{K}) = \lim_{\varepsilon\to 0^+}\mu^j(\mathcal{K},\varepsilon),$$
$$\bar{\alpha}^j(\mathcal{K}) = \sup_{t\in[0,j]}\alpha\bigl(\mathcal{K}(t)\bigr) := \sup_{t\in[0,j]}\alpha\bigl(\bigl\{y(t),\ y \in \mathcal{K}\bigr\}\bigr),$$

and

$$\alpha_n(\mathcal{K}) = \mu_0^j(\mathcal{K}) + \bar{\alpha}^j(\mathcal{K}).$$

The family of mapping $\{\alpha_n\}_{j\in\mathbb{N}}$, where $\alpha_n : M_\Omega \to \mathbb{R}^+$, satisfies conditions (i)–(iv) from Definition 4.

Definition 5. (See [14].) A nonempty subset $\mathcal{K} \subset \Omega$ is said to be bounded if

$$\sup_{y\in\mathcal{K}}\|y\|_j < \infty, \quad j \in \mathbb{N}.$$

Lemma 2. (*See* [7].) *If Λ is bounded subset of Banach space Ω, then for each $\varepsilon > 0$, there is a sequence $\{y_k\}_{i=1}^{\infty} \subset \Lambda$ such that*

$$\alpha(\Lambda) \leqslant 2\alpha\bigl(\{y_k\}_{i=1}^{\infty}\bigr) + \varepsilon.$$

Lemma 3. (*See* [26].) *If $\{y_i\}_{i=1}^{\infty} \subset L^1(\Theta)$ is uniformly integrable, then $\alpha(\{y_i\}_{i=1}^{\infty})$ is measurable for $j \in \mathbb{N}$, and*

$$\alpha\left(\left\{\int_0^t y_i(\varrho)\,\mathrm{d}\varrho\right\}_{i=1}^{\infty}\right) \leqslant 2\int_0^t \alpha\bigl(\bigl\{y_i(\varrho)\bigr\}_{i=1}^{\infty}\bigr)\,\mathrm{d}\varrho, \quad t \in [0,j].$$

Definition 6. (See [14].) Let Υ be a nonempty subset of a Fréchet space Ω, and let $\Phi : \Upsilon \to \Omega$ be a continuous operator, which transforms bounded subsets of onto bounded ones. One says that Φ satisfies the Darbo condition with constants $(k_j)_{j\in\mathbb{N}}$ with respect to a family of measures of noncompactness $(\alpha_j)_{j\in\mathbb{N}}$ if

$$\alpha_j\bigl(\Phi(\mathcal{K})\bigr) \leqslant k_j\alpha_j(\mathcal{K})$$

for each bounded set $\mathcal{K} \subset \Upsilon$ and $j \in \mathbb{N}$. If $k_j < 1$, $j \in \mathbb{N}$, then Φ is called a contraction with respect to $\{\alpha_j\}_{j\in\mathbb{N}}$.

3 Existence of mild solutions

For sake of simplicity, in the following, we always assume that $\sigma_1(\cdot) = \sigma_2(\cdot)$, and we note that the case $\sigma_1 \neq \sigma_2$ can be studied.

In this section, we present the main results of the existence of solutions for our problem.

Definition 7. A function $y \in C^1([-p, +\infty], E) \times \Upsilon$ is said to be a mild solution of (1)–(3) if y satisfies condition (2) for all $t \in [-p, 0]$, $\mu \in \Upsilon$ and y is solution of the following integral equation:

$$\begin{aligned} y(t,\mu) &= \mathcal{H}(t)\varphi(0,\mu) + \mathcal{G}(t)\varphi'(0,\mu) \\ &\quad + \int_0^t \mathcal{G}(t-\varrho)\, \hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)}, \mu)\,\mathrm{d}\varrho, \quad t \in \mathbb{R}_+,\ \mu \in \Upsilon. \end{aligned}$$

The hypotheses:

(A1) There exists a constant $\boldsymbol{I}_1 > 1$, where

$$\bigl\|\mathcal{G}(t)\bigr\|_{\mathfrak{F}(E)} \leqslant \boldsymbol{I}_1, \quad t \in \mathbb{R}_+.$$

(A2) There exists $\boldsymbol{I}_2 > 0$ such that

$$\bigl\|\mathcal{H}(t)\bigr\| \leqslant \boldsymbol{I}_2, \quad t \in \mathbb{R}_+.$$

(A3) The function $\hbar$ is random Carathéodory on $\mathbb{R}_+ \times C([-p,0],E) \times C([-p,0],E) \times \Upsilon$.

