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Existence of sunny nonexpansive retractions and approximation of fixed points of a representation of nonexpansive mappings

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Abstract. This paper presents an implicit scheme for a representation of nonexpansive mappings on a closed convex subset of a smooth uniformly convex Banach space with respect to a left-regular sequence of means defined on a subset of $l^\infty(S)$. The main results are to establish an existence theorem of a sunny nonexpansive retraction and to create an algorithm for finding a common fixed point of a representation of nonexpansive mappings in Banach spaces.

Keywords: Opial's condition, sunny nonexpansive retraction, representation of nonexpansive mappings.

1 Introduction

The nonlinear ergodic theorem established by Baillon [2] in a Hilbert space for finding nonexpansive retractions is as follows. Let C denotes a nonempty closed convex subset of a Hilbert space H, and T denotes a nonexpansive mapping of C into itself. If $\mathrm{Fix}(T)$, the set of fixed points of T, is nonempty, then for each $x \in C$, the Cesaro means $S_n x = (1/n)\sum_{k=1}^{n} T^k x$ converge weakly to some $y \in \mathrm{Fix}(T)$. In this theorem, setting $y = Px$ for each $x \in C$, P represents a nonexpansive retraction of C onto $\mathrm{Fix}(T)$ such that $PT^n = T^n P = P$ for all positive integers n and $Px \in \overline{\mathrm{co}}\{T^n x,\, n = 1, 2, \dots\}$ for each $x \in C$. Takahashi [17] further established the existence of such retractions, termed as ergodic retractions, for noncommutative semigroups of nonexpansive mappings

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in a Hilbert space: If S is an amenable semigroup, C is a closed convex subset of a Hilbert space H, and $\mathcal{S} = \{T_s,\ s \in S\}$ is a nonexpansive semigroup on C such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, then there exists a nonexpansive retraction P from C onto $\mathrm{Fix}(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{\mathrm{co}}\{T_tx,\ t \in S\}$ for each $x \in C$. These findings were extended to uniformly convex Banach spaces for commutative semigroups in [9] and for amenable semigroups in [11, 12]. For further related results, readers are directed to works such as [6–8, 15]. In this paper, we establish an existence theorem for finding sunny nonexpansive retractions in a smooth and reflexive Banach space.

This paper presents an implicit scheme for a representation of nonexpansive mappings on a closed convex subset of a smooth and uniformly convex Banach space and is a contribution to ongoing research in functional analysis [4, 5, 7, 8, 13, 14, 16]. The study focuses on establishing an existence theorem for a sunny nonexpansive retraction and devising an algorithm to compute common fixed points for this representation. Throughout this paper, unless otherwise stated, S will denote a semigroup, E is a Banach space, C is a nonempty closed convex subset of E, and E^* is the dual space of E. Our main theorems elucidate the conditions under which such retractions exist and provide insights into their uniqueness. We also explore the convergence properties of implicit sequences in the context of this representation. Overall, this work contributes to offering practical implications for computing fixed points in various mathematical contexts.

1.1 Preliminaries

A mapping $T : C \to C$ is called nonexpansive, provided $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$, and a mapping g is a β-contraction on E, provided $\|g(x) - g(y)\| \leqslant \beta\|x - y\|$, $x, y \in E$, with $0 \leqslant \beta < 1$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping J from E into E^* for an element $x \in E$ is defined as

$$J(x) = \{g \in E^*\colon\ \langle x, g\rangle = \|x\|^2 = \|g\|^2\}.$$

Let the single-valued normalized duality mapping be denoted by j, and suppose that $U = \{x \in E\colon\ \|x\| = 1\}$. Now E is called smooth, provided the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. From Corollary 2.6.9 in [1] a Banach space E is smooth, provided the mapping $J : E \to E^*$ is single valued. The strong convergence (respectively, the weak convergence) of a sequence $\{x_n\}$ to x in E is denoted by $x_n \to x$ (respectively, $x_n \rightharpoonup x$), and also, the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* is denoted by $x_n^* \overset{*}{\rightharpoonup} x^*$. The duality mapping J is called weakly sequentially continuous if $x_n \rightharpoonup x$ implies $Jx_n \overset{*}{\rightharpoonup} Jx$ [1].

