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Fractional elliptic obstacle systems with multivalued terms and nonlocal operators*

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Abstract. In this paper, we introduce and study a fractional elliptic obstacle system, which is composed of two elliptic inclusions with fractional (p_i, q_i)-Laplace operators, nonlocal functions, and multivalued terms. The weak solution of fractional elliptic obstacle system is formulated by a fully nonlinear coupled system driven by two nonlinear and nonmonotone variational inequalities with constraints. The nonemptiness and compactness of solution set in the weak sense are proved via employing a surjectivity theorem to the multivalued operators formulated by the sum of a multivalued pseudomonotone operator and a maximal monotone operator.

Keywords: fractional (p, q)-Laplacian, nonlocal operator, multivalued term, obstacle problem, pseudomonotone operator, existence, compactness.

1 Introduction

Obstacle problems for nonlocal operators appear in optimal control, mathematical finance, biology, elasticity, and other applied sciences; see, for instance, the books [18, 30]. Originally, the study of obstacle problems is due the pioneering contribution by Stefan [36] in which the temperature distribution in a homogeneous medium undergoing

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a phase change, typically a body of ice at zero degrees centigrade submerged in water, was studied.

In the current paper, we are interested in the study of the fractional Dirichlet elliptic inclusion system with fractional (p_i, q_i)-Laplace operators for $i = 1, 2$, nonlocal functions, convex subdifferential operators, multivalued mappings, and obstacle effect. To this end, let Ω be a bounded, open, and connected domain in $\mathbb{R}^N$ ($N \geqslant 2$) with smooth boundary $\partial\Omega$. For $i = 1, 2$, let $1 < q_i < p_i < +\infty$, $0 < t_1 < t_2 < 1$, and $a_i : L^{p^*_{i,s_i}}(\Omega) \to (0, +\infty)$ and $b_i : L^{p^*_{i,s_i}}(\Omega) \to [0, +\infty)$ be given nonlocal functions, where p^*_{i,s_i} is the fractional critical Sobolev exponent of p_i corresponding to s_i, defined by (5). We consider the following problem:

$$\begin{aligned} &a_1(u)(-\Delta)^{s_1}_{p_1} u + b_1(u)(-\Delta)^{t_1}_{q_1} u + \partial_c\phi(x, u) + U(x, u, v) \ni f(x) \quad \text{in } \Omega, \\ &a_2(u)(-\Delta)^{s_2}_{p_2} v + b_2(v)(-\Delta)^{t_2}_{q_2} v + \partial_c\psi(x, v) + R(x, v, u) \ni g(x) \quad \text{in } \Omega, \\ &u(x) = 0 \quad v(x) = 0, \quad \text{in } \Omega^{\complement}, \\ &u(x) \leqslant \Phi(x), \quad v(x) \leqslant \Psi(x) \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $U : \Omega\times\mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}}$ and $R : \Omega\times\mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}}$ are two given multivalued maps, $\partial_c\phi$ (resp. $\partial_c\psi$) stands for the convex subdifferential operator of convex function $s \mapsto \phi(x, s)$ (resp. $s \mapsto \psi(x, s)$), and $\Phi : \Omega \to [0, +\infty)$ and $\Psi : \Omega \to [0, +\infty)$ are two given obstacle functions. Here the symbol $(-\Delta)^l_r$ with $(r, l) \in \{(p_1, s_1), (p_2, s_2), (q_1, t_1), (q_2, t_2)\}$ is the fractional r-Laplace operator defined, up to a normalization constant depending on N, r and l, by setting

$$(-\Delta)^l_r u(x) := 2 \lim_{\delta\to 0} \int\limits_{\mathbb{R}^N\setminus B_\delta(x)} \frac{|u(x) - u(y)|^{r-2}(u(x) - u(y))}{|x - y|^{N+lr}} \,\mathrm{d}y \tag{2}$$

for a.a. $x \in \mathbb{R}^N$ and any function $u : \mathbb{R}^N \to \mathbb{R}$ sufficiently smooth. Concerning the fractinoal elliptic operator defined in (2), it can see that unlike local (partial) differential operators, such as Laplacian and p-Laplacian, (2) could be used to study the complicated problems, which have nonlocal behavior and singular characteristic, for instance, nonlocal Signorini contact mechanic problems (i.e., contact models with unilateral contact boundary condition), fractional viscoelastic constitutive laws, diffusion model along a comb structure, option price, and so forth (for more details, we refer to [10, Sect. 1]).

In the meanwhile, it should be mentioned that (1) is a generalized elliptic inclusion system, which contains several important and interesting problems as special case. For example, when U is independent of v, then system (1) becomes the following elliptic obstacle inclusion involving nonlocal terms a_1, b_1, fractional (p_1, q_1)-Laplace operator, a convex subdifferential term, and an abstract multivalued mapping U:

$$\begin{aligned} &a_1(u)(-\Delta)^{s_1}_{p_1} u + b_1(u)(-\Delta)^{t_1}_{q_1} u + \partial_c\phi(x, u) + U(x, u) \ni f(x) \quad \text{in } \Omega, \\ &u(x) = 0 \quad \text{in } \Omega^{\complement}, \qquad u(x) \leqslant \Phi(x) \quad \text{in } \Omega. \end{aligned} \tag{3}$$

The above elliptic inclusion system (3) has not been studied yet. But its particular case with $a_1 \equiv 1$, $b_1 \equiv 0$, and $U(x, u) = \partial j(x, u)$, where $\partial j(x, u)$ is the generalized Clarke subdifferential operator of $u \mapsto j(x,u)$, namely,

$$(-\Delta)_{p_1}^{s_1} u + \partial_c \phi(x,u) + \partial j(x,u) \ni f(x) \quad \text{in } \Omega,$$
$$u(x) = 0 \quad \text{in } \Omega^{\complement},$$

was studied recently by Migórski Nguyen, and Zeng [22] with bilateral obstacle conditions (that is, there are two obstacle functions Ψ, Φ such that $\Psi(x) \leqslant u(x) \leqslant \Phi(x)$ for a.e. $x \in \Omega$). Another motivation of this paper is that, from the applications point of view, system (1) contains two nonlocal elliptic inclusions with obstacle effect, such problems could be a powerful mathematical model to describe the stationary behavior of double-species growth problem with coexistence or competition effect and growth constraints (i.e., the obstacle constraints); see [21, 35].

In general, the novelty of the present work is the fact that several interesting and challenging phenomena are considered in one problem. To be more precise, problem (1) contains the following effects:

(i) elliptic inclusion system with fractional (p_i, q_i)-Laplace operators;
(ii) nonlocal functions;
(iii) convex subdifferential operators;
(iv) multivalued mappings;
(v) obstacle effect.

The partial differential equations with space-fractional differential operators are very powerful and have lots of applications to different nonlinear problems including phase transitions, thin obstacle problem, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, ultra-relativistic limits of quantum mechanics, multiple scattering, minimal surfaces, material science, water waves, and so on. The starting point in the study of nonlocal problems is due to the pioneering papers of Caffarelli et al. [7–9]. We refer to Di Nezza, Palatucci, and Valdinoci [27] for a comprehensive introduction to the study of space-fractional differential equations. Based on this, several other works have been published in the nonlocal framework; see [2, 4, 27, 32].

In particular, the fractional obstacle problem appears in many contexts, including the pricing of American options with jump processes (see [14] and the Appendix of [3] for an informal discussion) and the study of the regularity of minimizers of nonlocal interaction energies in kinetic equations (see [11]). While the obstacle problem for the fractional Laplacian is nonlocal and nonsmooth, it admits a local formulation thanks to the extension method (see [9, 25]). For more details about the fractional obstacle problem, we refer the readers to the important papers of Caffarelli, Figalli, Salsa, and Silvestre (see, e.g., [6, 31, 34]); see also [17] for the analysis of families of bilateral obstacle problems involving fractional-type energies in aperiodic settings, the paper [29] for the fractional obstacle problems with drift (it is nowadays pretty well understood); see the exhaustive lecture notes [31] and the work by Motreanu et al. [26], and the references therein.

Equations involving the obstacle problems for (p,q)-Laplace operators are a rarity in the literature. In fact, results on this subject are few. In particular, in [12, 19], the authors studied the regularity results for obstacle problems in the case of standard (p,q)-growth conditions of integer. Thus, the problem discussed here is new as the consideration of fractional Dirichlet elliptic inclusion system with fractional (p_i,q_i)-Laplace operators for $i=1,2$, nonlocal functions, convex subdifferential operators, multivalued mappings, and obstacle effect handled with surjectivity theorem is not found anywhere in the literature to our knowledge.

Finally, we mention some interesting phenomena such as the combination of an obstacle effect along with partial differential operators; see [1, 13, 16, 37–41]. On this subject, Zeng et al. [37] introduced and investigated a class of complicated implicit obstacle problems driven by the fractional (p,q)-Laplace operator and two multivalued terms, which contains several interesting and challenging untreated problems. Using nonsmooth analysis, Tychonoff's fixed point theorem for multivalued operators, and variational approach, they gave a general framework to examine the existence of a weak solution to the implicit obstacle problems.

The paper is organized as follows. Section 2 recalls some necessary preliminary materials, which will be used in Section 3 from time to time. Under very general assumptions on the data, in Section 3, we employ a surjectivity theorem for multivalued pseudomonotone operators to prove that the weak solution set of problem (1) is nonempty and compact no matter what a and b are coercive.

