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Impact of multiple time delays on bifurcation of a class of fractional nearest-neighbor coupled neural networks*

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Abstract. In this paper, the impacts of multiple time delays on bifurcation of a class of fractional nearest-neighbor coupled neural networks are considered. Firstly, the sum of time delays is selected as a parameter, and the fractional nearest-neighbor coupled neural network model is linearized to obtain the corresponding characteristic equation. Then, utilizing stability and bifurcation theory of fractional-order delay differential equations, we investigate the effect of time delays on the system's stability and bifurcations. The results show that when the time lag exceeds the critical value, the system will lose stability and generate Hopf bifurcation. Finally, the correctness of the conclusions in this paper is verified through numerical simulation.

Keywords: nearest-neighbor coupled neural networks, fractional order, Hopf bifurcation, stability, delay.

1 Introduction

Coupling refers to the level of correlation between two or more subsystems (or units) within a system. Specifically, it measures the extent to which a change in one subsystem affects another subsystem or multiple subsystems, and how such changes impact the

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overall characteristics of the system. The degree of coupling is a critical determinant of stability and flexibility of a system and can be categorized as either tight or loose. Tight coupling denotes a close relationship among subsystems with their functions being interdependent. Conversely, loose coupling describes a weak internal linkage among subsystems with their functions being largely unrelated.

Coupled systems have been extensively utilized in the design and operation of complex systems. For instance, Shen et al. [19] proposed a method for loosely coupled system integration, which can be applied in facility management and decision support to advance multidisciplinary design optimization and intelligent development. Chen et al. [4] developed a dual-coupling system for electric vehicles charging on the road, which allows for individual control of charging sections by coupling circuits, reducing power loss and addressing the current issues of short mileage, high cost, and low charging rate. In [21], a new dynamic mathematical model is designed for studying coupled hydro-generator shaft-foundation systems, based on nonlinear differential equations and the Runge–Kutta algorithm of the nonlinear response of the stimulating subsystem. This coupled model outperforms the classical model in terms of sensitivity and performance. For additional examples of applications of coupled systems, please refer to papers [3, 25].

Coupled systems are also widely used in modern science and technology, including computer network systems [23], robot systems [13], sensor systems [14], virtual reality systems [2], and artificial neural network systems [12], among others. Neurons in the human brain, for example, form a complex and highly interconnected large-scale information network through coupling behavior, sending electrical signals to collect, process, and deliver information [20]. Artificial neural networks are computational or mathematical models that imitate the structure and function of biological neural networks, originating from the working principle of human brain neurons. Research on neural networks can be traced back to the 1940s when scholars first proposed a mathematical model based on the interaction between neurons [6]. The perceptron, a typical structure of neural networks, was subsequently developed by Rosenblatt [7], and further research on multilayer perceptrons and the backpropagation algorithm for training followed. Due to the limitations of computer technology and algorithms at that time, research on neural networks stagnated for a time. However, the rapid development of science and technology and the rise of deep learning since 2000 have stimulated renewed interest in neural networks, resulting in the invention of new models such as convolutional neural network (CNN) [1], Hopfield network [10], bidirectional associative memory network (BAM) [22], nearest-neighbor coupling neural network (NNCN) [8], and others. These models have made significant breakthroughs in image processing, speech recognition, and data prediction [18, 33].

With the advancement of computer technology and information technology, the application of artificial neural networks in various fields has gained significant attention. Therefore, it is of utmost importance to establish a neural network model that is suitable for practical applications, based on the principles of neuron function. In practical applications, time delay reflects the time of information transmission and processing in the system, as well as the system's inertia and feedback mechanism. Time delay often affects the performance and stability of the system, causing dynamic behaviors such as periodic solutions, bifurcations, and chaos [27]. For example, in 2015, Mao and Wang studied the dynamic behavior of a four-neuron coupled neural network model with multiple delays, analyzed the local stability, and provided the sufficient conditions for the occurrence of Hopf bifurcations [15]. Xu discussed the stability at the zero equilibrium point of a class of multidelay simplified BAM neural networks in [29] and established the global existence conditions of periodic orbits using relevant theories. In [11], Zhang and Wei studied the dynamic behavior of fractional-order financial systems with time-delay feedback and analyzed the impact of time-delay feedback on system bifurcations. However, most neural network models focus only on the connection and time delay between adjacent nodes, and not on the delay in transmission between nonadjacent nodes. The nearest-neighbor coupled neural network model connects each node to its neighbors or nonadjacent nodes, which leads to an increase in the number of time delays during information transmission, thereby more accurately reflecting the overall characteristics of the neural network, and aligning with practical applications.

In 2020, Wang and Xiao et al. [24] proposed a class of nearest-neighbor coupled neural networks with six neurons and twelve time delays as follows:

$$
\begin{aligned}
\dot{\mu}_1(t) &= -\varphi_1\mu_1(t) + \eta_1 h_1\big(\mu_2(t-\xi_2)\big) + \eta_2 l_2\big(\mu_5(t-\xi_7)\big),\\
\dot{\mu}_2(t) &= -\varphi_2\mu_2(t) + \eta_3 h_3\big(\mu_3(t-\xi_3)\big) + \eta_4 l_4\big(\mu_6(t-\xi_{10})\big),\\
\dot{\mu}_3(t) &= -\varphi_3\mu_3(t) + \eta_5 h_5\big(\mu_4(t-\xi_4)\big) + \eta_6 l_6\big(\mu_1(t-\xi_9)\big),\\
\dot{\mu}_4(t) &= -\varphi_4\mu_4(t) + \eta_7 h_7\big(\mu_5(t-\xi_5)\big) + \eta_8 l_8\big(\mu_2(t-\xi_{12})\big),\\
\dot{\mu}_5(t) &= -\varphi_5\mu_5(t) + \eta_9 h_9\big(\mu_6(t-\xi_6)\big) + \eta_{10} l_{10}\big(\mu_3(t-\xi_8)\big),\\
\dot{\mu}_6(t) &= -\varphi_6\mu_6(t) + \eta_{11} h_{11}(\mu_1(t-\xi_1)\big) + \eta_{12} l_{12}\big(\mu_4(t-\xi_{11})\big),
\end{aligned}
\tag{1}
$$

where μ_p ($p = 1, 2, \dots, 6$) is the state of neuron p at time t, $\varphi_p > 0$ is the self-feedback coefficient, $\eta_i > 0$ ($i = 1, 2, \dots, 12$) is the connection weight between neurons, h_{2p-1}, h_{2p} are the activation functions, and ξ_i is the delay between different neurons.