A family $\mathcal{S} = \{T_s,\ s \in S\}$ of mappings from C into itself is called a representation of S, provided:

(i) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;
(ii) for every $s \in S$, the mapping $T_s : C \to C$ is nonexpansive.

The set of common fixed points of $\mathcal{S}$ is denoted by $\mathrm{Fix}(\mathcal{S})$, i.e., $\mathrm{Fix}(\mathcal{S}) = \bigcap_{s\in S}\{x \in C: T_s x = x\}$.

The space of all bounded real-valued functions defined on S with supremum norm is denoted by $l^\infty(S)$. Also, l_s and r_s in $l^\infty(S)$ are defined as $(l_t g)(s) = g(ts)$ and $(r_t g)(s) = g(st)$ for all $s \in S$, $t \in S$, and $g \in l^\infty(S)$.

Suppose that X is a subspace of $l^\infty(S)$ containing 1, and let X^* be its topological dual space. An element m of X^* is said to be a mean on X, provided $\|m\| = m(1) = 1$. For $m \in X^*$ and $g \in X$, $m_t(g(t))$ is often written instead of $m(g)$. Suppose that X is left invariant (respectively, right invariant), i.e., $l_t(X) \subset X$ (respectively, $r_t(X) \subset X$) for each $s \in S$. A mean m on X is called left invariant (respectively, right invariant), provided $m(l_t g) = m(g)$ (respectively, $m(r_t g) = m(g)$) for each $t \in S$ and $g \in X$. X is called left (respectively, right) amenable if X possesses a left- (respectively, right-) invariant mean. Now X is amenable if X is both left and right amenable.

Let D be a directed set in X, and let $\{m_\alpha,\ \alpha \in D\}$ [1, p. 5, Sect. 1.1]. A net $\{m_\alpha,\ \alpha \in D\}$ of means on X is called left regular, provided

$$\lim_{\alpha\in D} \|l_t^* m_\alpha - m_\alpha\| = 0$$

for every $t \in S$, where l_t^* is the adjoint operator of l_t.

Let E be a reflexive Banach space. Let g be a function on S into E such that the weak closure of $\{g(s),\ s \in S\}$ is weakly compact, and suppose that X is a subspace of $l^\infty(S)$ containing all the functions $s \to \langle g(s), x^*\rangle$ with $x^* \in E^*$. We know from [9] that, for any $m \in X^*$, there exists a unique element g_m in E such that $\langle g_m, x^*\rangle = m_s\langle f(s), x^*\rangle$ for all $x^* \in E^*$. We denote such a g_m by $\int g(s)\,\mathrm{d}m(s)$. Moreover, if m is a mean on X, then from [10] $\int g(s)\,\mathrm{d}m(s) \in \mathrm{cl}(\mathrm{co}\{g(s),\ s \in S\})$, where $\mathrm{cl}(\mathrm{co}\{g(s),\ s \in S\})$ denotes the closure of the convex hull of $\{g(s),\ s \in S\}$.

Remark 1. From Theorem 4.1.6 in [18] every uniformly convex Banach space is strictly convex and reflexive.

Remark 2. For details on retractions and sunny nonexpansive retract concepts, we refer the reader to [1, 18]. We know from [1, 18] that if E is a smooth Banach space, F is a nonempty convex subset of E, G is a nonempty subset of F, and R is a retraction from F onto G, then R is sunny and nonexpansive if and only if for each $x \in F$ and $z \in G$,

$$\big\langle x - Rx,\ J(z - Rx)\big\rangle \leqslant 0.$$

Remark 3. For details on demiclosed mappings at a point and the Opial condition for a vector space, we refer the reader to [1, 18].

In this paper, we denote by B_r an open ball of radius r centered at 0, and for $\epsilon > 0$ and a mapping $T : C \to C$, we denote by $F_\epsilon(T; G)$ the set of ϵ-approximate fixed points of T for a subset G of C, i.e., $F_\epsilon(T; G) = \{x \in G: \|x - Tx\| \leqslant \epsilon\}$.

2 Main results

In this section, our goal is to prove a strong convergence scheme for a representation of nonexpansive mappings. The following two results, Theorems 1 and 2, will enable us to prove Theorem 3.

