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Large deviations for stochastic predator–prey model with Lévy noise

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Abstract. This paper discusses the large deviations for stochastic predator–prey model driven by multiplicative Lévy noise. Using Galerkin approximation, we initially prove the existence and uniqueness of solution. Due to the equivalence between Laplace principle and large deviation principle under a Polish space, the method of weak convergence has been followed in order to establish our results for this coupled system of equations.

Keywords: large deviation principle, predator–prey model, weak convergence, Lévy noise.

1 Introduction

Biological models play a crucial role in maintaining ecological balance by determining the existence, behavior and exit time of various species in the environment. Many population models depict the ideal behavior of variety of species by analyzing their mutations, interdependence among them and their death rate. The initial descriptions of the predator–prey population was given by the model constructed independently by Lotka (1925) and Volterra (1926). Though this model proved to be a kick-start for many such biological models, its shortcomings like exclusion of many unavoidable factors were inevitable. Later, many more illustrious models were developed based on this model by bringing in more realistic factors into consideration.

The earlier models denoting the predator–prey population available in the literature were deterministic in nature. The prevalence of many random factors, which are not deterministic yet has the capability of destabilizing or affecting the ecology, were neglected in these models thus failing to completely model the phenomenon or process. This concern can be mitigated by considering stochastic versions of the models, which are prone to more accuracy than their deterministic counterparts.

In our case, a stochastic predator–prey model seems to be efficient in use since randomness occurs naturally in these systems. Fluctuations in a stable environment caused due to random factors like a natural calamity reiterates the necessity of opting a stochastic model. These disturbances can be modeled using noise terms.

In literature, there are abundant works on stochastic differential equations perturbed by Brownian motion. The choice of noise used is completely based on the characteristics of disturbances affecting the systems. The occurrence of flood or an earthquake can cause sudden discrete jumps in the population, and hence combination of continuous random process and random jumps are more appropriate for our system. These combined effects are modeled using Lévy noise [1] in the system considered in this work.

Large deviations is the analysis of decay of tail probabilities of those events with very less probability of occurrence but has a massive impact on the event of occurrence. The concept of large deviations has gained popularity over the recent years only because it has become evident that the knowledge of dynamics of such rare events have proven to be actually essential in many ways. It is used to gain knowledge about the impact, either positive or negative created due to rare events with the help of a rate function. Though many other theorems like law of large numbers and central limit theorem talk about the deviations, the Large Deviation Principle (LDP) deals with the deviations of greater order than those considered in other theorems.

Formally put forward by Varadhan [16], this theory was developed over the years. Out of the many available methods, we employ the weak convergence method given by Dupuis and Ellis [7]. In [4], Budhiraja and Dupuis instituted a variational representation for positive functional of Brownian motion to the study of large deviations for varied differential equations. Large deviations for equations perturbed by Wiener noise has an extensive literature [5, 10, 13]. In [15], LDP is analysed for tidal dynamics equation. For equations perturbed by Lévy noise, the available literature is not wide enough as in the previous case. The theory of large deviations for stochastic partial differential equations with pure jump noise is outlined in [3], and [9] establishes the principle for shell model of turbulence. Stochastic partial differential equations with Neumann boundary conditions is rarely dealt with. Few works on it include [2], which proves existence of unique solution for stochastic Landau–Lifshitz–Bloch equation, and [11], which establishes LDP for the same.

The predator–prey model considered here is an extension from the model in [8]. The deterministic model considered with Holling type III functional response is

$$\frac{\partial u_1}{\partial t} - \eta_1 \Delta u_1 = u_1(\alpha - u_1) - \frac{\beta u_1^2 u_2}{1 + u_1^2},$$
$$\frac{\partial u_2}{\partial t} - \eta_2 \Delta u_2 = \frac{\gamma u_1^2 u_2}{1 + u_1^2} - \delta u_2$$

with $u_{1,0}$ and $u_{2,0}$ being the initial population equipped with homogeneous Neumann boundary conditions. For detailed study of impact on functional responses on predator–prey models, we refer to [12]. In [14], the LDP is established for the above equation perturbed by Brownian motion. In this paper, we consider this model perturbed with Lévy noise, prove the existence and uniqueness of the solution using Galerkin approximation technique and then establish LDP for the same. The study of LDP for population models prevails as an essential element required to maintain ecological balance. LDP can be used to handle the effects of random events in population dynamics like evolution of predators, sudden damage of prey food due to flood or any event causing catastrophic effects. It aids in these situations by providing the rate at which these events deviates. The analytical proof for this principle is being done in this paper from which numerical results can be generated in future.

The method of weak convergence is used in order to establish LDP in which we consider a compact set of stochastic controls and prove that the stochastic control version of our system converges to its deterministic one whenever the noise parameter tends to zero.

2 Mathematical formulation

Consider $D \subset \mathbb{R}^2$ to be the bounded domain on which the model is defined, and let $[0, T]$ for a finite T be the interval on which the populations are studied. The stochastic predator–prey model (u_1 – prey, and u_2 – predator) with noise perturbed by a small parameter $\varepsilon > 0$ is given by

$$\begin{aligned}\frac{\partial u_1^\varepsilon}{\partial t} - \eta_1 \Delta u_1^\varepsilon &= u_1^\varepsilon\bigl(\alpha - u_1^\varepsilon\bigr) - \frac{\beta (u_1^\varepsilon)^2 u_2^\varepsilon}{1 + (u_1^\varepsilon)^2} + \sqrt{\varepsilon}\sigma_1\bigl(t, u_1^\varepsilon\bigr)\,\mathrm{d}W_1(t)\\ &\quad + \varepsilon \int_Z g_1\bigl(u_1^\varepsilon, z\bigr)\,\tilde{N}_1(\mathrm{d}t, \mathrm{d}z),\\ \frac{\partial u_2^\varepsilon}{\partial t} - \eta_2 \Delta u_2^\varepsilon &= \frac{\gamma (u_1^\varepsilon)^2 u_2^\varepsilon}{1 + (u_1^\varepsilon)^2} - \delta u_2^\varepsilon + \sqrt{\varepsilon}\sigma_2\bigl(t, u_2^\varepsilon\bigr)\,\mathrm{d}W_2(t)\\ &\quad + \varepsilon \int_Z g_2\bigl(u_2^\varepsilon, z\bigr)\,\tilde{N}_2(\mathrm{d}t, \mathrm{d}z)\end{aligned}$$

with $u_{1,0}^\varepsilon$ and $u_{2,0}^\varepsilon$ being the initial populations equipped with Neumann boundary conditions $\partial u_1^\varepsilon/\partial\nu = 0$ and $\partial u_2^\varepsilon/\partial\nu = 0$, where ν is an outward normal to the boundary ∂D. Here η_1 and η_2 are diffusion coefficients of prey and predator, respectively. Also, α is the carrying capacity of prey, δ is the death rate of predator, β and γ are ratios depending upon factors like the intrinsic growth rate, number of newly born predators for each captured prey and death rate of predators. For $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$, the abstract formulation of stochastic predator–prey model driven by Lévy noise is given by

$$\mathrm{d}u^\varepsilon + Au^\varepsilon\,\mathrm{d}t = f\bigl(u^\varepsilon\bigr)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma\bigl(t, u^\varepsilon\bigr)\,\mathrm{d}W(t) + \varepsilon \int_Z g\bigl(u^\varepsilon, z\bigr)\,\tilde{N}(\mathrm{d}t, \mathrm{d}z), \tag{1}$$

$$u^\varepsilon(0) = \bigl(u_1^\varepsilon(0), u_2^\varepsilon(0)\bigr) = u_0, \qquad \frac{\partial u^\varepsilon}{\partial \nu} = 0.$$

Here

- $W(t)$=$(W_1(t), W_2(t))$ – independent Wiener process;
- $\tilde{N}$ = $(\tilde{N}_1, \tilde{N}_2)$ – compensated Poisson random measure;

- The linear operator A and nonlinear functional f is given by

$$A = \begin{pmatrix} -\eta_1\Delta - \alpha & 0 \\ 0 & -\eta_2\Delta + \delta \end{pmatrix}, \qquad f(u) = \begin{pmatrix} -u_1^2 - \frac{\beta u_1^2 u_2}{1+u_1^2} \\ \frac{\gamma u_1^2 u_2}{1+u_1^2} \end{pmatrix}, \quad u = (u_1, u_2);$$

- The functions $\sigma(t, u^\varepsilon) = (\sigma_1(t, u_1^\varepsilon), \sigma_2(t, u_2^\varepsilon))$ and $g(u^\varepsilon, z) = (g_1(u_1^\varepsilon, z), g_1(u_1^\varepsilon, z))$ are noise coefficients subject to conditions stated later.

The deterministic equation corresponding to (1) is

$$\mathrm{d}u^0 + Au^0\,\mathrm{d}t = f\big(u^0\big)\,\mathrm{d}t$$

with $u(0) = u_0$. In this paper, let us denote the Lebesgue space $H = \mathbb{L}^2(D)$ and the Sobolev space $V = \mathbb{H}^1(D)$ with the norms defined, respectively, as follows. For $u \in H$ and $v \in V$,

$$|u|_H^2 = |u|^2 = \int_D \big|u(x)\big|^2\,\mathrm{d}x, \qquad |v|_V^2 = |v|^2 + |\nabla v|^2.$$

Similarly, the Sobolev space $\mathbb{H}^2(D)$ is equipped with $|u|_{\mathbb{H}^2}^2 = |u|^2 + |\nabla u|^2 + |\Delta u|^2$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be the complete filtered probability space on which the independent Wiener process W is defined. Here W is an $\mathbb{L}^2(D)$-valued process. Let Q be its covariance operator, which is a linear, symmetric, positive operator in H such that $H_0 = Q^{1/2}H$ is a Hilbert space with inner product

$$(\phi, \psi)_0 = \big(Q^{-1/2}\phi, Q^{-1/2}\psi\big) \quad \text{for } \phi, \psi \in H_0.$$

Let $\mathcal{L}_Q$ be the space of linear operators S such that $SQ^{1/2}$ is a Hilbert–Schmidt operator from H to H with the norm $|S|_{\mathcal{L}_Q} = \mathrm{tr}(SQS^*)$. For a locally compact Polish space Z, let N_i, $i = 1, 2$, be Poisson random measures defined on $[0, T] \times Z$, independent of W_i. Then let $\tilde{N}_i$ be the compensated Poisson measure with the compensator λ_i for $i = 1, 2$. Let

$$\mathbb{H}_\lambda^2\big([0,T] \times Z; H\big) = \Bigg\{ g : [0,T] \times Z \to H\colon\ g \text{ is measurable and } \int_0^T \int_Z \mathbf{E}\big(\big|g(t,z,\omega)\big|_H^2\big)\,\lambda(\mathrm{d}z)\,\mathrm{d}t < \infty \Bigg\}.$$

The space $\mathcal{D}([0,T]; \mathbb{H}^1)$ denotes the space of càdlàg functions from $[0,T]$ to $\mathbb{H}^1$. The assumptions on the noise coefficients σ and g are as follows.