2 Mathematical background

Given a bounded, open, and connected domain $\Omega\subset\mathbb{R}^N$ $(N\geqslant 2)$ with smooth boundary $\Gamma:=\partial\Omega$ for any $1<p<+\infty$ and $0<s<1$ such that $sp<N$, we adopt the symbol $W^{s,p}(\Omega)$ to stand for the fractional Sobolev space defined by

$$W^{s,p}(\Omega):=\left\{u\in L^p(\Omega)\colon \int\limits_\Omega\int\limits_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}\,\mathrm{d}x\,\mathrm{d}y<+\infty\right\},$$

where $L^p(\Omega):=L^p(\Omega;\mathbb{R})$ is the usual Lebesgue space equipped with the norm $\|\cdot\|_p$ given by

$$\|u\|_p:=\left(\int\limits_\Omega |u|^p\,\mathrm{d}x\right)^{1/p}\quad\text{for all } u\in L^p(\Omega).$$

Essential speaking, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, which is an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)}:=\|u\|_p+[u]_{W^{s,p}(\Omega)}\quad\text{for all } u\in W^{s,p}(\Omega),$$

become a Banach space. Here the term $[u]_{W^{s,p}(\Omega)}$ is the so-called Gagliardo seminorm of u formulated by

$$[u]_{W^{s,p}(\Omega)}:=\left(\int\limits_\Omega\int\limits_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\quad\text{for all } u\in W^{s,p}(\Omega),$$

whereas the fractional Sobolev space $W^{s,p}(\Omega)$ is not suffice to study the problems with the condition $u = 0$ outside Ω, i.e., fractional Dirichlet problems. So, a subspace $V_{p,s}$ of $W^{s,p}(\Omega)$ with the condition that for each $u \in V_{p,s}$, $u = 0$ holds outside Ω, will be introduced latter.

Let $1 < q_i < p_i < +\infty$ and $0 < t_i < s_i < 1$ be such that $p_i s_i < N$ for $i = 1, 2$. In what follows, we denote by $M(\mathbb{R}^N)$ the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ and introduce the function space X_{p_1,s_1} given by

$$X_{p_1,s_1} := \left\{ u \in M\left(\mathbb{R}^N\right) \colon \|u\|_{p_1} + \left(\int\limits_{\mathcal{Q}} \frac{|u(x) - u(y)|^{p_1}}{|x - y|^{N+s_1 p_1}} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/p_1} < +\infty \right\},$$

where $\mathcal{Q} := \mathbb{R}^{2N} \setminus (\Omega^{\complement} \times \Omega^{\complement})$, which was firstly introduced by Servadei and Valdinoci [33]. From [4] we can see that the function space X_{p_1,s_1} endowed with the norm becomes a Banach space as well

$$\|u\|_{X_{p_1,s_1}} := \|u\|_{p_1} + [u]_{X_{p_1,s_1}} \quad \text{for all } u \in X_{p_1,s_1},$$

where the term $[u]_{X_{p_1,s_1}}$ is defined by

$$[u]_{X_{p_1,s_1}} := \left(\int\limits_{\mathcal{Q}} \frac{|u(x) - u(y)|^{p_1}}{|x - y|^{N+s_1 p_1}} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/p_1}. \tag{4}$$

Because problem (1) contains extended Dirichlet boundary conditions, the main functional space to be considered is the one V_{p_1,s_1}, which is the closure of $C_c^\infty(\Omega)$ in X_{p_1,s_1}, thus, $V_{p_1,s_1} := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{X_{p_1,s_1}}}$. It is not difficult to prove that V_{p_1,s_1} endowed with the norm

$$\|u\|_{V_{p_1,s_1}} := [u]_{X_{p_1,s_1}} \quad \text{for all } u \in V_{p_1,s_1}$$

becomes a uniformly convex Banach space (see, e.g., [4]), so, it is reflexive as well. Additionally, it should be noticed that the integral in (4) can be extended to $\mathbb{R}^{2N}$ because $u(x) = 0$ for a.a. $x \in \mathbb{R}^N \setminus \Omega$. Likewise, we can also introduce the function spaces V_{p_2,s_2}, V_{q_1,t_1}, and V_{q_2,t_2}.

Let $1 < p < +\infty$ and $0 < s < 1$. Throughout this paper, we use the symbol p_s^* to stand for the fractional Sobolev critical exponent of p corresponding to s formulated by

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geqslant N. \end{cases} \tag{5}$$

Next, let us recall some important embedding results, which will be used in Section 3, from time to time, to prove the main results in this paper. The following proposition is a direct consequence of Proposition 2.1 of Nezza, Palatucci, and Valdinoci [27].

Proposition 1. *Let $1 \leqslant q \leqslant p < +\infty$, $0 < s_2 \leqslant s_1 < 1$, and let $\Omega \subset \mathbb{R}^N$ be an open set. Then there exists a constant $C := C(|\Omega|, N, s_1, s_2, p, q) > 0$ such that*

$$\|u\|_{W^{s_2,q}(\Omega)} \leqslant C\|u\|_{W^{s_1,p}(\Omega)} \quad \text{for all } u \in W^{s_1,p}(\Omega),$$

i.e., the embedding of $W^{s_1,p}(\Omega)$ to $W^{s_2,q}(\Omega)$ is continuous.

Employing [4] and [5, Lemma 2.6], we have the following embedding results.

Lemma 1. *Let* $1 \leqslant q \leqslant p < +\infty$, $0 < s_2 \leqslant s_1 < 1$, *and let* $\Omega \subset \mathbb{R}^N$ *be an open set. Then the following statements hold*:

(i) *the embedding of* V_{p,s_1} *to* V_{q,s_2} *is continuous*;
(ii) *the embedding of* V_{p,s_1} *to* $L^m(\Omega)$ *is continuous for all* $1 \leqslant m \leqslant p^*_{s_1}$;
(iii) *the embedding of* V_{p,s_1} *to* $L^m(\Omega)$ *is compact for all* $1 \leqslant m < p^*_{s_1}$.

In what follows, we define the constant $\lambda_{s,p} > 0$ by

$$\lambda_{s,p} := \inf_{u \in V_{p,s} \setminus \{0\}} \frac{\|u\|^p_{V_{p,s}}}{\|u\|^p_p}. \tag{6}$$

In the entire paper, the symbols "$\xrightarrow{\text{w}}$" and "$\to$" stand for the weak and the strong convergences, respectively, to various function spaces. We denote by $p' > 1$ the conjugate of $p > 1$, namely, $1/p + 1/p' = 1$.

Let us consider the nonlinear function $\mathcal{A}_{s_1,p} : V_{p,s_1} \to V^*_{p,s_1}$ defined by

$$\langle \mathcal{A}_{s_1,p}(u), v \rangle_{V_{p,s_1}} := \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps_1}} \,\mathrm{d}x\,\mathrm{d}y \tag{7}$$

for all $u, v \in V_{p,s_1}$, where $\langle \cdot, \cdot \rangle_{V_{p,s_1}}$ is the duality pairing between V_{p,s_1} and its dual space V^*_{p,s_1}. We are now in a position to deliver several important properties of operator $\mathcal{A}_{s_1,p}$ by the following proposition in which its proof can be found in Proposition 2.8 of Zeng, Bahrouni, and Rădulescu [37].

Proposition 2. *The operator* $\mathcal{A}_{s_1,p}$ *defined by* (7) *is bounded, continuous, monotone (hence maximal monotone), and of type* (S_+), *that is,*

$$u_n \xrightarrow{\text{w}} u \quad \text{in } V_{p,s_1} \quad \text{and} \quad \limsup_{n\to\infty} \langle \mathcal{A}_{s_1,p} u_n, u_n - u \rangle_{V_{p,s_1}} \leqslant 0$$
$$\Longrightarrow \quad u_n \to u \quad \text{in } V_{p,s_1}.$$

Let E be a Banach space. A function $j : E \to \mathbb{R}$ is said to be locally Lipschitz at $x \in E$ if there is a neighborhood $O(x)$ of x and a constant $L_x > 0$ such that

$$\bigl|j(y) - j(z)\bigr| \leqslant L_x \|y - z\|_E \quad \text{for all } y, z \in O(x).$$

We denote by

$$j^0(x; y) := \limsup_{z \to x,\, \lambda \downarrow 0} \frac{j(z + \lambda y) - j(z)}{\lambda}$$

the generalized directional derivative of j at the point x in the direction y, and $\partial j : E \to 2^{E^*}$ given by

$$\partial j(x) := \bigl\{\xi \in E^* \colon j^0(x; y) \geqslant \langle \xi, y \rangle_{E^* \times E},\ y \in E\bigr\} \quad \text{for all } x \in E$$

is the generalized gradient of j at x in the sense of Clarke.

We end this section to recall a surjectivity theorem for multivalued pseudomonotone operators, which will play a critical role to prove the existence of a weak solution to problem (1).

Theorem 1. *Let X be a real reflexive Banach space with its dual space X^*, let $G : D(G) \subset X \subset 2^{X^*}$ be a maximal monotone operator, let $F : D(F) = X \to 2^{X^*}$ be a bounded multivalued pseudomonotone operator, and let $L \in X^*$. Assume that there exist $u_0 \in X$ and a constant $r \geqslant \|u_0\|_X$ such that $D(G) \cap B_r(0) \neq \emptyset$, and*

$$\langle \xi + \eta - L,\, u - u_0 \rangle_{X^* \times X} > 0$$

for all $u \in D(G)$ with $\|u\|_X = r$, for all $\xi \in F(u)$, and for all $\eta \in G(u)$. Then the inclusion

$$F(u) + G(u) \ni L$$

has a solution in $D(G)$.