In recent years, fractional calculus has emerged as an interdisciplinary subject and become a hot research topic in various branches of mathematics. Compared with traditional integer-order calculus, fractional-order calculus exhibits long memory and nonlocality characteristics, which provide more effective tools for studying dynamic changes in systems. As a result, it has been widely applied in various fields, including image encryption [31], fluid mechanics [9], medical research [32], and neural networks [26], among others. The fractional-order time-delay mathematical model is a dynamical system that combines fractional-order calculus and time-delay differential equation theory, enabling more realistic and accurate analyses of dynamic behavior in practical applications. Consequently, the study of fractional-order time-delay differential systems has attracted the attention of many scholars with many excellent results emerging. For example, Xu investigated the dynamic behavior of fractional-order BAM neural networks with time-delays, deriving sufficient conditions for system-generated Hopf bifurcations, which verified the importance of time delays on the dynamic behavior of fractional-order systems [28, 30].

Based on the aforementioned analysis, this paper aims to improve and further promote model (1) by studying a class of fractional-order nearest-neighbor coupled neural network models with multiple delays as follows:

$$
\begin{aligned}
D^q\mu_1(t) &= -\varphi_1\mu_1(t) + \eta_1 h_1\big(\mu_2(t-\xi_2)\big) + \eta_2 l_2\big(\mu_5(t-\xi_7)\big),\\
D^q\mu_2(t) &= -\varphi_2\mu_2(t) + \eta_3 h_3\big(\mu_3(t-\xi_3)\big) + \eta_4 l_4\big(\mu_6(t-\xi_{10})\big),\\
D^q\mu_3(t) &= -\varphi_3\mu_3(t) + \eta_5 h_5\big(\mu_4(t-\xi_4)\big) + \eta_6 l_6\big(\mu_1(t-\xi_9)\big),\\
D^q\mu_4(t) &= -\varphi_4\mu_4(t) + \eta_7 h_7\big(\mu_5(t-\xi_5)\big) + \eta_8 l_8\big(\mu_2(t-\xi_{12})\big),\\
D^q\mu_5(t) &= -\varphi_5\mu_5(t) + \eta_9 h_9\big(\mu_6(t-\xi_6)\big) + \eta_{10} l_{10}\big(\mu_3(t-\xi_8)\big),\\
D^q\mu_6(t) &= -\varphi_6\mu_6(t) + \eta_{11} h_{11}\big(\mu_1(t-\xi_1)\big) + \eta_{12} l_{12}\big(\mu_4(t-\xi_{11})\big),
\end{aligned}
\tag{2}
$$

where $q \in (0,1]$, D^q represents the Caputo-type derivative and is the original model for $q = 1$. μ_p ($p = 1,2,\dots,6$) is the state of neuron p at time t, $\varphi_p > 0$ is the self-feedback coefficient, $\eta_i > 0$ ($i = 1,2,\dots,12$) is the connection weight between neurons, h_{2p-1}, l_{2p} are the activation functions, and ξ_i is the delay between different neurons. For more parameter information, please refer to paper [24].

For the further development of later research, the following assumptions are necessary.

(H1) $h_{2p-1}(\cdot) \in (R,R)$, $h_{2p-1}(0) = 0$, $h'_{2p-1}(0) \neq 0$; $l_{2p}(\cdot) \in (R,R)$, $l_{2p}(0) = 0$, $l'_{2p}(0) \neq 0$ ($p = 1,2,\dots,6$).

In this paper, we mainly consider the impacts of multiple time delays on bifurcation of a class of fractional nearest-neighbor coupled neural networks. We demonstrate that the equilibrium point of system (2) loses its stability, and Hopf bifurcation emerges when the sum of the delay passes through a critical value.

It is worth mentioning that so far, some research has mainly focused on integer-order systems, while research on the Hopf bifurcation problem of fractional-order systems, especially, fractional-order coupled systems, is rare. The key technique to study the stability and bifurcation of fractional-order coupled systems is to linearize the system by Laplace transform and consider the influence of delays in this article.

The analysis for the rest of this paper will be carried out in the following order. In Section 2, the basics of fractional differential systems are introduced. In Section 3, the sufficient conditions for the stability and bifurcation of the multidelay fractional neighbor coupled neural network model at the equilibrium point are analyzed. In Section 4, the correctness of the obtained conclusions is verified through numerical simulation examples. Section 5 finishes this paper with a conclusion.

2 Preliminaries

In this section, the basic definitions and lemmas of fractional differential systems will be given. There are three classic definition methods for fractional derivatives, which are Riemann–Liouville fractional derivatives, Caputo fractional derivatives, and Grünwald–Letnikov fractional derivatives. Because the Caputo fractional derivative can adjust the value of the order at any time according to the needs, it is more flexible in calculation and application. So this article will use the Caputo fractional derivative.

Definition 1. (See [17].) The Caputo fractional derivative is defined as

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int\limits_a^t \frac{f^{(n)}\,\mathrm{d}s}{(t-s)^{q+1-n}}, $$

where $n-1 < q < n \in \mathbb{Z}^+$, $\Gamma(s) = \int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$.

Caputo fractional derivatives can be obtained by Laplace transform as follows:

$$ L\bigl\{D^q f(t); s\bigr\} = s^p F(s) - \sum_{l=0}^{n-1} s^{q-u-1} f^{(l)}(0) \quad \bigl(n-1 < q < n \in \mathbb{Z}^+\bigr), $$

where $f^{(u)}(0) = 0$ $(u = 1, 2, \dots, n)$, $L\{D^q f(t); s\} = s^q F(s)$.

Lemma 1. (*See* [16].) *Consider the fractional-order system*

$$ \begin{aligned} D^q y(t) &= g\bigl(t, y(t)\bigr), \\ y(0) &= y_0, \end{aligned} \tag{3} $$

where $q \in (0,1]$ *and* $g(t, y(t)) : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$. *The equilibrium point of system* (3) *is locally asymptotically stable if all eigenvalues* λ *of the Jacobian matrix* $\partial g(t,y)/\partial y$ *evaluated near the equilibrium point satisfy* $|\arg(\lambda)| > q\pi/2$.

