Nonlinear Analysis: Modelling and Control, Vol. 29, No. 5, 1026–1050
https://doi.org/10.15388/namc.2024.29.36106

A coupled fractional conformable Langevin differential system and inclusion on the circular graph*

Lihong Zhang, Xuehui Liu, Guotao Wang[1]

School of Mathematics and Computer Science,
Shanxi Normal University,
Taiyuan, Shanxi 030031, China
zhanglih149@126.com; lxh17636647620@163.com;
wgt2512@163.com

Received: February 3, 2024 / **Revised:** April 30, 2024 / **Published online:** August 27, 2024

Abstract. In this paper, we study a class of coupled fractional conformable Langevin differential system and inclusion on the circular graph. On the one hand, the existence and uniqueness of solutions of this coupled fractional conformable Langevin differential system are studied by fixed point theorems. On the other hand, in the multivalued case, the existence of at least one solution of the fractional conformable Langevin differential inclusion on the circular graph is discussed and the sufficient conditions are established by using Leray–Schauder nonlinear alternative and Covitz–Nadler fixed point theorem.

Keywords: circular graph, fractional conformable derivative, Langevin differential equation and inclusion, fixed point theorem.

1 Introduction and preliminaries

Fractional calculus is a branch of mathematics that extends the traditional calculus theory. It allows the calculation of fractional derivatives and fractional integrals, can capture the long-range dependence of time and space more accurately when describing nonlocal behaviors, can be used to model fractional differential equations [6, 13, 27], and can also be applied to physics, engineering, biology and other fields [10, 18], especially, nonlinear dynamics, signal processing and image processing. In recent years, fractional calculus is often used to describe nonlinear, multiscale and time-varying systems, and it is also widely used in deep learning such as fractional neural network and fractional convolutional neural network.

What's more, fractional differential equations occupy an important position in mathematical theory and have a wide range of applications in many fields such as natural

*This work is supported by NSF of Shanxi Province, China (No. 20210302123339).
[1]Corresponding author.

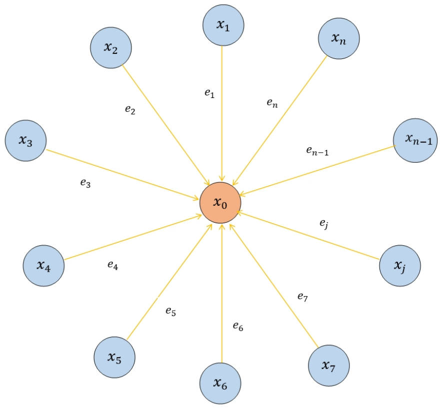

Figure 1. n-edge star graph $\mathbb{G}$.

science and engineering technology [9, 15]. Fractional differential equations can be used to simulate and predict dynamic behaviors in tumor cell proliferation and hemodynamics [2], in addition, it can also be used to describe time-dependent behaviors of nonlinear materials and elastic and plastic behaviors of nanostructures [20]. More results on fractional differential equations can be found in [3,14,28]. In 2017, Jarad et al. [16] proposed a new fractional derivative called the fractional conformable derivative. As described by Jarad et al., this fractional conformable derivative contains some classical fractional derivatives. For more information, please refer to [16].

In recent years, the research of fractional calculus has been further developed and has become a new interdisciplinary research field. As one of the representative models of fractional calculus, fractional Langevin equation is of great significance to deeply explore the principle and application of fractional calculus. Fractional Langevin equation is widely used in various fields [22, 26]. In the field of biomedical engineering, it is used to establish the model of heart disease to study the complex dynamic phenomena in cardiovascular system. In the field of robot control, it is used to study the motion control and trajectory tracking of robots, so as to improve the motion accuracy and stability of robots [29]. With the wide application of fractional Langevin equation, the solutions of its initial value problem and boundary value problem have attracted more attention from scholars [1, 4, 24, 25].

In 2019, Mehandiratta, Mehra and Leugering [19] studied a star graph with n edges and $n+1$ vertices (Fig. 1) with $V(\mathbb{G}) = \{x_0, x_1, \dots x_n\}$ and $E(\mathbb{G}) = \{e_1 = \overrightarrow{x_0 x_1},\, e_2 = \overrightarrow{x_2 x_0},\, \dots,\, e_n = \overrightarrow{x_n x_0}\}$, where $l_j = |\overrightarrow{x_j x_0}|$ is regarded as the length of e_j connecting vertices x_j to x_0 ($j = 1, \dots, n$). They considered the following system:

$$
\begin{aligned}
&{}^{c}D_0^{v}\vartheta_j(t) = \varphi_j\bigl(t, \vartheta_j(t), {}^{c}D_0^{\alpha}\vartheta_j(t)\bigr), \quad t \in (0, l_j),\ j = 1, 2, \dots, n, \\
&\vartheta_j(0) = 0, \qquad \vartheta_j(l_j) = \vartheta_i(l_i) \quad (i \neq j), \qquad \sum_{j=1}^{n} y_j'(l_j) = 0
\end{aligned} \tag{1}
$$

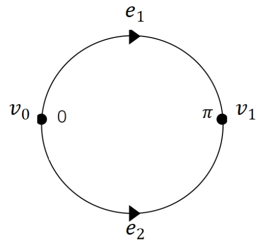

Figure 2. A circular.

in which $v \in (1,2]$, $\alpha \in (0, v-1]$, $\varphi_j \in C([0,l_j]\times\mathbb{R}^2,\mathbb{R})$, and ${}^cD_0^v$ is the vth Caputo derivative. The authors established transformations $w = t/l_j \in [0,1]$ and $y(w) = \vartheta(t) = \vartheta(l_j w)$ for $t \in (0,l_j]$ to prove the relation ${}^cD_0^v\vartheta(t) = l_j^{-v}({}^cD_0^v y(w))$. By the above transformations over the interval $[0,1]$, system (1) can be rewritten into the following form:

$$\begin{aligned} &{}^cD_0^v y_j(w) = l_j^v h_j\bigl(w, y_j(w), l_j^{-v}\,{}^cD_0^v y_j(w)\bigr), \quad w \in [0,1],\\ &y_j(0) = 0, \qquad y_j(1) = y_i(1) \quad (i \neq j), \qquad \sum_{j=1}^n l_j^{-1} y_j'(1) = 0, \end{aligned} \tag{2}$$

where $y_j(w) = u_i(l_j w)$ and $h_j(w,u,\tilde{y}) = \varphi_j(l_j w, u, \tilde{y})$ for $j \in \mathbb{N}_1^n := \{1,2,\dots,n\}$.

Inspired by the above research, this paper studies a class of coupled fractional conformable Langevin differential system and inclusion on the circular graph. Circular network structure is widely used in neural network, image processing, computer vision and bioinformatics [12, 17]. For example, recurrent neural network is a kind of neural network with a circular structure that can be used to process temporal data. It has a wide range of applications in natural language processing, speech recognition, music generation, etc. [12]. In addition, in the field of bioinformatics, circular networks are often used to demonstrate the relationships between complex systems such as genes, proteins, metabolic pathways and interaction networks [17].

In order to reduce the complexity of the presentation, we study the fractional boundary value problem on the simplest circular graph. In particular, the circular graph used in this paper is an example that includes a circular network structure. We construct a circular graph $G = (V,E)$, which consists of a set of nodes $V = \{v_0, v_1\}$ and a set of arcs $E = \{e_1 = \overrightarrow{v_0v_1},\, e_2 = \overrightarrow{v_0v_1}\}$ connecting these nodes. The arcs e_1 and e_2 are parameterized by the interval $(0,\pi)$, so e_1 and e_2 form a circular with a length of 2π (Fig. 2).

To be precise, we study the fractional conformable Langevin differential system on the circular graph as follows:

$$\begin{aligned} &{}_0^{C\beta}\mathcal{D}^\alpha\bigl({}_0^{C\gamma}\mathcal{D}^\alpha + \lambda_j\bigr)u_j(t) = g_j\bigl(t, u_j(t)\bigr), \quad t \in [0,\pi],\\ &u_1(0) = u_2(0) = 0, \qquad u_1(\pi) = u_2(\pi),\\ &{}_0^{C\gamma}\mathcal{D}^\alpha u_1(0) + {}_0^{C\gamma}\mathcal{D}^\alpha u_2(0) = 0, \end{aligned} \tag{3}$$

where $\alpha, \beta, \gamma \in (0,1]$, $\gamma < \beta$, $g_j \in C([0,\pi] \times \mathbb{R}, \mathbb{R})$, $\lambda_j \in \mathbb{R}^+$ and $j = 1, 2$. ${}_0^{C\beta}D^\alpha$ and ${}_0^{C\gamma}D^\alpha$ represent fractional conformable derivatives of Caputo type of order β and γ, respectively.

The multivalued problem corresponding to (3) is also studied in this paper by

$$\begin{aligned} &{}_0^{C\beta}\mathcal{D}^\alpha\big({}_0^{C\gamma}\mathcal{D}^\alpha + \lambda_j\big)u_j(t) \in F_j\big(t, u_j(t)\big), \quad t \in [0,\pi], \\ &u_1(0) = u_2(0) = 0, \qquad u_1(\pi) = u_2(\pi), \\ &{}_0^{C\gamma}\mathcal{D}^\alpha u_1(0) + {}_0^{C\gamma}\mathcal{D}^\alpha u_2(0) = 0, \end{aligned} \tag{4}$$

where $j = 1, 2$, $F_j : [0,\pi] \times \mathbb{R} \to \mathrm{P}(\mathbb{R})$ ($\mathrm{P}(\mathbb{R})$ represents a family of all nonempty subsets of $\mathbb{R}$) denotes a multivalued map.