(A4) There exist a continuous function $p : \mathbb{R}_+ \times \Upsilon \to \mathbb{R}_+$ with $p(\cdot,\mu) \in L^1_{\mathrm{Loc}}(\mathbb{R}_+;\mathbb{R}_+)$ and a continuous nondecreasing function $\lambda : \mathbb{R}_+ \to (0,\infty)$ such that for any $\mu \in \Upsilon$,

$$\bigl\|\hbar(t, y, \varkappa, \mu)\bigr\| \leqslant p(t,\mu)\lambda\bigl(\|y\| + \|\varkappa\|\bigr)$$

for a.e. $t \in \mathbb{R}_+$ and $y, \varkappa \in C([-p,0],E)$.

(A5) For each bounded and measurable sets $\mathcal{K}, D \subset C([-p,0],E)$ and for any $\mu \in \Upsilon$, we have

$$\alpha\big(\hbar(t,\mathcal{K},D,\mu)\big) \leqslant p(t,\mu) \sup_{\tau\in[-p,0]} \alpha\big(\mathcal{K}(\tau)+D(\tau)\big), \quad \text{a.e. } t \in \mathbb{R}_+,$$

where α is a Kuratowski measure of noncompactness on the Banach space E.

(A6) There exists a random function $R : \Upsilon \to (0,\infty)$ such that

$$\max(\boldsymbol{I}_2,\boldsymbol{I}_1)\big[\|\varphi(0,\mu)\| + \|\varphi'(0,\mu)\| + \lambda\big(R_j(\mu)\big)p_j^*(\mu)\big] \leqslant R_j(\mu).$$

For $j \in \mathbb{N}$, let

$$p_j^*(\mu) = \int_0^j p(\varrho,\mu)\,\mathrm{d}\varrho$$

and define on $C([-p,\infty),E)$ the family of measure of noncompactness by

$$\alpha_j(D) = \sup_{t\in[-p,j]} \alpha\big(D(t)\big) + \mu_0^j(D),$$

where $D(t) = \{\varkappa(t) \in E; \varkappa \in D\}$, $t \in [-p,j]$.

Theorem 1. *Assume that* (A1)–(A6) *are satisfied, and*

$$\max\{\boldsymbol{I}_1,\boldsymbol{I}_2\}\big[4p_j^*(\mu)\big] < 1 \tag{6}$$

for each $j \in \mathbb{N}$. *Then* (1)–(3) *has at least one random mild solution.*

Proof. Consider the operators $\aleph, \aleph' : \Upsilon \times C^1 \to C^1$ defined by

$$\begin{aligned}\big(\aleph(\mu)y\big)(t) &= \mathcal{H}(t)\varphi(0,\mu) + \mathcal{G}(t)\varphi'(0,\mu)\\ &\quad + \int_0^t \mathcal{G}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)},\mu)\ \mathrm{d}\varrho, \quad t \in \mathbb{R}_+,\end{aligned}$$

and

$$\begin{aligned}\big(\aleph'(\mu)y\big) &= \mathcal{G}(t)\varphi(0,\mu) + \mathcal{H}(t)\varphi'(0,\mu)\\ &\quad + \int_0^t \mathcal{H}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)},\mu)\ \mathrm{d}\varrho, \quad t \in \mathbb{R}_+.\end{aligned}$$

Since the function $\hbar$ is continuous on $\mathbb{R}_+$, then $\aleph(\mu)$ and $\aleph'(\mu)$ define the mappings $\aleph, \aleph' : \Upsilon \times C^1 \to C^1$. Thus, y is a random solution for (1)–(3) if and only if $y = (\aleph(\mu))y$. We will demonstrate that $\aleph$ and $\aleph'$ verify all requirements of Darbo's fixed point theorem for Fréchet spaces [14].

Step 1. $\aleph(\mu)$ and $\aleph'(\mu)$ are a random operators with stochastic domain on C^1. Since $\hbar(t, y, \varkappa, \mu)$ is random Carathéodory, $\mu \to \hbar(t, y, \varkappa, \mu)$ is measurable in view of Definition 1. Therefore, the map

$$\mu \to \mathcal{H}(t)\varphi(0,\mu) + \mathcal{G}(t)\varphi'(0,\mu) + \int_0^t \mathcal{G}(t-\varrho)\,\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)}, \mu)\,\mathrm{d}\varrho$$

is measurable. Thus, $\aleph$ and $\aleph'$ are a random operators on $\Upsilon \times C^1$ into C^1.