Lemma 2. (*See* [5].) *Consider a fractional linear system with several state variables*

$$ \begin{aligned} D^{q_1}\rho_1(t) &= \chi_{11}\rho_1(t-\tau_{11}) + \chi_{12}\rho_2(t-\tau_{12}) + \cdots + \chi_{1n}\rho_n(t-\tau_{1n}), \\ D^{q_2}\rho_2(t) &= \chi_{21}\rho_1(t-\tau_{21}) + \chi_{22}\rho_2(t-\tau_{22}) + \cdots + \chi_{2n}\rho_n(t-\tau_{2n}), \\ &\dots, \\ D^{q_n}\rho_n(t) &= \chi_{n1}\rho_1(t-\tau_{n1}) + \chi_{n2}\rho_2(t-\tau_{n2}) + \cdots + \chi_{nn}\rho_n(t-\tau_{nn}), \end{aligned} \tag{4} $$

where $q_i \in (0,1]$ $(i = 1, 2, \dots, n)$. *Let*

$$ \Delta(\lambda) = \begin{bmatrix} \lambda^{q_1} - \chi_{11}\mathrm{e}^{-\lambda\tau_{11}} & -\chi_{12}\mathrm{e}^{-\lambda\tau_{12}} & \cdots & -\chi_{1n}\mathrm{e}^{-\lambda\tau_{1n}} \\ -\chi_{21}\mathrm{e}^{-\lambda\tau_{21}} & \lambda^{q_2} - \chi_{22}\mathrm{e}^{-\lambda\tau_{22}} & \cdots & -\chi_{2n}\mathrm{e}^{-\lambda\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\chi_{n1}\mathrm{e}^{-\lambda\tau_{n1}} & -\chi_{n2}\mathrm{e}^{-\lambda\tau_{n2}} & \cdots & \lambda^{q_n} - \chi_{nn}\mathrm{e}^{-\lambda\tau_{nn}} \end{bmatrix}. $$

Then, the zero solution of system (4) *is globally asymptotically stable if the real parts of all roots of* $\det(\Delta(\lambda)) = 0$ *are negative.*

3 Stability of equilibrium point and analysis of Hopf bifurcation

From (H1) it can be obtained that the equilibrium point of system (2) is zero. Now the linear transformation of system (2) can be obtained

$$\begin{aligned}
D^q\mu_1(t) &= -\varphi_1\mu_1(t) + \gamma_1\mu_2(t-\xi_2) + \eta_2\mu_5(t-\xi_7),\\
D^q\mu_2(t) &= -\varphi_2\mu_2(t) + \gamma_3\mu_3(t-\xi_3) + \eta_4\mu_6(t-\xi_{10}),\\
D^q\mu_3(t) &= -\varphi_3\mu_3(t) + \gamma_5\mu_4(t-\xi_4) + \eta_6\mu_1(t-\xi_9),\\
D^q\mu_4(t) &= -\varphi_4\mu_4(t) + \gamma_7\mu_5(t-\xi_5) + \eta_8\mu_2(t-\xi_{12}),\\
D^q\mu_5(t) &= -\varphi_5\mu_5(t) + \gamma_9\mu_6(t-\xi_6) + \eta_{10}\mu_3(t-\xi_8),\\
D^q\mu_6(t) &= -\varphi_6\mu_6(t) + \gamma_{11}\mu_1(t-\xi_1) + \eta_{12}\mu_4(t-\xi_{11})
\end{aligned}$$

in which $\gamma_{2p-1} = \eta_{2p-1}h'_{2p-1}(0)$, $\gamma_{2p} = \eta_{2p}l'_{2p}(0)$ $(p = 1, 2, \dots, 6)$.

By Lemma 2 the characteristic equation of system (2) is

$$\begin{vmatrix}
s^q+\varphi_1 & -\gamma_1 e^{-s\xi_2} & 0 & 0 & -\gamma_2 e^{-s\xi_7} & 0\\
0 & s^q+\varphi_2 & -\gamma_3 e^{-s\xi_3} & 0 & 0 & -\gamma_4 e^{-s\xi_{10}}\\
-\gamma_6 e^{-s\xi_9} & 0 & s^q+\varphi_3 & -\gamma_5 e^{-s\xi_4} & 0 & 0\\
0 & -\gamma_8 e^{-s\xi_{12}} & 0 & s^q+\varphi_4 & -\gamma_7 e^{-s\xi_5} & 0\\
0 & 0 & -\gamma_{10} e^{-s\xi_8} & 0 & s^q+\varphi_5 & -\gamma_9 e^{-s\xi_6}\\
-\gamma_{11} e^{-s\xi_1} & 0 & 0 & -\gamma_{12} e^{-s\xi_{11}} & 0 & s^q+\varphi_6
\end{vmatrix} = 0. \qquad (5)$$

Based on [24], we can carry on the following assumption.

(H2) $$\begin{aligned}\xi &= \xi_1+\xi_2+\xi_{10} = \xi_1+\xi_6+\xi_7 = \xi_2+\xi_3+\xi_9 = \xi_3+\xi_4+\xi_{12}\\
&= \xi_4+\xi_5+\xi_8 = \xi_5+\xi_6+\xi_{11} = \xi_7+\xi_8+\xi_9 = \xi_{10}+\xi_{11}+\xi_{12}.\end{aligned}$$