For the rest of this article, we arrange as follows. In Section 2, some related preliminary concepts and lemmas are reviewed. Section 3 is devoted to proving the existence and uniqueness of the single-valued problem. Section 4 proves the existence results of the multivalued problem. Finally, in Section 5, several examples are provided to verify the reliability of the proposed results.

2 Some preliminary concepts and lemmas

Definition 1. (See [16].) Let $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$, the fractional conformable integral operator is defined as

$${}_a^\beta\mathcal{I}^\alpha h(t) = \frac{1}{\Gamma(\beta)} \int_a^t \left(\frac{(t-a)^\alpha - (v-a)^\alpha}{\alpha}\right)^{\beta-1} h(v) \frac{\mathrm{d}v}{(v-a)^{1-\alpha}}.$$

Definition 2. [16] Let $\alpha > 0$, $\mathrm{Re}(\beta) > 0$ and $n = [\mathrm{Re}(\beta)] + 1$. If $h \in C^n_{\alpha,a}$, the βth Caputo fractional conformable derivative of the function h is defined as

$${}_a^{C\beta}\mathcal{D}^\alpha h(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \left(\frac{(t-a)^\alpha - (v-a)^\alpha}{\alpha}\right)^{n-\beta-1} \frac{{}_a^n\mathcal{T}^\alpha h(v)}{(v-a)^{1-\alpha}}\, \mathrm{d}v,$$

where

$${}_a^n\mathcal{T}^\alpha = \underbrace{{}_a\mathcal{T}^\alpha\, {}_a\mathcal{T}^\alpha \cdots {}_a\mathcal{T}^\alpha}_{n \text{ times}}, \qquad {}_a\mathcal{T}^\alpha h(v) = (v-a)^{1-\alpha} h'(v).$$

Lemma 1. (*See* [16].) *Let* $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$. *Then*

$${}_a^\beta\mathcal{I}^\alpha\big({}_a^\gamma\mathcal{I}^\alpha\big)h(t) = {}_a^{\beta+\gamma}\mathcal{I}^\alpha h(t).$$

Lemma 2. (*See* [16].) *Let* $h \in C^n_{\alpha,a}[a,b]$, $\beta \in \mathbb{C}$. *Then*

$${}_a^\beta\mathcal{I}^\alpha\big({}_a^{C\beta}\mathcal{D}^\alpha h(t)\big) = h(t) - \sum_{k=0}^{n-1} \frac{{}_a^k\mathcal{T}^\alpha h(a)(v-a)^{\alpha k}}{k!\alpha^k}.$$

Next, we give a linear variation lemma, which provides an auxiliary result for the transformation of our boundary value problem.

Lemma 3. *Let* $\mathfrak{h}_j \in C([0,\pi],\mathbb{R})$, $\beta,\gamma \in (0,1]$, $\gamma < \beta$, $j = 1,2$. *Then* u_j *satisfy the following problem*

$$\begin{aligned} &{}_0^{C\beta}\mathcal{D}^\alpha\big({}_0^{C\gamma}\mathcal{D}^\alpha + \lambda_j\big)u_j(t) = \mathfrak{h}_j(t), \quad t \in [0,\pi],\\ &u_1(0) = u_2(0) = 0, \qquad u_1(\pi) = u_2(\pi),\\ &{}_0^{C\gamma}\mathcal{D}^\alpha u_1(0) + {}_0^{C\gamma}\mathcal{D}^\alpha u_2(0) = 0 \end{aligned} \tag{5}$$

if and only if

$$\begin{aligned} u_j(t) &= \frac{1}{\Gamma(\gamma+\beta)}\int_0^t\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\mathfrak{h}_j(s)\frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)}\int_0^t\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1}u_j(s)\frac{\mathrm{d}s}{s^{1-\alpha}}\\ &\quad + \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\big(\mathfrak{h}_1(s) - \mathfrak{h}_2(s)\big)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.\\ &\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1}u_2(s)\frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1}u_1(s)\frac{\mathrm{d}s}{s^{1-\alpha}}\right]. \end{aligned} \tag{6}$$

Proof. Suppose that u_j satisfy problem (5). Integrating ${}_0^{\beta}\mathcal{I}^\alpha$ on both sides of Eq. (5) and according to Lemma 2, we have

$${}_0^{\beta}\mathcal{I}^\alpha\big({}_0^{C\beta}\mathcal{D}^\alpha\big({}_0^{C\gamma}\mathcal{D}^\alpha + \lambda_j\big)u_j(t)\big) = \big({}_0^{C\gamma}\mathcal{D}^\alpha + \lambda_j\big)u_j(t) - c_0^{(j)}.$$

Consequently,

$${}_0^{C\gamma}\mathcal{D}^\alpha u_j(t) = {}_0^{\beta}\mathcal{I}^\alpha\mathfrak{h}_j(t) - \lambda_j u_j(t) + c_0^{(j)}. \tag{7}$$

Continuing to taking integral ${}_0^{\gamma}\mathcal{I}^\alpha$ on both sides of Eq. (7), we find that

$$u_j(t) = {}_0^{\gamma+\beta}\mathcal{I}^\alpha\mathfrak{h}_j(t) - \lambda_j{}_0^{\gamma}\mathcal{I}^\alpha u_j(t) + \frac{t^{\alpha\gamma}}{\alpha^\gamma\Gamma(\gamma+1)}c_0^{(j)} + c_1^{(j)}. \tag{8}$$

From the condition $u_1(0) = u_2(0) = 0$ we get $c_1^{(j)} = 0$. Further, using boundary conditions $u_1(\pi) = u_2(\pi)$ and ${}_0^{C\gamma}\mathcal{D}^\alpha u_1(0) + {}_0^{C\gamma}\mathcal{D}^\alpha u_2(0) = 0$, we obtain

$$\begin{aligned} c_0^{(j)} &= \frac{(-1)^j\alpha^\gamma\Gamma(\gamma+1)}{2\pi^{\alpha\gamma}}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\big(\mathfrak{h}_1(s) - \mathfrak{h}_2(s)\big)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.\\ &\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1}u_2(s)\frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1}u_1(s)\frac{\mathrm{d}s}{s^{1-\alpha}}\right]. \end{aligned}$$

By plugging the values of $c_0^{(j)}$ into (8), we get (6). Further, the inverse of this lemma can be obtained by direct calculation. □

3 The single-valued problem

In this section, we will obtain the main results of problem (3) on a circular graph. Firstly, we consider the Banach spaces $M_j = \{u_j\colon\ u_j \in C[0,\pi]\}$ with $\|u_j\|_{M_j} = \sup_{t\in[0,\pi]} |u_j(t)|$ for $j = 1,2$. Then the product space $M = (M_1, M_2)$ equipped with the norm $\|u\|_M = \|u_1\|_{M_1} + \|u_2\|_{M_2}$ is a Banach space.

Here by considering Lemma 3, define $\mathscr{A}$ on M by

$$\mathscr{A}(u_1,u_2)(t) := \big((\mathscr{A}_1 u_1)(t), (\mathscr{A}_2 u_2)(t)\big),$$

where $\mathscr{A}_j : M_j \to M_j$ ($j = 1,2$),

$$\begin{aligned}
&\mathscr{A}_j u_j(t) \\
&= \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} g_j\big(s, u_j(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\quad + \frac{(-1)^j}{2} \left(\frac{t}{\pi}\right)^{\alpha\gamma} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(g_1\big(s,u_1(s)\big) - g_2\big(s,u_2(s)\big)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\
&\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right].
\end{aligned}$$

3.1 Existence

Lemma 4. (*See* [23].) *Let $\mathcal{M}$ be a Banach space, and let $\mathscr{A}$ be a completely continuous mapping on $\mathcal{M}$. Then $\{x \in \mathcal{M}\colon\ x = \lambda\mathscr{A}x,\ \lambda \in (0,1)\}$ is unbounded, or there exists a fixed point of $\mathscr{A}$ in $\mathcal{M}$.*

Theorem 1. *Assume that*

(H1) *the functions $g_j : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ ($j = 1,2$) are continuous, and there exist positive functions $\mathfrak{p}_j(t), \mathfrak{q}_j(t) \in L^1[0,\pi]$ ($j = 1,2$) such that*

$$\big|g_j\big(t, u_j(t)\big)\big| \leqslant \mathfrak{p}_j(t) + \mathfrak{q}_j(t)\big|u_j(t)\big|$$

for all $t \in [0,\pi]$, $u_j \in \mathbb{R}$.