The functions $\sigma \in C([0,T] \times V; \mathcal{L}_Q(H_0; H))$ and $g \in \mathbb{H}_\lambda^2([0,T] \times Z; H)$ satisfy:

(A1) For all $t \in [0,T]$, there exists a constant $K_1 > 0$ such that for all $u \in V$,

$$\big|\sigma(t,u)\big|_{\mathcal{L}_Q}^2 + \int_Z \big|g(u,z)\big|^2\,\lambda(\mathrm{d}z) \leqslant K_1\big(1 + |\nabla u|^2\big).$$

(A2) For all $t \in [0,T]$, there exists a constant $K_2 > 0$ such that for all $u, v \in V$,

$$\left|\sigma(t,u) - \sigma(t,v)\right|_{\mathcal{L}_Q}^2 + \int\limits_Z \left|g(u,z) - g(v,z)\right|^2 \lambda(\mathrm{d}z) \leqslant K_2\left(\left|\nabla(u-v)\right|^2\right).$$

3 Existence results

We now establish the existence results for the solution of (1). A weak solution of (1) is a process

$$u^\varepsilon(t,\omega) \in \mathbb{L}^p\big(\Omega;\, \mathcal{D}\big([0,T];\mathbb{H}^1\big) \cap \mathbb{L}^2\big((0,T);\mathbb{H}^2\big)\big) \quad \text{for } p > 2,$$

which satisfies the initial condition u_0, and for $\psi \in D(A)$,

$$\begin{aligned}
&\big(u^\varepsilon(t),\psi\big) - (u_0,\psi) + \int\limits_0^t \big[\big(Au^\varepsilon(s),\psi\big) - \big(f\big(u^\varepsilon(s)\big),\psi\big)\big]\,\mathrm{d}s \\
&\quad = \sqrt{\varepsilon}\int\limits_0^t \big(\sigma\big(s,u^\varepsilon(s)\big)\,\mathrm{d}W,\psi\big) + \varepsilon\int\limits_0^t\int\limits_Z \big(g\big(u^\varepsilon(s),z\big),\psi\big)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}$$

The properties of the linear operator A and the nonlinear functional f required for further estimations are given below. For detailed proof, one can refer to [14].

Lemma 1. *The operator A satisfies for $u \in \mathbb{H}^1$,*

$$(Au,u) \geqslant \eta|\nabla u|^2 - \alpha|u_1|^2, \tag{2}$$

where $\eta = \eta_1 \wedge \eta_2$.

Lemma 2. *For $u, v \in \mathbb{H}^1$,*

(i) *Boundedness*:

$$\big(f(u),u\big) \leqslant \frac{\beta}{2}|u_1|^2 + \left(\frac{\beta}{2} + \gamma\right)|u_2|^2; \tag{3}$$

(ii) *Lipschitz continuity*:

$$\begin{aligned}
2\big(f(u) - f(v),\, z\big) &\leqslant \eta|\nabla z|^2 + \frac{4}{\eta}\big(1+\beta^2\big)\big(|u|^2 + |v|^2\big)|z_1|^2 + \frac{2\beta^2 C^2}{\eta}|z|^2 \\
&\quad + \frac{\gamma^2}{\eta}\big(|u_2|^2 + |v_2|^2\big)|z|^2 + \frac{8\gamma^2 C_a}{\eta}|z_2|^2,
\end{aligned} \tag{4}$$

where $z = u - v$ and $C_a = (\text{area}(D))^{1/2}$.

We now prove the existence of weak solution of (1) with pathwise uniqueness in $\mathbb{X} = \mathcal{D}([0,T];\mathbb{H}^1) \cap \mathbb{L}^2((0,T);\mathbb{H}^2)$.

Theorem 1. *Let* (A1)–(A2) *and* $\mathbf{E}|u_0|^2 < \infty$ *hold. Then there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in [0, \varepsilon_0]$*, there exists a pathwise unique weak solution* u^ε *for the stochastic equation* (1) *in* $\mathbb{X} = \mathcal{D}([0,T];\mathbb{H}^1) \cap \mathbb{L}^2((0,T);\mathbb{H}^2)$ *with* $u^\varepsilon(0) = u_0 \in H$ *such that*

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\left|u^\varepsilon(t)\right|^2 + \eta\mathbf{E}\left(\int_0^T\left|\nabla u^\varepsilon(s)\right|^2\mathrm{d}s\right) \leqslant C\left(1+\mathbf{E}|u_0|^2\right),$$

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\left|\nabla u^\varepsilon(t)\right|^2 + \eta\mathbf{E}\left(\int_0^T\left|\Delta u^\varepsilon(s)\right|^2\mathrm{d}s\right) \leqslant C\left(1+\mathbf{E}|u_0|^2\right),$$

where C *is an appropriate constant.*

We use Galerkin approximation to prove the existence. Let a complete orthonormal basis of the space H be $\{\varphi_n\}_{n\geqslant 1}$ such that $\varphi_n \in D(A)$. For any $n \geqslant 1$, let $H_n = \operatorname{span}(\varphi_1, \dots, \varphi_n) \in D(A)$ and $P_n : H \to H_n$ be an orthogonal projection onto H_n, which contracts H and V norms. Let $W_n = P_n W$, $\sigma_n = P_n\sigma$ and $g_n = P_n g$. Then for $\psi \in H_n$, consider the equation in H_n

$$\begin{aligned}\left(\mathrm{d}u_n^\varepsilon, \psi\right) &= \left(-Au_n^\varepsilon + f\left(u_n^\varepsilon\right), \psi\right)\mathrm{d}t + \sqrt{\varepsilon}\left(\sigma_n\left(t, u_n^\varepsilon\right)\mathrm{d}W_n, \psi\right) \\ &\quad + \varepsilon\int_Z \left(g_n\left(u_n^\varepsilon, z\right), \psi\right)\tilde{N}(\mathrm{d}t, \mathrm{d}z)\end{aligned} \tag{5}$$

with $u_n^\varepsilon(0) = P_n u_0$. The Lipschitz property satisfied by the coefficients assures well-posedness, and hence, there exists a maximal solution to (5), that is, a stopping time $\tau_n^\varepsilon \leqslant T$ such that for $t < \tau_n^\varepsilon$, (5) holds, and for $t \uparrow \tau_n^\varepsilon < T$, $|u_n^\varepsilon(t)| \to \infty$. We now prove $\tau_n^\varepsilon = T$ and estimate u_n^ε for all n and $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$. For $N > 0$, take

$$\tau_N = \inf\left\{t\colon \left|u_n^\varepsilon(t)\right| + \left|\nabla u_n^\varepsilon(t)\right| \geqslant N\right\} \wedge T.$$

Then on $\{\tau_N = T\}$, $u_n^\varepsilon \in \mathcal{D}([0,T], H_n)$ a.s. We require the following lemma proved in [6] for further proofs.

Lemma 3. *Let* X*,* Y *and* I *be nondecreasing and nonnegative processes,* φ *be a nonnegative process and* Z *be nonnegative integrable random variable. Assume that* $\int_0^T \varphi(t)\,\mathrm{d}t \leqslant C$ *a.s., and that there exist positive constants* $a, b \leqslant 1/(2(1 + C\mathrm{e}^C))$*,* $m \leqslant \alpha/(2(1 + C\mathrm{e}^C))$ *and* $\tilde{C}$ *such that for* $0 \leqslant t \leqslant T$*,*

$$X(t) + aY(t) \leqslant Z(t) + \int_0^T \varphi(s)X(s)\,\mathrm{d}s + I(t) \quad \textit{a.s.,}$$

$$\mathbf{E}\big(I(t)\big) \leqslant b\mathbf{E}\big(X(t)\big) + m\mathbf{E}\big(Y(t)\big) + \tilde{C}.$$

Then if $X \in \mathbb{L}^\infty([0,T] \times \Omega)$*, we have for* $t \in [0,T]$*,*

$$\mathbf{E}\big[X(t) + aY(t)\big] \leqslant 2\big(1 + C\mathrm{e}^C\big)\big(\mathbf{E}(Z) + \tilde{C}\big).$$

Proposition 1. *For an integer $p \geqslant 1$ and $\mathbf{E}|u_0|^{2p} < \infty$, there exists $\varepsilon_{0,p}$ such that for $0 \leqslant \varepsilon \leqslant \varepsilon_{0,p}$, $\tau_n^\varepsilon = T$, and there exists a unique solution $u_n^\varepsilon \in \mathcal{D}([0,T], H_n)$ satisfying*

$$\mathbf{E} \sup_{0\leqslant t\leqslant T} \left|u_n^\varepsilon(t)\right|^{2p} + \eta \mathbf{E}\left(\int_0^T \left|\nabla u_n^\varepsilon(s)\right|^2 \mathrm{d}s \right)^p \leqslant C\left(1 + \mathbf{E}|u_0|^{2p}\right),$$

$$\mathbf{E} \sup_{0\leqslant t\leqslant T} \left|\nabla u_n^\varepsilon(t)\right|^{2p} + \eta \mathbf{E}\left(\int_0^T \left|\nabla u_n^\varepsilon(s)\right|^{2p-2} \left|\Delta u_n^\varepsilon(s)\right|^2 \mathrm{d}s \right)^p \leqslant C\left(1 + \mathbf{E}|u_0|^{2p}\right),$$

where C is an appropriate constant.

Proof. For $t \in [0,T]$ and τ_N, applying Itô's formula for $|u_n^\varepsilon|^2$,

$$\begin{aligned}
\left|u_n^\varepsilon(t\wedge\tau_N)\right|^2 = |P_n u_0|^2 &+ 2 \int_0^{t\wedge\tau_N} \left(-Au_n^\varepsilon(s) + f\left(u_n^\varepsilon(s)\right), u_n^\varepsilon(s)\right) \mathrm{d}s \\
&+ 2\sqrt{\varepsilon} \int_0^{t\wedge\tau_N} \left(\sigma_n\left(s, u_n^\varepsilon(s)\right) \mathrm{d}W_n, u_n^\varepsilon(s)\right) \\
&+ \varepsilon \int_0^{t\wedge\tau_N} \left|\sigma_n\left(s, u_n^\varepsilon(s)\right)\right|_{\mathcal{L}_Q}^2 \mathrm{d}s + \varepsilon \int_0^{t\wedge\tau_N}\int_Z \left|g_n\left(u_n^\varepsilon(s), z\right)\right|^2 N(\mathrm{d}s, \mathrm{d}z) \\
&+ 2\varepsilon \int_0^{t\wedge\tau_N}\int_Z \left(g_n\left(u_n^\varepsilon(s), z\right), u_n^\varepsilon(s)\right) \tilde{N}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Applying Itô's formula for $|u_n^\varepsilon(t)|^p$,

$$\left|u_n^\varepsilon(t\wedge\tau_N)\right|^{2p} = |P_n u_0|^{2p} + I_1 + I_2 + I_3 + J_1,$$

where

$$I_1 = 2p \int_0^{t\wedge\tau_N} \left(-Au_n^\varepsilon(s) + f\left(u_n^\varepsilon(s)\right),\ u_n^\varepsilon(s)\right) \left|u_n^\varepsilon(s)\right|^{2(p-1)} \mathrm{d}s,$$