3 Main results

This section is devoted to explore the existence of a weak solution and the compactness of solution set of problem (1). Our method is based on a surjectivity theorem for multivalued mappings formulated by the sum of a multivalued pseudomonotone operator and a multivalued maximal monotone operator.

To this end, we impose the following hypotheses on the data of problem (1).

(H_{a_1}) $a_1 : L^{p^*_{1,s_1}}(\Omega) \to (0, +\infty)$ is bounded and continuous in V_{p_1,s_1} such that $c_{a_1} := \inf_{u \in V_{p_1,s_1}} a_1(u) > 0$, where p^*_{1,s_1} is the fractional Sobolev critical exponent of p_1 associated with s_1 (see (5) for $p = p_1$ and $s = s_1$).

(H_{b_1}) $b_1 : L^{p^*_{1,s_1}}(\Omega) \to [0, +\infty)$ is bounded and continuous.

(H_{a_2}) $a_2 : L^{p^*_{2,s_2}}(\Omega) \to (0, +\infty)$ is bounded and continuous in V_{p_2,s_2} such that $c_{a_2} := \inf_{\in V_{p_2,s_2}} a_2(v) > 0$.

(H_{b_2}) $b_2 : L^{p^*_{2,s_2}}(\Omega) \to [0, +\infty)$ is bounded and continuous.

(H_U) The multivalued mapping $U : \Omega \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that $0 \notin U(x, 0, 0)$ for a.a. $x \in \Omega$, and fulfills the following conditions:

(i) for all $(u, v) \in \mathbb{R} \times \mathbb{R}$ and a.a. $x \in \Omega$, the set $U(x, u, v)$ is nonempty, bounded, closed, and convex in $\mathbb{R}$;

(ii) for all $(u, v) \in \mathbb{R} \times \mathbb{R}$, the multivalued function $x \mapsto U(x, u, v)$ admits a measurable selection, and $\mathbb{R} \times \mathbb{R} \ni (u, v) \mapsto U(x, u, v) \subset \mathbb{R}$ is u.s.c. for a.a. $x \in \Omega$;

(iii) there exist constants $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4 \geqslant 0$ and a function $\delta_1 \in L^{r_1/(r_1-1)}(\Omega)_+$ such that

$$\bigl|U(x, u, v)\bigr| \leqslant \alpha_1 |u|^{\beta_1} + \alpha_2 |v|^{\beta_2} + \alpha_3 |u|^{\beta_3} |v|^{\beta_4} + \delta_1(x)$$

for a.a. $x \in \Omega$ and for all $(u, v) \in \mathbb{R} \times \mathbb{R}$, where $1 < r_1 < p^*_{1,s_1}$.

(H_R) The multivalued mapping $R : \Omega \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is such that $0 \notin R(x,0,0)$ for a.a. $x \in \Omega$, and fulfills the following conditions:

(i) for all $(u,v) \in \mathbb{R} \times \mathbb{R}$ and a.a. $x \in \Omega$, the set $R(x,v,u)$ is nonempty, bounded, closed, and convex in $\mathbb{R}$;

(ii) for all $(u,v) \in \mathbb{R} \times \mathbb{R}$, the multivalued function $x \mapsto R(x,v,u)$ admits a measurable selection, and $\mathbb{R} \times \mathbb{R} \ni (u,v) \mapsto R(x,v,u) \subset \mathbb{R}$ is u.s.c. for a.a. $x \in \Omega$;

(iii) there exist constants $\gamma_1, \gamma_2, \gamma_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \geqslant 0$ and a function $\delta_2 \in L^{r_2/(r_2-1)}(\Omega)_+$ such that

$$\big|R(x,v,u)\big| \leqslant \gamma_1|v|^{\zeta_1} + \gamma_2|u|^{\zeta_2} + \gamma_3|v|^{\zeta_3}|u|^{\zeta_4} + \delta_2(x)$$

for a.a. $x \in \Omega$ and for all $(u,v) \in \mathbb{R} \times \mathbb{R}$, where $1 < r_2 < p^*_{2,s_2}$.

(H_ϕ) The function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $x \mapsto \phi(x,r)$ is measurable on Ω for all $r \in \mathbb{R}$;

(ii) $r \mapsto \phi(x,r)$ is convex and l.s.c. for a.a. $x \in \Omega$;

(iii) for each function $u \in L^{p^*_{1,s_1}}(\Omega)$, the function $x \mapsto \phi(x,u(x))$ belongs to $L^1(\Omega)$.

(H_ψ) The function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $x \mapsto \psi(x,r)$ is measurable on Ω for all $r \in \mathbb{R}$;

(ii) $r \mapsto \psi(x,r)$ is convex and l.s.c. for a.a. $x \in \Omega$;

(iii) for each function $v \in L^{p^*_{2,s_2}}(\Omega)$, the function $x \mapsto \psi(x,u(x))$ belongs to $L^1(\Omega)$.

Moreover, we need the following compatibility conditions.

(H_0) The following inequalities hold:

$$\text{(i)}\ \beta_1 \leqslant r_1 - 1, \qquad \text{(ii)}\ \beta_2 \leqslant \frac{r_2}{r_1'}, \qquad \text{(iii)}\ \frac{\beta_3}{r_1} + \frac{\beta_4}{r_2} \leqslant \frac{1}{r_1'},$$

$$\text{(iv)}\ \zeta_1 \leqslant \frac{r_1}{r_2'}, \qquad \text{(v)}\ \zeta_2 \leqslant r_2 - 1, \qquad \text{(vi)}\ \frac{\zeta_3}{r_1} + \frac{\zeta_4}{r_2} \leqslant \frac{1}{r_2'}.$$

(H_1) $f \in L^{p_1'}(\Omega)$, $g \in L^{p_2'}(\Omega)$, and there exist a constants $\pi \geqslant 0$ and a function $\omega \in L^1(\Omega)_+$ satisfying the following inequality for a.a. $x \in \Omega$ and for all $(u,v) \in \mathbb{R} \times \mathbb{R}$:

$$|\theta u| + |\sigma v| \leqslant \pi_2\big(|u|^{p_1} + |v|^{p_2}\big) + \omega(x) \tag{8}$$

for all $\theta \in U(x,u,v)$ and all $\sigma \in R(x,v,u)$.

The weak solutions to problem (1) are understood by the following way.

Definition 1. Let $K_1 := \{u \in V_{p_1,s_1}\colon u(x) \leqslant \Phi(x)$ for a.a. $x \in \Omega\}$ and $K_2 := \{v \in V_{p_2,s_2}\colon u(x) \leqslant \Psi(x)$ for a.a. $x \in \Omega\}$. We say that a pair of functions $(u,v) \in K_1 \times K_2$ is a weak solution of problem (1) if there exist functions $\xi \in L^{r_1'}(\Omega)$ and $\eta \in L^{r_2'}(\Omega)$ such that $\xi(x) \in U(x,u(x),v(x))$ and $\eta(x) \in R(x,v(x),u(x))$ for a.a. $x \in \Omega$ and the following inequalities are satisfied:

$$\begin{aligned} &a_1(u)\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p_1-2}(u(x)-u(y))[(w(x)-u(x))-(w(y)-u(y))]}{|x-y|^{N+s_1p_1}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + b_1(u)\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{q_1-2}(u(x)-u(y))[(w(x)-u(x))-(w(y)-u(y))]}{|x-y|^{N+t_1q_1}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\Omega} \big(\xi(x)-f(x)\big)\big(w(x)-u(x)\big)\,\mathrm{d}x + \int_{\Omega} \phi(x,w)\,\mathrm{d}x - \int_{\Omega} \phi(x,u)\,\mathrm{d}x \\ &\quad \geqslant 0 \quad \text{for all } w \in K_1 \end{aligned}$$

and

$$\begin{aligned} &a_2(v)\int_{\mathcal{Q}} \frac{|v(x)-v(y)|^{p_2-2}(v(x)-v(y))[(z(x)-v(x))-(z(y)-v(y))]}{|x-y|^{N+s_2p_2}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + b_2(v)\int_{\mathcal{Q}} \frac{|v(x)-v(y)|^{q_2-2}(v(x)-v(y))[(z(x)-v(x))-(v(y)-z(y))]}{|x-y|^{N+t_2q_2}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\Omega} \big(\eta(x)-g(x)\big)\big(z(x)-v(x)\big)\,\mathrm{d}x + \int_{\Omega} \psi(x,z)\,\mathrm{d}x - \int_{\Omega} \psi(x,v)\,\mathrm{d}x \\ &\quad \geqslant 0 \quad \text{for all } z \in K_2. \end{aligned}$$

Remark 1. We mention that

(i) Hypotheses (H_{a_1}) and (H_{a_2}) are mild assumptions, which indicate that a_1 and a_2 are nondegenerate and continuous. But hypotheses (H_{b_1}) and (H_{b_2}) show that b_1 and b_2 are degenerate, which means that the terms $b_1(u)(-\Delta)_{q_1}^{t_1}u$ and $b_2(v)(-\Delta)_{q_2}^{t_2}v$ could be vanished. In fact, there are a plenty of functions, which satisfy hypotheses (H_{a_1}), (H_{a_2}), (H_{b_1}), and (H_{b_2}), for example,

- the following functions satisfy hypothesis (H_{a_1}):

$$a_1(u) := \mathrm{e}^{\|u\|_{V_{p_1,s_1}}}, \qquad a_1(u) := c_{a_1} + \|u\|_{V_{p_1,s_1}},$$
$$a_1(u) := c_{a_1} + \ln\big(1 + \|u\|_{V_{p_1,s_1}}\big)$$

  for all $u \in V_{p_1,s_1}$, for some $c_{a_1} > 0$.
- the following functions satisfy hypothesis (H_{b_1}):

$$b_1(u) := \|u\|_{q_1}, \qquad b_1(u) := \|u\|_{V_{q_1,t_1}}, \qquad b(u) := \ln(1 + \|u\|_{p_1})$$

  for all $u \in V_{p_1,s_1}$.