If

$$\delta^* = P(\mathfrak{q}_1^* + \mathfrak{q}_2^*) + N(\lambda_1 + \lambda_2) < 1,$$

where

$$P = \frac{2}{\Gamma(\gamma+\beta+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)}, \qquad N = \frac{2}{\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma}, \tag{9}$$

and

$$\mathfrak{p}_j^* = \|\mathfrak{p}_j\|_{L^1}, \qquad \mathfrak{q}_j^* = \|\mathfrak{q}_j\|_{L^1}.$$

Then on the circular graph, problem (3) *has at least one solution.*

Proof. We first verify that $\mathscr{A}$ is completely continuous. Continuity of g_j ($j = 1, 2$) implies that $\mathscr{A}: M \to M$ is continuous. Let Θ be a bounded subset of M_j, and let $\Theta = \{u_j \in M_j\colon \|u_j\|_{M_j} \leqslant \varepsilon_j\}$. Then, for any $u = (u_1, u_2) \in \Theta$, we find that

$$\begin{aligned}
&\big|\mathscr{A}_j u_j(t)\big| \\
&\quad\leqslant \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big|g_j\big(s, u_j(s)\big)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\qquad + \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_j(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\qquad + \frac{1}{2}\left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\big|g_1\big(s, u_1(s)\big)\big| + \big|g_2\big(s, u_2(s)\big)\big|\big) \frac{\mathrm{d}s}{s^{1-\alpha}}\right. \\
&\qquad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_2(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} + \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_1(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}}\right] \\
&\quad\leqslant \frac{1}{\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \big(\mathfrak{p}_j^* + \mathfrak{q}_j^* \|u_j\|_{M_j}\big) + \frac{\lambda_j}{\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \|u_j\|_{M_j} \\
&\qquad + \frac{1}{2\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \sum_{i=1}^2 \big(\mathfrak{p}_i^* + \mathfrak{q}_i^* \|u_i\|_{M_i}\big) \\
&\qquad + \frac{1}{2\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \sum_{i=1}^2 \lambda_i \|u_i\|_{M_i}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathscr{A} u\|_M &= \sum_{j=1}^2 \|\mathscr{A}_j u_j\|_{M_j} \\
&\leqslant \frac{2}{\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \sum_{j=1}^2 (\mathfrak{p}_j^* + \mathfrak{q}_j^* \|u_j\|_{M_j}) \\
&\quad + \frac{2}{\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \sum_{j=1}^2 \lambda_j \|u_j\|_{M_j} \\
&= P(\mathfrak{p}_1^* + \mathfrak{p}_2^*) + \big(P(\mathfrak{q}_1^* + \mathfrak{q}_2^*) + N(\lambda_1 + \lambda_2)\big) \sum_{j=1}^2 \varepsilon_j.
\end{aligned}$$

From this we deduced that the operator $\mathscr{A}$ is uniformly bounded. In order to prove the equicontinuity of the operator $\mathscr{A}$, let $u = (u_1, u_2) \in \Theta$, $t_1, t_2 \in [0, \pi]$ and $t_1 < t_2$. Then

$$\begin{aligned}
&\big|(\mathscr{A}_j u_j)(t_2) - (\mathscr{A}_j u_j)(t_1)\big| \\
&\quad \leqslant \frac{1}{\Gamma(\beta+\gamma)} \int_0^{t_1} \left(\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)^{\beta+\gamma-1} - \left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)^{\beta+\gamma-1}\right) \big|g_j\big(s, u_j(s)\big)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\quad + \frac{1}{\Gamma(\beta+\gamma)} \int_{t_1}^{t_2} \left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)^{\beta+\gamma-1} \big|g_j\big(s, u_j(s)\big)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\quad + \frac{\lambda_j}{\Gamma(\gamma)} \int_0^{t_1} \left(\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} - \left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1}\right) \big|u_j(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\quad + \frac{\lambda_j}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_j(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\quad + \frac{t_2^{\alpha\gamma} - t_1^{\alpha\gamma}}{2\pi^{\alpha\gamma}} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \left(\big|g_1\big(s, u_1(s)\big)\big| + \big|g_2\big(s, u_2(s)\big)\big|\right) \frac{\mathrm{d}s}{s^{1-\alpha}}\right. \\
&\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_2(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} + \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_1(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}}\right] \\
&\quad \leqslant \frac{1}{\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \big(t_2^{\alpha(\beta+\gamma)} - t_1^{\alpha(\beta+\gamma)}\big) \big(\mathfrak{p}_j^* + \mathfrak{q}_j^* \|u_j\|_{M_j}\big) \\
&\quad + \frac{\lambda_j}{\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \big(t_2^{\alpha\gamma} - t_1^{\alpha\gamma}\big) \big(\mathfrak{p}_j^* + \mathfrak{q}_j^* \|u_j\|_{M_j}\big) \\
&\quad + \frac{t_2^{\alpha\gamma} - t_1^{\alpha\gamma}}{2\pi^{\alpha\gamma}} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \left(\big|g_1\big(s, u_1(s)\big)\big| + \big|g_2\big(s, u_2(s)\big)\big|\right) \frac{\mathrm{d}s}{s^{1-\alpha}}\right. \\
&\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_2(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} + \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_1(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}}\right].
\end{aligned}$$

The right-side converges to zero (independently of $u \in \Theta$) as $t_1 \to t_2$. Hence, $\|\mathscr{A} u(t_2) - \mathscr{A} u(t_1)\|_M \to 0$ as $t_1 \to t_2$. This shows that $\mathscr{A}$ is an equicontinuous operator on M. Applying the Arzelà–Ascoli theorem, we obtain that $\mathscr{A}$ is completely continuous. Consider a subset of M

$$\Lambda = \big\{(u_1, u_2) \in M\colon\ (u_1, u_2) = \mu \mathscr{A}(u_1, u_2),\ \mu \in (0, 1)\big\}.$$

We prove the boundedness of Λ. Let $(u_1, u_2) \in \Lambda$. Then

$$(u_1, u_2) = \mu \mathscr{A}(u_1, u_2),$$

and $u_j(t) = \mu\mathscr{A}_j u_j$ for all $t \in [0,\pi]$ and $j = 1,2$. Thus,

$$\begin{aligned}\left|u_j(t)\right| \leqslant \mu\Bigg\{ &\frac{1}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\big(\mathfrak{p}_j^* + \mathfrak{q}_j^*\|u_j\|_{M_j}\big) \\ &+ \frac{\lambda_j}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\pi^{\alpha\gamma}\|u_j\|_{M_j} \\ &+ \frac{1}{2\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\sum_{i=1}^{2}\big(\mathfrak{p}_i^* + \mathfrak{q}_i^*\|u_i\|_{M_i}\big) \\ &+ \frac{1}{2\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\pi^{\alpha\gamma}\sum_{i=1}^{2}\lambda_i\|u_i\|_{M_i}\Bigg\}.\end{aligned}$$

This implies that

$$\begin{aligned}\|u\|_M &= \sum_{j=1}^{2}\|u_j\|_{M_j} \\ &\leqslant \frac{2}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\sum_{j=1}^{2}\big(\mathfrak{p}_j^* + \mathfrak{q}_j^*\|u_j\|_{M_j}\big) \\ &\quad + \frac{2}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\pi^{\alpha\gamma}\sum_{j=1}^{2}\lambda_j\|u_j\|_{M_j} \\ &= P(\mathfrak{p}_1^* + \mathfrak{p}_2^*) + \big(P(\mathfrak{q}_1^* + \mathfrak{q}_2^*) + N(\lambda_1+\lambda_2)\big)\|u\|_M.\end{aligned}$$

It follows from $\delta^* < 1$ that Λ is bounded. By using Lemma 4, we conclude that $\mathscr{A}$ has a fixed point, which is a solution of problem (3). □

3.2 Uniqueness

Lemma 5. (*See* [23].) *Suppose that $\mathscr{B}$ is a Banach space, $M \subset \mathscr{B}$ is closed, and $\mathscr{A} : M \to M$ is a strict contraction, i.e., $|\mathscr{A}p - \mathscr{A}q| \leqslant \lambda|p-q|$ for some $\lambda \in (0,1)$ and all $p, q \in M$. Then a unique fixed point p^* of $\mathscr{A}$ exists.*

Theorem 2. *Suppose that there exist constants $\rho_j > 0$ such that*

(H2) *for all $u_j, y_j \in \mathbb{R}$ $(j = 1,2)$ and $t \in [0,\pi]$,*

$$\big|g_j(t,u_j) - g_j(t,y_j)\big| \leqslant \rho_j|u_j - y_j|.$$

Then on each edge of the circular graph, problem (3) *has a unique solution if*

$$\sigma^* = P(\rho_1+\rho_2) + N(\lambda_1+\lambda_2) < 1.$$

Proof. Use Banach's fixed point theorem to transform problem (3) into a fixed point problem. Then our main goal is to prove that operator $\mathscr{A}$ has a fixed point, where operator $\mathscr{A} : M \to M$.