Because of (H2), Eq. (5) is equivalent to the following equation:

$$\begin{aligned}
&s^{6q} + P_5 s^{5q} + P_4 s^{4q} + P_3 s^{3q} + P_2 s^{2q} + P_1 s^q + P_0\\
&\quad - \left(Q_3 s^{3q} + Q_2 s^{2q} + Q_1 s^q + Q_0\right)e^{-s\xi} + J_0 e^{-2s\xi} = 0,
\end{aligned} \qquad (6)$$

where

$$\begin{aligned}
P_5 &= \varphi_1+\varphi_2+\varphi_3+\varphi_4+\varphi_5+\varphi_6,\\
P_4 &= \varphi_1(\varphi_2+\varphi_3+\varphi_4+\varphi_5+\varphi_6) + \varphi_2(\varphi_3+\varphi_4+\varphi_5+\varphi_6)\\
&\quad + \varphi_3(\varphi_4+\varphi_5+\varphi_6) + \varphi_4(\varphi_5+\varphi_6) + \varphi_5\varphi_6,\\
P_3 &= \varphi_1\varphi_2(\varphi_3+\varphi_4+\varphi_5+\varphi_6) + \varphi_1\varphi_3(\varphi_4+\varphi_5+\varphi_6) + \varphi_1\varphi_4(\varphi_5+\varphi_6)\\
&\quad + \varphi_2\varphi_3(\varphi_4+\varphi_5+\varphi_6) + \varphi_2\varphi_4(\varphi_5+\varphi_6) + \varphi_3\varphi_4(\varphi_5+\varphi_6)\\
&\quad + \varphi_5\varphi_6(\varphi_1+\varphi_2+\varphi_3+\varphi_4),\\
P_2 &= \varphi_1\varphi_2\varphi_3(\varphi_4+\varphi_5+\varphi_6) + \varphi_1\varphi_2\varphi_4(\varphi_5+\varphi_6) + \varphi_1\varphi_3\varphi_4(\varphi_5+\varphi_6)\\
&\quad + \varphi_2\varphi_3\varphi_4(\varphi_5+\varphi_6) + \varphi_5\varphi_6(\varphi_1+\varphi_3)(\varphi_2+\varphi_4) + (\varphi_1\varphi_3+\varphi_2\varphi_4),\\
P_1 &= \varphi_1\varphi_2\varphi_3(\varphi_4\varphi_5+\varphi_4\varphi_6+\varphi_5\varphi_6) + \varphi_4\varphi_5\varphi_6(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3),\\
P_0 &= \varphi_1\varphi_2\varphi_3\varphi_4\varphi_5\varphi_6,
\end{aligned}$$

$$
\begin{aligned}
Q_3 &= \gamma_1\gamma_3\gamma_6 + \gamma_1\gamma_4\gamma_{11} + \gamma_3\gamma_5\gamma_8 + \gamma_2\gamma_6\gamma_{10} + \gamma_2\gamma_9\gamma_{11} \\
&\quad + \gamma_5\gamma_7\gamma_{10} + \gamma_4\gamma_8\gamma_{12} + \gamma_7\gamma_9\gamma_{12}, \\
Q_2 &= \gamma_1\gamma_3\gamma_6(\varphi_4 + \varphi_5 + \varphi_6) + \gamma_1\gamma_4\gamma_{11}(\varphi_3 + \varphi_4 + \varphi_5) + \gamma_3\gamma_5\gamma_8(\varphi_1 + \varphi_5 + \varphi_6) \\
&\quad + \gamma_2\gamma_6\gamma_{10}(\varphi_2 + \varphi_4 + \varphi_6) + \gamma_2\gamma_9\gamma_{11}(\varphi_2 + \varphi_3 + \varphi_4) + \gamma_5\gamma_7\gamma_{10}(\varphi_1 + \varphi_2 + \varphi_6) \\
&\quad + \gamma_4\gamma_8\gamma_{12}(\varphi_1 + \varphi_3 + \varphi_5) + \gamma_7\gamma_9\gamma_{12}(\varphi_1 + \varphi_2 + \varphi_3), \\
Q_1 &= \gamma_1\gamma_3\gamma_6(\varphi_4\varphi_5 + \varphi_4\varphi_6 + \varphi_5\varphi_6) + \gamma_1\gamma_4\gamma_{11}(\varphi_3\varphi_4 + \varphi_3\varphi_5 + \varphi_4\varphi_5) \\
&\quad + \gamma_3\gamma_5\gamma_8(\varphi_1\varphi_5 + \varphi_1\varphi_6 + \varphi_5\varphi_6) + \gamma_2\gamma_6\gamma_{10}(\varphi_2\varphi_4 + \varphi_2\varphi_6 + \varphi_4\varphi_6) \\
&\quad + \gamma_2\gamma_9\gamma_{11}(\varphi_2\varphi_3 + \varphi_2\varphi_4 + \varphi_3\varphi_4) + \gamma_5\gamma_7\gamma_{10}(\varphi_1\varphi_2 + \varphi_1\varphi_6 + \varphi_2\varphi_6) \\
&\quad + \gamma_4\gamma_8\gamma_{12}(\varphi_1\varphi_3 + \varphi_1\varphi_5 + \varphi_3\varphi_5) + \gamma_7\gamma_9\gamma_{12}(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3), \\
Q_0 &= \gamma_1\gamma_3\gamma_6\varphi_4\varphi_5\varphi_6 + \gamma_1\gamma_4\gamma_{11}\varphi_3\varphi_4\varphi_5 + \gamma_3\gamma_5\gamma_8\varphi_1\varphi_5\varphi_6 + \gamma_2\gamma_6\gamma_{10}\varphi_2\varphi_4\varphi_6 \\
&\quad + \gamma_2\gamma_9\gamma_{11}\varphi_2\varphi_3\varphi_4 + \gamma_5\gamma_7\gamma_{10}\varphi_1\varphi_2\varphi_6 + \gamma_4\gamma_8\gamma_{12}\varphi_1\varphi_3\varphi_5 + \gamma_7\gamma_9\gamma_{12}\varphi_1\varphi_2\varphi_3, \\
J_0 &= (\gamma_6\gamma_{12} - \gamma_5\gamma_{11})(\gamma_1\gamma_3\gamma_7\gamma_9 - \gamma_1\gamma_4\gamma_7\gamma_{10} + \gamma_2\gamma_4\gamma_8\gamma_{10} - \gamma_2\gamma_3\gamma_8\gamma_9).
\end{aligned}
$$

Multiply the left and right sides of Eq. (6) by $\mathrm{e}^{s\xi}$ at the same time to get

$$
\begin{aligned}
&\big(s^{6q} + P_5 s^{5q} + P_4 s^{4q} + P_3 s^{3q} + P_2 s^{2q} + P_1 s^q + P_0\big)\mathrm{e}^{s\xi} \\
&\quad - \big(Q_3 s^{3q} + Q_2 s^{2q} + Q_1 s^q + Q_0\big) + J_0 \mathrm{e}^{-s\xi} = 0. \qquad (7)
\end{aligned}
$$