$$\begin{aligned}
I_2 = \varepsilon p &\int_0^{t\wedge\tau_N} \left|\sigma_n\left(s, u_n^\varepsilon(s)\right)\right|_{\mathcal{L}_Q}^2 \left|u_n^\varepsilon(s)\right|^{2(p-1)} \mathrm{d}s \\
&+ 2p(p-1)\varepsilon \int_0^{t\wedge\tau_N} \left|\Pi_n \sigma_n\left(s, u_n^\varepsilon(s)\right) u_n^\varepsilon(s)\right|_{\mathcal{L}_Q}^2 \left|u_n^\varepsilon(s)\right|^{2(p-2)} \mathrm{d}s,
\end{aligned}$$

$$I_3 = \int_0^{t\wedge\tau_N}\int_Z \left[\left|u_n^\varepsilon(s-) + \varepsilon g_n\left(u_n^\varepsilon(s-),\ z\right)\right|^{2p} - \left|u_n^\varepsilon(s-)\right|^{2p}\right] \tilde{N}(\mathrm{d}s, \mathrm{d}z)$$

$$+ \int_0^{t\wedge\tau_N}\int_Z \bigl[\bigl|u_n^\varepsilon(s-) + \varepsilon g_n\bigl(u_n^\varepsilon(s-),\, z\bigr)\bigr|^{2p} - \bigl|u_n^\varepsilon(s-)\bigr|^{2p}\bigr]\,\lambda(\mathrm{d}z)\,\mathrm{d}s$$

$$- 2p\varepsilon \int_0^{t\wedge\tau_N}\int_Z \bigl(g_n\bigl(u_n^\varepsilon(s-), z\bigr),\, u_n^\varepsilon(s-)\bigr)\bigl|u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\lambda(\mathrm{d}z)\,\mathrm{d}s,$$

$$J_1 = 2p\sqrt{\varepsilon} \int_0^{t\wedge\tau_N} \bigl(\sigma_n\bigl(s, u_n^\varepsilon(s)\bigr)\,\mathrm{d}W_n,\, u_n^\varepsilon(s)\bigr)\bigl|u_n^\varepsilon(s)\bigr|^{2(p-1)}.$$

Considering each term separately, using (2) and (3),

$$I_1 \leqslant 2p \int_0^{t\wedge\tau_N} \bigl[-\eta\bigl|\nabla u_n^\varepsilon(s)\bigr|^2 + c_1\bigl|u_n^\varepsilon(s)\bigr|^2\bigr]\bigl|u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\mathrm{d}s,$$

where $c_1 = \alpha + \beta + \gamma$. Using (A1),

$$\begin{aligned} I_2 &= \varepsilon p \int_0^{t\wedge\tau_N} \bigl[\bigl|\sigma_n\bigl(s, u_n^\varepsilon(s)\bigr)\bigr|^2_{\mathcal{L}_Q} + 2(p-1)\bigl|\Pi_n\sigma_n\bigl(s, u_n^\varepsilon(s)\bigr)\bigr|^2_{\mathcal{L}_Q}\bigr]\bigl|u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\mathrm{d}s \\ &\leqslant \varepsilon p(2p-1)K_1 \int_0^{t\wedge\tau_N} \bigl(1 + \bigl|\nabla u_n^\varepsilon(s)\bigr|^2\bigr)\bigl|u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\mathrm{d}s. \end{aligned}$$

Using Taylor's formula and Cauchy–Schwarz inequality,

$$\begin{aligned} I_3 = &\int_0^{t\wedge\tau_N}\int_Z \bigl[\bigl|u_n^\varepsilon(s-) + \varepsilon g_n\bigl(u_n^\varepsilon(s-), z\bigr)\bigr|^{2p} - \bigl|u_n^\varepsilon(s-)\bigr|^{2p}\bigr]\,N(\mathrm{d}s, \mathrm{d}z) \\ &+ 2p\varepsilon \int_0^{t\wedge\tau_N}\int_Z \bigl(g_n\bigl(u_n^\varepsilon(s-), z\bigr), u_n^\varepsilon(s-)\bigr)\bigl|u_n^\varepsilon(s-)\bigr|^{2(p-1)}\,\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &- 2p\varepsilon \int_0^{t\wedge\tau_N}\int_Z \bigl(g_n\bigl(u_n^\varepsilon(s-), z\bigr), u_n^\varepsilon(s)\bigr)\bigl|u_n^\varepsilon(s-)\bigr|^{2(p-1)}\,N(\mathrm{d}s, \mathrm{d}z) \\ \leqslant\, &2p(2p-1)\varepsilon \int_0^{t\wedge\tau_N}\int_Z \bigl|u_n^\varepsilon(s-)\bigr|^{2(p-1)}\bigl|g_n\bigl(u_n^\varepsilon(s-), z\bigr)\bigr|^2\,N(\mathrm{d}s, \mathrm{d}z) \\ &+ 2p\varepsilon \int_0^{t\wedge\tau_N}\int_Z \bigl(g_n\bigl(u_n^\varepsilon(s-), z\bigr), u_n^\varepsilon(s-)\bigr)\bigl|u_n^\varepsilon(s-)\bigr|^{2(p-1)}\,\tilde{N}(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

Combining all the estimates, taking supremum and then taking expectation, we get

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\left|u_n^\varepsilon(s)\right|^{2p}+2p\eta\mathbf{E}\int_0^{t\wedge\tau_N}\left|\nabla u_n^\varepsilon(s)\right|^2\left|u_n^\varepsilon(s)\right|^{2(p-1)}\,\mathrm{d}s\\
&\quad\leqslant \mathbf{E}|P_nu_0|^{2p}+2pc_1\mathbf{E}\int_0^{t\wedge\tau_N}\left|u_n^\varepsilon(s)\right|^{2p}\,\mathrm{d}s+\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\{J_1+J_2\}\\
&\qquad+3p(2p-1)\varepsilon K_1\mathbf{E}\int_0^{t\wedge\tau_N}\left[\left|u_n^\varepsilon(s)\right|^{2(p-1)}\left(1+\left|\nabla u_n^\varepsilon(s)\right|^2\right)\right]\mathrm{d}s,
\end{aligned}$$

where

$$J_2=2p\varepsilon\int_0^{t\wedge\tau_N}\int_Z\left(g_n\left(u_n^\varepsilon(s),z\right),u_n^\varepsilon(s)\right)\left|u_n^\varepsilon(s)\right|^{2(p-1)}\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

Using Burkholder–Davis–Gundy inequality, Young's inequality and (A1), for $C_1>0$,

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\{J_1\}\\
&\quad\leqslant 2p\sqrt{\varepsilon}C_1\mathbf{E}\left\{\int_0^{t\wedge\tau_N}\left|\sigma_n\left(s,u_n^\varepsilon(s)\right)\right|_{\mathcal{L}_Q}^2\left|u_n^\varepsilon(s)\right|^{4p-2}\,\mathrm{d}s\right\}^{1/2}\\
&\quad\leqslant \delta\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\left|u_n^\varepsilon(s)\right|^{2p}+\frac{p^2\varepsilon C_1^2K_1}{\delta}\mathbf{E}\int_0^{t\wedge\tau_N}\left(1+\left|\nabla u_n^\varepsilon(s)\right|^2\right)\left|u_n^\varepsilon(s)\right|^{2(p-1)}\,\mathrm{d}s.
\end{aligned}$$

Similarly using Burkholder–Davis–Gundy inequality, Young's inequality and (A1),

$$\begin{aligned}
\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\{J_2\}&\leqslant \delta\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\left|u_n^\varepsilon(s)\right|^{2p}\\
&\quad+\frac{p^2\varepsilon^2C_1^2K_1}{\delta}\mathbf{E}\int_0^{t\wedge\tau_N}\left(1+\left|\nabla u_n^\varepsilon(s)\right|^2\right)\left|u_n^\varepsilon(s)\right|^{2(p-1)}\,\mathrm{d}s.
\end{aligned}$$

Comparing with Lemma 3, assign

$$X(t)=\sup_{0\leqslant s\leqslant t}\left|u_n^\varepsilon(s\wedge\tau_N)\right|^{2p},\qquad Y(t)=\int_0^{t\wedge\tau_N}\left|\nabla u_n^\varepsilon(s)\right|^2\left|u_n^\varepsilon(s)\right|^{2(p-1)}\,\mathrm{d}s,$$

$$Z(t)=|u_0|^{2p}+3p(2p-1)\varepsilon K_1\int_0^{\tau_N}\left|u_n^\varepsilon(s)\right|^{2(p-1)}\,\mathrm{d}s,$$

$$I(t) = \sup_{0\leqslant s\leqslant t}\{J_1 + J_2\}, \qquad \varphi(t) = 2pc_1, \qquad \int\limits_0^T \varphi(t)\,\mathrm{d}t = C = 2pc_1T,$$

$$a = p\eta, \quad b = 2\delta = \frac{1}{2(1+C\mathrm{e}^C)}, \qquad m \leqslant ab = \frac{p\eta}{2(1+C\mathrm{e}^C)} = \frac{p^2C_1^2K_1}{\delta}(\varepsilon+\varepsilon^2),$$

$$\tilde{C} = \frac{p^2C_1^2K_1}{\delta}\big(\varepsilon+\varepsilon^2\big)\int\limits_0^t \big|u_n^\varepsilon(s)\big|^{2(p-1)}\,\mathrm{d}s.$$

Using induction argument and Lemma 3 for $t = T$, it is easy to prove that for

$$\varepsilon_{0,p} = 1 \wedge \frac{\eta}{3K_1(2p-1)} \wedge \frac{\eta}{8pC_1^2K_1(1+C\mathrm{e}^C)^2}$$

and for $0 \leqslant \varepsilon \leqslant \varepsilon_{0,p}$, we have

$$\sup_n \mathbf{E}\Bigg(\sup_{0\leqslant s\leqslant \tau_N}\big|u_n^\varepsilon(s)\big|^{2p} + p\eta\int\limits_0^{\tau_N}\big|\nabla u_n^\varepsilon(s)\big|^2\big|u_n^\varepsilon(s)\big|^{2(p-1)}\,\mathrm{d}s\Bigg) \leqslant C(p,T,K_1).$$