Firstly, we will prove that $\mathscr{A}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{u \in M\colon \|u\|_M \leqslant r\}$. Let $r \geqslant PQ/(1-\sigma^*)$ and $\max_{t\in[0,\pi]}|g_j(t,0)| = Q < \infty$. By (H2), we have

$$\begin{aligned}\left|g_j\big(t,u_j(t)\big)\right| &\leqslant \left|g_j\big(t,u_j(t)\big)-g_j(t,0)\right| + \left|g_j(t,0)\right| \\ &\leqslant \rho_j\left|u_j(t)\right| + Q \leqslant \rho_j\|u_j\|_{M_j} + Q.\end{aligned}$$

For any $u_j \in \mathcal{B}_r$, we obtain

$$\begin{aligned}&\left|\big(\mathscr{A}_j u_j\big)(t)\right| \\ &\quad\leqslant \frac{1}{\Gamma(\gamma+\beta)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\left|g_j\big(s,u_j(s)\big)\right|\frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\qquad+\frac{\lambda_j}{\Gamma(\gamma)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\left|u_j(s)\right|\frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\qquad+\frac{1}{2}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\left(\left|g_1\big(s,u_1(s)\big)\right|+\left|g_2\big(s,u_2(s)\big)\right|\right)\frac{\mathrm{d}s}{s^{1-\alpha}}\right. \\ &\qquad\left.+\frac{\lambda_2}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\left|u_2(s)\right|\frac{\mathrm{d}s}{s^{1-\alpha}}+\frac{\lambda_1}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\left|u_1(s)\right|\frac{\mathrm{d}s}{s^{1-\alpha}}\right] \\ &\quad\leqslant \frac{1}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\big(\rho_j\|u_j\|_{M_j}+Q\big)+\frac{\lambda_j}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\pi^{\alpha\gamma}\|u_j\|_{M_j} \\ &\qquad+\frac{1}{2\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\sum_{i=1}^2\big(\rho_i\|u_i\|_{M_i}+Q\big) \\ &\qquad+\frac{1}{2\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\pi^{\alpha\gamma}\sum_{i=1}^2\lambda_i\|u_i\|_{M_i}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\mathscr{A}u\|_M &= \sum_{j=1}^2\|\mathscr{A}_j u_j\|_{M_j} \\ &\leqslant \frac{2}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\sum_{j=1}^2(\rho_j\|u_j\|_{M_j}+Q) \\ &\quad+\frac{2}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\pi^{\alpha\gamma}\sum_{j=1}^2\lambda_j\|u_j\|_{M_j} \\ &= P\sum_{j=1}^2(\rho_j\|u_j\|_{M_j}+Q)+N\sum_{j=1}^2\lambda_j\|u_j\|_{M_j}\leqslant PQ+\sigma^* r\leqslant r,\end{aligned}$$

and thus, $\mathscr{A}\mathcal{B}_r \subset \mathcal{B}_r$.

Next, in order to prove that $\mathscr{A}$ is a contraction, let $u = (u_1, u_2), y = (y_1, y_2) \in M$. For $t \in [0, \pi]$, we have

$$\begin{aligned}
&\left|(\mathscr{A}_j u_j)(t) - (\mathscr{A}_j y_j)(t)\right| \\
&\quad \leqslant \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \left|g_j\bigl(s, u_j(s)\bigr) - g_j\bigl(s, y_j(s)\bigr)\right| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\qquad + \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \left|u_j(s) - y_j(s)\right| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\qquad + \frac{1}{2}\left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \sum_{i=1}^{2} \bigl(\left|g_i\bigl(s, u_i(s)\bigr) - g_i\bigl(s, y_i(s)\bigr)\right|\bigr) \frac{\mathrm{d}s}{s^{1-\alpha}}\right. \\
&\qquad + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \left|u_2(s) - y_2(s)\right| \frac{\mathrm{d}s}{s^{1-\alpha}} \\
&\qquad \left. + \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \left|u_1(s) - y_1(s)\right| \frac{\mathrm{d}s}{s^{1-\alpha}}\right] \\
&\quad \leqslant \frac{1}{\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \rho_j \|u_j - y_j\|_{M_j} + \frac{\lambda_j}{\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \|u_j - y_j\|_{M_j} \\
&\qquad + \frac{1}{2\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \sum_{i=1}^{2} \rho_i \|u_i - y_i\|_{M_i} \\
&\qquad + \frac{1}{2\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \sum_{i=1}^{2} \lambda_i \|u_i - y_i\|_{M_i}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathscr{A}u - \mathscr{A}y\|_M &= \sum_{j=1}^{2} \|\mathscr{A}_j u_j - \mathscr{A}_j y_j\|_{M_j} \\
&\leqslant \frac{2}{\Gamma(\beta+\gamma+1)} \left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \sum_{j=1}^{2} \rho_j \|u_j - y_j\|_{M_j} \\
&\quad + \frac{2}{\Gamma(\gamma+1)} \left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \sum_{j=1}^{2} \lambda_j \|u_j - y_j\|_{M_j} \\
&\leqslant P(\rho_1 + \rho_2) \sum_{j=1}^{2} \|u_j - y_j\|_{M_j} + N(\lambda_1 + \lambda_2) \sum_{j=1}^{2} \|u_j - y_j\|_{M_j} \\
&= \bigl(P(\rho_1 + \rho_2) + N(\lambda_1 + \lambda_2)\bigr) \|u - y\|_M.
\end{aligned}$$

$\mathscr{A}$ is a contraction due to $\sigma^* < 1$. According to Lemma 5, we concluded that operator $\mathscr{A}$ has a unique fixed point, which is the unique solution of problem (3). □

4 The multivalued problem

For a normed space $(\mathrm{D}, \|\cdot\|)$, one can define:

- $\mathrm{P}_{\mathrm{cl}}(\mathrm{D}) = \{\mathrm{Y} \in \mathrm{P}(\mathrm{D})\colon \mathrm{Y} \text{ is closed}\}$,
- $\mathrm{P}_{\mathrm{cp}}(\mathrm{D}) = \{\mathrm{Y} \in \mathrm{P}(\mathrm{D})\colon \mathrm{Y} \text{ is compact}\}$,
- $\mathrm{P}_{\mathrm{cp,c}}(\mathrm{D}) = \{\mathrm{Y} \in \mathrm{P}(\mathrm{D})\colon \mathrm{Y} \text{ is compact and convex}\}$,
- $\mathrm{P}_{\mathrm{cl,b}}(\mathrm{D}) = \{\mathrm{Y} \in \mathrm{P}(\mathrm{D})\colon \mathrm{Y} \text{ is closed and bounded}\}$.

For each $u_j \in M_j$ $(j = 1, 2)$, define the set of a selection of F_j by

$$S_{F_j,u_j} = \bigl\{\nu_j \in L^1\bigl([0,\pi],\mathbb{R}\bigr)\colon \nu_j(t) \in F_j\bigl(t, u_j(t)\bigr) \text{ on } [0,\pi]\bigr\}.$$

Definition 3. If there exist functions $\nu_j \in L^1([0,\pi],\mathbb{R})$ with $\nu_j(t) \in F_j(t, u_j(t))$ for almost every $t \in [0,\pi]$ and $j = 1, 2$ such that u_j satisfy the differential equations ${}^{C\beta}_{0}\mathcal{D}^{\alpha}({}^{C\gamma}_{0}\mathcal{D}^{\alpha} + \lambda_j)u_j(t) = \nu_j(t)$ on $[0,\pi]$ and boundary conditions $u_1(0) = u_2(0) = 0$, $u_1(\pi) = u_2(\pi)$ and ${}^{C\gamma}_{0}\mathcal{D}^{\alpha}u_1(0) + {}^{C\gamma}_{0}\mathcal{D}^{\alpha}u_2(0) = 0$, then the continuous function u is a solution of problem (4).

We define $Gr(G) = \{(p,q) \in P \times Q,\ q \in G(p)\}$ as a graph of G and review two important lemmas.

Lemma 6. (*See* [8].) *If $G : P \to \mathrm{P}_{\mathrm{cl}}(Q)$ is u.s.c., then $Gr(G)$ is a closed subset of $P \times Q$, i.e., for every sequence $\{p_n\}_{n\in\mathbb{N}} \subset P$ and $\{q_n\}_{n\in\mathbb{N}} \subset Q$, if when $n \to \infty$, $p_n \to p_*$, $q_n \to q_*$ and $q_n \in G(p_n)$, then $q_* \in G(p_*)$. Inversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 7. (*See* [21].) *Suppose there exists a linear continuous mapping $\mathfrak{B}$ from $L^1([0,\pi],\mathbb{R})$ to $C([0,\pi],\mathbb{R})$ and an L^1-Carathéodory multivalued map $\mathrm{F} : [0,\pi] \times \mathbb{R} \to \mathrm{P}_{\mathrm{cp,c}}(\mathbb{R})$. Then*

$$\mathfrak{B} \circ S_{\mathrm{F}} : C\bigl([0,\pi],\mathbb{R}\bigr) \to \mathrm{P}_{\mathrm{cp,c}}\bigl(C\bigl([0,\pi],\mathbb{R}\bigr)\bigr),$$
$$u \mapsto (\mathfrak{B} \circ S_{\mathrm{F}})(u) = \mathfrak{B}(S_{\mathrm{F},u})$$

is said to be a closed graph operator in $C([0,\pi],\mathbb{R}) \times C([0,\pi],\mathbb{R})$.