- the following functions satisfy hypothesis (H_{a_2}):

$$a_2(v) := c_{a_2} + \|v\|_{V_{p_2,s_2}} + \|v\|_{V_{p_2,s_2}}, \quad a_2(v) := c_{a_2} + \|v\|_{p_2,\Omega}\|v\|_E,$$
$$a_2(v) := \mathrm{e}^{\|v\|_E} + \ln\bigl(1 + \|v\|_{q_2,\Omega}\bigr)$$

for all $v \in V_{p_2,s_2}$, for some $c_{a_2} > 0$.
- the following functions satisfy hypothesis (H_{b_2}):

$$b_2(v) := \bigl|\cos\bigl(\|v\|_{V_{q_2,t_2}}\bigr)\bigr|, \qquad b_2(v) := \frac{1}{\|v\|_{V_{p_2,s_2}} + 1}, \qquad b_2(v) := \mathrm{e}^{-\|v\|_{q_2}}$$

for all $v \in V_{p_2,s_2}$.

(ii) let U and R be formulated by $U(x,s,t) = \partial j_1(x,s,t)$ and $R(x,s,t) = \partial j_2(x,s,t)$ for all $(x,s,t) \in \Omega \times \mathbb{R}^2$ such that

$$|\xi| \leqslant \delta_{j_1}(x) + \alpha_1|s|^{p_1-1} + \alpha_2|t|^{p_2-1} + \alpha_3|s|^{\beta_1}|t|^{\beta_2},$$
$$|\eta| \leqslant \delta_{j_2}(x) + \gamma_1|s|^{p_1-1} + \gamma_2|t|^{p_2-1} + \gamma_3|s|^{\zeta_1}|t|^{\zeta_2}$$

for all $\xi \in \partial j_1(x,s,t)$, $\eta \in \partial j_2(x,s,t)$, a.e. $x \in \Omega$, and $(t,s) \in \mathbb{R}^2$, where $\delta_{j_1} \in L^{p_1'}(\Omega)_+$, $\delta_{j_2} \in L^{p_2'}(\Omega)_+$, and $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\gamma_1,\gamma_2,\gamma_3,\zeta_1,\zeta_2 \geqslant 0$ are such that $\beta_1/p_1 + \beta_2/p_2 \leqslant p_1 - 1/p_1$ and $\zeta_1/p_1 + \zeta_2/p_2 \leqslant p_2 - 1/p_2$. Then hypotheses ($\mathrm{H}_U$) and ($\mathrm{H}_R$) are satisfied. Here $\partial j_1(x,s,t)$ and $\partial j_2(x,s,t)$ are the generalized Clarke subdifferential operators of locally Lipschitz functions $s \mapsto j_1(x,s,t)$ and $t \mapsto j_2(x,s,t)$, respectively.

(iii) It can be observed that the functions $\phi(u) = \int_\Omega \delta_1(x)|u(x)|\,\mathrm{d}x$ and $\psi(v) = \int_\Omega \delta_2(x)|v(x)|\,\mathrm{d}x$ for all $u \in V_{p_1,s_1}$ and $v \in V_{p_2,s_2}$ fulfill hypotheses (H_ϕ) and (H_ψ) with $p_1 = p_2 = 2$.

(iv) Hypotheses (H_0) are called compatible conditions in which the similar inequalities have been used by Marino and Winkert in [20] for the study of a double phase system with single-valued convection terms.

(v) In inequality (8), there is no cross term (i.e., uv). Indeed, if $l_1, l_2 \geqslant 0$ and $\pi_1' > 0$ are such that $l_1 + l_2 \leqslant \min\{p_1,p_2\}$, then we can utilize Young inequality to reformulate the following inequality with cross term to (8) for some $\pi > 0$:

$$\theta u + \sigma v \leqslant \pi_1'\bigl(|u|^{p_1} + |v|^{p_2} + |u|^{l_1}|v|^{l_2}\bigr)$$

for all $\theta \in U(x,u,v)$, all $\sigma \in R(x,v,u)$, a.a. $x \in \Omega$, and all $(u,v) \in \mathbb{R}\times\mathbb{R}$.

Next, we consider the Nemitskii operators $\mathcal{U} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_1'}(\Omega)}$ and $\mathcal{R} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_2'}(\Omega)}$ of U and R, respectively, which are given by

$$\mathcal{U}(u,v) := \bigl\{\xi \in L^{r_1'}(\Omega)\colon\ \xi(x) \in U\bigl(x,u(x),v(x)\bigr) \text{ for a.a. } x \in \Omega\bigr\}$$

and

$$\mathcal{R}(u,v) := \bigl\{\xi \in L^{r_2'}(\Omega)\colon\ \eta(x) \in R(x,v(x),u(x)) \text{ for a.a. } x \in \Omega\bigr\},$$

respectively.

We are now in a position to verify that $\mathcal{U} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_1'}(\Omega)}$ and $\mathcal{R} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_2'}(\Omega)}$ are well defined and strongly-weakly u.s.c.

Lemma 2. *Suppose that* (H_U), (H_R), *and* (H_0) *are fulfilled. Then the following statements hold*:

(i) $\mathcal{U} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_1'}(\Omega)}$ *and* $\mathcal{R} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_2'}(\Omega)}$ *are well defined such that for each* $(u,v) \in V_{p_1,s_1} \times V_{p_2,s_2}$, *the sets* $\mathcal{U}(u,v) \subset L^{r_1'}(\Omega)$ *and* $\mathcal{R}(u,v) \subset L^{r_2'}(\Omega)$ *are nonempty, bounded, closed, and convex*;

(ii) $\mathcal{U} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_1'}(\Omega)}$ *and* $\mathcal{R} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_2'}(\Omega)}$ *are strongly-weakly u.s.c.*

Proof. (i) By hypothesis (H_U)(ii) (i.e., for all $(u,v) \in \mathbb{R} \times \mathbb{R}$, the multivalued function $x \mapsto U(x,u,v)$ admits a measurable selection, and $\mathbb{R}\times\mathbb{R} \ni (u,v) \mapsto U(x,u,v) \subset \mathbb{R}$ is u.s.c. for a.a. $x \in \Omega$), we can apply Yankov–von Neumann–Aumann selection theorem (see [28, Thm. 2.7.25]) to conclude that for every $(u,v) \in V_{p_1,s_1} \times V_{p_2,s_2}$, the multifunctions $\Omega \ni x \mapsto U(x,u(x),v(x)) \subset \mathbb{R}$ and $\Omega \ni x \mapsto R(x,v(x),u(x)) \subset \mathbb{R}$ are both measurable in Ω and have measurable selections $\xi : \Omega \to \mathbb{R}$ and $\eta : \Omega \to \mathbb{R}$ such that $\xi(x) \in U(x,u(x),v(x))$ and $\eta(x) \in R(x,v(x),u(x))$ for a.a. $x \in \Omega$. On the other side, it follows from condition (H_U)(iii) and elementary inequality $(|a|+|b|)^\delta \leqslant 2^{\delta-1}(|a|^\delta + |b|^\delta)$ for all $a,b \in \mathbb{R}$ and $\delta \geqslant 1$ that

$$\begin{aligned} \|\xi\|_{r_1'}^{r_1'} &= \int_\Omega \big|\xi(x)\big|^{r_1'}\,\mathrm{d}x \leqslant \int_\Omega \big(\alpha_1|u|^{\beta_1} + \alpha_2|v|^{\beta_2} + \alpha_3|u|^{\beta_3}|v|^{\beta_4} + \delta_1(x)\big)^{r_1'}\,\mathrm{d}x \\ &\leqslant C_0\big(\|u\|_{\beta_1 r_1'}^{\beta_1 r_1'} + \|v\|_{\beta_2 r_2'}^{\beta_2 r_2'} + \|u\|_{r_1}^{\beta_3 r_1'}\|v\|_{(r_1/\beta_3 r_1')'\beta_4 r_1'}^{\beta_4 r_1'} + \|\delta_1\|_{r_1'}^{r_1'}\big) \end{aligned} \tag{9}$$

for some $C_0 > 0$. Invoking the compatibility condition (H_0), it yields

$$\left(\frac{r_1}{\beta_3 r_1'}\right)'\beta_4 r_1' \leqslant r_2.$$

The latter, combined with the continuity of embeddings of V_{p_i,s_i} to $L^{p_i}(\Omega)$ for $i = 1,2$ (see Lemma 1(ii)), implies that there is a constant $C_1 > 0$ such that

$$\|\xi\|_{r_1'}^{r_1'} \leqslant C_1\big(1 + \|u\|_{p_1,s_1}^{\max\{r_1,p_1\}} + \|v\|_{p_2,s_2}^{\max\{r_2,p_2\}}\big).$$

We have proved that $\mathcal{U} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_1'}(\Omega)}$ is well defined and bounded. Remembering that $U(x,s,t)$ is bounded, closed, and convex in $\mathbb{R}$ for all $(s,t) \in \mathbb{R} \times \mathbb{R}$ and a.a. $x \in \Omega$ (see hypothesis (H_U)(i)), this reveals that for each $(u,v) \in V_{p_1,s_1} \times V_{p_2,s_2}$, the set $\mathcal{U}(u,v)$ is closed and convex in $L^{r_1'}(\Omega)$ as well. Using the same arguments, we also can prove the same conclusion to $\mathcal{R} : V_{p_1,s_1} \times V_{p_2,s_2} \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to 2^{L^{r_2'}(\Omega)}$.