Applying Itô's formula for $|\nabla u_n^\varepsilon|^{2p}$,

$$\big|\nabla u_n^\varepsilon(t\wedge\tau_N)\big|^{2p} = |P_n\nabla u_0|^{2p} + I_4 + I_5 + I_6 + J_3,$$

where

$$I_3 = 2p\int\limits_0^{t\wedge\tau_N}\big(-Au_n^\varepsilon(s) + f\big(u_n^\varepsilon(s)\big),\,\nabla u_n^\varepsilon(s)\big)\big|\nabla u_n^\varepsilon(s)\big|^{2(p-1)}\,\mathrm{d}s,$$

$$I_4 = \varepsilon p\int\limits_0^{t\wedge\tau_N}\big|\sigma_n\big(s,u_n^\varepsilon(s)\big)\big|_{\mathcal{L}_Q}^2\big|\nabla u_n^\varepsilon(s)\big|^{2(p-1)}\,\mathrm{d}s$$

$$+ 2p(p-1)\varepsilon\int\limits_0^{t\wedge\tau_N}\big|\varPi_n\sigma_n\big(s,u_n^\varepsilon(s)\big)u_n^\varepsilon(s)\big|_{\mathcal{L}_Q}^2\big|\nabla u_n^\varepsilon(s)\big|^{2(p-2)}\,\mathrm{d}s,$$

$$I_5 = \int\limits_0^{t\wedge\tau_N}\int\limits_Z\big[\big|\nabla u_n^\varepsilon(s-) + \varepsilon g_n(u_n^\varepsilon(s-),\,z)\big|^{2p} - \big|\nabla u_n^\varepsilon(s-)\big|^{2p}\big]\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)$$

$$+ \int\limits_0^{t\wedge\tau_N}\int\limits_Z\big[\big|\nabla u_n^\varepsilon(s-) + \varepsilon g_n\big(u_n^\varepsilon(s-),\,z\big)\big|^{2p} - \big|\nabla u_n^\varepsilon(s-)\big|^{2p}\big]\,\lambda(\mathrm{d}z)\,\mathrm{d}s$$

$$- 2p\varepsilon\int\limits_0^{t\wedge\tau_N}\int\limits_Z\big(g_n\big(u_n^\varepsilon(s-),z\big),\nabla u_n^\varepsilon(s-)\big)\big|\nabla u_n^\varepsilon(s)\big|^{2(p-1)}\,\lambda(\mathrm{d}z)\,\mathrm{d}s,$$

$$J_3 = 2p\sqrt{\varepsilon} \int_0^{t\wedge\tau_N} \bigl(\sigma_n\bigl(s, u_n^\varepsilon(s)\bigr)\,\mathrm{d}W_n,\ \nabla u_n^\varepsilon(s)\bigr)\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2(p-1)}.$$

By reducing as before we get

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2p} + 2p\eta\,\mathbf{E}\int_0^{t\wedge\tau_N}\bigl|\Delta u_n^\varepsilon(s)\bigr|^2\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\mathrm{d}s \\
&\quad\leqslant \mathbf{E}\bigl|P_n\nabla u_0\bigr|^{2p} + 2pc_1\mathbf{E}\int_0^{t\wedge\tau_N}\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2p}\,\mathrm{d}s + \mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\{J_3+J_4\} \\
&\qquad + 3p(2p-1)\varepsilon K_1\mathbf{E}\int_0^{t\wedge\tau_N}\bigl[\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2(p-1)}\bigl(1+\bigl|\nabla u_n^\varepsilon(s)\bigr|^2\bigr)\bigr]\,\mathrm{d}s,
\end{aligned}$$

where

$$J_4 = 2p\varepsilon\int_0^{t\wedge\tau_N}\int_Z \bigl(g_n\bigl(u_n^\varepsilon(s), z\bigr), \nabla u_n^\varepsilon(s)\bigr)\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\tilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Using a similar computation, we can get

$$\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2p} + 2p\eta\mathbf{E}\int_0^{t\wedge\tau_N}\bigl|\Delta u_n^\varepsilon(s)\bigr|^2\bigl|\nabla u_n^\varepsilon(s)\bigr|^{2(p-1)}\,\mathrm{d}s \leqslant C.$$

Here $\tau_N \to \tau_n^\varepsilon$ as $N \to \infty$, and for $\{\tau_n^\varepsilon < T\}$, $\sup_{0\leqslant s\leqslant\tau_N}|u_n^\varepsilon(s)| \to \infty$. Hence $\mathbf{P}(\tau_n^\varepsilon < T) = 0$, and so for large N, $\tau_N = T$ and $u_n^\varepsilon \in \mathcal{D}([0,T], H_n)$. Hence the proof. □

The following lemma is an essential version of Itô's formula.

Lemma 4. *Let* (A1) *and* $u_0 \in \mathbb{L}^4(\Omega, H)$ *hold and* $\rho' : [0,T]\times\Omega \to [0,+\infty)$ *be adapted such that for every* ω*, the map* $t \to \rho'(t,\omega) \in \mathbb{L}^1([0,T])$ *and* $\rho = \int_0^T \rho'(s)\,\mathrm{d}s$*. For* $i = 1, 2$*, let* $\phi_i \in \mathcal{D}([0,T], \mathbb{H}^1) \cap \mathbb{L}^2([0,T], \mathbb{H}^2)$ *with* $\phi_i(0) = u_0$ *be such that*

$$\begin{aligned}
\mathrm{d}\phi_i(t) &= \bigl[-A\phi_i(t) + f\bigl(\phi_i(t)\bigr)\bigr]\,\mathrm{d}t + \sqrt{\varepsilon}\sigma_i\bigl(t, \phi_i(t)\bigr)\,\mathrm{d}W \\
&\quad + \varepsilon\int_Z g_i\bigl(\phi_i(t), z\bigr)\,\tilde{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Then for every $t \in [0,T]$ *and* $\Phi = \phi_1 - \phi_2$*, we have for* $c_1, c_2 > 0$*,*

$$\begin{aligned}
&\mathrm{e}^{-\rho(t)}\bigl|\Phi(t)\bigr|^2 \\
&\quad\leqslant I(t) + \int_0^t \mathrm{e}^{-\rho(s)}\bigl\{\varepsilon\bigl|\sigma_1\bigl(s, \phi_1(s)\bigr) - \sigma_2\bigl(s, \phi_2(s)\bigr)\bigr|_{\mathcal{L}_Q}^2 - \eta\bigl|\nabla\Phi(s)\bigr|^2\bigr\}\,\mathrm{d}s
\end{aligned}$$

$$+ \int_0^t e^{-\rho(s)} \left[c_1 + c_2\left(\left|\phi_1(s)\right|^2 + \left|\phi_2(s)\right|^2\right) - \rho'(s)\right] \left|\Phi(s)\right|^2 ds$$
$$+ \int_0^t \int_Z e^{-\rho(s)} \varepsilon \left|g_1\left(\phi_1(s), z\right) - g_2\left(\phi_2(s), z\right)\right|^2 N(ds, dz),$$

where

$$I(t) = 2\sqrt{\varepsilon} \int_0^t e^{-\rho(s)} \left(\left[\sigma_1\left(s, \phi_1(s)\right) - \sigma_2\left(s, \phi_2(s)\right)\right] dW, \Phi(s)\right)$$
$$+ 2\varepsilon \int_0^t \int_Z e^{-\rho(s)} \left(g_1\left(\phi_1(s), z\right) - g_2\left(\phi_2(s), z\right), \Phi(s)\right) \tilde{N}(ds, dz).$$

Proof. Itô's formula gives

$$e^{-\rho(t)} \left|\Phi(t)\right|^2 = \int_0^t e^{-\rho(s)} \left\{-\rho'(s)\left|\Phi(s)\right|^2 + \varepsilon\left|\sigma_1\left(s, \phi_1(s)\right) - \sigma_2\left(s, \phi_2(s)\right)\right|^2_{\mathcal{L}_Q}\right\} ds$$
$$+ 2\int_0^t e^{-\rho(s)} \left(-A\Phi(s) + \left[f\left(\phi_1(s)\right) - f\left(\phi_2(s)\right)\right], \Phi(s)\right) ds$$
$$+ \int_0^t \int_Z e^{-\rho(s)} \varepsilon \left|g_1\left(\phi_1(s), z\right) - g_2\left(\phi_2(s), z\right)\right|^2 N(ds, dz) + I(t).$$

Using Lemma 2, (4) and Young's inequality,

$$2\int_0^t e^{-\rho(s)} \left(-A\Phi(s) + \left[f\left(\phi_1(s)\right) - f\left(\phi_2(s)\right)\right], \Phi(s)\right) ds$$
$$\leqslant \int_0^t e^{-\rho(s)} \left(-\eta\left|\nabla\Phi(s)\right|^2 + c_1\left|\Phi(s)\right|^2 + c_2\left(\left|\phi_1(s)\right|^2 + \left|\phi_2(s)\right|^2\right)\left|\Phi(s)\right|^2\right) ds,$$

where $c_1 = 2\alpha + (2\beta^2 C^2/\eta) + (8\gamma^2 C_a/\eta)$ and $c_2 = (4/\eta)(1+\beta^2) + (\gamma^2/\eta)$. Using this in the above estimate, we get the required result. □

Proof of Theorem 1. Let $\Omega_T = [0, T] \times \Omega$, $\varepsilon = \varepsilon_{0,2} \wedge (\eta/K_2)$ and $F(u) = -Au + f(u)$.

Step 1. From Proposition 1 we can conclude that there exist a subsequence $\{u_n^\varepsilon\}_{n\geqslant 0}$ and processes $u^\varepsilon \in \mathbb{L}^2(\Omega_T, \mathbb{H}^2) \cap \mathbb{L}^4(\Omega, \mathcal{D}([0, T], \mathbb{H}^1))$, $F^\varepsilon \in \mathbb{L}^2(\Omega_T, V')$, $S^\varepsilon \in \mathbb{L}^2(\Omega_T, \mathcal{L}_Q)$ and $G^\varepsilon \in \mathbb{H}^2_\lambda([0, T] \times Z; H)$ such that

(P1) $u_n^\varepsilon \to u^\varepsilon$ weakly in $\mathbb{L}^2(\Omega_T, \mathbb{H}^2)$;
(P2) u_n^ε is weak star converging to u^ε in $\mathbb{L}^4(\Omega, \mathcal{D}([0,T], \mathbb{H}^1))$;
(P3) $F(u_n^\varepsilon) \to F^\varepsilon$ in $\mathbb{L}^2(\Omega_T, V')$;
(P4) $\sigma_n(u_n^\varepsilon) \to S^\varepsilon$ in $\mathbb{L}^2(\Omega_T, \mathcal{L}_Q)$;
(P5) $g_n(u_n^\varepsilon, \cdot) \to G^\varepsilon$ in $\mathbb{H}^2_\lambda([0,T] \times Z; H)$.