Remark 1. (See [21].) A multivalued map $\mathrm{F} : [0,\pi] \times \mathbb{R} \to \mathrm{P}_{\mathrm{cp,c}}(\mathbb{R})$ is said to be L^1-Carathéodory if:

(i) for each $u \in \mathbb{R}$, $t \mapsto \mathrm{F}(t,u)$ is measurable;
(ii) for almost all $t \in [0,\pi]$, $u \mapsto \mathrm{F}(t,u)$ is upper semicontinuous;
(iii) there exists $\varphi_\beta \in L^1([0,\pi],\mathbb{R}^+)$ such that

$$\|\mathrm{F}(t,u)\| = \sup\{|\omega|\colon \omega \in \mathrm{F}(t,u)\} \leqslant \varphi_\beta(t)$$

for each $\beta > 0$, $u \in \mathbb{R}$ with $\|u\| \leqslant \beta$ and any $t \in [0,\pi]$.

4.1 The Carathéodory case

For the case where the multivalued map has convex values, we establish an existence result by the following lemma.

Lemma 8. (*See* [11].) *Assume that* $\mathfrak{B}$ *is a Banach space,* $\mathcal{Q}$ *is a closed convex subset of* $\mathfrak{B}$*, and let* W *be an open subset of* $\mathcal{Q}$ *and* $0 \in W$*. If* $\mathrm{F} : \overline{W} \to \mathrm{P}_{\mathrm{cp}}(\mathcal{Q})$ *is an upper semicontinuous compact map, then either*

(i) F *has a fixed point in* $\overline{W}$ *or*
(ii) *there a* $x \in \partial W$ *and* $\epsilon \in (0,1)$ *with* $x \in \epsilon \mathrm{F}(x)$.

Theorem 3. *Suppose that*:

(G1) $F_j : [0,\pi] \times \mathbb{R} \to \mathrm{P}_{\mathrm{cp,c}}(\mathbb{R})$ *are* L^1*-Carathéodory*;
(G2) *for each* $(t, u_j) \in [0,\pi] \times \mathbb{R}$, $\mathcal{H}_j \in L^1([0,\pi], \mathbb{R}^+)$,
$$\big\|F_j\big(t, u_j(t)\big)\big\|_{\mathrm{P}} := \sup\big\{|\nu_j|\colon \nu_j \in F_j\big(t, u_j(t)\big)\big\} \leqslant \mathcal{H}_j(t)\phi_j\big(\|u\|_M\big),$$
where ϕ_j *are continuous nondecreasing functions:* $\mathbb{R}^+ \to \mathbb{R}^+$;
(G3) *there exists a constant* $\mathscr{L} > 0$ *such that*
$$P\sum_{j=1}^{2} \mathcal{H}_j^* \phi_j(\mathscr{L}) + N\mathscr{C}(\lambda_1 + \lambda_2) < \mathscr{L},$$
where P *and* N *are defined by* (9)*, and* $\mathcal{H}_j^* = \|\mathcal{H}_j\|_{L^1}$.

Then, for all $j = 1, 2$*, the multivalued problem* (4) *has at least one solution on the circular graph.*

Proof. We define an operator $\mathcal{T} : M \to \mathrm{P}(M)$, where $\mathcal{T}_j : M_j \to \mathrm{P}(M_j)$,

$$\begin{aligned}
\mathcal{T}_j(u_j) = \Bigg\{ h_j \in M_j\colon\ h_j(t) = {} & \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \nu_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\
& - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\
& + \frac{(-1)^j}{2} \left(\frac{t}{\pi}\right)^{\alpha\gamma} \Bigg[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\nu_1(s) - \nu_2(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \\
& + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\
& - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \Bigg] \Bigg\}
\end{aligned}$$

for $t \in [0,\pi]$ and $\nu_j \in S_{F_j,u_j}$. Obviously, the fixed point of $\mathcal{T}$ is a solution of problem (4).

We split the proof into the following steps.

Step 1. For each $u_j \in M_j$, $\mathcal{T}_j(u_j)$ are convex.

Since S_{F_j,u_j} are convex, this step is obvious, and we omit the proof.

Step 2. $\mathcal{T}$ maps bounded sets to bounded sets in M.

Let $B_r := \{u = (u_1, u_2) \in M = (M_1, M_2)\colon \|u\|_M \leqslant r\}$, $r > 0$. Then, for each $h_j \in \mathcal{T}_j(u_j)$, $u_j \in B_r$, there exist $\nu_j \in S_{F_j,u_j}$ such that

$$\begin{aligned} h_j(t) &= \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \nu_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\quad + \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\nu_1(s) - \nu_2(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\ &\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right]. \end{aligned}$$

Further, for $t \in [0, \pi]$, we have

$$\begin{aligned} &\big|h_j(t)\big| \\ &\quad \leqslant \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big|\nu_j(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} + \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_j(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\qquad + \frac{1}{2}\left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\big|\nu_1(s)\big| + \big|\nu_2(s)\big|\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\ &\qquad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_2(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}} + \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} \big|u_1(s)\big| \frac{\mathrm{d}s}{s^{1-\alpha}}\right] \\ &\quad \leqslant \frac{1}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \mathcal{H}_j^* \phi_j(r) + \frac{\lambda_j}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \|u_j\|_{M_j} \\ &\qquad + \frac{1}{2\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \sum_{i=1}^2 \mathcal{H}_i^* \phi_i(r) + \frac{1}{2\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \sum_{i=1}^2 \lambda_i \|u_i\|_{M_i}. \end{aligned}$$

Hence,

$$\begin{aligned} \|h\|_M \leqslant{}& \frac{2}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma} \pi^{\alpha(\beta+\gamma)} \sum_{j=1}^2 \mathcal{H}_j^* \phi_j(r) \\ &+ \frac{2}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma} \pi^{\alpha\gamma} \sum_{j=1}^2 \lambda_j \|u_j\|_{M_j} \end{aligned}$$

$$\begin{aligned}
&\leqslant P\sum_{j=1}^{2}\mathcal{H}_j^*\phi_j(r)+N\sum_{j=1}^{2}\lambda_j\sum_{j=1}^{2}\|u_j\|_{M_j}\\
&=P\sum_{j=1}^{2}\mathcal{H}_j^*\phi_j(r)+N(\lambda_1+\lambda_2)r.
\end{aligned}$$

Step 3. A bounded sets is mapped by $\mathcal{T}$ to an equicontinuous sets of M.
Let $t_1, t_2 \in [0, \pi]$, $t_1 < t_2$ and $u \in B_r$. Then, for any $h_j \in \mathcal{T}_j(u_j)$, we have

$$\begin{aligned}
&\big|h_j(t_2)-h_j(t_1)\big|\\
&\quad\leqslant \frac{1}{\Gamma(\beta+\gamma)}\int_0^{t_1}\left(\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)^{\beta+\gamma-1}-\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)^{\beta+\gamma-1}\right)\big|\nu_j(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\\
&\qquad+\frac{1}{\Gamma(\beta+\gamma)}\int_{t_1}^{t_2}\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)^{\beta+\gamma-1}\big|\nu_j(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\\
&\qquad+\frac{\lambda_j}{\Gamma(\gamma)}\int_0^{t_1}\left(\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}-\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\right)\big|u_j(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\\
&\qquad+\frac{\lambda_j}{\Gamma(\gamma)}\int_{t_1}^{t_2}\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\big|u_j(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\\
&\qquad+\frac{t_2^{\alpha\gamma}-t_1^{\alpha\gamma}}{2\pi^{\alpha\gamma}}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^{\pi}\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\left(\big|\nu_1(s)\big|+\big|\nu_2(s)\big|\right)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.\\
&\qquad\left.+\frac{\lambda_2}{\Gamma(\gamma)}\int_0^{\pi}\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\big|u_2(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}+\frac{\lambda_1}{\Gamma(\gamma)}\int_0^{\pi}\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\big|u_1(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\right]\\
&\quad\leqslant \frac{1}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\left(t_2^{\alpha(\beta+\gamma)}-t_1^{\alpha(\beta+\gamma)}\right)\mathcal{H}_j^*\phi_j(r)\\
&\qquad+\frac{\lambda_j}{\Gamma(\gamma+1)}\left(\frac{1}{\alpha}\right)^{\gamma}\left(t_2^{\alpha\gamma}-t_1^{\alpha\gamma}\right)\mathcal{H}_j^*\phi_j(r)\\
&\qquad+\frac{t_2^{\alpha\gamma}-t_1^{\alpha\gamma}}{2\pi^{\alpha\gamma}}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^{\pi}\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\left(\big|\nu_1(s)\big|+\big|\nu_2(s)\big|\right)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.\\
&\qquad\left.+\frac{\lambda_2}{\Gamma(\gamma)}\int_0^{\pi}\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\big|u_2(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}+\frac{\lambda_1}{\Gamma(\gamma)}\int_0^{\pi}\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}\big|u_1(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\right].
\end{aligned}$$

Therefore, $|h_j(t_2)-h_j(t_1)| \to 0$ as $t_1 \to t_2$, independently of $u \in B_r$. By the application of the Arzelà–Ascoli theorem, $\mathcal{T} : M \to \mathrm{P}(M)$ is completely continuous.

Next, we prove that $\mathcal{T}$ has a closed graph, which means that $\mathcal{T}$ is an upper semicontinuous multivalued map by Lemma 6.

Step 4. $\mathcal{T}$ has a closed graph.