(ii) Indeed, if we can verify that for each weakly closed set $D \subset L^{r_1'}(\Omega)$ with $\mathcal{U}^-(D) \neq \emptyset$, the set $\mathcal{U}^-(D)$ is closed in $V_{p_1,r_1} \times V_{p_2,r_2}$, then via employing [23, Prop. 3.9] directly we have that $\mathcal{U}$ is strongly-weakly closed.

To this end, let $D \subset L^{r_1'}(\Omega)$ be such that $\mathcal{U}^-(D) \neq \emptyset$, and let $\{(u_n, v_n)\} \subset \mathcal{U}^-(D)$ be such that $(u_n, v_n) \to (u, v)$ in $V_{p_1,r_1} \times V_{p_2,r_2}$. Then we can find a sequence $\{\xi_n\}$ with $\xi_n \in \mathcal{U}(u_n, v_n) \cap D$. From (9) we can see that $\{\xi_n\}$ is bounded in $L^{r_1'}(\Omega)$. Without any loss of generality, there exists a function $\xi \in L^{r_1'}(\Omega)$ satisfying $\xi_n \xrightarrow{w} \xi$ in $L^{r_1'}(\Omega)$. Employing Mazur's theorem, we can find a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of convex combinations of $\{\xi_n\}_{n\in\mathbb{N}}$ satisfying

$$\varphi_n \to \xi \quad \text{in } L^{r_1'(\cdot)}(\Omega).$$

So, passing to a subsequence if necessary, one has $\varphi_n(x) \to \xi(x)$ for a.a. $x \in \Omega$. The convexity of U gives

$$\varphi_n(x) \in U(x, u_n, v_n) \quad \text{for a.a. } x \in \Omega. \tag{10}$$

Keeping in mind that $U(x, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is u.s.c. (see (H_U)(ii)), we use [15, Prop. 4.1.9] to infer that $U(x, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a closed mapping for a.a. $x \in \Omega$. Letting $\lim_{n\to\infty}$ to (10), we have

$$\xi(x) \in U\big(x, u(x), v(x)\big) \quad \text{for a.a. } x \in \Omega.$$

This means that $\xi \in \mathcal{U}(u, v)$. The latter together with the weak closedness of D implies $(u, v) \in \mathcal{U}^-(D)$. Therefore, $\mathcal{U}$ is strongly-weakly u.s.c. Similarly, we also can verify that $\mathcal{R}$ is strongly-weakly u.s.c. □

The existence result to problem (1) is stated by the following theorem, which reveals that the weak solution set of problem (1) is nonempty and compact in $V_{p_1,r_1} \times V_{p_2,r_2}$.

Theorem 2. *Assume that* (H_{a_1}), (H_{b_1}), (H_{a_2}), (H_{b_2}), (H_U), (H_R), (H_ϕ), (H_ψ), (H_0), *and* (H_1) *are fulfilled. Let, in addition, one of the following condition hold*:

(i) *a_1 and a_2 are coercive in V_{p_1,s_1} and V_{p_2,s_2}, respectively,*

(ii) *the following inequality is satisfied*:

$$\min\{c_{a_1} - \pi\lambda_{s_1,p_1}^{-1},\, c_{a_2} - \pi\lambda_{s_2,p_2}^{-1}\} > 0. \tag{11}$$

Then the solution set of problem (1) *is nonempty and compact in $V_{p_1,s_1} \times V_{p_2,s_2}$, where the constant λ_{s_i,p_i} is given in* (6) *for $s = s_i$ and $p = p_i$ with $i = 1, 2$.*

Proof. We denote by $i_1 : V_{p_1,s_1} \to L^{r_1}(\Omega)$ and $i_2 : V_{p_2,s_2} \to L^{r_2}(\Omega)$ the embedding operators of V_{p_1,s_1} to $L^{r_1}(\Omega)$ and of V_{p_2,s_2} of $L^{r_2}(\Omega)$, respectively. Let us consider the functions $\varphi_1 : V_{p_1,s_1} \to \overline{\mathbb{R}}$ and $\varphi_2 : V_{p_2,s_2} \to \overline{\mathbb{R}}$ defined by

$$\varphi_1(u) := \int_\Omega \phi\big(x, u(x)\big)\,\mathrm{d}x + I_{K_1}(u) \quad \text{for all } u \in V_{p_1,s_1}$$

and

$$\varphi_2(v) := \int_\Omega \psi\big(x, v(x)\big)\,\mathrm{d}x + I_{K_2}(v) \quad \text{for all } v \in V_{p_2,s_2},$$

respectively, where I_{K_i} is the indicator function of K_i for $i = 1, 2$. By the definition of convex subgradient, it is not hard to prove that $(u, v) \in K_1 \times K_2$ is a weak solution of problem (1) if and only if it solves the following inclusion problem:

$$(f, g) \in \big(\mathcal{F}(u, u) + i_2^* \mathcal{U}(u, v) + \partial_c \varphi_1(u), \mathcal{G}(v, v) + i_2^* \mathcal{R}(u, v) + \partial_c \varphi_2(v)\big) \tag{12}$$

in $V_{p_1,s_1}^* \times V_{p_2,s_2}^*$, where $i_j^* : L^{r_j'}(\Omega) \to V_{p_j,s_j}^*$ and $\partial_c \varphi_j$ are the dual mappings of i_j and the convex subdifferential of φ_j for $j = 1, 2$, and the nonlinear functions $\mathcal{F}$: $V_{p_1,s_1} \times V_{p_1,s_1} \to V_{p_1,s_1}^*$ and $\mathcal{G} : V_{p_2,s_2} \times V_{p_2,s_2} \to V_{p_2,s_2}^*$ are defined by

$$\begin{aligned}
&\big\langle \mathcal{F}(s, u), w \big\rangle_{V_{p_1,s_1}} \\
&\quad := a_1(s) \int\limits_{\mathcal{Q}} \frac{|u(x) - u(y)|^{p_1 - 2}(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N + s_1 p_1}} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + b_1(s) \int\limits_{\mathcal{Q}} \frac{|u(x) - u(y)|^{q_1 - 2}(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N + t_1 q_1}} \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

for all $s, u, w \in V_{p_1,s_1}$ and

$$\begin{aligned}
&\big\langle \mathcal{G}(t, v), z \big\rangle_{V_{p_2,s_2}} \\
&\quad := a_2(t) \int\limits_{\mathcal{Q}} \frac{|v(x) - v(y)|^{p_2 - 2}(v(x) - v(y))(z(x) - z(y))}{|x - y|^{N + s_2 p_2}} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + b_2(t) \int\limits_{\mathcal{Q}} \frac{|v(x) - v(y)|^{q_2 - 2}(v(x) - v(y))(z(x) - z(y))}{|x - y|^{N + t_2 q_2}} \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

for all $t, v, z \in V_{p_2,s_2}$, respectively. By this essential property, we are going to use the surjectivity theorem, Theorem 1, to examine the existence of a weak solution to inclusion (12). To this end, we have the following claims.

Claim 1. $(\partial_c \varphi_1(\cdot), \partial_c \varphi_2(\cdot)) : V_{p_1,s_1} \times V_{p_2,s_2} \to 2^{V_{p_1,s_1}^* \times V_{p_2,s_2}^*}$ is maximal monotone.

Let us consider the function $\varphi : V_{p_1,s_1} \times V_{p_2,s_2} \to \overline{\mathbb{R}}$ defined by

$$\varphi(u, v) = \varphi_1(u) + \varphi_2(v) \quad \text{for all } (u, v) \in V_{p_1,s_1} \times V_{p_2,s_2}.$$

From hypotheses (H_ϕ) and (H_ψ) it is easily to get that φ is a proper convex and l.s.c. function with $D(\varphi) = K_1 \times K_2$. A simple calculation gives $\partial_c \varphi(u, v) = (\partial_c \varphi_1(v), \partial_c \varphi_2(v))$. So, we conclude that $(\partial_c \varphi_1(\cdot), \partial_c \varphi_2(\cdot)) : V_{p_1,s_1} \times V_{p_2,s_2} \to 2^{V_{p_1,s_1}^* \times V_{p_2,s_2}^*}$ is maximal monotone.

Claim 2. The multivalued mapping

$$V_{p_1,s_1} \times V_{p_2,s_2} \ni (u, v) \mapsto \big(\mathcal{F}(u, u) + i_2^* \mathcal{U}(u, v),\ \mathcal{G}(v, v) + i_2^* \mathcal{R}(u, v)\big) \subset V_{p_1,s_1}^* \times V_{p_2,s_2}^*$$

is generalized pseudomonotone.