(P1)–(P2) hold as a direct consequence of Proposition 1. To prove (P3), consider $\psi \in \mathbb{L}^2(\Omega_T, \mathbb{H}^2)$. Then

$$\begin{aligned}
&\mathbf{E}\int_0^T \big(F\big(u_n^\varepsilon(s)\big), \psi(s)\big)\,\mathrm{d}s \\
&\quad= \int_0^T \big[\big(-A\big(u_n^\varepsilon(s)\big), \psi(s)\big) + \big(f\big(u_n^\varepsilon(s)\big), \psi(s)\big)\big]\,\mathrm{d}s \\
&\quad\leqslant \int_0^T \big[-\eta\big(\nabla u_n^\varepsilon(s), \nabla\psi(s)\big) + (\alpha+\beta+\gamma)\big|u_n^\varepsilon(s)\big|\big|\psi(s)\big|\big]\,\mathrm{d}s.
\end{aligned}$$

Using (P1), we can prove (P3). From (A1)

$$\begin{aligned}
&\mathbf{E}\int_0^T \big|\sigma_n\big(s, u_n^\varepsilon(s)\big)\big|^2_{\mathcal{L}_Q}\,\mathrm{d}s + \mathbf{E}\int_0^T\int_Z \big|g_n\big(u_n^\varepsilon(s), z\big)\big|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s \\
&\quad\leqslant K_1\mathbf{E}\int_0^T \big(1 + \big|\nabla u_n^\varepsilon(s)\big|^2\big)\,\mathrm{d}s < \infty.
\end{aligned}$$

This implies (P4) and (P5). Since as $n \to \infty$, $P_n u_0 = u_n^\varepsilon(0) \to u_0$ in H, we have that u^ε satisfies the equation

$$u^\varepsilon(t) = u_0 + \int_0^t F^\varepsilon(s)\,\mathrm{d}s + \sqrt{\varepsilon}\int_0^t S^\varepsilon(s)\,\mathrm{d}W + \varepsilon\int_0^t\int_Z G^\varepsilon(s)\,\tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{6}$$

Step 2. It remains to prove that $F^\varepsilon(s) = F(u^\varepsilon(s))$, $S^\varepsilon(s) = \sigma(s, u^\varepsilon(s))$ and $G^\varepsilon(s) = g(u^\varepsilon(s), z)$. For $\psi \in \mathcal{D}(\Omega_T, \mathbb{H}^1)$, take

$$r(t) = \int_0^t \big[c_1 + c_2\big(\big|u^\varepsilon(s)\big|^2 + \big|\psi(s)\big|^2\big)\big]\,\mathrm{d}s < \infty \quad \text{for all } t \in [0,T].$$

By Fatou's lemma

$$\mathbf{E}\big\{\mathrm{e}^{-r(T)}\big|u^\varepsilon(T)\big|^2\big\} \leqslant \liminf_n \mathbf{E}\big\{\mathrm{e}^{-r(T)}\big|u_n^\varepsilon(T)\big|^2\big\}. \tag{7}$$

Applying Itô's formula to (5),

$$
\begin{aligned}
&\mathrm{e}^{-r(T)}\big|u_n^\varepsilon(T)\big|^2 \\
&\quad= |P_n u_0|^2 + \int_0^T \mathrm{e}^{-r(s)}\big\{-r'(s)\big(u_n^\varepsilon(s), u_n^\varepsilon(s)\big) + 2\big(F\big(u_n^\varepsilon(s)\big), u_n^\varepsilon(s)\big)\big\}\,\mathrm{d}s \\
&\qquad+ \int_0^T \mathrm{e}^{-r(s)}\varepsilon\big|\sigma_n\big(s, u_n^\varepsilon(s)\big)\big|_{\mathcal{L}_Q}^2\,\mathrm{d}s + \int_0^T\int_Z \mathrm{e}^{-r(s)}\varepsilon\big|g_n\big(u_n^\varepsilon(s), z\big)\big|^2\,N(\mathrm{d}s, \mathrm{d}z) \\
&\qquad+ I(t).
\end{aligned}
$$

Here since $I(t)$ is a local martingale with zero average,

$$
\begin{aligned}
\mathbf{E}\big\{I(t)\big\} &= 2\sqrt{\varepsilon}\,\mathbf{E}\int_0^T \mathrm{e}^{-r(s)}\big(\sigma_n\big(s, u_n^\varepsilon(s)\big)\,\mathrm{d}W_n, u_n^\varepsilon(s)\big) \\
&\quad+ 2\varepsilon\mathbf{E}\int_0^T\int_Z \mathrm{e}^{-r(s)}\big(g_n\big(u_n^\varepsilon(s), z\big), u_n^\varepsilon(s)\big)\,\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&= 0.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\mathbf{E}\big(\mathrm{e}^{-r(T)}\big|u_n^\varepsilon(T)\big|^2\big) \\
&\quad= \mathbf{E}|P_n u_0|^2 + \mathbf{E}\int_0^T \mathrm{e}^{-r(s)}\big\{-r'(s)\big(u_n^\varepsilon(s), u_n^\varepsilon(s)\big) + 2\big(F\big(u_n^\varepsilon(s)\big), u_n^\varepsilon(s)\big)\big\}\,\mathrm{d}s \\
&\qquad+ \varepsilon\mathbf{E}\int_0^T \mathrm{e}^{-r(s)}\big|\sigma_n\big(s, u_n^\varepsilon(s)\big)\big|_{\mathcal{L}_Q}^2\,\mathrm{d}s + \varepsilon\mathbf{E}\int_0^T\int_Z \mathrm{e}^{-r(s)}\big|g_n\big(u_n^\varepsilon(s), z\big)\big|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
$$

Similarly applying Itô's formula for (6),

$$
\begin{aligned}
&\mathbf{E}\big(\mathrm{e}^{-r(T)}\big|u^\varepsilon(T)\big|^2\big) \\
&\quad= \mathbf{E}|u_0|^2 + \mathbf{E}\int_0^T \mathrm{e}^{-r(s)}\big\{-r'(s)\big(u^\varepsilon(s), u^\varepsilon(s)\big) + 2\big(F^\varepsilon(s), u^\varepsilon(s)\big)\big\}\,\mathrm{d}s \\
&\qquad+ \varepsilon\mathbf{E}\int_0^T \mathrm{e}^{-r(s)}\big|S^\varepsilon(s)\big|_{\mathcal{L}_Q}^2\,\mathrm{d}s + \varepsilon\mathbf{E}\int_0^T\int_Z \mathrm{e}^{-r(s)}\big|G^\varepsilon(s)\big|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
$$

From (7)

$$\begin{aligned}
&\mathbf{E}\Bigg[\int_0^T \mathrm{e}^{-r(s)}\Big\{-r'(s)\big(u^\varepsilon(s),u^\varepsilon(s)\big)+2\big(F^\varepsilon(s),u^\varepsilon(s)\big)\Big\}\,\mathrm{d}s \\
&\qquad+\varepsilon\int_0^T \mathrm{e}^{-r(s)}\big|S^\varepsilon(s)\big|_{\mathcal{L}_Q}^2\,\mathrm{d}s+\varepsilon\int_0^T\!\!\int_Z \mathrm{e}^{-r(s)}\big|G^\varepsilon(s)\big|^2\,\lambda(\mathrm{d}z)\mathrm{d}s\Bigg] \\
&\quad\leqslant \liminf_n \mathbf{E}\Bigg[\int_0^T \mathrm{e}^{-r(s)}\Big\{-r'(s)\big(u_n^\varepsilon(s),u_n^\varepsilon(s)\big)+2\big(F\big(u_n^\varepsilon(s)\big),u_n^\varepsilon(s)\big)\Big\}\,\mathrm{d}s \\
&\qquad+\varepsilon\int_0^T \mathrm{e}^{-r(s)}\big|\sigma_n\big(s,u_n^\varepsilon(s)\big)\big|_{\mathcal{L}_Q}^2\,\mathrm{d}s+\varepsilon\int_0^T\!\!\int_Z \mathrm{e}^{-r(s)}\big|g_n\big(u_n^\varepsilon(s),z\big)\big|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s\Bigg]. \qquad (8)
\end{aligned}$$

Using properties of A and f, we derive that

$$\begin{aligned}
&\mathbf{E}\Bigg[\int_0^T \mathrm{e}^{-r(s)}\Big\{-r'(s)\big|u_n^\varepsilon(s)-\psi(s)\big|^2+2\big(F\big(u_n^\varepsilon\big)-F(\psi),\,u_n^\varepsilon(s)-\psi(s)\big)\Big\}\,\mathrm{d}s \\
&\qquad+\varepsilon\int_0^T \mathrm{e}^{-r(s)}\bigg\{\big|\sigma_n\big(s,u_n^\varepsilon\big)-\sigma_n(s,\psi)\big|_{\mathcal{L}_Q}^2+\int_Z\big|g_n\big(u_n^\varepsilon,z\big)-g_n(\psi,z)\big|^2\,\lambda(\mathrm{d}z)\bigg\}\,\mathrm{d}s\Bigg] \\
&\quad\leqslant 0.
\end{aligned}$$

Subtracting the above estimate from RHS in (8) and applying limit, we get

$$\begin{aligned}
&\mathbf{E}\Bigg[\int_0^T \mathrm{e}^{-r(s)}\big\{-r'(s)\big(u^\varepsilon(s),u^\varepsilon(s)\big)+2\big(F^\varepsilon(s),u^\varepsilon(s)\big)\big\}\,\mathrm{d}s \\
&\qquad+\varepsilon\int_0^T \mathrm{e}^{-r(s)}\big|S^\varepsilon(s)\big|_{\mathcal{L}_Q}^2\,\mathrm{d}s+\varepsilon\int_0^T\!\!\int_Z \mathrm{e}^{-r(s)}\big|G^\varepsilon(s)\big|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s\Bigg] \\
&\quad\leqslant \mathbf{E}\Bigg[\int_0^T \mathrm{e}^{-r(s)}\big\{2\big(F^\varepsilon(s),\psi(s)\big)+2\big(F\big(\psi(s)\big),\,u^\varepsilon(s)-\psi(s)\big)\big\}\,\mathrm{d}s \\
&\qquad-\int_0^T \mathrm{e}^{-r(s)}r'(s)\big(2u^\varepsilon(s)-\psi(s),\,\psi(s)\big)\,\mathrm{d}s
\end{aligned}$$

$$+ \varepsilon \int_0^T e^{-r(s)} \big(2S^\varepsilon(s) - \sigma\big(s, \psi(s)\big), \, \sigma\big(s, \psi(s)\big)\big) \, ds$$
$$+ \varepsilon \int_0^T \int_Z e^{-r(s)} \big(2G^\varepsilon(s) - g\big(\psi(s), z\big), \, g\big(\psi(s), \, z\big)\big) \, \lambda(dz) \, ds \bigg].$$

Then rearranging the terms, we get

$$\mathbf{E}\bigg[\int_0^T e^{-r(s)} \big\{ 2\big(F^\varepsilon(s) - F(\psi(s)), \, u^\varepsilon(s) - \psi(s)\big)\big\} \, ds$$
$$- \int_0^T e^{-r(s)} r'(s) \big|u^\varepsilon(s) - \psi(s)\big|^2 \, ds$$
$$+ \varepsilon \int_0^T e^{-r(s)} \big|S^\varepsilon(s) - \sigma\big(s, \psi(s)\big)\big|^2_{\mathcal{L}_Q} \, ds$$
$$+ \varepsilon \int_0^T \int_Z e^{-r(s)} \big|G^\varepsilon(s) - g\big(\psi(s), z\big)\big|^2 \, \lambda(dz) \, ds \bigg]$$
$$\leqslant 0.$$

Taking $\psi = u^\varepsilon$ in the above inequality, we get $S^\varepsilon(t) = \sigma(t, u^\varepsilon(t))$ and $G^\varepsilon(t) = g(u^\varepsilon(t), z)$. For some $\mu > 0$ and $\tilde{\psi} \in \mathbb{L}^\infty(\Omega_T, H)$, taking $\psi = u^\varepsilon - \mu\tilde{\psi}$, we get

$$\mathbf{E}\bigg[\int_0^T e^{-r(s)} \big\{ 2\mu\big(F^\varepsilon(s) - F\big(\psi(s)\big), \, \tilde{\psi}(s)\big) - \mu^2 r'(s) \big|\tilde{\psi}(s)\big|^2 \big\} \, ds \bigg] \leqslant 0. \tag{9}$$