Theorem 1. *Let S be a semigroup and E be a real smooth and reflexive Banach space, and suppose that C is a nonempty closed convex subset of E. Let X be a left-invariant subspace of $l^\infty(S)$ such that $1 \in X$, and $t \mapsto \langle T_t x, x^* \rangle$ belongs to X for each $x \in C$ and $x^* \in E^*$. Assume that X is left amenable, and let $\mathcal{S} = \{T_s : C \to C,\ s \in S\}$ be a representation of S of nonexpansive mappings such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. If J is weakly sequentially continuous, then $\mathrm{Fix}(\mathcal{S})$ is a sunny nonexpansive retract of C, and the sunny nonexpansive retraction of C onto $\mathrm{Fix}(\mathcal{S})$ is unique.*

Proof. From Theorem 3.2.8 in [1] E satisfies the Opial condition. Consider a sequence z_n in C as follows:

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T_\mu z_n \quad (n \in \mathbb{N}),$$

where $x \in C$ is fixed, and μ is a left-invariant mean on X. We claim the mapping N_n given by

$$N_n z := \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T_\mu z \quad (z \in C)$$

is a contraction. To see this, put $\beta_n = (1 - 1/n)$. Then $0 \leqslant \beta_n < 1$ ($n \in \mathbb{N}$), and we have

$$\begin{aligned}\|N_n z - N_n y\| &= \left(1 - \frac{1}{n}\right)\|T_\mu z - T_\mu y\| \\ &\leqslant \left(1 - \frac{1}{n}\right)\|z - y\| = \beta_n \|z - y\|.\end{aligned}$$

Therefore, by the Banach contraction principle [18], there exists a unique point $z_n \in C$ such that $N_n z_n = z_n$. Note, we have

$$\lim_{n\to\infty} \|z_n - T_\mu z_n\| = 0. \tag{1}$$

Now, we show that $\{z_n\}$ is bounded. Let $p \in \mathrm{Fix}(\mathcal{S})$. From (ii) of Theorem 2.1 in [15] $T_\mu p = p$, and we have

$$\begin{aligned}\|z_n - p\|^2 &= \left\langle \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T_\mu z_n - p,\ J(z_n - p) \right\rangle \\ &= \frac{1}{n}\langle x - p,\ J(z_n - p)\rangle + \left\langle \left(1 - \frac{1}{n}\right)(T_\mu z_n - T_\mu p),\ J(z_n - p) \right\rangle \\ &\leqslant \left(1 - \frac{1}{n}\right)\|z_n - p\|^2 + \frac{1}{n}\langle x - p,\ J(z_n - p)\rangle.\end{aligned}$$

Thus,

$$\|z_n - p\|^2 \leqslant \big\langle x - p,\ J(z_n - p)\big\rangle. \tag{2}$$

Hence,

$$\|z_n - p\| \leqslant \|x - p\|.$$

That is, the sequence $\{z_n\}$ is bounded.

We next show that the weak limit set of $\{z_n\}$ (denoted by $\omega_\omega\{z_n\}$) is a subset of $\mathrm{Fix}(\mathcal{S})$. Let $x^* \in \omega_\omega\{z_n\}$, and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \rightharpoonup x^*$. We need to show that $x^* \in \mathrm{Fix}(\mathcal{S})$. Note from Corollary 5.2.10 in [1], $I - T_t$ is demiclosed at zero for each $t \in S$. Hence, from (1) we conclude that $x^* \in \mathrm{Fix}(\mathcal{S})$. Therefore, $\omega_\omega\{z_n\} \subseteq \mathrm{Fix}(\mathcal{S})$.

Now note since $\{z_n\}$ is bounded and E is reflexive, from Theorem 1.9.21 in [1] $\{z_n\}$ is a weakly compact subset of E. Hence, by Proposition 1.7.2 in [1], we can select a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $\{z_{n_j}\}$ weakly converges to a point z. Consequently, $z \in \mathrm{Fix}(\mathcal{S})$. Let $\{z_{n_i}\}$ and $\{z_{n_j}\}$ be subsequences of $\{z_n\}$ such that $\{z_{n_i}\}$ and $\{z_{n_j}\}$ converge weakly to y and z, respectively. Therefore, $y, z \in \mathrm{Fix}(\mathcal{S})$. Since J is weakly sequentially continuous, from (2) we have that $\{z_{n_i}\}$ and $\{z_{n_j}\}$ converge strongly to y and z, respectively, since from (2) we have

$$\begin{aligned}\lim_{i\to\infty} \|z_{n_i} - y\|^2 &\leqslant \lim_{i\to\infty} \big\langle x - y,\ J(z_{n_i} - y)\big\rangle \\ &= \big\langle x - y,\ J(y - y)\big\rangle = 0.\end{aligned}$$