When $\xi > 0$, let $s = \mathrm{i}\nu = \nu(\cos\pi 2 + \mathrm{i}\sin\pi 2)$ $(\nu > 0)$ be a pure imaginary root of Eq. (7). We can get

$$
\begin{aligned}
&\bigg(\nu^{6q}(\cos 3q\pi + \mathrm{i}\sin 3q\pi) + P_5\nu^{5q}\bigg(\cos\frac{5q\pi}{2} + \mathrm{i}\sin\frac{5q\pi}{2}\bigg) \\
&\quad + P_4\nu^{4q}(\cos 2q\pi + \mathrm{i}\sin 2q\pi) + P_3\nu^{3q}\bigg(\cos\frac{3q\pi}{2} + \mathrm{i}\sin\frac{3q\pi}{2}\bigg) \\
&\quad + P_2\nu^{2q}(\cos q\pi + \mathrm{i}\sin q\pi) + P_1\nu^q\bigg(\cos\frac{q\pi}{2} + \mathrm{i}\sin\frac{q\pi}{2}\bigg) + P_0\bigg) \\
&\quad \times (\cos\nu\xi + \mathrm{i}\sin\nu\xi) \\
&\quad - \bigg(Q_3\nu^{3q}\bigg(\cos\frac{3q\pi}{2} + \mathrm{i}\sin\frac{3q\pi}{2}\bigg) + Q_2\nu^{2q}(\cos q\pi + \mathrm{i}\sin q\pi) \\
&\quad + Q_1\nu^q\bigg(\cos\frac{q\pi}{2} + \mathrm{i}\sin\frac{q\pi}{2}\bigg) + Q_0\bigg) + J_0(\cos\nu\xi - \mathrm{i}\sin\nu\xi) = 0.
\end{aligned}
$$

Separate the real and imaginary parts to get

$$
\begin{aligned}
&\bigg(\nu^{6q}\cos 3q\pi + P_5\nu^{5q}\cos\frac{5q\pi}{2} + P_4\nu^{4q}\cos 2q\pi \\
&\quad + P_3\nu^{3q}\cos\frac{3q\pi}{2} + P_2\nu^{2q}\cos q\pi + P_1\nu^q\cos\frac{q\pi}{2} + P_0\bigg)\cos\nu\xi \\
&\quad - \bigg(\nu^{6q}\sin 3q\pi + P_5\nu^{5q}\sin\frac{5q\pi}{2} + P_4\nu^{4q}\sin 2q\pi
\end{aligned}
$$

$$
\begin{aligned}
&+ P_3\nu^{3q}\sin\frac{3q\pi}{2} + P_2\nu^{2q}\sin q\pi + P_1\nu^q\sin\frac{q\pi}{2}\biggr)\sin\nu\xi \\
&- \biggl(Q_3\nu^{3q}\cos\frac{3q\pi}{2} + Q_2\nu^{2q}\cos q\pi + Q_1\nu^q\cos\frac{q\pi}{2} + Q_0\biggr) \\
&+ J_0\cos\nu\xi = 0
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
&\biggl(\nu^{6q}\cos 3q\pi + P_5\nu^{5q}\cos\frac{5q\pi}{2} + P_4\nu^{4q}\cos 2q\pi \\
&\quad + P_3\nu^{3q}\cos\frac{3q\pi}{2} + P_2\nu^{2q}\cos q\pi + P_1\nu^q\cos\frac{q\pi}{2} + P_0\biggr)\sin\nu\xi \\
&\quad + \biggl(\nu^{6q}\sin 3q\pi + P_5\nu^{5q}\sin\frac{5q\pi}{2} + P_4\nu^{4q}\sin 2q\pi \\
&\quad + P_3\nu^{3q}\sin\frac{3q\pi}{2} + P_2\nu^{2q}\sin q\pi + P_1\nu^q\sin\frac{q\pi}{2}\biggr)\cos\nu\xi \\
&\quad - \biggl(Q_3\nu^{3q}\sin\frac{3q\pi}{2} + Q_2\nu^{2q}\sin q\pi + Q_1\nu^q\sin\frac{q\pi}{2}\biggr) \\
&\quad - J_0\sin\nu\xi = 0.
\end{aligned} \tag{9}
$$

Let

$$
\begin{aligned}
A_{11} &= \nu^{6q}\cos 3q\pi + P_5\nu^{5q}\cos\frac{5q\pi}{2} + P_4\nu^{4q}\cos 2q\pi \\
&\quad + P_3\nu^{3q}\cos\frac{3q\pi}{2} + P_2\nu^{2q}\cos q\pi + P_1\nu^q\cos\frac{q\pi}{2} + P_0, \\
A_{12} &= \nu^{6q}\sin 3q\pi + P_5\nu^{5q}\sin\frac{5q\pi}{2} + P_4\nu^{4q}\sin 2q\pi \\
&\quad + P_3\nu^{3q}\sin\frac{3q\pi}{2} + P_2\nu^{2q}\sin q\pi + P_1\nu^q\sin\frac{q\pi}{2}, \\
B_{11} &= Q_3\nu^{3q}\cos\frac{3q\pi}{2} + Q_2\nu^{2q}\cos q\pi + Q_1\nu^q\cos\frac{q\pi}{2} + Q_0, \\
B_{12} &= Q_3\nu^{3q}\sin\frac{3q\pi}{2} + Q_2\nu^{2q}\sin q\pi + Q_1\nu^q\sin\frac{q\pi}{2}.
\end{aligned} \tag{10}
$$

From (8), (9), and (10) we get

$$
\begin{aligned}
(A_{11} + J_0)\cos\nu\xi - A_{12}\sin\nu\xi &= B_{11}, \\
(A_{11} - J_0)\sin\nu\xi + A_{12}\cos\nu\xi &= B_{12}.
\end{aligned} \tag{11}
$$

By (11) we can obtain

$$
\begin{aligned}
\sin\nu\xi &= \frac{A_{11}B_{12} - A_{12}B_{11} + J_0B_{12}}{A_{11}^2 + A_{12}^2 - J_0^2}, \\
\cos\nu\xi &= \frac{A_{11}B_{11} + A_{12}B_{12} - J_0B_{11}}{A_{11}^2 + A_{12}^2 - J_0^2}.
\end{aligned} \tag{12}
$$