Let $u_{j_n} \to u_{j_*}$, $h_{j_n} \in \mathcal{T}_j(u_{j_n})$ and $h_{j_n} \to h_{j_*}$. Then we need to show that $h_{j_*} \in \mathcal{T}_j(u_{j_*})$. Since $h_{j_n} \in \mathcal{T}_j(u_{j_n})$, there exist $\omega_{j_n} \in S_{F_j,u_{j_n}}$ such that, for each $t \in [0,\pi]$,

$$\begin{aligned} h_{j_n}(t) &= \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \omega_{j_n}(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\quad + \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\omega_{1_n}(s) - \omega_{2_n}(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\ &\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right]. \end{aligned}$$

Thus, we need to prove that there exist $\omega_{j_*} \in S_{F_j,u_{j_*}}$ such that, for each $t \in [0,\pi]$,

$$\begin{aligned} h_{j_*}(t) &= \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \omega_{j_*}(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\quad + \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\omega_{1_*}(s) - \omega_{2_*}(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\ &\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right]. \end{aligned}$$

Define the linear operator $\Omega : L^1([0,\pi],\mathbb{R}) \to M$ as follows:

$$\begin{aligned} \nu_j \to \Omega(\nu_j)(t) &= \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \nu_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\ &\quad + \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\nu_1(s) - \nu_2(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\ &\quad \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right]. \end{aligned}$$

Observe that $\|h_{j_n} - h_{j_*}\|_{M_j} \to 0$ as $n \to \infty$. Hence, by Lemma 7 we get that $\Omega \circ S_{F_j}$ is a closed graph operator. Further, we get $h_{j_n}(t) \in \Omega(S_{F_j,u_{j_n}})$. Since $u_{j_n} \to u_{j_*}$, we have

$$h_{j_*}(t) = \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \omega_{j_*}(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}}$$
$$+ \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\omega_{1_*}(s) - \omega_{2_*}(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}}\right.$$
$$\left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right]$$

for some $\omega_{j_*} \in S_{F_j,u_{j_*}}$.

Step 5. We show that there exists an open set $\mathscr{D} \subseteq M$ such that, for all $u \in \partial\mathscr{D}$ and any $\epsilon \in (0,1)$, $u \notin \epsilon\mathcal{T}(u)$. Let $u_j \in \epsilon\mathcal{T}_j(u_j)$ and $\epsilon \in (0,1)$. Then there exists $\nu_j \in L^1([0,\pi],\mathbb{R})$, where $\nu_j \in S_{F_j,u_j}$ such that, for $t \in [0,\pi]$,

$$u_j(t) = \epsilon\left\{\frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \nu_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right.$$
$$+ \frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\nu_1(s) - \nu_2(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}}\right.$$
$$\left.\left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}}\right]\right\}.$$

Then, similar to the second step, we get

$$P\sum_{j=1}^2 \mathcal{H}_j^* \phi_j\big(\|u\|_M\big) + N\sum_{j=1}^2 \lambda_j \|u\|_M \geqslant \|u\|_M.$$

By (G3), there exists $\mathscr{L}$ such that $\|u\|_M \neq \mathscr{L}$. Set

$$\mathscr{D} = \big\{u \in M\colon\ \|u\|_M < \mathscr{L}\big\}.$$

Note that the operator $\mathcal{T} : \overline{\mathscr{D}} \to \mathrm{P}(M)$ is a multivalued map that is upper semicontinuous, compact and contains convex closed values. By definition of $\mathscr{D}$, there is no $u \in \partial\mathscr{D}$ such that $u \in \epsilon\mathcal{T}(u)$ for some $\epsilon \in (0,1)$. Therefore, through Lemma 8, we can get that $\mathcal{T}$ has a fixed point $u \in \overline{\mathscr{D}}$, which satisfies multivalued problem (4). □

4.2 The Lipschitz case

Our next existence result is that F_j are nonconvex, we will use Covitz–Nadler fixed point theorem to solve this problem.

Lemma 9. (*See* [7].) *Let* (B, d) *be a complete metric space. If* $A : B \to \mathrm{P_{cl}}(B)$ *is a contraction, then* $\mathrm{Fix}\, A \neq \emptyset$.

Theorem 4. *Suppose that*:

(S1) $F_j : [0, \pi] \times \mathbb{R} \to \mathrm{P_{cp}}(\mathbb{R})$ *such that* $F_j(\cdot, u_j) : [0, \pi] \to \mathrm{P_{cp}}(\mathbb{R})$ *are measurable for each* $u_j \in \mathbb{R}$;

(S2) *there exist positive functions* $m_j \in L^1[0, \pi]$ *such that*

$$H_d\big(F_j(t, u_j), F_j(t, \hat{u}_j)\big) \leqslant m_j(t)|u_j - \hat{u}_j|$$

for almost all $t \in [0, \pi]$ *with* $d(0, F_j(t, 0)) \leqslant m_j(t)$, $u_j, \hat{u}_j \in \mathbb{R}$.

Then, for all $j = 1, 2$, *problem* (4) *has at least one solution on the circular graph if*

$$\theta^* := P(m_1^* + m_2^*) < 1,$$

where P *is given by* (9), *and* $m_j^* = \|m_j\|_{L^1}$.

Proof. For each $u_j \in M_j$, by assumption (S1), the sets S_{F_j, u_j} are nonempty. According to [7, Thm. III.6], F_j have a measurable selection. Now we will show that $\mathcal{T}(u) \in \mathrm{P_{cl}}(M)$ for each $u \in M$. Let $\{z_{j_n}\}_{n \geqslant 0} \in \mathcal{T}_j(u_j)$ such that $z_{j_n} \to z_j (n \to \infty)$ in M_j. Then we have $z_j \in M_j$, and there exist $\vartheta_{j_n} \in S_{F_j, u_{j_n}}$ such that, for each $t \in [0, \pi]$,

$$\begin{aligned} u_{j_n}(t) = {} & \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \vartheta_{j_n}(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \\ & + \frac{(-1)^j}{2} \left(\frac{t}{\pi}\right)^{\alpha\gamma} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \big(\vartheta_{1_n}(s) - \vartheta_{2_n}(s)\big) \frac{\mathrm{d}s}{s^{1-\alpha}} \right. \\ & \left. + \frac{\lambda_2}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_2(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_1}{\Gamma(\gamma)} \int_0^\pi \left(\frac{\pi^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_1(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \right]. \end{aligned}$$

ϑ_{j_n} converge to ϑ_j in $L^1([0, \pi], \mathbb{R})$ because F_j have compact values. Hence, $\vartheta_j \in S_{F_j, u_j}$, and for each $t \in [0, \pi]$, we have

$$\begin{aligned} & z_{j_n}(t) \to z_j(t) \\ & \quad = \frac{1}{\Gamma(\gamma+\beta)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma+\beta-1} \vartheta_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} - \frac{\lambda_j}{\Gamma(\gamma)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{\gamma-1} u_j(s) \frac{\mathrm{d}s}{s^{1-\alpha}} \end{aligned}$$

$$+\frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\left(\vartheta_1(s)-\vartheta_2(s)\right)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.$$
$$\left.+\frac{\lambda_2}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_2(s)\frac{\mathrm{d}s}{s^{1-\alpha}}-\frac{\lambda_1}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_1(s)\frac{\mathrm{d}s}{s^{1-\alpha}}\right].$$

Consequently, $z_j \in \mathcal{T}_j(u_j)$.

In the next step, we prove that, for each $u, \hat{u} \in M$,

$$H_d\big(\mathcal{T}(u), \mathcal{T}(\hat{u})\big) \leqslant \theta^*\|u-\hat{u}\|_M, \quad \theta^* < 1.$$

Let $u, \hat{u} \in M$ and $h_{j_1} \in \mathcal{T}_j(u_j)$. Then there exist $\vartheta_{j_1}(t) \in F_j(t, u_j(t))$ such that, for each $t \in [0,\pi]$,

$$h_{j_1}(t)=\frac{1}{\Gamma(\gamma+\beta)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\vartheta_{j_1}(s)\frac{\mathrm{d}s}{s^{1-\alpha}}-\frac{\lambda_j}{\Gamma(\gamma)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_j(s)\frac{\mathrm{d}s}{s^{1-\alpha}}$$
$$+\frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\left(\vartheta_{1_1}(s)-\vartheta_{2_1}(s)\right)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.$$
$$\left.+\frac{\lambda_2}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_2(s)\frac{\mathrm{d}s}{s^{1-\alpha}}-\frac{\lambda_1}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_1(s)\frac{\mathrm{d}s}{s^{1-\alpha}}\right].$$

Using (S2), we obtain

$$H_d\big(F_j(t,u_j), F_j(t,\hat{u}_j)\big) \leqslant m_j(t)\big|u_j(t)-\hat{u}_j(t)\big|.$$

Hence, there exist $w_j \in F_j(t, \hat{u}_j(t))$ such that

$$\big|\vartheta_{j_1}(t)-w_j(t)\big| \leqslant m_j(t)\big|u_j(t)-\hat{u}_j(t)\big|, \quad t\in[0,\pi].$$

Define $\mathcal{Q}: [0,\pi] \to \mathrm{P}(\mathbb{R})$ by

$$\mathcal{Q}(t)=\big\{w_j\in\mathbb{R}\colon \big|\vartheta_{j_1}(t)-w_j(t)\big| \leqslant m_j(t)\big|u_j(t)-\hat{u}_j(t)\big|\big\}.$$

The multivalued operators $\mathcal{Q}(t)\cap F_j(t,\hat{u}_j(t))$ are measurable according to [5, Prop. III.4]. Thus, there exist functions $\vartheta_{j_2}(t)$, which are measurable selections for $\mathcal{Q}$. So $\vartheta_{j_2}(t) \in F_j(t,\hat{u}_j(t))$, and for each $t\in[0,\pi]$, $|\vartheta_{j_1}(t)-\vartheta_{j_2}(t)| \leqslant m_j(t)|u_j(t)-\hat{u}_j(t)|$.