Hence, $z_{n_i} \to y$ and, similarly, $z_{n_j} \to z$. Now, for each $z \in \mathrm{Fix}(\mathcal{S})$ and $n \in \mathbb{N}$,

$$\big\langle z_n - x,\ J(z_n - z)\big\rangle \leqslant 0 \tag{3}$$

since for each $z \in \mathrm{Fix}(\mathcal{S})$, we have

$$\begin{aligned}\big\langle z_n - x,\ J(z_n - z)\big\rangle &= \left\langle \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T_\mu z_n - x,\ J(z_n - z)\right\rangle \\ &= (n-1)\big\langle T_\mu z_n - z_n,\ J(z_n - z)\big\rangle \\ &= (n-1)\big\langle T_\mu z_n - T_\mu z,\ J(z_n - z)\big\rangle \\ &\quad + (n-1)\big\langle z - z_n,\ J(z_n - z)\big\rangle \\ &\leqslant (n-1)\big(\|T_\mu z_n - T_\mu z\|\|z_n - z\| - \|z_n - z\|^2\big) \\ &\leqslant (n-1)\big(\|z_n - z\|^2 - \|z_n - z\|^2\big) = 0.\end{aligned}$$

Further,

$$\begin{aligned}&\big|\big\langle z_{n_i} - x,\ J(z_{n_i} - z)\big\rangle - \big\langle y - x,\ J(y - z)\big\rangle\big| \\ &\quad = \big|\big\langle z_{n_i} - x,\ J(z_{n_i} - z)\big\rangle - \big\langle y - x,\ J(z_{n_i} - z)\big\rangle \\ &\qquad + \big\langle y - x,\ J(z_{n_i} - z)\big\rangle - \big\langle y - x,\ J(y - z)\big\rangle\big| \\ &\quad \leqslant \big|\big\langle z_{n_i} - y,\ J(z_{n_i} - z)\big\rangle\big| + \big|\big\langle y - x,\ J(z_{n_i} - z) - J(y - z)\big\rangle\big| \\ &\quad \leqslant \|z_{n_i} - y\|\big\|J(z_{n_i} - z)\big\| + \big|\big\langle y - x,\ J(z_{n_i} - z) - J(y - z)\big\rangle\big| \\ &\quad \leqslant \|z_{n_i} - y\|M + \big|\big\langle y - x,\ J(z_{n_i} - z) - J(y - z)\big\rangle\big|,\end{aligned}$$

where M is an upper bound for $\{J(z_{n_i} - z)\}_{i\in\mathbb{N}}$. Hence, we have

$$\lim_{i\to\infty} \langle z_{n_i} - x,\, J(z_{n_i} - z)\rangle = \langle y - x,\, J(y - z)\rangle. \tag{4}$$

Now, since J is weakly sequentially continuous, from (4) and (3) we have

$$\langle y - x,\, J(y - z)\rangle = \lim_{i\to\infty} \langle z_{n_i} - x,\, J(z_{n_i} - z)\rangle \leqslant 0.$$

Similarly, $\langle z - x,\, J(z - y)\rangle \leqslant 0$, and thus, we have

$$\begin{aligned}\|y - z\|^2 &= \langle y - z,\, J(y - z)\rangle \\ &= \langle y - x,\, J(y - z)\rangle + \langle x - z,\, J(y - z)\rangle \\ &= \langle y - x,\, J(y - z)\rangle + \langle z - x,\, J(z - y)\rangle \\ &\leqslant 0,\end{aligned}$$

so $y = z$. Thus, $\{z_n\}$ weakly converges to an element of $\mathrm{Fix}(\mathcal{S})$.