Now, take

$$
\begin{aligned}
&a_6 = \cos 3q\pi, && a_5 = P_5 \cos\frac{5q\pi}{2}, && a_4 = P_4 \cos 2q\pi, && a_3 = P_3 \cos\frac{3q\pi}{2},\\
&a_2 = P_2 \cos q\pi, && a_1 = P_1 \cos\frac{q\pi}{2}, && a_0 = P_0, &&\\
&b_6 = \sin 3q\pi, && b_5 = P_5 \sin\frac{5q\pi}{2}, && b_4 = P_4 \sin 2q\pi, && b_3 = P_3 \sin\frac{3q\pi}{2},\\
&b_2 = P_2 \sin q\pi, && b_1 = P_1 \sin\frac{q\pi}{2}, && b_0 = 0, &&\\
&c_3 = Q_3 \cos\frac{3q\pi}{2}, && c_2 = Q_2 \cos q\pi, && c_1 = Q_1 \cos\frac{q\pi}{2}, && c_0 = Q_0,\\
&d_3 = Q_3 \sin\frac{3q\pi}{2}, && d_2 = Q_2 \sin q\pi, && d_1 = Q_1 \sin\frac{q\pi}{2}, && d_0 = 0,\\
&e_0 = J_0, && e_i = 0 \quad (i > 0). && &&
\end{aligned}
$$

According to the above form, (10) can be transformed into

$$
\begin{aligned}
A_{11} &= a_6\nu^{6q} + a_5\nu^{5q} + a_4\nu^{4q} + a_3\nu^{3q} + a_2\nu^{2q} + a_1\nu^{q} + a_0,\\
A_{12} &= b_6\nu^{6q} + b_5\nu^{5q} + b_4\nu^{4q} + b_3\nu^{3q} + b_2\nu^{2q} + b_1\nu^{q},\\
B_{11} &= c_3\nu^{3q} + c_2\nu^{2q} + c_1\nu^{q} + c_0,\\
B_{12} &= d_3\nu^{3q} + d_2\nu^{2q} + d_1\nu^{q}.
\end{aligned} \tag{13}
$$

For $\sin^2 \nu\xi + \cos^2 \nu\xi = 1$, it can be obtained

$$
\begin{aligned}
&(A_{11}B_{12} - A_{12}B_{11} + J_0B_{12})^2 + (A_{12}B_{12} + A_{11}B_{11} - J_0B_{11})^2\\
&\quad = \left(A_{11}^2 + A_{12}^2 - J_0^2\right)^2.
\end{aligned} \tag{14}
$$

Let

$$
\begin{aligned}
\beta_k &= \sum_{i=0}^{6} (a_i a_{k-i} + b_i b_{k-i} - e_i e_{k-i}) \quad (0 \leqslant k \leqslant 12),\\
\delta_l &= \sum_{i=0}^{6} (a_i c_{\iota-i} + b_i d_{\iota-i} - c_i e_{\iota-i}) - \sum_{\iota-i=1} b_i d_{\iota-i} + \sum_{\iota-i=1} b_i b_{\iota-i} \quad (0 \leqslant \iota \leqslant 9),\\
\varepsilon_r &= \sum_{i=0}^{6} (b_i c_{r-i} - a_i d_{r-i} + e_i d_{r-i}) - \sum_{r-i=1} (e_i d_{r-i} - a_i d_{r-i})\\
&\quad + \sum_{r-i=1} (e_i b_{r-i} - a_i b_{r-i}) \quad (0 \leqslant r \leqslant 9).
\end{aligned}
$$

Substituting (13) into (14), we can get

$$
\begin{aligned}
&\alpha_0\nu^{24q} + \alpha_1\nu^{23q} + \alpha_2\nu^{22q} + \alpha_3\nu^{21q} + \alpha_4\nu^{20q} + \alpha_5\nu^{19q} + \alpha_6\nu^{18q}\\
&\quad + \alpha_7\nu^{17q} + \alpha_8\nu^{16q} + \alpha_9\nu^{15q} + \alpha_{10}\nu^{14q} + \alpha_{11}\nu^{13q} + \alpha_{12}\nu^{12q}
\end{aligned}
$$

$$+\alpha_{13}\nu^{11q}+\alpha_{14}\nu^{10q}+\alpha_{15}\nu^{9q}+\alpha_{16}\nu^{8q}+\alpha_{17}\nu^{7q}+\alpha_{18}\nu^{6q}$$
$$+\alpha_{19}\nu^{5q}+\alpha_{20}\nu^{4q}+\alpha_{21}\nu^{3q}+\alpha_{22}\nu^{2q}+\alpha_{23}\nu^{q}+\alpha_{24}=0 \tag{15}$$