In what follows, we denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between $V_{p_1,s_1}\times V_{p_2,s_2}$ and $V^*_{p_1,s_1}\times V^*_{p_2,s_2}$ formulated by

$$\langle\langle(u^*,v^*),(u,v)\rangle\rangle := \langle u^*,u\rangle_{V_{p_1,s_1}} + \langle v^*,v\rangle_{V_{p_2,s_2}}$$

for all $(u,v)\in V_{p_1,s_1}\times V_{p_2,s_2}$ and $(u^*,v^*)\in V^*_{p_1,s_1}\times V^*_{p_2,s_2}$. Assume that sequences $\{(u_n,v_n)\}\subset V_{p_1,s_1}\times V_{p_2,s_2}$, $\{(u^*_n,v^*_n)\}\subset V^*_{p_1,s_1}\times V^*_{p_2,s_2}$, and $(u,v)\in V_{p_1,s_1}\times V_{p_2,s_2}$ satisfy the following conditions:

$$\begin{aligned}
&(u^*_n,v^*_n)\in\big(\mathcal{F}(u_n,u_n)+i^*_1\mathcal{U}(u_n,v_n),\ \mathcal{G}(v_n,v_n)+i^*_2\mathcal{R}(u_n,v_n)\big)\quad\text{for every } n\in\mathbb{N},\\
&(u_n,v_n)\xrightarrow{\mathrm{w}}(u,v)\quad\text{in } V_{p_1,s_1}\times V_{p_2,s_2},\\
&(u^*_n,v^*_n)\xrightarrow{\mathrm{w}}(u^*,v^*)\quad\text{in } V^*_{p_1,s_1}\times V^*_{p_2,s_2},\\
&\limsup_{n\to\infty}\langle\langle(u^*_n,v^*_n),\,(u_n,v_n)-(u,v)\rangle\rangle\leqslant 0.
\end{aligned}\tag{13}$$

Hence, for every $n\in\mathbb{N}$, we are able to find functions $\xi_n\in L^{r'_1}(\Omega)$ and $\eta_n\in L^{r'_2}(\Omega)$ such that

$$\begin{aligned}
&\xi_n\in\mathcal{U}(u_n,v_n), &&u^*_n=\mathcal{F}(u_n,u_n)+i^*_1\xi_n,\\
&\eta_n\in\mathcal{R}(u_n,v_n), &&v^*_n=\mathcal{G}(v_n,v_n)+i^*_2\eta_n.
\end{aligned}$$

Using Lemma 2, we can show the boundedness of $\{\xi_n\}\subset L^{r'_1}(\Omega)$ and $\{\eta_n\}\subset L^{r'_2}(\Omega)$. Passing to a subsequence if necessary, one has

$$\xi_n\xrightarrow{\mathrm{w}}\xi\quad\text{in } L^{r'_1}(\Omega)\quad\text{and}\quad\eta_n\xrightarrow{\mathrm{w}}\eta\quad\text{in } L^{r'_2}(\Omega)\tag{14}$$

for some $(\xi,\eta)\in L^{r'_1}(\Omega)\times L^{r'_2}(\Omega)$. Recalling that the embeddings of V_{p_i,s_i} to $L^{r_i}(\Omega)$ is compact for $i=1,2$, we have $u_n\to u$ in $L^{r_1}(\Omega)$ and $v_n\to v$ in $L^{r_2}(\Omega)$. Hence,

$$\begin{aligned}
&\lim_{n\to\infty}\langle\langle(i^*_1\xi_n,i^*_2\eta_n),(u_n-u,v_n-v)\rangle\rangle\\
&\quad=\lim_{n\to\infty}\langle i^*_1\xi_n,\,u_n-u\rangle_{V_{p_1,s_1}}+\lim_{n\to\infty}\langle i^*_2\eta_n,\,v_n-v\rangle_{V_{p_2,s_2}}\\
&\quad=\lim_{n\to\infty}\langle\xi_n,\,u_n-u\rangle_{L^{r'_1}(\Omega)\times L^{r_1}(\Omega)}+\lim_{n\to\infty}\langle\eta_n,\,v_n-v\rangle_{L^{r'_2}(\Omega)\times L^{r_2}(\Omega)}=0.
\end{aligned}\tag{15}$$

Taking into account the last inequality of (13) and (15), it yields

$$\begin{aligned}
0&\geqslant\limsup_{n\to\infty}\langle\langle(u^*_n,v^*_n),(u_n,v_n)-(u,v)\rangle\rangle\\
&=\limsup_{n\to\infty}\big(\langle\mathcal{F}(u_n,u_n),\,u_n-u\rangle_{V_{p_1,s_1}}+\langle\mathcal{G}(v_n,v_n),\,v_n-v\rangle_{V_{p_2,s_2}}\big)\\
&\quad-\lim_{n\to\infty}\langle\langle(i^*_1\xi_n,i^*_2\eta_n),(u_n-u,\,v_n-v)\rangle\rangle\\
&=\limsup_{n\to\infty}\bigg(a_1(u_n)\int_{\mathcal{Q}}\frac{|u_n(x)-u_n(y)|^{p_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p_1}}\\
&\qquad\times\big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$+ b_1(u_n) \int_{\mathcal{Q}} \frac{|u_n(x) - u_n(y)|^{q_1-2}(u_n(x) - u_n(y))}{|x-y|^{N+t_1 q_1}} \times \big(u_n(x) - u_n(y) - u(x) + u(y)\big) \, \mathrm{d}x \, \mathrm{d}y$$

$$+ a_2(v_n) \int_{\mathcal{Q}} \frac{|v_n(x) - v_n(y)|^{p_1-2}(v_n(x) - v_n(y))}{|x-y|^{N+s_2 p_2}} \times \big(v_n(x) - v_n(y) - v(x) + v(y)\big) \, \mathrm{d}x \, \mathrm{d}y$$

$$+ b_2(v_n) \int_{\mathcal{Q}} \frac{|v_n(x) - v_n(y)|^{q_2-2}(v_n(x) - v_n(y))}{|x-y|^{N+t_2 q_2}} \times \big(v_n(x) - v_n(y) - v(x) + v(y)\big) \, \mathrm{d}x \, \mathrm{d}y \bigg).$$

Applying the boundedness of a_1, b_1, a_2, and b_2, as well as the convergence $(u_n, v_n) \xrightarrow{\mathrm{w}} (u, v)$ in $V_{p_1,s_1} \times V_{p_2,s_2}$, it is found that

$$\lim_{n\to\infty} a_1(u_n) \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^{p_1-2}(u(x) - u(y))}{|x-y|^{N+s_1 p_1}} \times \big(u_n(x) - u_n(y) - u(x) + u(y)\big) \, \mathrm{d}x \, \mathrm{d}y = 0,$$

$$\lim_{n\to\infty} b_1(u_n) \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^{q_1-2}(u(x) - u(y))}{|x-y|^{N+t_1 q_1}} \times \big(u_n(x) - u_n(y) - u(x) + u(y)\big) \, \mathrm{d}x \, \mathrm{d}y = 0,$$

$$\lim_{n\to\infty} a_2(v_n) \int_{\mathcal{Q}} \frac{|v(x) - v(y)|^{p_2-2}(v(x) - v(y))}{|x-y|^{N+s_2 p_2}} \times \big(v_n(x) - v_n(y) - v(x) + v(y)\big) \, \mathrm{d}x \, \mathrm{d}y = 0,$$

$$\lim_{n\to\infty} b_2(v_n) \int_{\mathcal{Q}} \frac{|v(x) - v(y)|^{q_2-2}(v(x) - v(y))}{|x-y|^{N+t_2 q_2}} \times \big(v_n(x) - v_n(y) - v(x) + v(y)\big) \, \mathrm{d}x \, \mathrm{d}y = 0.$$

So, we have

$$\limsup_{n\to\infty} a_1(u_n) \int_{\mathcal{Q}} \frac{|u_n(x) - u_n(y)|^{p_1-2}(u_n(x) - u_n(y))}{|x-y|^{N+s_1 p_1}} \times \big(u_n(x) - u_n(y) - u(x) + u(y)\big) \, \mathrm{d}x \, \mathrm{d}y \leqslant 0 \tag{16}$$

and

$$\limsup_{n\to\infty} a_2(v_n) \int_{\mathcal{Q}} \frac{|v_n(x) - v_n(y)|^{p_1-2}(v_n(x) - v_n(y))}{|x-y|^{N+s_2 p_2}} \times \big(v_n(x) - v_n(y) - v(x) + v(y)\big) \, \mathrm{d}x \, \mathrm{d}y \leqslant 0. \tag{17}$$

From (16) we get

$$0 \geqslant \limsup_{n\to\infty} \int\limits_{Q} \frac{|u_n(x)-u_n(y)|^{p_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p_1}} \times \big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \liminf_{n\to\infty}\left(a_1(u_n)-\frac{c_{a_1}}{2}\right) \int\limits_{Q} \frac{|u_n(x)-u_n(y)|^{p_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p_1}} \times \big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \limsup_{n\to\infty}\frac{c_{a_1}}{2} \int\limits_{Q} \frac{|u_n(x)-u_n(y)|^{p_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p_1}} \times \big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \liminf_{n\to\infty}\left(a_1(u_n)-\frac{c_{a_1}}{2}\right) \int\limits_{Q} \frac{|u(x)-u(y)|^{p_1-2}(u(x)-u(y))}{|x-y|^{N+s_1p_1}} \times \big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \limsup_{n\to\infty}\frac{c_{a_1}}{2} \int\limits_{Q} \frac{|u_n(x)-u_n(y)|^{p_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p_1}} \times \big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \limsup_{n\to\infty}\frac{c_{a_1}}{2} \int\limits_{Q} \frac{|u_n(x)-u_n(y)|^{p_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p_1}} \times \big(u_n(x)-u_n(y)-u(x)+u(y)\big)\,\mathrm{d}x\,\mathrm{d}y.$$

We are now in a position to invoke Proposition 2 to conclude that $u_n \to u$ in V_{p_1,s_1}. Likewise, by (17), we also have $v_n \to v$ in V_{p_2,s_2}.