Therefore, a mapping P of C into itself can be defined by $Px = \text{weak} - \lim_n z_n$. Then, since $Px \in \mathrm{Fix}(\mathcal{S})$ and J is weakly sequentially continuous, we have from (2) that

$$\begin{aligned}\lim_{n\to\infty} \|z_n - Px\|^2 &\leqslant \lim_{n\to\infty} \langle x - Px,\, J(z_n - Px)\rangle \\ &= \langle x - Px,\, J(Px - Px)\rangle = 0.\end{aligned}$$

Hence, $z_n \to Px$. Then from the condition that the duality mapping is weakly sequentially continuous we have, for each $z \in \mathrm{Fix}(\mathcal{S})$,

$$\langle x - Px,\, J(z - Px)\rangle = \lim_{n\to\infty} \langle z_n - x,\, J(z_n - z)\rangle \leqslant 0. \tag{5}$$

It follows from Lemma 5.1.6 in [18] that P is a sunny nonexpansive retraction of C onto $\mathrm{Fix}(\mathcal{S})$. We note that P is unique. Indeed, let R be another sunny nonexpansive retraction of C onto $\mathrm{Fix}(\mathcal{S})$. Then from [18, p.199] we have, for each $x \in C$ and $z \in \mathrm{Fix}(\mathcal{S})$,

$$\langle x - Rx,\, J(z - Rx)\rangle \leqslant 0. \tag{6}$$

Putting $z = Rx$ in (5) and $z = Px$ in (6), we have $\langle x - Px,\, J(Rx - Px)\rangle \leqslant 0$ and $\langle x - Rx,\, J(Px - Rx)\rangle \leqslant 0$, and hence, $\langle Rx - Px,\, J(Rx - Px)\rangle \leqslant 0$. This implies $Rx = Px$, so the proof is completed. □

Theorem 2. *Let S be a semigroup and E be a real uniformly convex and smooth Banach space, and suppose that C is a nonempty closed convex subset of E. Suppose that $\mathcal{S} = \{T_s,\, s \in S\}$ is a representation of S of nonexpansive mappings from C into itself such that* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$*. Let X be a left-invariant subspace of $l^\infty(S)$ such that $1 \in X$, and $t \mapsto \langle T_t x, x^*\rangle$ belongs to X for each $x \in C$ and $x^* \in E^*$. If μ is a left-invariant mean on X and if J is weakly sequentially continuous, then* $\mathrm{Fix}(T_\mu) = T_\mu(C) = \mathrm{Fix}(\mathcal{S})$*, and there exists a unique sunny nonexpansive retraction from C onto* $\mathrm{Fix}(\mathcal{S})$.

Proof. Let $\epsilon > 0$ be arbitrary. From Theorem 3.2.8 in [1] E satisfies the Opial condition. Consider $t \in S$ and let $x \in C$. Let p be an arbitrary element of $\mathrm{Fix}(\mathcal{S})$. Set $D = \{y \in C\colon \|y - p\| \leqslant \|x - p\|\}$. Note that D is a bounded closed convex set, $x \in D$, and $T_t(D) \subset D$. Using Theorem 1.2 and Corollary 1.1 in [3], since μ is left invariant, as in the proof of Theorem 1 in [11], we have that

$$T_\mu x \in F_\epsilon(T_t; D).$$

Therefore, we have $T_t T_\mu x = T_\mu x$, and hence, $\mathrm{Fix}(T_\mu) = T_\mu(C) = \mathrm{Fix}(\mathcal{S})$. The remaining part is similar to that in the proof of Theorem 1. □

In what follows, the index set for n is the natural numbers, i.e., $n \in \mathbb{N}^+$.

Theorem 3. *Let S be a semigroup and E be a real uniformly convex and smooth Banach space, and suppose C is a nonempty closed convex subset of E. Suppose that $\mathcal{S} = \{T_s,\ s \in S\}$ is a representation of S of nonexpansive mappings on C into itself such that the weak closure of $\{T_t x,\ t \in S\}$ is weakly compact for each $x \in C$ and $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Let X be a left-invariant subspace of $l^\infty(S)$ such that $1 \in X$, and $t \mapsto \langle T_t x, x^*\rangle$ belongs to X for each $x \in C$ and $x^* \in E^*$. Suppose that $\{\mu_n\}$ is a left-regular sequence of means on X and that f is an α-contraction on C. Let ϵ_n be a sequence in $(0,1)$ such that $\lim_n \epsilon_n = 0$, and let the duality mapping J be weakly sequentially continuous. Then there exists a unique sunny nonexpansive retraction P of C onto $\mathrm{Fix}(\mathcal{S})$ and $x \in C$ such that the following sequence $\{z_n\}$, generated by*

$$z_n = \epsilon_n f(z_n) + (1 - \epsilon_n) T_{\mu_n} z_n \quad (n \in \mathbb{N}), \tag{7}$$

strongly converges to Px.