Let us define

$$h_{j_2}(t)=\frac{1}{\Gamma(\gamma+\beta)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\vartheta_{j_2}(s)\frac{\mathrm{d}s}{s^{1-\alpha}}-\frac{\lambda_j}{\Gamma(\gamma)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_j(s)\frac{\mathrm{d}s}{s^{1-\alpha}}$$
$$+\frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\left[\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\big(\vartheta_{1_2}(s)-\vartheta_{2_2}(s)\big)\frac{\mathrm{d}s}{s^{1-\alpha}}\right.$$
$$\left.+\frac{\lambda_2}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_2(s)\frac{\mathrm{d}s}{s^{1-\alpha}}-\frac{\lambda_1}{\Gamma(\gamma)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma-1}u_1(s)\frac{\mathrm{d}s}{s^{1-\alpha}}\right].$$

Thus,

$$\begin{aligned}&\big|h_{j_1}(t)-h_{j_2}(t)\big|\\&\quad=\frac{1}{\Gamma(\gamma+\beta)}\int_0^t\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\big|\vartheta_{j_1}(s)-\vartheta_{j_2}(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\\&\qquad+\frac{(-1)^j}{2}\left(\frac{t}{\pi}\right)^{\alpha\gamma}\frac{1}{\Gamma(\gamma+\beta)}\int_0^\pi\left(\frac{\pi^\alpha-s^\alpha}{\alpha}\right)^{\gamma+\beta-1}\sum_{i=1}^2\big|\vartheta_{i_1}(s)-\vartheta_{i_2}(s)\big|\frac{\mathrm{d}s}{s^{1-\alpha}}\\&\quad\leqslant\frac{1}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}m_j^*\|u_j-\hat{u}_j\|_{M_j}\\&\qquad+\frac{1}{2\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\sum_{i=1}^2m_i^*\|u_i-\hat{u}_i\|_{M_i}.\end{aligned}$$

Hence,

$$\begin{aligned}\|h_1-h_2\|_M&=\sum_{j=1}^2\|h_{j_1}-h_{j_2}\|_{M_j}\\&\leqslant\frac{2}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\alpha}\right)^{\beta+\gamma}\pi^{\alpha(\beta+\gamma)}\sum_{j=1}^2m_j^*\|u_j-\hat{u}_j\|_{M_j}\\&\leqslant P(m_1^*+m_2^*)\|u-\hat{u}\|_M.\end{aligned}$$

Swapping the roles of u and $\hat{u}$, we get

$$H_d\big(\mathcal{T}(u),\mathcal{T}(\hat{u})\big)\leqslant P(m_1^*+m_2^*)\|u-\hat{u}\|_M.$$

Therefore, $\mathcal{T}$ is a contraction. $\mathcal{T}$ has a fixed point u by Lemma 9, which satisfies problem (4). □

5 Some examples

Several examples are provided in this section to verify the theoretical results.

Example 1. Consider the following fractional conformable Langevin differential system:

$$\begin{aligned} {}_0^C{}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{3}}\left({}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}} + \frac{1}{20}\right)u_1(t) &= g_1\bigl(t, u_1(t)\bigr), \\ {}_0^C{}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{3}}\left({}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}} + \frac{1}{30}\right)u_2(t) &= g_2\bigl(t, u_2(t)\bigr) \end{aligned} \tag{10}$$

with boundary value conditions

$$\begin{aligned} &{}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}}u_1(0) = {}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}}u_2(0) = 0, \\ &u_1(\pi) = u_2(\pi), \qquad u_1(0) + u_2(0) = 0, \end{aligned} \tag{11}$$

where $\alpha = 1/3$, $\beta = 1/4$, $\gamma = 1/5$, $\lambda_1 = 1/20$, $\lambda_2 = 1/30$, and we can compute that $P \approx 4.3956$, $N \approx 2.9286$. Define continuous functions g_1, g_2: $[0, \pi] \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} g_1\bigl(t, u_1(t)\bigr) &= \frac{t}{10} + \frac{\sin u_1(t)}{2(t+3)^3}, \\ g_2\bigl(t, u_2(t)\bigr) &= \sin t + \frac{u_2(t)}{3(t+2)^2}. \end{aligned} \tag{12}$$

Let $u_1, u_2 \in \mathbb{R}$, it is clear that

$$\begin{aligned} \bigl|g_1\bigl(t, u_1(t)\bigr)\bigr| &\leqslant \frac{t}{10} + \frac{1}{2(t+3)^3}\bigl|u_1(t)\bigr|, \\ \bigl|g_2\bigl(t, u_2(t)\bigr)\bigr| &\leqslant \sin t + \frac{1}{3(t+2)^2}\bigl|u_2(t)\bigr| \end{aligned}$$

in which $\mathfrak{q}_1(t) = 1/(2(t+3)^3)$, $\mathfrak{q}_2(t) = 1/(3(t+2)^2)$. Hence, $\mathfrak{q}_1^* = 1/54$, $\mathfrak{q}_2^* = 1/12$. Further, we get

$$\delta^* = P(\mathfrak{q}_1^* + \mathfrak{q}_2^*) + N(\lambda_1 + \lambda_2) \approx 0.6918 < 1.$$

Using Theorem 1, we can conclude that the fractional conformable Langevin differential system (10)–(12) has a solution.

Example 2. Consider the following fractional conformable Langevin differential system:

$$\begin{aligned} {}_0^C{}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{3}}\left({}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}} + \frac{1}{40}\right)u_1(t) &= g_1\bigl(t, u_1(t)\bigr), \\ {}_0^C{}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{3}}\left({}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}} + \frac{1}{30}\right)u_2(t) &= g_2\bigl(t, u_2(t)\bigr) \end{aligned} \tag{13}$$

with boundary value conditions

$$\begin{aligned} &{}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}}u_1(0) = {}_0^C{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}}u_2(0) = 0, \\ &u_1(\pi) = u_2(\pi), \qquad u_1(0) + u_2(0) = 0, \end{aligned} \tag{14}$$

where $\alpha = 1/3$, $\beta = 1/4$, $\gamma = 1/5$, $\lambda_1 = 1/40$, $\lambda_2 = 1/30$, and we can compute that $P \approx 4.3956$, $N \approx 2.9286$. Define continuous functions g_1, g_2: $[0, \pi] \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} g_1\big(t, u_1(t)\big) &= \frac{1}{\sqrt{t}} + \frac{|u_1(t)|}{50 + |u_1(t)|}\mathrm{e}^{-|t+1|} + \frac{1}{3}, \\ g_2\big(t, u_2(t)\big) &= \frac{1}{5}t^3 + \frac{3|u_2(t)|}{100 + |u_2(t)|}\mathrm{e}^{-|3t-1|}. \end{aligned} \tag{15}$$

For $u_j, y_j \in \mathbb{R} (j = 1, 2)$, we estimate

$$\begin{aligned} \big|g_1\big(t, u_1(t)\big) - g_1\big(t, y_1(t)\big)\big| &\leqslant \left|\left(\frac{|u_1(t)|}{50 + |u_1(t)|} - \frac{|y_1(t)|}{50 + |y_1(t)|}\right)\mathrm{e}^{-|t+1|}\right| \\ &\leqslant \frac{1}{50}\big|u_1(t) - y_1(t)\big|, \\ \big|g_2\big(t, u_2(t)\big) - g_2\big(t, y_2(t)\big)\big| &\leqslant \left|\left(\frac{3|u_2(t)|}{100 + |u_2(t)|} - \frac{3|y_2(t)|}{100 + |y_2(t)|}\right)\mathrm{e}^{-|3t-1|}\right| \\ &\leqslant \frac{3}{100}\big|u_2(t) - y_2(t)\big|. \end{aligned}$$

Hence, $\rho_1 = 1/50$, $\rho_2 = 3/100$. Therefore, we see that

$$\sigma^* = P(\rho_1 + \rho_2) + N(\lambda_1 + \lambda_2) \approx 0.2856 < 1.$$

By utilizing Theorem 2, we deduce that the fractional conformable Langevin differential system (13)–(15) has a unique solution.