Let $\Lambda : \Upsilon \to \mathcal{P}(C^1)$ be the ball given by

$$\begin{aligned}\Lambda(\mu) &= \mathcal{K}_{R_j}(\mu) = \mathcal{K}\big(0, R_j(\mu)\big)\\ &= \big\{y \in C^1\colon \|y\|_j + \|y'\|_j \leqslant R_j(\mu)\big\}, \quad \mu \in \Upsilon,\ j \in \mathbb{N}.\end{aligned}$$

Then $\Lambda(\mu)$ is a bounded, closed, convex, and solid for all $\mu \in \Upsilon$. Consequently, Λ is measurable by Lemma [15]. Let $\mu \in \Upsilon$ be fixed, then from (A1)–(A4), for any $j \in \mathbb{N}$ and each $y, y \in \mu(\mu)$ and $t \in [0, j]$, we get

$$\begin{aligned}\big\|\big(\aleph(\mu)y\big)(t)\big\| &\leqslant \bigg\|\mathcal{H}(t)\varphi(0,\mu) + \mathcal{G}(t)\varphi'(0,\mu)\\ &\quad + \int_0^t \mathcal{G}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)}, \mu)\ \mathrm{d}\varrho\bigg\|\\ &\leqslant \boldsymbol{I}_2\big\|\varphi(0,\mu)\big\| + \boldsymbol{I}_1\big\|\varphi'(0,\mu)\big\| + \boldsymbol{I}_1\int_0^t p(\varrho,\mu)\lambda\big(\|y\|_j + \|y'\|_j\big)\,\mathrm{d}\varrho\\ &\leqslant \boldsymbol{I}_2\big\|\varphi\big(0,\mu\big)\big\| + \boldsymbol{I}_1\big\|\varphi'\big(0,\mu\big)\big\| + \boldsymbol{I}_1\lambda\big(R_j(\mu)\big)\int_0^t p(\varrho,\mu)\,\mathrm{d}\varrho\\ &\leqslant \boldsymbol{I}_2\big\|\varphi(0,\mu)\big\| + \boldsymbol{I}_1\big\|\varphi'(0,\mu)\big\| + \boldsymbol{I}_1\lambda\big(R_j(\mu)\big)p_j^*(\mu)\\ &\leqslant R_j(\mu)\end{aligned}$$

and

$$\begin{aligned}\big\|\big(\aleph'(\mu)y\big)(t)\big\| &\leqslant \bigg\|\mathcal{G}(t)\varphi(0,\mu) + \mathcal{H}(t)\varphi'(0,\mu)\\ &\quad + \int_0^t \mathcal{H}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)}, \mu)\ \mathrm{d}\varrho\bigg\|\\ &\leqslant \boldsymbol{I}_1\big\|\varphi(0,\mu)\big\| + \boldsymbol{I}_2\big\|\varphi'(0,\mu)\big\| + \boldsymbol{I}_2\int_0^t p(\varrho,\mu)\lambda\big(\|y\|_j + \|y'\|_j\big)\,\mathrm{d}\varrho\end{aligned}$$

$$\begin{aligned}
&\leqslant \boldsymbol{I}_1\big\|\varphi(0,\mu)\big\| + \boldsymbol{I}_2\big\|\varphi'(0,\mu)\big\| + \boldsymbol{I}_2\lambda\big(R_j(\mu)\big)\int_0^t p(\varrho,\mu)\,\mathrm{d}\varrho \\
&\leqslant \boldsymbol{I}_1\big\|\varphi(0,\mu)\big\| + \boldsymbol{I}_2\big\|\varphi'(0,\mu)\big\| + \boldsymbol{I}_2\lambda\big(R_j(\mu)\big)p_j^*(\mu) \\
&\leqslant R_j(\mu).
\end{aligned}$$

Therefore, $\aleph$ and $\aleph'$ are random operators with stochastic domain Λ, and $\aleph(\mu) : \Lambda(\mu) \to \aleph(\mu)$, $\aleph'(\mu) : \Lambda(\mu) \to \aleph'(\mu)$. Furthermore, $\aleph(\mu)$ and $\aleph'(\mu)$ map bounded sets into bounded sets in C^1.