in which

$$\begin{aligned}
\alpha_0 &= \beta_{12}^2, \qquad \alpha_1 = 2\beta_{11}\beta_{12},\\
\alpha_2 &= \beta_{11}^2+2\beta_{10}\beta_{12}, \qquad \alpha_3 = 2(\beta_9\beta_{12}+\beta_{10}\beta_{11}),\\
\alpha_4 &= \beta_{10}^2+2(\beta_8\beta_{12}+\beta_9\beta_{11}), \qquad \alpha_5 = 2(\beta_7\beta_{12}+\beta_8\beta_{11}+\beta_9\beta_{10}),\\
\alpha_6 &= \left(\beta_9^2-\delta_9^2-\varepsilon_9^2\right)+2(\beta_6\beta_{12}+\beta_7\beta_{11}+\beta_8\beta_{10}),\\
\alpha_7 &= 2(\beta_5\beta_{12}+\beta_6\beta_{11}+\beta_7\beta_{10}+\beta_8\beta_9)-2(\delta_8\delta_9+\varepsilon_8\varepsilon_9),\\
\alpha_8 &= \beta_8^2+2(\beta_4\beta_{12}+\beta_5\beta_{11}+\beta_6\beta_{10}+\beta_7\beta_9)-2(\delta_7\delta_9+\varepsilon_7\varepsilon_9),\\
\alpha_9 &= 2(\beta_3\beta_{12}+\beta_4\beta_{11}+\beta_5\beta_{10}+\beta_6\beta_9+\beta_7\beta_8)\\
&\quad -2(\delta_6\delta_9+\delta_7\delta_8+\varepsilon_6\varepsilon_9+\varepsilon_7\varepsilon_8),\\
\alpha_{10} &= \left(\beta_7^2-\delta_7^2-\varepsilon_7^2\right)+2(\beta_2\beta_{12}+\beta_3\beta_{11}+\beta_4\beta_{10}+\beta_5\beta_9+\beta_6\beta_8)\\
&\quad -2(\delta_5\delta_9+\delta_6\delta_8+\varepsilon_5\epsilon_9+\varepsilon_6\varepsilon_8),\\
\alpha_{11} &= 2(\beta_2\beta_{11}+\beta_3\beta_{10}+\beta_4\beta_9+\beta_5\beta_8+\beta_6\beta_7)\\
&\quad -2(\delta_4\delta_9+\delta_5\delta_8+\delta_6\delta_7+\varepsilon_4\varepsilon_9+\varepsilon_5\varepsilon_8+\varepsilon_6\varepsilon_7),\\
\alpha_{12} &= \left(\beta_6^2-\delta_6^2-\varepsilon_6^2\right)+2(\beta_1\beta_{11}+\beta_2\beta_{10}+\beta_3\beta_9+\beta_4\beta_8+\beta_5\beta_7+\beta_6\beta_8)\\
&\quad -2(\delta_3\delta_9+\delta_4\delta_8+\delta_5\delta_7+\delta_6\delta_8+\varepsilon_3\varepsilon_9+\varepsilon_4\varepsilon_8+\varepsilon_5\varepsilon_7),\\
\alpha_{13} &= 2(\beta_0\beta_{11}+\beta_1\beta_{10}+\beta_2\beta_9+\beta_3\beta_8+\beta_4\beta_7+\beta_5\beta_6)\\
&\quad -2(\delta_2\delta_9+\delta_3\delta_8+\delta_4\delta_7+\delta_5\delta_6+\varepsilon_2\varepsilon_9+\varepsilon_3\varepsilon_8+\varepsilon_4\varepsilon_7+\varepsilon_5\varepsilon_6),\\
\alpha_{14} &= \left(\beta_5^2-\delta_5^2-\varepsilon_5^2\right)+2(\beta_0\beta_{10}+\beta_1\beta_9+\beta_2\beta_8+\beta_3\beta_7+\beta_4\beta_6)\\
&\quad -2(\delta_1\delta_9+\delta_2\delta_8+\delta_3\delta_7+\delta_4\delta_6+\varepsilon_1\varepsilon_9+\varepsilon_2\varepsilon_8+\varepsilon_3\varepsilon_7+\varepsilon_4\varepsilon_6),\\
\alpha_{15} &= 2(\beta_0\beta_9+\beta_1\beta_8+\beta_2\beta_7+\beta_3\beta_6+\beta_4\beta_5)\\
&\quad -2(\delta_0\delta_9+\delta_1\delta_8+\delta_2\delta_7+\delta_3\delta_6+\delta_4\delta_5+\varepsilon_1\varepsilon_8+\varepsilon_2\varepsilon_7+\varepsilon_3\varepsilon_6+\varepsilon_4\varepsilon_5),\\
\alpha_{16} &= \left(\beta_4^2-\delta_4^2-\varepsilon_4^2\right)+2(\beta_0\beta_8+\beta_1\beta_7+\beta_2\beta_6+\beta_3\beta_5)\\
&\quad -2(\delta_0\delta_8+\delta_1\delta_7+\delta_2\delta_6+\delta_3\delta_5+\varepsilon_1\varepsilon_7+\varepsilon_2\varepsilon_6+\varepsilon_3\varepsilon_5),\\
\alpha_{17} &= 2(\beta_0\beta_7+\beta_1\beta_6+\beta_2\beta_5+\beta_3\beta_4)\\
&\quad -2(\delta_0\delta_7+\delta_1\delta_6+\delta_2\delta_5+\delta_3\delta_4+\varepsilon_1\varepsilon_6+\varepsilon_2\varepsilon_5+\varepsilon_3\varepsilon_4),\\
\alpha_{18} &= \left(\beta_3^2-\delta_3^2-\varepsilon_3^2\right)+2(\beta_0\beta_6+\beta_1\beta_5+\beta_2\beta_4)\\
&\quad -2(\delta_0\delta_6+\delta_1\delta_5+\delta_2\delta_4+\varepsilon_1\varepsilon_5),\\
\alpha_{19} &= 2(\beta_0\beta_5+\beta_1\beta_4+\beta_2\beta_3)-2(\delta_0\delta_5+\delta_1\delta_4+\delta_2\delta_3+\varepsilon_1\varepsilon_4+\varepsilon_2\varepsilon_3),\\
\alpha_{20} &= \left(\beta_2^2-\delta_2^2-\varepsilon_2^2\right)+2(\beta_0\beta_4+\beta_1\beta_3)-2(\delta_0\delta_4+\delta_1\delta_3+\varepsilon_1\varepsilon_3),\\
\alpha_{21} &= 2(\beta_0\beta_3+\beta_1\beta_2)-2(\delta_0\delta_3+\delta_1\delta_2+\varepsilon_1\varepsilon_2),\\
\alpha_{22} &= \left(\beta_1^2-\delta_1^2-\varepsilon_1^2\right)+2(\beta_0\beta_2-\delta_0\delta_2),\\
\alpha_{23} &= 2(\beta_0\beta_1-\delta_0\delta_1), \qquad \alpha_{24} = \left(\beta_0^2-\delta_0^2\right).
\end{aligned} \tag{16}$$

Let

$$\begin{aligned}\varpi(\nu) &= \alpha_0\nu^{24q} + \alpha_1\nu^{23q} + \alpha_2\nu^{22q} + \alpha_3\nu^{21q} + \alpha_4\nu^{20q} + \alpha_5\nu^{19q} + \alpha_6\nu^{18q} \\ &\quad + \alpha_7\nu^{17q} + \alpha_8\nu^{16q} + \alpha_9\nu^{15q} + \alpha_{10}\nu^{14q} + \alpha_{11}\nu^{13q} + \alpha_{12}\nu^{12q} \\ &\quad + \alpha_{13}\nu^{11q} + \alpha_{14}\nu^{10q} + \alpha_{15}\nu^{9q} + \alpha_{16}\nu^{8q} + \alpha_{17}\nu^{7q} + \alpha_{18}\nu^{6q} \\ &\quad + \alpha_{19}\nu^{5q} + \alpha_{20}\nu^{4q} + \alpha_{21}\nu^{3q} + \alpha_{22}\nu^{2q} + \alpha_{23}\nu^{q} + \alpha_{24}. \end{aligned} \tag{17}$$

Lemma 3. *Let the following assumption holds*:

(H3) $\beta_0^2 - \delta_0^2 < 0$.