From (4) we have

$$\big(F\big(\psi(s)\big) - F\big(u^\varepsilon(s)\big), \, \tilde{\psi}(s)\big)$$
$$\leqslant \mu^2 \big[-\eta \big|\nabla\tilde{\psi}(s)\big|^2 + c_1 \big|\tilde{\psi}(s)\big|^2 + c_2 \big(\big|\psi(s)\big|^2 + \big|u^\varepsilon(s)\big|^2\big) \big|\tilde{\psi}(s)\big|^2 \big].$$

Then dividing (9) by μ and letting $\mu \to 0$,

$$\mathbf{E}\bigg[\int_0^T e^{-r(s)} \big\{ 2\big(F^\varepsilon(s) - F\big(u^\varepsilon(s)\big), \, \tilde{\psi}(s)\big)\big\} \, ds \bigg] \leqslant 0.$$

Since $\tilde{\psi}$ is arbitrary, $F^\varepsilon(t) = F(u^\varepsilon(t))$. Hence, u^ε satisfies

$$u^\varepsilon(t) = u_0 + \int_0^t F\big(u^\varepsilon(s)\big) \, ds + \sqrt{\varepsilon} \int_0^t \sigma\big(s, u^\varepsilon(s)\big) \, dW + \varepsilon \int_0^t \int_Z g\big(u^\varepsilon(s), z\big) \, \tilde{N}(ds, dz).$$

Hence the existence of solution is proved.

Step 3. To prove uniqueness, consider $\psi^\varepsilon \in \mathbb{X}$. Let $\tau_N = \tilde{\tau}_N \wedge \bar{\tau}_N \to T$ as $N \to \infty$, where $\tilde{\tau}_N = \inf\{t\colon |u^\varepsilon(t)| + |\nabla u^\varepsilon(t)| \geqslant N\}$ and $\bar{\tau}_N = \inf\{t\colon |\psi^\varepsilon(t)| + |\nabla \psi^\varepsilon(t)| \geqslant N\}$. Then $\vartheta = u^\varepsilon - \psi^\varepsilon$ satisfies

$$\mathrm{d}\vartheta(t) = \bigl[F\bigl(u^\varepsilon(t)\bigr) - F\bigl(\psi^\varepsilon(t)\bigr)\bigr]\,\mathrm{d}t + \sqrt{\varepsilon}\bigl[\sigma\bigl(t, u^\varepsilon(t)\bigr) - \sigma\bigl(t, \psi^\varepsilon(t)\bigr)\,\mathrm{d}W\bigr] + \varepsilon \int_Z \bigl[g\bigl(u^\varepsilon(t), z\bigr) - g\bigl(\psi^\varepsilon(t), z\bigr)\bigr]\,\tilde{N}(\mathrm{d}t, \mathrm{d}z).$$

For $a = c_2$, let $\rho'(t) = a(|\psi^\varepsilon(s)|^2 + |u^\varepsilon(s)|^2)$. Itô's formula in Lemma 4 for $\phi_1 = u^\varepsilon$ and $\phi_2 = \psi^\varepsilon$ and (A2) gives

$$\begin{aligned}
&\mathrm{e}^{-\rho(t\wedge\tau_N)}\bigl|\vartheta(t\wedge\tau_N)\bigr|^2 \\
&\quad\leqslant I(t\wedge\tau_N) + \int_0^{t\wedge\tau_N} \mathrm{e}^{-\rho(s)}\bigl\{\varepsilon K_2 - \eta\bigr\}\bigl|\nabla\vartheta(s)\bigr|^2\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_N} \mathrm{e}^{-\rho(s)}\bigl[c_1 + c_2\bigl(\bigl|u^\varepsilon(s)\bigr|^2 + \bigl|\psi^\varepsilon(s)\bigr|^2\bigr) - \rho'(s)\bigr]\bigl|\vartheta(s)\bigr|^2\,\mathrm{d}s,
\end{aligned}$$

where

$$\begin{aligned}
I(t\wedge\tau_N) &= 2\sqrt{\varepsilon}\int_0^{t\wedge\tau_N} \mathrm{e}^{-\rho(s)}\bigl(\bigl[\sigma\bigl(s, u^\varepsilon(s)\bigr) - \sigma\bigl(s, \psi^\varepsilon(s)\bigr)\bigr]\,\mathrm{d}W, \vartheta(s)\bigr) \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau_N}\int_Z \mathrm{e}^{-\rho(s)}\bigl(g\bigl(u^\varepsilon(s), z\bigr) - g\bigl(\psi^\varepsilon(s), z\bigr), \vartheta(s)\bigr)\,\tilde{N}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Here

$$\begin{aligned}
\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N} I(s) &\leqslant \delta\mathbf{E}\sup_{0\leqslant s\leqslant t\wedge\tau_N}\bigl[\mathrm{e}^{-\rho(s)}\bigl|\vartheta(s)\bigr|^2\bigr] \\
&\quad + \frac{C_1(\varepsilon + \varepsilon^2)K_2}{\delta}\mathbf{E}\int_0^{t\wedge\tau_N} \mathrm{e}^{-\rho(s)}\bigl|\nabla\vartheta(s)\bigr|^2\,\mathrm{d}s.
\end{aligned}$$

Therefore by using Lemma 3 with $Z(t) = 0$ and $\tilde{C} = 0$ we get

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\bigl\{\mathrm{e}^{-\rho(t\wedge\tau_N)}\bigl|\vartheta(t\wedge\tau_N)\bigr|^2\bigr\} = 0.$$

Hence $|\vartheta(t)|^2 = 0$ for all $t \in [0, T]$ since $\tau_N \to T$ as $N \to \infty$. □

4 Large deviation principle

Let O be a locally compact Polish space and $O_T = [0, T] \times O$ corresponding to O for finite $T > 0$. Define

$$\mathbb{M}(O) = \bigl\{\mu \text{ on } \bigl(O, \mathcal{B}(O)\bigr)\colon\ \mu(K) < \infty \text{ for compact } K \subset O\bigr\}.$$

Let $\mathcal{M} = \mathbb{M}(O_T)$ and P be the probability measure on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. Then $\mathcal{B}(\mathcal{M})$ is a Polish space. Let

$$\mathbb{V} = C\big([0,T]; H\big) \times \mathcal{M},$$
$$\mathcal{G}_t = \sigma\big\{N(s,Z)\colon 0 \leqslant s \leqslant t,\ Z \in \mathcal{B}(O_T)\big\} \quad \text{for } t > 0.$$

Let $\tilde{P}$ be the probability measure on the space $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$, $\{F_t\}$ be completion of $\{\mathcal{G}_t\}$ and $\mathcal{P}$ be predictable σ-field with respect to it. Define the class $\mathcal{A}$ and $l : [0,\infty) \to [0,\infty)$ such that

$$\mathcal{A} = \big\{\psi\colon O_T \times \mathbb{V} \to [0,\infty)\colon \psi \text{ is } \big(\mathcal{P} \times \mathcal{B}(O)\big)/\mathcal{B}[0,\infty)\text{-measurable}\big\},$$
$$l(a) = a \log a - a + 1.$$

Let X be a locally compact Polish space with $X_T = [0,T] \times X$. For $\psi \in \mathcal{A}$, define N^{ψ} as

$$N^{\psi}(t,Z) = \int\limits_{[0,T]\times Z} \int\limits_0^{\infty} \mathbf{1}_{[0,\psi(s,z)]}(r)\, \tilde{N}(\mathrm{d}s,\mathrm{d}z)\,\mathrm{d}r, \quad t \in [0,T],\ Z \in \mathcal{B}(X).$$

For $\psi \in \mathcal{A}$, define

$$\tilde{L_T}(\psi) = \int\limits_0^T \int\limits_Z l\big(\psi(t,z,\omega)\big)\, \lambda(\mathrm{d}z)\,\mathrm{d}t.$$

Define $\mathcal{P}_2 = \{\phi\colon \phi \text{ is } \mathcal{P}/\mathcal{B}(\mathbb{R})\text{-measurable}, \int_0^T |\phi(s)|_0^2\,\mathrm{d}s < \infty\}$ and $\mathcal{U}(H) = \mathcal{P}_2 \times \mathcal{A}$. For $\phi \in \mathcal{P}_2$, consider

$$\bar{L_T}(\phi) = \frac{1}{2}\int\limits_0^T \big|\phi(s)\big|_0^2\,\mathrm{d}s.$$

For $N \in \mathbb{N}$, define

$$\tilde{S_N}(\psi) = \big\{\psi : O_T \to [0,\infty)\colon \tilde{L_T}(\psi) \leqslant N\big\},$$
$$\bar{S_N}(H_0) = \big\{\phi \in \mathbb{L}^2\big([0,T]; H_0\big)\colon \bar{L_T}(\phi) \leqslant N\big\}.$$

Define a compact set $\{\lambda_T^g\colon g \in \tilde{S_N}\}$ in $\mathcal{M}$, where

$$\lambda_T^g = \int\limits_0^T \int\limits_Z g(s,z)\, \lambda(\mathrm{d}z)\,\mathrm{d}s, \quad Z \in \mathcal{B}(O_T).$$

Let $\mathcal{U} = \mathcal{P}_2(H_0) \times \mathcal{A}$ and $\mathbb{S} = \bigcup_{N\geqslant 1} S_N$, where $S_N = \bar{S_N}(H_0) \times \tilde{S_N}$. Here take $\mathcal{U}^N = \{\xi = (\phi,\psi) \in \mathcal{U},\ \xi(\omega) \in S_N\}$. The following theorem states the postulates to be proved in order to establish large deviation principle. Let X_0 be a Polish space.

Theorem 2. *Suppose there exists $\mathcal{G}^0 : X_0 \times \mathbb{V} \to \mathbb{X}$, a measurable map such that*

(i) *For $M < \infty$ and $\{\xi^\varepsilon = (\phi^\varepsilon, \psi^\varepsilon) \in \mathcal{U}\colon \xi^\varepsilon(\omega) \in S_M$ for a.e. $\omega\} \subset \mathcal{U}^M$, if $(\phi^\varepsilon, \psi^\varepsilon) \to (\phi, \psi)$ in distribution in S_M as $\varepsilon \to 0$, then*
$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot) + \int_0^{\cdot} \phi^\varepsilon \,\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\psi^\varepsilon}\right) \to \mathcal{G}^0\left(\int_0^{\cdot} \phi(s)\,\mathrm{d}s, \lambda_T^\psi\right).$$

(ii) *For every finite M, $K_M = \{\mathcal{G}^0(\int_0^{\cdot} \phi(s)\,\mathrm{d}s, \lambda_T^\psi)\colon (\phi,\psi) \in \mathcal{U}^M\}$ is a compact subset in $\mathbb{X}$.*

Then the family of solutions $\{u^\varepsilon, \varepsilon > 0\}$ satisfies the Laplace principle with the rate function I given by
$$I(g) = \inf_{(\phi,\psi)\in S_g}\left\{\frac{1}{2}\int_0^T\int_{\mathcal{Z}} \left|l\big(\psi(s,z)\big)\right|^2 \lambda(\mathrm{d}z)\,\mathrm{d}s + \frac{1}{2}\int_0^T \left|\phi(s)\right|_0^2 \mathrm{d}s\right\},$$
where $S_g = \{(\phi,\psi) \in \bigcup_{M>1} S_M\colon g = \mathcal{G}^0(\int_0^{\cdot} \phi(s)\,\mathrm{d}s, \lambda_T^\psi)\}$, where infimum over the empty set is taken as ∞.