Proof. From Theorem 3.2.8 in [1] E satisfies the Opial condition, and from Theorem 2.2.8 in [1] E is reflexive. The proof will be divided into six steps.

Step 1. The existence of the element z_n (satisfying (7)) in C.

This follows immediately from the fact that for every $n \in \mathbb{N}$, the mapping N_n, given by

$$N_n x := \epsilon_n f(x) + (1 - \epsilon_n) T_{\mu_n} x \quad (x \in C),$$

is a contraction since if we put $\beta_n = (1 + \epsilon_n(\alpha - 1))$, then $0 \leqslant \beta_n < 1$ $(n \in \mathbb{N})$, and we have

$$\begin{aligned}\|N_n x - N_n y\| &\leqslant \epsilon_n \big\|f(x) - f(y)\big\| + (1 - \epsilon_n)\|T_{\mu_n} x - T_{\mu_n} y\| \\ &\leqslant \epsilon_n \alpha \|x - y\| + (1 - \epsilon_n)\|x - y\| \\ &= \big(1 + \epsilon_n(\alpha - 1)\big)\|x - y\| \\ &= \beta_n \|x - y\|.\end{aligned}$$

Therefore, by the Banach contraction principle [18], there exists a unique point $z_n \in C$ such that $N_n z_n = z_n$.

Step 2. $\{z_n\}$ is bounded.

Let $p \in \operatorname{Fix}(\mathcal{S})$. From (ii) of Theorem 2.1 in [15] $T_{\mu_n} p = p$ for each $n \in \mathbb{N}$, and we have

$$\begin{aligned}\|z_n - p\|^2 &= \bigl\langle \epsilon_n f(z_n) + (1-\epsilon_n) T_{\mu_n} z_n - p,\ J(z_n - p)\bigr\rangle \\ &= \epsilon_n \bigl\langle f(z_n) - f(p),\ J(z_n - p)\bigr\rangle + \epsilon_n \bigl\langle f(p) - p,\ J(z_n - p)\bigr\rangle \\ &\quad + \bigl\langle (1-\epsilon_n)\bigl(T_{\mu_n} z_n - T_{\mu_n} p\bigr),\ J(z_n - p)\bigr\rangle \\ &\leqslant \epsilon_n \alpha \|z_n - p\|^2 + (1-\epsilon_n)\|z_n - p\|^2 + \epsilon_n \bigl\langle f(p) - p,\ J(z_n - p)\bigr\rangle.\end{aligned}$$

Thus,

$$\|z_n - p\|^2 \leqslant \frac{1}{1-\alpha} \bigl\langle f(p) - p,\ J(z_n - p)\bigr\rangle,$$

so,

$$\|z_n - p\| \leqslant \frac{1}{1-\alpha} \bigl\| f(p) - p \bigr\|.$$

That is, the sequence $\{z_n\}$ is bounded.

Step 3. $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$ for all $t \in S$. Let $t \in S$ and p be an arbitrary element of $\operatorname{Fix}(\mathcal{S})$. Set $D = \{y \in C\colon \|y - p\| \leqslant \|f(p) - p\|/(1-\alpha)\}$. We note that D is a bounded closed convex set, $\{z_n\} \subset D$, and $T_t(D) \subset D$. Let $\epsilon > 0$. From Theorem 1.2 in [3] there exists a $\delta > 0$ such that

$$\operatorname{cl}\bigl(\operatorname{co} F_\delta(T_t; D)\bigr) + B_\delta \subset F_\epsilon(T_t; D). \tag{8}$$