Step 2. $\aleph(\mu), \aleph'(\mu) : \mathcal{K}_{R_j}(\mu) \to \mathcal{K}_{R_j}(\mu)$ are continuous. Let $\{y^k\}_{k\in\mathbb{N}}$ be a sequence such that $y^k \to y$ in $\mathcal{K}_{R_j}(\mu)$. Then for each $t \in [0, j]$ and $\mu \in \Upsilon$, we have

$$\begin{aligned}
&\big\|\big(\aleph(\mu)y^k(t)\big) - \big(\aleph(\mu)y\big)(t)\big\| \\
&\quad = \left\|\int_0^t \mathcal{G}(t-\varrho)\big[\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)},\mu\big) - \hbar\big(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)})\big]\,\mathrm{d}\varrho\right\| \\
&\quad \leqslant \big\|\mathcal{G}(t-\varrho)\big\|_{\mathfrak{F}(E)} \\
&\qquad \times \int_0^t \big\|\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)},\mu\big) - \hbar\big(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)},\mu\big)\big\|\,\mathrm{d}\varrho \\
&\quad \leqslant \boldsymbol{I}_1 \int_0^t \big\|\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)},\mu\big) - \hbar\big(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)},\mu\big)\big\|\,\mathrm{d}\varrho.
\end{aligned}$$

Since $y^k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\big\|\aleph\big(y^k\big) - \aleph(y)\big\|_j \to 0 \quad \text{as } k \to +\infty$$

and

$$\begin{aligned}
&\big\|\big(\aleph'(\mu)y^k(t)\big) - \big(\aleph'(\mu)y\big)\big\| \\
&\quad = \left\|\int_0^t \mathcal{H}(t-\varrho)\big[\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)},\mu\big) - \hbar\big(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)})\big]\,\mathrm{d}\varrho\right\| \\
&\quad \leqslant \big\|\mathcal{H}(t-\varrho)\big\|_{\mathfrak{F}(E)} \\
&\qquad \times \int_0^t \big\|\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)},\mu\big) - \hbar\big(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)},\mu\big)\big\|\,\mathrm{d}\varrho \\
&\quad \leqslant \boldsymbol{I}_2 \int_0^t \big\|\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)},\mu\big) - \hbar\big(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)},\mu\big)\big\|\,\mathrm{d}\varrho.
\end{aligned}$$

Since $y^k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\left\|\aleph'\left(y^k\right) - \aleph'(y)\right\|_j \to 0 \quad \text{as } k \to +\infty.$$

As a consequence of Steps 1 and 2, we can conclude that $\aleph(\mu) : \Lambda(\mu) \to \aleph(\mu)$ and $\aleph'(\mu) : \Lambda(\mu) \to \aleph'(\mu)$ are continuous random operators with stochastic domain Λ, and $\aleph(\mu)(\Lambda(\mu))$ and $\aleph'(\mu)(\Lambda(\mu))$ are bounded.

Step 3. For each bounded subset D of $\Lambda(\mu)$,

$$\alpha_j\left(\aleph(\mu)(D)\right) \leqslant l_j \alpha_j(D)$$

and

$$\alpha_j\left(\aleph'(\mu)(D)\right) \leqslant l_j \alpha_j(D).$$

From Lemmas 2 and 3, for any $D \subset \mathcal{K}_{R_j}(\mu)$ and any $\epsilon > 0$, there exist a sequence $\{y_k\}_{k=1}^{\infty} \subset D$ such that for all $t \in [0, j]$ and $\mu \in \Upsilon$, we have