Using Lemma 2(ii) and (14) gives that $\xi \in \mathcal{U}(u,v)$ and $\eta \in \mathcal{R}(u,v)$. Observe that a_1, a_2, b_1, and b_2 are continuous, so, $\mathcal{F}$ and $\mathcal{G}$ are both continuous. Then one has

$$\begin{aligned} &u_n^* = \mathcal{F}(u_n,u_n) + i_1^*\xi_n \xrightarrow{\mathrm{w}} u^* = \mathcal{F}(u,u) + i_1^*\xi \quad \text{in } V_{p_1,s_1}^*,\\ &v_n^* = \mathcal{G}(v_n,v_n) + i_2^*\eta_n \xrightarrow{\mathrm{w}} v^* = \mathcal{G}(v,v) + i_2^*\eta \quad \text{in } V_{p_2,s_2}^*,\\ &(u^*,v^*) \in \big(\mathcal{F}(u,u) + i_1^*\mathcal{U}(u,v),\ \mathcal{G}(v,v) + i_2^*\mathcal{R}(u,v)\big),\\ &\big\langle\!\big\langle (u_n^*,v_n^*),(u_n,v_n)\big\rangle\!\big\rangle \to \big\langle\!\big\langle (u^*,v^*),(u,v)\big\rangle\!\big\rangle. \end{aligned}$$

This means that $V_{p_1,s_1}\times V_{p_2,s_2} \ni (u,v) \mapsto (\mathcal{F}(u,u)+i_1^*\mathcal{U}(u,v), \mathcal{G}(v,v)+i_2^*\mathcal{R}(u,v)) \subset V_{p_1,s_1}^* \times V_{p_2,s_2}^*$ is generalized pseudomonotone.

The boundedness of a_1, a_2, b_1, and b_2 and Lemma 2 imply that $V_{p_1,s_1} \times V_{p_2,s_2} \ni (u,v) \mapsto (\mathcal{F}(u,u) + i_1^*\mathcal{U}(u,v), \mathcal{G}(v,v) + i_2^*\mathcal{R}(u,v)) \subset V_{p_1,s_1}^* \times V_{p_2,s_2}^*$ is a bounded mapping. Whereas the closedness of and convexity of $(\mathcal{F}(u,u) + i_1^*\mathcal{U}(u,v),\ \mathcal{G}(v,v) + i_2^*\mathcal{G}(u,v))$ as well as Proposition 3.58(ii) of [23] imply that $V_{p_1,s_1} \times V_{p_2,s_2} \ni (u,v) \mapsto (\mathcal{F}(u,u) + i_1^*\mathcal{U}(u,v), \mathcal{G}(v,v) + i_2^*\mathcal{R}(u,v)) \subset V_{p_1,s_1}^* \times V_{p_2,s_2}^*$ is pseudomonotone.

Claim 3. $V_{p_1,s_1} \times V_{p_2,s_2} \ni (u,v) \mapsto (\mathcal{F}(u,u) + i_1^*\mathcal{U}(u,v), \mathcal{G}(v,v) + i_2^*\mathcal{R}(u,v)) \subset V_{p_1,s_1}^* \times V_{p_2,s_2}^*$ is coercive.

For any $(u,v) \in V_{p_1,s_1} \times V_{p_2,s_2}$ and $(\xi,\eta) \in (\mathcal{U}(u,v), \mathcal{R}(u,v))$ fixed, we use hypothesis (H_1) to get

$$\begin{aligned}
&\big\langle\!\big\langle \big(\mathcal{F}(u,u) + i_1^*\xi,\, \mathcal{G}(v,v) + i_2^*\eta\big), (u,v)\big\rangle\!\big\rangle \\
&\quad = a_1(u)\|u\|_{V_{p_1,s_1}}^{p_1} + b_1(u)\|\nabla u\|_{V_{q_1,t_1}}^{q_1} + a_2(v)\|v\|_{V_{p_2,s_2}}^{p_2} + b_2(v)\|v\|_{V_{q_2,t_2}}^{q_2} \\
&\qquad + \int_\Omega \xi(x)u(x)\,\mathrm{d}x + \int_\Omega \eta(x)v(x)\,\mathrm{d}x \\
&\quad \geqslant a_1(u)\|u\|_{V_{p_1,s_1}}^{p_1} + b_1(u)\|\nabla u\|_{V_{q_1,t_1}}^{q_1} + a_2(v)\|v\|_{V_{p_2,s_2}}^{p_2} + b_2(v)\|v\|_{V_{q_2,t_2}}^{q_2} \\
&\qquad - \int_\Omega \pi\big(|u|^{p_1} + |v|^{p_2}\big)\,\mathrm{d}x - \int_\Omega \omega(x)\,\mathrm{d}x \\
&\quad \geqslant \big(a_1(u) - \pi\lambda_{s_1,p_1}^{-1}\big)\|u\|_{V_{p_1,s_1}}^{p_1} + b_1(u)\|\nabla u\|_{V_{q_1,t_1}}^{q_1} \\
&\qquad + \big(a_2(v) - \pi\lambda_{s_2,p_2}^{-1}\big)\|v\|_{V_{p_2,s_2}}^{p_2} + b_2(v)\|v\|_{V_{q_2,t_2}}^{q_2} - \|\omega\|_{1,\Omega}.
\end{aligned}$$

If a_1 and a_2 are coercive in V_{p_1,s_1} and V_{p_2,s_2}, respectively, then we have

$$\big\langle\!\big\langle \big(\mathcal{F}(u,u) + i_1^*\xi,\, \mathcal{G}(v,v) + i_2^*\eta\big), (u,v)\big\rangle\!\big\rangle \to \infty \quad \text{as } \|u\|_{V_{p_1,s_1}} + \|v\|_{V_{p_2,s_2}} \to +\infty.$$

This means that $V_{p_1,s_1} \times V_{p_2,s_2} \ni (u,v) \mapsto (\mathcal{F}(u,u) + i_1^*\mathcal{U}(u,v), \mathcal{G}(v,v) + i_2^*\mathcal{R}(u,v)) \subset V_{p_1,s_1}^* \times V_{p_2,s_2}^*$ is coercive. But, when inequality (11) holds, we also have

$$\begin{aligned}
&\big\langle\!\big\langle (\mathcal{F}(u,u) + i_1^*\xi,\, \mathcal{G}(v,v) + i_2^*\eta), (u,v)\big\rangle\!\big\rangle \\
&\quad \geqslant \big(c_{a_1} - \pi\lambda_{s_1,p_1}^{-1}\big)\|u\|_{V_{p_1,s_1}}^{p_1} + b_1(u)\|\nabla u\|_{V_{q_1,t_1}}^{q_1} \\
&\qquad + \big(c_{a_2} - \pi\lambda_{s_2,p_2}^{-1}\big)\|v\|_{V_{p_2,s_2}}^{p_2} + b_2(v)\|v\|_{V_{q_2,t_2}}^{q_2} - \|\omega\|_{1,\Omega} \\
&\quad \to +\infty \quad \text{as } \|u\|_{V_{p_1,s_1}} + \|v\|_{V_{p_2,s_2}} \to +\infty.
\end{aligned}$$

The Claim 3 follows.

By Claims 1–3, we have verified the conditions of Theorem 1. Applying this theorem, it is admitted that the solution set of problem (12) is nonempty. Therefore, problem (1) has at least one weak solution.

Finally, we shall show the compactness of solution set to problem (1). Assume that $\{(u_n,v_n)\}$ is a solution sequence of (1). So, we are able to find two sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfying $\xi_n \in \mathcal{U}(u_n,v_n)$, $\eta_n \in \mathcal{R}(u_n,v_n)$, and

$$\begin{aligned}
&a_1(u_n)\int_{\mathcal{Q}} \frac{|u_n(x) - u_n(y)|^{p_1-2}(u_n(x) - u_n(y))}{|x-y|^{N+s_1p_1}} \\
&\qquad \times \big[\big(w(x) - u_n(x)\big) - \big(w(y) - u_n(y)\big)\big]\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$+ b_1(u_n) \int_{\mathcal{Q}} \frac{|u_n(x)-u_n(y)|^{q_1-2}(u_n(x)-u_n(y))}{|x-y|^{N+t_1q_1}} \times \big[\big(w(x)-u_n(x)\big)-\big(w(y)-u_n(y)\big)\big]\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_{\Omega} \big(\xi_n(x)-f(x)\big)\big(w(x)-u_n(x)\big)\,\mathrm{d}x + \int_{\Omega} \phi(x,w)\,\mathrm{d}x - \int_{\Omega} \phi(x,u_n)\,\mathrm{d}x \geqslant 0 \tag{18}$$

for all $w \in K_1$ and

$$a_2(v_n) \int_{\mathcal{Q}} \frac{|v_n(x)-v_n(y)|^{p_2-2}(v_n(x)-v_n(y))}{|x-y|^{N+s_2p_2}} \times \big[\big(z(x)-v_n(x)\big)-\big(z(y)-v_n(y)\big)\big]\,\mathrm{d}x\,\mathrm{d}y$$

$$+ b_2(v_n) \int_{\mathcal{Q}} \frac{|v_n(x)-v_n(y)|^{q_2-2}(v_n(x)-v_n(y))}{|x-y|^{N+t_2q_2}} \times \big[\big(z(x)-v_n(x)\big)-\big(v_n(y)-z(y)\big)\big]\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_{\Omega} \big(\eta_n(x)-g(x)\big)\big(z(x)-v_n(x)\big)\,\mathrm{d}x + \int_{\Omega} \psi(x,z)\,\mathrm{d}x - \int_{\Omega} \psi(x,v_n)\,\mathrm{d}x \geqslant 0 \tag{19}$$

for all $z \in K_2$. By Claim 3, it is not hard to prove that following the inequality holds:

$$0 \geqslant \big(a_1(u_n) - \pi\lambda_{s_1,p_1}^{-1}\big)\|u_n\|_{V_{p_1,s_1}}^{p_1} + b_1(u_n)\|\nabla u_n\|_{V_{q_1,t_1}}^{q_1} + \big(a_2(v_n) - \pi\lambda_{s_2,p_2}^{-1}\big)\|v_n\|_{V_{p_2,s_2}}^{p_2} + b_2(v_n)\|v_n\|_{V_{q_2,t_2}}^{q_2} - \|\omega\|_{1,\Omega}.$$