The stochastic control equation with respect to (1) is
$$\begin{aligned}\mathrm{d}u^\varepsilon_{\xi^\varepsilon} &= \left[-Au^\varepsilon_{\xi^\varepsilon} + f\big(u^\varepsilon_{\xi^\varepsilon}\big) + \sigma\big(t, u^\varepsilon_{\xi^\varepsilon}\big)\phi^\varepsilon\right]\mathrm{d}t + \sqrt{\varepsilon}\sigma\big(t, u^\varepsilon_{\xi^\varepsilon}\big)\,\mathrm{d}W \\ &\quad + \int_Z g(u^\varepsilon_{\xi^\varepsilon}, z) l\big(\psi^\varepsilon\big)\,\lambda(\mathrm{d}z)\,\mathrm{d}t + \varepsilon\int_Z g\big(u^\varepsilon_{\xi^\varepsilon}, z\big)\,\tilde{N}(\mathrm{d}t, \mathrm{d}z).\end{aligned} \tag{10}$$

Lemma 5. *Assume that* (A1)–(A2) *hold. There exists a unique strong solution for stochastic control equation* (10) *with $u^\varepsilon_{\xi^\varepsilon}(0) = u_0$ for $\xi^\varepsilon \in \mathcal{U}^M$, $M \in (0,\infty)$, and $\varepsilon > 0$ satisfying the estimate*
$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\left|u_{\xi^\varepsilon}(t)\right|^2 + \eta\mathbf{E}\int_0^t \left|\nabla u_{\xi^\varepsilon}(s)\right|^2 \mathrm{d}s \leqslant K,$$
$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\left|\nabla u_{\xi^\varepsilon}(t)\right|^2 + \eta\mathbf{E}\int_0^t \left|\Delta u_{\xi^\varepsilon}(s)\right|^2 \mathrm{d}s \leqslant K$$
in $\mathbb{X}$, where K is an appropriate constant.

The solution of (10) can be expressed as $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}W(\cdot) + \int_0^{\cdot} \phi^\varepsilon\,\mathrm{d}s,\, \varepsilon N^{\varepsilon^{-1}\psi^\varepsilon})$. The estimates for the solution is evaluated as done earlier.

Theorem 3. *Assume that* (A1)–(A2) *hold. For $\xi \in \mathcal{A}$, the deterministic controlled equation*
$$\mathrm{d}u_\xi = \left[-Au_\xi + f(u_\xi) + \sigma(t, u_\xi)\phi\right]\mathrm{d}t + \int_Z g(u_\xi, z) l(\psi)\,\lambda(\mathrm{d}z)\,\mathrm{d}t \tag{11}$$
with the initial condition $u_\xi(0) = u_0$ has a unique strong solution in $\mathbb{X}$.

The solution of above system can be expressed as $\mathcal{G}^0(\int_0^\cdot \phi(s)\,\mathrm{d}s, \lambda_T^\psi)$. The proof of Theorem 2 consists essentially to prove the weak convergence of the solution of (10) to the solution of (11) as $\varepsilon \to 0$.

4.1 Compactness. Proof of postulate (ii) of Theorem 2

Theorem 4. *For fixed finite $M > 0$ and $\xi \in \mathcal{A}$, consider the set*

$$K_M = \{u_\xi \in \mathbb{X} = \mathcal{D}([0,T];\mathbb{H}^1) \cap \mathbb{L}^2((0,T);\mathbb{H}^2)\},$$

where u_ξ is the solution of (11). *Then the set K_M is compact in $\mathbb{X}$.*

Proof. Let $\{u_{\xi_n}\} \in K_M$ to be the solution of (11), where the control $\xi = (\phi,\psi)$ is replaced by $\xi_n = (\phi_n,\psi_n) \in \mathcal{S}_M$ for $n \in \mathbb{N}$. There exists a subsequence of $\xi_n \in \mathcal{S}_M$, also denoted by ξ_n, converging weakly to ξ since S_M is weakly compact. In order to prove $u_{\xi_n} \to u_\xi$ weakly, it is enough to prove that $w_n = (u_{\xi_n} - u_\xi)$ tends to 0 as $n \to \infty$. The difference w_n satisfies the equation

$$\begin{aligned}\mathrm{d}w_n &= \bigl[-Aw_n + f(u_{\xi_n}) - f(u_\xi) + \sigma(t,u_{\xi_n})\phi_n - \sigma(t,u_\xi)\phi\bigr]\,\mathrm{d}t \\ &\quad + \int_Z \bigl[g(u_{\xi_n},z)l\bigl(\psi_n(t,z)\bigr) - g(u_\xi,z)l\bigl(\psi(t,z)\bigr)\bigr]\,\lambda(\mathrm{d}z)\,\mathrm{d}t.\end{aligned}$$

Taking inner product with w_n and integrating,

$$\begin{aligned}&\bigl|w_n(t)\bigr|^2 + 2\eta\int_0^t \bigl|\nabla w_n(s)\bigr|^2\,\mathrm{d}s \\ &\quad = 2\int_0^t \bigl(f(u_{\xi_n}(s)) - f\bigl(u_\xi(s)\bigr),\, w_n\bigr)\,\mathrm{d}s \\ &\qquad + 2\int_0^t \bigl(\sigma\bigl(s,u_{\xi_n}(s)\bigr)\phi_n(s) - \sigma\bigl(s,u_\xi(s)\bigr)\phi(s), w_n\bigr)\,\mathrm{d}s \\ &\qquad + 2\int_0^t\int_Z \bigl(w_n, \bigl[g\bigl(u_{\xi_n}(s),z\bigr)l\bigl(\psi_n(s,z)\bigr) - g\bigl(u_\xi(s),z\bigr)l\bigl(\psi(s,z)\bigr)\bigr]\bigr)\,\lambda(\mathrm{d}z)\,\mathrm{d}s \\ &\qquad + 2\alpha\int_0^t |w_{1,n}|^2\,\mathrm{d}s.\end{aligned}$$

Using the property of the nonlinear operator f, we get

$$\begin{aligned}&2\int_0^t \bigl(f\bigl(u_{\xi_n}(s)\bigr) - f\bigl(u_\xi(s)\bigr), w_n(s)\bigr)\,\mathrm{d}s \\ &\quad \leqslant \int_0^t \biggl[\eta|\nabla w_n|^2 + \frac{4}{\eta}(1+\beta^2)\bigl(|u_{\xi_n}|^2 + |u_\xi|^2\bigr)|w_{1,n}|^2 + \frac{2\beta^2C^2}{\eta}|w_n|^2\end{aligned}$$

$$+ \frac{8\gamma^2 C_a}{\eta}|w_{2,n}|^2 + \frac{\gamma^2}{\eta}|w_n|^2\big(|u_{2,\xi_n}|^2 + |u_{2,\xi}|^2\big)\bigg]\,\mathrm{d}s$$

$$\leqslant \int_0^t \bigg[\eta|\nabla w_n|^2 + \left(\frac{4}{\eta}(1+\beta^2) + \frac{\gamma^2}{\eta}\right)\big(|u_{\xi_n}|^2 + |u_\xi|^2\big)|w_n|^2 + \left(\frac{2\beta^2 C^2}{\eta} + \frac{8\gamma^2 C_a^2}{\eta}\right)|w_n|^2\bigg]\,\mathrm{d}s.$$

Using Young's inequality and (A2),

$$\begin{aligned}
&2\int_0^t \big(\sigma\big(s, u_{\xi_n}(s)\big)\phi_n(s) - \sigma\big(s, u_\xi(s)\big)\phi(s), w_n\big)\,\mathrm{d}s \\
&\quad = 2\int_0^t \big(\sigma\big(s, u_{\xi_n}(s)\big)\phi_n(s) - \sigma\big(s, u_\xi(s)\big)\phi_n(s) \\
&\qquad + \sigma\big(s, u_\xi(s)\big)\phi_n(s) - \sigma\big(s, u_\xi(s)\big)\phi(s), w_n\big)\,\mathrm{d}s \\
&\quad \leqslant \int_0^t \big|\sigma\big(s, u_{\xi_n}(s)\big) - \sigma\big(s, u_\xi(s)\big)\big|_{\mathcal{L}_Q}\big|\phi_n(s)\big|_0 |w_n|\,\mathrm{d}s \\
&\qquad + 2\int_0^t \big|\sigma\big(s, u_\xi(s)\big)\big(\phi_n(s) - \phi(s)\big)\big||w_n|\,\mathrm{d}s \\
&\quad \leqslant \int_0^t \frac{\eta}{4}|\nabla w_n|^2 + \left(\frac{4K_2}{\eta}\big|\phi_n(s)\big|_0^2 + 1\right)|w_n|^2\,\mathrm{d}s \\
&\qquad + \int_0^t \big|\sigma\big(s, u_\xi(s)\big)\big(\phi_n(s) - \phi(s)\big)\big|^2\,\mathrm{d}s.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&2\int_0^t\int_Z \big(w_n, \big[g\big(u_{\xi_n}(s), z\big)l\big(\psi_n(s,z)\big) - g\big(u_\xi(s), z\big)l\big(\psi(s,z)\big)\big]\big)\,\lambda(\mathrm{d}z)\,\mathrm{d}s \\
&\quad \leqslant \int_0^t \frac{\eta}{4}|\nabla w_n|^2 + \left(\frac{4K_2}{\eta}\big|l\big(\psi_n(s)\big)\big|^2 + 1\right)|w_n|^2\,\mathrm{d}s \\
&\qquad + \int_0^t\int_Z \big|g\big(u_\xi(s), z\big)\big[l\big(\psi_n(s)\big) - l\big(\psi(s)\big)\big]\big|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}$$

By substitution we get

$$\begin{aligned}
&\left|w_n(t)\right|^2 + \frac{\eta}{2}\int_0^t \left|\nabla w_n(s)\right|^2 \mathrm{d}s \\
&\quad\leqslant \int_0^t D_1\left|w_n(s)\right|^2 \mathrm{d}s + \int_0^t D_2\left(\left|u_{\xi_n}(s)\right|^2 + \left|u_\xi(s)\right|^2\right)\left|w_n(s)\right|^2 \mathrm{d}s \\
&\qquad + \int_0^t \left|\sigma\left(s, u_\xi(s)\right)\left[\phi_n(s) - \phi(s)\right]\right|^2 \mathrm{d}s \\
&\qquad + \int_0^t \int_Z \left|g\left(u_\xi(s), z\right)\left[l\left(\psi_n(s)\right) - l\left(\psi(s)\right)\right]\right|^2 \lambda(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}$$