From Corollary 1.1 in [3] there is a natural number N such that

$$\left\| \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y - T_t \left(\frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \right) \right\| \leqslant \delta \tag{9}$$

for $s \in S$ and $y \in D$. Let $M_0 = \|f(p) - p\|/(1-\alpha) + \|p\|$. Therefore, $\sup_{y\in D} \|y\| \leqslant M_0$. From the condition that $\{\mu_n\}$ is strongly left regular there is a $N_0 \in \mathbb{N}$ such that $\|\mu_n - l^*_{t^i} \mu_n\| \leqslant \delta / M_0$ for $n \geqslant N_0$ and $i = 1, 2, \dots, N$. Therefore, as in the proof of Theorem 3 in [11], we have

$$\begin{aligned}&\sup_{y\in D} \left\| T_{\mu_n} y - \int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \mu_n(s)\, \mathrm{d}\mu_n \right\| \\ &\quad \leqslant \max_{i=1,2,\dots,N} \|\mu_n - l^*_{t^i} \mu_n\| (M_0) \leqslant \delta \quad (n \geqslant N_0).\end{aligned} \tag{10}$$

Hence from Theorem 2.1 in [15] we have

$$\int \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y \mu_n(s)\, \mathrm{d}\mu_n \in \operatorname{cl}\left(\operatorname{co}\left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i}(T_s y),\ s \in S \right\} \right). \tag{11}$$

Therefore, it follows from (9)–(11) that

$$\begin{aligned}
T_{\mu_n}y &= \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\mu_n(s)\,\mathrm{d}\mu_n + \left(T_{\mu_n}y - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\mu_n(s)\,\mathrm{d}\mu_n\right) \\
&\in \mathrm{cl}\left(\mathrm{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y,\ s\in S\right\}\right) + B_\delta \\
&\subset \mathrm{cl}\big(\mathrm{co}\,F_\delta(T_t;D)\big) + B_\delta \subset F_\epsilon(T_t;D)
\end{aligned}$$

for all $y\in D$ and $n\geqslant N_0$. Thus, we have $\limsup_{n\to\infty}\sup_{y\in D}\|T_t(T_{\mu_n}y)-T_{\mu_n}y\|\leqslant\epsilon$. Since $\epsilon>0$ is arbitrary, we conclude that

$$\limsup_{n\to\infty}\sup_{y\in D}\big\|T_t(T_{\mu_n}y)-T_{\mu_n}y\big\| = 0. \tag{12}$$

Suppose that $t\in S$ and $\epsilon>0$. Then there is a $\delta>0$, which satisfies (8). Take $L_0 = 2\|f(p)-p\|/(1-\alpha)$. Since $\lim_n \epsilon_n = 0$, from (12) there is a natural number N_1 such that $T_{\mu_n}y\in F_\delta(T_t;D)$ for all $y\in D$ and $\epsilon_n<\delta/(2L_0)$ for all $n\geqslant N_1$. Since $p\in\mathrm{Fix}(\mathcal{S})$ and $\{z_n\}\subset D$, we have

$$\begin{aligned}
&\epsilon_n\big\|f(z_n)-T_{\mu_n}z_n\big\| \\
&\quad\leqslant \epsilon_n\big(\big\|f(z_n)-f(p)\big\| + \big\|f(p)-p\big\| + \big\|T_{\mu_n}p - T_{\mu_n}z_n\big\|\big) \\
&\quad\leqslant \epsilon_n\big(\alpha\|z_n-p\| + \big\|f(p)-p\big\| + \|z_n-p\|\big) \\
&\quad\leqslant \epsilon_n\left(\frac{1+\alpha}{1-\alpha}\big\|f(p)-p\big\| + \big\|f(p)-p\big\|\right) \\
&\quad= \epsilon_n L_0 \leqslant \frac{\delta}{2}
\end{aligned}$$

for all $n\geqslant N_1$. It can also be observed that

$$\begin{aligned}
z_n &= \epsilon_n f(z_n) + (1-\epsilon_n)T_{\mu_n}z_n \\
&= T_{\mu_n}z_n + \epsilon_n\big(f(z_n)-T_{\mu_n}z_n\big) \\
&\in F_\delta(T_t;D) + B_{\delta/2} \subseteq F_\delta(T_t;D) + B_\delta \\
&\subseteq F_\epsilon(T_t;D)
\end{aligned}$$

for all $n\geqslant N_1$. Thus, $\|z_n - T_t z_n\|\leqslant\epsilon$ $(n\geqslant N_1)$. Since $\epsilon>0$ is arbitrary, we have $\lim_{n\to\infty}\|z_n-T_t z_n\| = 0$.