$$\begin{aligned}
&\alpha\big(\big(\aleph(\mu)D\big)(t)\big) \\
&\quad = \alpha\Bigg(\Bigg\{\mathcal{H}(t)\varphi(0,\mu) + \mathcal{G}(t)\varphi'(0,\mu) \\
&\qquad + \int_0^t \mathcal{G}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)}, \mu)\,\mathrm{d}\varrho,\ y, y' \in D\Bigg\}\Bigg) \\
&\quad \leqslant \alpha\Bigg(\Bigg\{\int_0^t \mathcal{G}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho),\mu}, \mu)\,\mathrm{d}\varrho,\ y, y' \in D\Bigg\}\Bigg) \\
&\quad \leqslant 2\alpha\Bigg(\Bigg\{\int_0^t \mathcal{G}(t-\varrho)\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}, \mu\big)\,\mathrm{d}\varrho\Bigg\}_{k=1}^{\infty}\Bigg) + \epsilon \\
&\quad \leqslant 4\int_0^t \alpha\big(\left\|\mathcal{G}(t-\varrho)\right\|_{\mathfrak{F}(E)}\big\{\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}, \mu\big)\big\}_{k=1}^{\infty}\big)\,\mathrm{d}\varrho + \epsilon \\
&\quad \leqslant 4\boldsymbol{I}_1\int_0^t \alpha\big\{\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}, \mu\big)\big\}_{k=1}^{\infty}\big)\,\mathrm{d}\varrho + \epsilon \\
&\quad \leqslant 4\boldsymbol{I}_1\int_0^t p(\varrho,\mu)\alpha\big(\big\{y^k_{\sigma(\varrho,y^k_\varrho,\mu)}\big\}_{k=1}^{\infty} + \big\{y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}\big\}_{k=1}^{\infty}\big)\,\mathrm{d}\varrho + \epsilon \\
&\quad \leqslant 4\boldsymbol{I}_1\alpha(D)\int_0^t p(\varrho,\mu)\,\mathrm{d}\varrho + \epsilon \leqslant 4\boldsymbol{I}_1 p_j^*(\mu)\alpha_j(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha\big(\big(\aleph(\mu)D\big)(t)\big) \leqslant 4\boldsymbol{I}_1 p_j^*(\mu)\alpha_j(D).$$

Thus,

$$\alpha_j\big(\aleph(\mu)D\big) \leqslant 4\boldsymbol{I}_1 p_j^*\alpha_j(D).$$

On the other hand, we have

$$\begin{aligned}
&\alpha\big(\big(\aleph'(\mu)D\big)(t)\big)\\
&\quad= \alpha\left(\left\{\mathcal{G}(t)\varphi(0,\mu) + \mathcal{H}(t)\varphi'(0,\mu)\right.\right.\\
&\qquad \left.\left. + \int_0^t \mathcal{H}(t-\varrho)\,\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho,\mu)}, \mu)\,\mathrm{d}\varrho,\ y, y' \in D\right\}\right)\\
&\quad\leqslant \alpha\left(\left\{\int_0^t \mathcal{H}(t-\varrho)\hbar(\varrho, y_{\sigma(\varrho,y_\varrho,\mu)}, y'_{\sigma(\varrho,y_\varrho),\mu}, \mu)\,\mathrm{d}\varrho,\ y, y' \in D\right\}\right)\\
&\quad\leqslant 2\alpha\left(\left\{\int_0^t \mathcal{H}(t-\varrho)\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}, \mu\big)\,\mathrm{d}\varrho\right\}_{k=1}^{\infty}\right) + \epsilon\\
&\quad\leqslant 4\int_0^t \alpha\big(\big\|\mathcal{G}(t-\varrho)\big\|_{\mathfrak{F}(E)}\big\{\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}, \mu\big)\big\}_{k=1}^{\infty}\big)\,\mathrm{d}\varrho + \epsilon\\
&\quad\leqslant 4\boldsymbol{I}_2\int_0^t \alpha\big\{\hbar\big(\varrho, y^k_{\sigma(\varrho,y^k_\varrho,\mu)}, y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}, \mu\big)\big\}_{k=1}^{\infty}\mathrm{d}\varrho + \epsilon\\
&\quad\leqslant 4\boldsymbol{I}_2\int_0^t p(\varrho,\mu)\alpha\big(\big\{y^k_{\sigma(\varrho,y^k_\varrho,\mu)}\big\}_{k=1}^{\infty} + \big\{y'^k_{\sigma(\varrho,y^k_\varrho,\mu)}\big\}_{k=1}^{\infty}\big)\,\mathrm{d}\varrho + \epsilon\\
&\quad\leqslant 4\boldsymbol{I}_2\alpha(D)\int_0^t p(\varrho,\mu)\,\mathrm{d}\varrho + \epsilon \leqslant 4\boldsymbol{I}_2 p_j^*(\mu)\alpha_j(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha\big(\big(\aleph'(\mu)D\big)(t)\big) \leqslant 4\boldsymbol{I}_2 p_j^*(\mu)\alpha_j(D),$$

Thus,

$$\alpha_j\big(\aleph'(\mu)D\big) \leqslant 4\boldsymbol{I}_2 p_j^*\alpha_j(D).$$

Consequently, by Darbo's fixed point theorem for Fréchet spaces [14] we can conclude that $\aleph$ and $\aleph'$ have at least one fixed point in $\Lambda(\mu)$, which is a random mild solution of problem (1)–(3). $\square$