Here

$$\begin{aligned}
D_1 &= \left(\frac{4K_2}{\eta}\left[\left|l\left(\psi_n(s)\right)\right|^2 + \left|\phi_n(s)\right|_0^2\right] + 2\right) + \max\left\{2\alpha,\ \frac{8\gamma^2C^2}{\eta} + \frac{2\beta^2C^2}{\eta}\right\}, \\
D_2 &= \left(\frac{4}{\eta}\left(1+\beta^2\right) + \frac{\gamma^2}{\eta}\right).
\end{aligned}$$

Applying Gronwall's inequality as $n \to \infty$,

$$\left|w_n(t)\right|^2 + \frac{\eta}{2}\int_0^t \left|\nabla w_n(s)\right|^2 \mathrm{d}s \to 0.$$

This implies that K_M is a compact set in $\mathbb{X}$. □

4.2 Weak convergence. Proof of postulate (i) of Theorem 2

Theorem 5. *If $\{\xi^\varepsilon, \varepsilon > 0\} \in \mathcal{A}$ converges to ξ in distribution with respect to the weak topology in $\mathcal{A}$, then as $\varepsilon \to 0$,*

$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot) + \int_0^\cdot \phi^\varepsilon\,\mathrm{d}s,\ \varepsilon N^{\varepsilon^{-1}\psi^\varepsilon}\right) \to \mathcal{G}^0\left(\int_0^\cdot \phi(s)\,\mathrm{d}s,\ \lambda_T^\psi\right)$$

in distribution in $\mathbb{X}$.

Proof. It is enough to prove that $w^\varepsilon = u^\varepsilon_{\xi^\varepsilon} - u_\xi$ tends to zero as $\varepsilon \to 0$, where $u^\varepsilon_{\xi^\varepsilon}$ and u_ξ are the solutions (10) and (11), respectively. The equation satisfied by w^ε is

$$\begin{aligned}
\mathrm{d}w^\varepsilon &= \left[-Aw^\varepsilon + f\left(u^\varepsilon_{\xi^\varepsilon}\right) - f(u_\xi) + \sigma\left(t, u^\varepsilon_{\xi^\varepsilon}\right)\phi^\varepsilon - \sigma(t, u_\xi)\phi\right]\mathrm{d}t + \sqrt{\varepsilon}\sigma\left(t, u^\varepsilon_{\xi^\varepsilon}\right)\mathrm{d}W \\
&\quad + \int_Z \left[g\left(u^\varepsilon_{\xi^\varepsilon}, z\right)l\left(\psi^\varepsilon\right) - g(u_\xi, z)l(\psi)\right]\lambda(\mathrm{d}z)\,\mathrm{d}t + \varepsilon\int_Z g\left(u^\varepsilon_{\xi^\varepsilon}, z\right)\tilde{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Itô's formula gives

$$\begin{aligned}
&\left|w^\varepsilon(t)\right|^2 \\
&\quad = 2\int_0^t \left(\left[-Aw^\varepsilon + f\left(u^\varepsilon_{\xi^\varepsilon}(s)\right) - f\left(u_\xi(s)\right) + \sigma\left(s, u^\varepsilon_{\xi^\varepsilon}(s)\right)\phi^\varepsilon(s) - \sigma(s, u_\xi)\phi\right], w^\varepsilon\right) \mathrm{d}s \\
&\qquad + 2\int_0^t\int_Z \left(\left[g\left(u^\varepsilon_{\xi^\varepsilon}(s), z\right) l\left(\psi^\varepsilon(s)\right) - g\left(u_\xi(s), z\right) l\left(\psi(s)\right)\right],\ w^\varepsilon\right) \lambda(\mathrm{d}z)\,\mathrm{d}s \\
&\qquad + 2\sqrt{\varepsilon}\int_0^t \left(\sigma\left(s, u^\varepsilon_{\xi^\varepsilon}(s)\right)\mathrm{d}W, w^\varepsilon\right) + 2\varepsilon\int_0^t\int_Z \left(g\left(u^\varepsilon_{\xi^\varepsilon}(s), z\right), w^\varepsilon\right) \tilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\qquad + \varepsilon\int_0^t \left|\sigma\left(s, u^\varepsilon_{\xi^\varepsilon}(s)\right)\right|^2_{\mathcal{L}_Q} \mathrm{d}s + \varepsilon\int_0^t\int_Z \left|g\left(u^\varepsilon_{\xi^\varepsilon}(s), z\right)\right|^2 N(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Employing the same method as in previous estimates, the inequality reduces as

$$\begin{aligned}
&\left|w^\varepsilon(t)\right|^2 + \frac{\eta}{2}\int_0^t \left|\nabla w^\varepsilon(s)\right|^2 \mathrm{d}s \\
&\quad \leqslant 2\alpha\int_0^t \left|w^\varepsilon_1(s)\right|^2 \mathrm{d}s \\
&\qquad + \int_0^t D_2\left(\left|u^\varepsilon_\xi(s)\right|^2 + \left|u_\xi(s)\right|^2\right)\left|w^\varepsilon\right|^2 + \left|\sigma\left(s, u_\xi(s)\right)\left(\phi^\varepsilon(s) - \phi(s)\right)\right|^2 \mathrm{d}s \\
&\qquad + \int_0^t \left(\frac{4K_2}{\eta}\left[\left|l\left(\psi^\varepsilon(s)\right)\right|^2 + \left|\phi^\varepsilon(s)\right|_0^2\right] + 2\right)|w^\varepsilon|^2 + \left(\frac{2\beta^2 C^2}{\eta} + \frac{8\gamma^2 C_a^2}{\eta}\right)\left|w^\varepsilon\right|^2 \mathrm{d}s \\
&\qquad + 2\sqrt{\varepsilon}\int_0^t \left(\sigma\left(s, u^\varepsilon_{\xi^\varepsilon}(s)\right)\mathrm{d}W, w^\varepsilon\right) + 2\varepsilon\int_0^t\int_Z \left(g\left(u^\varepsilon_{\xi^\varepsilon}(s), z\right), w^\varepsilon\right) \tilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\qquad + \varepsilon K_1\int_0^t \left(1 + \left|\nabla u^\varepsilon_{\xi^\varepsilon}(s)\right|^2\right) \mathrm{d}s \\
&\qquad + \int_0^t\int_Z \left|g\left(u_\xi(s), z\right)\left[l\left(\psi^\varepsilon(s)\right) - l\left(\psi(s)\right)\right]\right|^2 \lambda(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}$$

Taking supremum over 0 to T, then taking expectation and using Burkholder–Davis–Gundy inequality, we get

$$2\mathbf{E}\sup_{0\leqslant t\leqslant T}\left\{\sqrt{\varepsilon}\int_0^t \bigl(\sigma\bigl(s,u^\varepsilon_{\xi^\varepsilon}(s)\bigr)\,\mathrm{d}W, w^\varepsilon(s)\bigr) + \varepsilon\int_0^t\int_Z \bigl(g\bigl(u^\varepsilon_{\xi^\varepsilon}(s),z\bigr), w^\varepsilon(s)\bigr)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\}$$
$$\leqslant \frac{1}{2}\mathbf{E}\Bigl\{\sup_{0\leqslant t\leqslant T}\bigl|w^\varepsilon(t)\bigr|^2\Bigr\} + \bigl(\varepsilon^2+\varepsilon\bigr)CK_1^2\,\mathbf{E}\int_0^T \bigl(1+\bigl|\nabla u^\varepsilon_{\xi^\varepsilon}(s)\bigr|^2\bigr)\,\mathrm{d}s.$$

For appropriate constant D_4,

$$\frac{1}{2}\mathbf{E}\sup_{0\leqslant t\leqslant T}\bigl|w^\varepsilon(t)\bigr|^2 + \frac{\eta}{2}\mathbf{E}\int_0^t \bigl|\nabla w^\varepsilon(s)\bigr|^2\,\mathrm{d}s$$
$$\leqslant \mathbf{E}\int_0^t D_4\bigl|w^\varepsilon(s)\bigr|^2\,\mathrm{d}s + \bigl(\varepsilon K_1 + \bigl(\varepsilon^2+\varepsilon\bigr)CK_1^2\bigr)\mathbf{E}\int_0^t \bigl(1+\bigl|\nabla u^\varepsilon_{\xi^\varepsilon}(s)\bigr|^2\bigr)\,\mathrm{d}s$$
$$+\,\mathbf{E}\int_0^t \bigl|\sigma\bigl(s,u_\xi(s)\bigr)\bigl(\phi^\varepsilon(s)-\phi(s)\bigr)\bigr|^2\,\mathrm{d}s$$
$$+\,\mathbf{E}\int_0^t\int_Z \bigl|g\bigl(u_\xi(s),z\bigr)\bigl[l\bigl(\psi^\varepsilon(s)\bigr)-l\bigl(\psi(s)\bigr)\bigr]\bigr|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s.$$

Applying Gronwall's inequality,

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}\bigl|w^\varepsilon(t)\bigr|^2 + \eta\,\mathbf{E}\int_0^t \bigl|\nabla w^\varepsilon(s)\bigr|^2\,\mathrm{d}s$$
$$\leqslant \exp(D_4T)\biggl\{\mathbf{E}\int_0^t \bigl|\sigma\bigl(s,u_\xi(s)\bigr)\bigl(\phi^\varepsilon(s)-\phi(s)\bigr)\bigr|^2\,\mathrm{d}s + C(\varepsilon)\,\mathbf{E}\int_0^t \bigl(1+\bigl|\nabla u^\varepsilon_{\xi^\varepsilon}(s)\bigr|^2\bigr)\,\mathrm{d}s$$
$$+\,\mathbf{E}\int_0^t\int_Z \bigl|g\bigl(u_\xi(s),z\bigr)\bigl[l\bigl(\psi^\varepsilon(s)\bigr)-l\bigl(\psi(s)\bigr)\bigr]\bigr|^2\,\lambda(\mathrm{d}z)\,\mathrm{d}s\biggr\},$$

where $C(\varepsilon) = (\varepsilon K_1 + (\varepsilon^2+\varepsilon)CK_1^2)$. Since $C(\varepsilon)\to 0$ when letting $\varepsilon\to 0$, we get $w^\varepsilon\to 0$. Hence the convergence is proved. □

Theorems 4 and 5 now guarantee that the solution of (1) satisfies the Laplace principle thereby satisfying the large deviation principle with the same rate function as well.

5 Conclusion

A stochastic version of predator–prey model with Holling type III functional response perturbed with Lévy noise is considered for which the principle of large deviations is studied. Foremost, the existence and uniqueness for this problem is analyzed using the technique of Galerkin approximations. Due to the equivalence with Laplace principle, LDP is proved rigorously using the weak convergence method by establishing the convergence between stochastic controlled problem to its deterministic counterpart in weaker sense. In order to do this, two postulates, namely, compactness and weak convergence are proved.

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