Example 3. Assume that (10) is replaced by

$$\begin{aligned} {}_{0}^{C}{}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{3}}\left({}_{0}^{C}{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}} + \frac{1}{20}\right)u_1(t) \in F_1\big(t, u_1(t)\big), \\ {}_{0}^{C}{}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{3}}\left({}_{0}^{C}{}^{\frac{1}{5}}\mathcal{D}^{\frac{1}{3}} + \frac{1}{30}\right)u_2(t) \in F_2\big(t, u_2(t)\big), \end{aligned} \tag{16}$$

where

$$\begin{aligned} F_1\big(t, u_1(t)\big) &= \left[0,\ \frac{\sin u_1(t)}{5(1 + 2t)} + \frac{3}{4}\right], \\ F_2\big(t, u_2(t)\big) &= \left[0,\ \frac{1}{3t + 1}\left(\frac{\sin u_2(t)}{9(\sin u_2(t) + 1)} + \frac{1}{8}\right)\right]. \end{aligned}$$

Observe that $F_j(t, u_j(t))$ $(j = 1, 2)$ are measurable sets. On the other hand,

$$\begin{aligned} H_d\big(F_1\big(t, u_1(t)\big), F_1\big(t, \hat{u}_1(t)\big)\big) &\leqslant \frac{1}{5(1 + 2t)}\big|u_1(t) - \hat{u}_1(t)\big|, \\ H_d\big(F_2\big(t, u_2(t)\big), F_2\big(t, \hat{u}_2(t)\big)\big) &\leqslant \frac{1}{9(3t + 1)}\big|u_2(t) - \hat{u}_2(t)\big|, \end{aligned}$$

where $m_j(t)$ ($j = 1, 2$) are defined by $m_1(t) = 1/(5(1 + 2t))$, $m_2(t) = 1/(9(3t + 1))$. Therefore, $m_1^* = 1/5$, $m_2^* = 1/9$. Since

$$\theta^* = P(m_1^* + m_2^*) \approx 0.6837 < 1,$$

according to Theorem 4, we obtain that under the boundary conditions (11), the Langevin fractional differential inclusion (16) has at least one solution on $[0, \pi]$.

6 Conclusion

In this paper, we mainly study a class of coupled fractional conformable Langevin differential and inclusion system on the circular graph. We obtain the existence and uniqueness of the solution for single-valued problems and the existence of the solution for multivalued problems by using fixed point theorems. So far, although some scholars have studied the fractional differential equation on the graph, the coupled fractional conformable Langevin differential and inclusion system on the circular graph is studied for the first time. There are still some topics worthy of our further research. For example, one can study the stability of fractional differential and inclusion systems on different graphs and their specific applications in chemical theory. In the future, we will consider numerical calculation and simulation of fractional differential and inclusion systems on graphs.

References

1. B. Ahmad, J.J. Nieto, A. Alsaedi, M. El-Shahed, A study of nonlinear Langevin equation involving two fractional orders in different intervals, *Nonlinear Anal., Real World Appl.*, **13**(2): 599–606, 2012, https://doi.org/10.1016/j.nonrwa.2011.07.052.
2. G. Alotta, M. Di Paola, F.P. Pinnola, M. Zingales, A fractional nonlocal approach to nonlinear blood flow in small-lumen arterial vessels, *Meccanica*, **55**:891–906, 2020, https://doi.org/10.1007/s11012-020-01144-y.
3. Z. Bai, S. Sun, Z. Du, Y. Chen, The Green function for a class of Caputo fractional differential equations with a convection term, *Fract. Calc. Appl. Anal.*, **23**(3):787–798, 2020, https://doi.org/10.1515/fca-2020-0039.
4. D. Baleanu, R. Darzi, B. Agheli, Existence results for Langevin equation involving Atangana-Baleanu fractional operators, *Mathematics*, **8**(3):408, 2020, https://doi.org/10.3390/math8030408.
5. C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lect. Notes Math., Vol. 580, Springer, Berlin, 1977, https://doi.org/10.1007/BFb0087685.
6. Y.-K. Chang, Y. Wei, Pseudo S-asymptotically Bloch type periodic solutions to fractional integro-differential equations with Stepanov-like force terms, *Z. Angew. Math. Phys.*, **73**(2):77, 2022, https://doi.org/10.1007/s00033-022-01722-y.
7. H. Covitz, S.B. Nadler, Multi-valued contraction mappings in generalized metric spaces, *Isr. J. Math.*, **8**:5–11, 1970, https://doi.org/10.1007/bf02771543.
8. K. Deimling, *Nonlinear Functional Analysis*, Dover Publications, Mineola, NY, 2010.

9. Y. Ding, Z. Wang, H. Ye, Optimal control of a fractional-order HIV-immune system with memory, *IEEE Trans. Control Syst. Technol.*, **20**(3):763–769, 2011, https://doi.org/10.1109/TCST.2011.2153203.

10. A.A. Elsadany, A.E. Matouk, Dynamical behaviors of fractional-order Lotka–Volterra predator–prey model and its discretization, *J. Appl. Math. Comput.*, **49**:269–283, 2015, https://doi.org/10.1007/s12190-014-0838-6.

11. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer Monogr. Math., Springer, New York, 2003, https://doi.org/10.1007/978-0-387-21593-8.

12. A. Graves, Long short-term memory, in *Supervised Sequence Labelling with Recurrent Neural Networks*, Stud. Comput. Intell., Vol. 385, Springer, Berlin, Heidelberg, 2012, pp. 37–45, https://doi.org/10.1007/978-3-642-24797-2_4.

13. L. Guo, L. Liu, Maximal and minimal iterative positive solutions for singular infinite-point p-Laplacian fractional differential equations, *Nonlinear Anal. Model. Control*, **23**(6):851–865, 2018, https://doi.org/10.15388/na.2018.6.3.

14. L. Guo, L. Liu, Y. Wu, Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters, *Nonlinear Anal. Model. Control*, **23**(2): 182–203, 2018, https://doi.org/10.15388/na.2018.2.3.

15. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000, https://doi.org/10.1142/3779.

16. F. Jarad, E. Uğurlu, T. Abdeljawad, D. Baleanu, On a new class of fractional operators, *Adv. Difference Equ.*, **2017**:1–16, 2017, https://doi.org/10.1186/s13662-017-1306-z.

17. E. Karatzas, M. Gkonta, J. Hotova, F.A. Baltoumas, P.I. Kontou, C.J. Bobotsis, P.G. Bagos, G.A. Pavlopoulos, VICTOR: A visual analytics web application for comparing cluster sets, *Comput. Biol. Med.*, **135**:104557, 2021, https://doi.org/10.1016/j.compbiomed.2021.104557.

18. K. Liu, Y. Chen, P.D. Domański, X. Zhang, A novel method for control performance assessment with fractional order signal processing and its application to semiconductor manufacturing, *Algorithms*, **11**(7):90, 2018, https://doi.org/10.3390/a11070090.

19. V. Mehandiratta, M. Mehra, G. Leugering, Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph, *J. Math. Anal. Appl.*, **477**(2):1243–1264, 2019, https://doi.org/10.1016/j.jmaa.2019.05.011.

20. M. Mozafarifard, D. Toghraie, Time-fractional subdiffusion model for thin metal films under femtosecond laser pulses based on Caputo fractional derivative to examine anomalous diffusion process, *Int. J. Heat Mass Transfer*, **153**:119592, 2020, https://doi.org/10.1016/j.ijheatmasstransfer.2020.119592.

21. S.K. Ntouyas, B. Ahmad, J. Tariboon, (k, ψ)-Hilfer nonlocal integro-multi-point boundary value problems for fractional differential equations and inclusions, *Mathematics*, **10**(15):2615, 2022, https://doi.org/10.3390/math10152615.

22. R. Satija, D.E. Makarov, Generalized Langevin equation as a model for barrier crossing dynamics in biomolecular folding, *J. Phys. Chem. B*, **123**(4):802–810, 2019, https://doi.org/10.1021/acs.jpcb.8b11137.

23. D.R. Smart, Fixed point theorems, in *Methods in Nonlinear Analysis*, Springer Monogr. Math., Springer, Berlin, Heidelberg, 2005, pp. 71–126, https://doi.org/10.1007/3-540-29232-2_2.
24. G. Wang, J. Qin, L. Zhang, D. Baleanu, Explicit iteration to a nonlinear fractional Langevin equation with non-separated integro-differential strip-multi-point boundary conditions, *Chaos Solitons Fractals*, **131**:109476, 2020, https://doi.org/10.1016/j.chaos.2019.109476.
25. G. Wang, L. Zhang, G. Song, Boundary value problem of a nonlinear Langevin equation with two different fractional orders and impulses, *Fixed Point Theory Appl.*, **2012**:1–17, 2012, https://doi.org/10.1186/1687-1812-2012-200.
26. S. Yang, M. Deng, R. Ren, Stochastic resonance of fractional-order Langevin equation driven by periodic modulated noise with mass fluctuation, *Adv. Difference Equ.*, **2020**:1–12, 2020, https://doi.org/10.1186/s13662-020-2492-7.
27. C. Zhai, L. Xu, Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter, *Commun. Nonlinear Sci. Numer. Simul.*, **19**(8):2820–2827, 2014, https://doi.org/10.1016/j.cnsns.2014.01.003.
28. X. Zhang, L. Liu, Y. Wu, Multiple positive solutions of a singular fractional differential equation with negatively perturbed term, *Math. Comput. Modelling*, **55**(3–4):1263–1274, 2012, https://doi.org/10.1016/j.mcm.2011.10.006.
29. F. Zouari, A. Ibeas, A. Boulkroune, J. Cao, M.M. Arefi, Adaptive neural output-feedback control for nonstrict-feedback time-delay fractional-order systems with output constraints and actuator nonlinearities, *Neural Netw.*, **105**:256–276, 2018, https://doi.org/10.1016/j.neunet.2018.05.014.