Step 4. The weak limit set of $\{z_n\}$, which is denoted by $\omega_\omega\{z_n\}$, is a subset of $\mathrm{Fix}(\mathcal{S})$.

Let $x^*\in\omega_\omega\{z_n\}$, and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j}\rightharpoonup x^*$. We need to show that $x^*\in\mathrm{Fix}(\mathcal{S})$. From Corollary 5.2.10 in [1] $I-T_t$ is demiclosed at zero for each $t\in S$, and hence, from Step 3 we conclude that $x^*\in\mathrm{Fix}(\mathcal{S})$.

Step 5. The existence of a unique sunny nonexpansive retraction P of C onto $\mathrm{Fix}(\mathcal{S})$, and $x\in C$, which satisfies

$$\Gamma := \limsup_n\big\langle x-Px,\ J(z_n-Px)\big\rangle \leqslant 0. \tag{13}$$

Let μ be a cluster point of $\{\mu_n\}$. It is clear that μ is an invariant mean. Hence, by Theorem 2, there exists a unique sunny nonexpansive retraction P of C onto $\mathrm{Fix}(\mathcal{S})$. By the Banach contraction mapping principle fP has a unique fixed point $x \in C$. We now prove that

$$\Gamma := \limsup_n \bigl\langle x - Px,\ J(z_n - Px)\bigr\rangle \leqslant 0.$$

Since E is reflexive, it follows from Step 2 and Theorem 1.9.21 in [1] that $\{z_n\}$ is weakly compact. Hence, from the definition of Γ and Proposition 1.7.2 in [1] we can choose a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ with the following properties:

(i) $\lim_j \langle x - Px,\ J(z_{n_j} - Px)\rangle = \Gamma$;
(ii) $\{z_{n_j}\}$ weakly converges to a point z;

and using Step 4, we have $z \in \mathrm{Fix}(\mathcal{S})$. Since E is smooth, by the weakly sequentially continuity of J, we have

$$\Gamma = \lim_j \bigl\langle x - Px,\ J(z_{n_j} - Px)\bigr\rangle = \bigl\langle x - Px,\ J(z - Px)\bigr\rangle \leqslant 0.$$

Step 6. $\{z_n\}$ strongly converges to Px.

We begin by showing that for each $n \in \mathbb{N}$,

$$\|z_n - Px\|^2 \leqslant \frac{2}{1-\alpha}\bigl\langle x - Px,\ J(z_n - Px)\bigr\rangle. \tag{14}$$

To see first note, since $fPx = x$, we have $(f - I)Px = x - Px$. Now from [18, p. 99] we have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\epsilon_n(\alpha - 1)\|z_n - Px\|^2 \\
&\quad\geqslant \bigl[\epsilon_n\alpha\|z_n - Px\| + (1-\epsilon_n)\|z_n - Px\|\bigr]^2 - \|z_n - Px\|^2 \\
&\quad\geqslant \bigl[\epsilon_n\bigl\|f(z_n) - f(Px)\bigr\| + (1-\epsilon_n)\|T_{\mu_n}z_n - Px\|\bigr]^2 - \|z_n - Px\|^2 \\
&\quad\geqslant 2\bigl\langle \epsilon_n\bigl(f(z_n) - f(Px)\bigr) + (1-\epsilon_n)(T_{\mu_n}z_n - Px) - (z_n - Px),\ J(z_n - Px)\bigr\rangle \\
&\quad= -2\epsilon_n\bigl\langle (f - I)Px,\ J(z_n - Px)\bigr\rangle \\
&\quad= -2\epsilon_n\bigl\langle x - Px,\ J(z_n - Px)\bigr\rangle,
\end{aligned}$$

and so,

$$\|z_n - Px\|^2 \leqslant \frac{2}{1-\alpha}\bigl\langle x - Px,\ J(z_n - Px)\bigr\rangle.$$

Therefore, from (13), (14) and that the fact that $Px \in \mathrm{Fix}(\mathcal{S})$ we have

$$\limsup_n \|z_n - Px\|^2 \leqslant \frac{2}{1-\alpha} \limsup_n \bigl\langle x - Px,\ J(z_n - Px)\bigr\rangle \leqslant 0.$$

That is, $z_n \to Px$. □

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