It follows from the coercivity of a_1 and a_2 or inequality (11) that sequence $\{(u_n, v_n)\}$ is bounded in $V_{p_1,s_1} \times V_{p_2,s_2}$. Because of the weak closedness of K_1 and K_2, without any loss of generality, there exists a pair of functions $(u, v) \in K_1 \times K_2$ such that $(u_n, v_n) \xrightarrow{\mathrm{w}} (u, v)$ in $V_{p_1,s_1} \times V_{p_2,s_2}$. Hence,

$$\big\langle\big\langle \big(\mathcal{F}(u_n,u_n) + i_1^*\xi_n, \mathcal{G}(v_n,v_n) + i_2^*\eta_n\big),\, (u_n - u,\, v_n - v)\big\rangle\big\rangle + \varphi(u_n,v_n) - \varphi(v,u) \leqslant 0. \tag{20}$$

Additionally, Lemma 2 and the boundedness of a_1, a_2, b_1, b_2 reveal that the sequence $\{(u_n^*, v_n^*)\}$ defined as $u_n^* := \mathcal{F}(u_n,u_n) + i_1^*\xi_n$ and $v_n^* := \mathcal{G}(v_n,v_n) + i_2^*\eta_n$ are bounded in $V_{p_1,s_1}^* \times V_{p_2,s_2}^*$. So, we can take a subsequence of $\{(u_n^*, v_n^*)\}$, still denoted by the same indices, such that $(u_n^*, v_n^*) \xrightarrow{\mathrm{w}} (u^*, v^*)$ in $V_{p_1,s_1}^* \times V_{p_2,s_2}^*$ for some $(u^*, v^*) \in V_{p_1,s_1}^* \times V_{p_2,s_2}^*$. Taking the upper limit as $n \to \infty$ to inequality (20), one has

$$\limsup_{n\to\infty} \big\langle\big\langle \big(\mathcal{F}(u_n,u_n) + i_1^*\xi_n,\, \mathcal{G}(v_n,v_n) + i_2^*\eta_n\big), (u_n - u,\, v_n - v)\big\rangle\big\rangle \leqslant 0.$$

Arguing as in the proof of Claim 2, it is not difficult to show that $u_n \to u$ in V_{p_1,s_1} and $v_n \to v$ in V_{p_2,s_2}. Therefore, from Lemma 2 we get $\xi_n \xrightarrow{w} \xi \in \mathcal{U}(u,v)$ in $L^{r_1}(\Omega)$ and $\eta_n \xrightarrow{w} \eta \in \mathcal{R}(u,v)$ in $L^{r_2}(\Omega)$. Passing to the upper limit as $n \to \infty$ to (18) and (19), we infer that $(u,v) \in V_{p_1,s_1} \times V_{p_2,s_2}$ is a weak solution of problem (1). Consequently, the solution set of problem (1) is compact in $V_{p_1,s_1} \times V_{p_2,s_2}$. □

Particularly, when the multivalued mappings U and R are specialized by

$$U(x,s,t) = l_1(s,t)\partial j_1(x,s,t) \quad \text{and} \quad R(x,t,s) = l_2(t,s)\partial j_2(x,t,s),$$

then problem (1) becomes

$$\begin{aligned} &a_1(u)(-\Delta)^{s_1}_{p_1} u + b_1(u)(-\Delta)^{t_1}_{q_1} u + \partial_c \phi(x,u) + l_1(u,v)\partial j_1(x,u,v) \ni f(x) \quad \text{in } \Omega,\\ &a_2(u)(-\Delta)^{s_2}_{p_2} v + b_2(v)(-\Delta)^{t_2}_{q_2} v + \partial_c \psi(x,v) + l_2(v,u)\partial j_2(x,v,u) \ni g(x) \quad \text{in } \Omega,\\ &u(x) = 0, \quad v(x) = 0 \quad \text{in } \Omega^{\complement}, \qquad u(x) \leqslant \Phi(x), \quad v(x) \leqslant \Psi(x) \quad \text{in } \Omega, \end{aligned} \tag{21}$$

where $\partial j_1(x,s,t)$ and $\partial j_2(x,t,s)$ are the generalized subdifferential operators in the Clarke sense of locally Lipschitz functions $s \mapsto j_1(x,s,t)$ and $t \mapsto j_2(x,t,s)$, respectively. Here the functions $l_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $l_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $j_1 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $j_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are supposed to satisfy the following properties.

(H_{j_1}) The functions $j_1 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $l_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are such that

(i) $x \mapsto j_1(x,s,t)$ is measurable in Ω for all $(s,t) \in \mathbb{R}^2$ with $x \mapsto j_1(x,0,0)$ belonging to $L^1(\Omega)$;

(ii) $s \mapsto j_1(x,s,t)$ is locally Lipschitz continuous for a.a. $x \in \Omega$ and $t \in \mathbb{R}$, $t \mapsto j_1(x,s,t)$ is continuous for a.a. $x \in \Omega$ and $s \in \mathbb{R}$, and the function $l_1 : \mathbb{R}^2 \to \mathbb{R}$ is continuous;

(iii) there exist a function $\delta_{j_1} \in L^{p_1'}(\Omega)_+$ and constants $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \geqslant 0$ with $\beta_1/p_1 + \beta_2/p_2 \leqslant (p_1 - 1)/p_1$ such that

$$\left|l_1(s,t)\xi\right| \leqslant \delta_{j_1}(x) + \alpha_1|s|^{p_1-1} + \alpha_2|t|^{p_2-1} + \alpha_3|s|^{\beta_1}|t|^{\beta_2}$$

for all $\xi \in \partial j_1(x,s,t)$, for a.a. $x \in \Omega$, and for all $(s,t) \in \mathbb{R}^2$.

(H_{j_2}) The functions $j_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $l_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are such that

(i) $x \mapsto j_2(x,s,t)$ is measurable in Ω for all $(s,t) \in \mathbb{R}^2$ with $x \mapsto j_2(x,0,0)$ belonging to $L^1(\Omega)$;

(ii) $s \mapsto j_2(x,s,t)$ is locally Lipschitz continuous for a.a. $x \in \Omega$ and $t \in \mathbb{R}$, $t \mapsto j_2(x,s,t)$ is continuous for a.a. $x \in \Omega$ and $s \in \mathbb{R}$, and the function $l_2 : \mathbb{R}^2 \to \mathbb{R}$ is continuous;

(iii) there exist a function $\delta_{j_2} \in L^{p_2'}(\Omega)_+$ and constants $\gamma_1, \gamma_2, \gamma_3, \zeta_1, \zeta_2 \geqslant 0$ with $\zeta_1/p_1 + \zeta_2/p_2 \leqslant (p_2 - 1)/p_2$ such that

$$\left|l_2(s,t)\eta\right| \leqslant \delta_{j_2}(x) + \gamma_1|s|^{p_1-1} + \gamma_2|t|^{p_2-1} + \gamma_3|s|^{\zeta_1}|t|^{\zeta_2}$$

for all $\eta \in \partial j_2(x,s,t)$, for a.a. $x \in \Omega$, and for all $s \in \mathbb{R}$.

Arguing as in the proof of Theorem 3.11 of [41] and Lemma 7 of [24], we have the following lemma.

Lemma 3. *Assume that* (H_{j_1}) *and* (H_{j_2}) *are fulfilled. Then the multivalued mappings* $U : \Omega \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ *and* $R : \Gamma_2 \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$*, defined by*

$$U(x,s,t) = l_1(s,t)\partial j_1(x,s,t) \quad \text{and} \quad R(x,t,s) = l_2(t,s)\partial j_2(x,t,s)$$

for all $(s,t) \in \mathbb{R}^2$*, for a.a.* $x \in \Omega$*, satisfy* (H_U) *and* (H_R)*, respectively, such that* (H_0) *and* (H_1) *hold.*

By Lemma 3 and Theorem 2, we have the following corollary.

Corollary. *Assume that* (H_{a_1})*,* (H_{b_1})*,* (H_{a_2})*,* (H_{b_2})*,* (H_{j_1})*,* (H_{j_2})*,* (H_{ϕ})*, and* (H_{ψ}) *are fulfilled. If, in addition,* a_1 *and* a_2 *are coercive in* V_{p_1,s_1} *and* V_{p_2,s_2}*, respectively, then the solution set of problem* (21) *is nonempty and compact in* $V_{p_1,s_1} \times V_{p_2,s_2}$.

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