4 An example

We consider the following abstract differential equation with state dependent delay:

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2}(t,\gamma,\mu) &= \frac{\partial^2 y}{\partial \gamma^2}(t,\gamma,\mu) + \hbar\bigl(t, y\bigl(t-\zeta\bigl(t,y(t,\mu),\mu\bigr)\bigr), \\
&\quad y'\bigl(t-\zeta\bigl(t,y(t,\mu),\mu\bigr),\gamma,\mu\bigr)\bigr), \quad t\in\mathbb{R}^+,\ \gamma\in[0,\pi],\ \mu\in\Upsilon, \\
y(t,0,\mu) &= y(t,\pi,\mu) = 0, \quad t\in\mathbb{R}^+,\ \mu\in\Upsilon, \\
y(\tau,\gamma,\mu) &= \varphi(\tau,\gamma,\mu), \quad \tau\in[-p,0],\ \mu\in\Upsilon, \\
y'(0,\mu) &= \varphi'(0,\mu),
\end{aligned} \tag{7}$$

where $\varphi\in C([-p,0];E)\times\Upsilon$, $\zeta\in C(\mathbb{R}^+\times\mathbb{R};\mathbb{R}_+)$, $\zeta(0,\varphi,\mu)=0$, and $\hbar$ is Carathéodory on $\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\times\Upsilon$. To make system (7) as problem (1)–(3), we need to define $\hbar:\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\times\Upsilon\to E$ and $\sigma:\mathbb{R}^+\times C([-p,0];\mathbb{R})\times\Upsilon\to\mathbb{R}$ by

$$\hbar(t,\lambda,\phi,\mu)(\gamma) = \hbar\bigl(t,\lambda(0,\gamma,\mu),\phi(0,\gamma,\mu),\mu\bigr)$$

and

$$\sigma(\varrho,\lambda,\mu) = \varrho - \zeta\bigl(\varrho,\lambda(0,\mu),\mu\bigr).$$

It is easy to show that $\hbar(\cdot)$ and $\sigma(\cdot)$ are Lipschitz, which implies that hypotheses (A3)–(A5) are satisfied.

Consider $E=L^2([0,\pi],\mathbb{R})$ and the domain

$$D(\Phi) := \bigl\{y\in E\colon\ y''\in E,\ y(0)=y(\pi)=0\bigr\}.$$

Let the operator $\Phi: D(\Phi)\subset E\to E$ be the operator given by $\Phi y = y''$. Clearly, Φ is the infinitesimal generator of a strongly continuous cosine family $(\mathcal{H}(t))_{t\in\mathbb{R}}$ on E. The spectrum of Φ is discrete with eigenvalues $-j^2$, $j\in\mathbb{N}$, and eigenvectors

$$z_j(\zeta) := \left(\frac{1}{\pi}\right)^{1/2}\sin(j\zeta).$$

The set of functions $\{z_j,\ j\in\mathbb{N}\}$ is an orthonormal basis of E. We note that

$$\mathcal{H}(t)y = \sum_{j=1}^{\infty}\cos(jt)\langle y,z_j\rangle z_j,$$

$$\mathcal{G}(t)y = \sum_{j=1}^{\infty}\frac{\sin(jt)}{j}\langle y,z_j\rangle z_j,$$

the sine family $\mathcal{G}(\cdot)$ is compact, $\|\mathcal{H}(t)\| = \|\mathcal{G}(t)\| = 1$ for all $t\in\mathbb{R}$. Thus, hypotheses (A1) and (A2) are satisfied.

Furthermore, by choosing a suitable function R we can verify that (A6) and (6) hold. Thus, Theorem 1 implies that (7) has at least one mild solution on $[-p,+\infty)$.

5 Conclusions

In the present research, we have investigated the random mild solutions for a general class of second-order abstract random differential equations with state-dependent delay. To achieve the desired results for the given problem, the fixed point approach was used, namely, the Darbo fixed point theorem for Fréchet spaces associated with the concept of measures of noncompactness. An example is provided to demonstrate how the major results can be applied. Our results in the given configuration are novel and contribute to the literature on this field of study. We believe that there exist numerous potential avenues for further exploration, including but not limited to coupled systems, problems involving infinite delays, extensions to fractional-order derivatives, and more. The limited number of publications on this class of differential equations suggests promising opportunities for additional research.

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