Then there is at least one pair of pure imaginary roots in Eq. (6).

Proof. Combining (16) and (17), we have $\lim_{\nu\to+\infty} \varpi(\nu) = +\infty$. Obviously, $\varpi(0) < 0$. According to the zero point theorem of continuous functions, we can obtain that Eq. (15) has at least one positive root, and this positive root is recorded as ν_k ($k = 1, 2, \dots, N$). Then substitute it into (12) to get

$$\xi_k^{(j)} = \frac{1}{\nu_k}\left[\arccos\left(-\frac{A_{11}B_{11} + A_{12}B_{12} - J_0B_{11}}{A_{11}^2 + A_{12}^2 - J_0^2}\right) + 2j\pi\right],$$

where $k = 1, 2, \dots, N$; $j = 0, 1, 2, \dots$.

Now define $\xi_0 = \min\{\xi_k^{(0)}\}$, and ξ_0 is the root of Eq. (6) to ν_0, that is, there are at least one pair of pure imaginary roots in characteristic equation (6). The proof is complete. □

We assume that the following condition holds.

(H4) $\kappa_R\lambda_R + \kappa_I\lambda_I/(\lambda_{\mathbb{R}}^2 + \lambda_I^2) > 0$, where

$$\begin{aligned}\kappa_R = &\bigg(6qw_0^{6q-1}\cos\frac{(6q-1)\pi}{2} + 5qP_5w_0^{5q-1}\cos\frac{(5q-1)\pi}{2} \\ &+ 4qP_4w_0^{4q-1}\cos\frac{(4q-1)\pi}{2} + 3qP_3w_0^{3q-1}\cos\frac{(3q-1)\pi}{2} \\ &+ 2qP_2w_0^{2q-1}\cos\frac{(2q-1)\pi}{2} + qP_1w_0^{q-1}\cos\frac{(q-1)\pi}{2}\bigg)\cos(w_0\xi_0) \\ &- \bigg(6qw_0^{6q-1}\sin\frac{(6q-1)\pi}{2} + 5qP_5w_0^{5q-1}\sin\frac{(5q-1)\pi}{2} \\ &+ 4qP_4w_0^{4q-1}\sin\frac{(4q-1)\pi}{2} + 3qP_3w_0^{3q-1}\sin\frac{(3q-1)\pi}{2} \\ &+ 2qP_2w_0^{2q-1}\sin\frac{(2q-1)\pi}{2} + qP_1w_0^{q-1}\sin\frac{(q-1)\pi}{2}\bigg)\sin(w_0\xi_0) \\ &- \bigg(3qQ_3w_0^{3q-1}\cos\frac{(3q-1)\pi}{2} + 2qQ_2w_0^{2q-1}\cos\frac{(2q-1)\pi}{2} \\ &+ qQ_1w_0^{q-1}\cos\frac{(q-1)\pi}{2}\bigg),\end{aligned}$$

$$
\begin{aligned}
\kappa_I = {} & \bigg(6qw_0^{6q-1}\cos\frac{(6q-1)\pi}{2} + 5qP_5w_0^{5q-1}\cos\frac{(5q-1)\pi}{2} \\
& + 4qP_4w_0^{4q-1}\cos\frac{(4q-1)\pi}{2} + 3qP_3w_0^{3q-1}\cos\frac{(3q-1)\pi}{2} \\
& + 2qP_2w_0^{2q-1}\cos\frac{(2q-1)\pi}{2} + qP_1w_0^{q-1}\cos\frac{(q-1)\pi}{2}\bigg)\sin(w_0\xi_0) \\
& + \bigg(6qw_0^{6q-1}\sin\frac{(6q-1)\pi}{2} + 5qP_5w_0^{5q-1}\sin\frac{(5q-1)\pi}{2} \\
& + 4qP_4w_0^{4q-1}\sin\frac{(4q-1)\pi}{2} + 3qP_3w_0^{3q-1}\sin\frac{(3q-1)\pi}{2} \\
& + 2qP_2w_0^{2q-1}\sin\frac{(2q-1)\pi}{2} + qP_1w_0^{q-1}\sin\frac{(q-1)\pi}{2}\bigg)\cos(w_0\xi_0) \\
& - \bigg(3qQ_3w_0^{3q-1}\sin\frac{(3q-1)\pi}{2} + 2qQ_2w_0^{2q-1}\sin\frac{(2q-1)\pi}{2} \\
& + qQ_1w_0^{q-1}\sin\frac{(q-1)\pi}{2}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_R = {} & \bigg(w_0^{6q}\cos\frac{6q\pi}{2} + P_5w_0^{5q}\cos\frac{5q\pi}{2} + P_4w_0^{4q}\cos\frac{4q\pi}{2} \\
& + P_3w_0^{3q}\cos\frac{3q\pi}{2} + P_2w_0^{2q}\cos\frac{2q\pi}{2} + P_1w_0^{q}\cos\frac{q\pi}{2} + P_0\bigg)w_0\sin(w_0\xi_0) \\
& + \bigg(w_0^{6q}\sin\frac{6q\pi}{2} + P_5w_0^{5q}\sin\frac{5q\pi}{2} + P_4w_0^{4q}\sin\frac{4q\pi}{2} \\
& + P_3w_0^{3q}\sin\frac{3q\pi}{2} + P_2w_0^{2q}\sin\frac{2q\pi}{2} + P_1w_0^{q}\sin\frac{q\pi}{2} + P_0\bigg)w_0\cos(w_0\xi_0) \\
& + J_0w_0\sin(w_0\xi_0)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_I = {} & -\bigg(w_0^{6q}\cos\frac{6q\pi}{2} + P_5w_0^{5q}\cos\frac{5q\pi}{2} + P_4w_0^{4q}\cos\frac{4q\pi}{2} \\
& + P_3w_0^{3q}\cos\frac{3q\pi}{2} + P_2w_0^{2q}\cos\frac{2q\pi}{2} + P_1w_0^{q}\cos\frac{q\pi}{2} + P_0\bigg)w_0\cos(w_0\xi_0) \\
& + \bigg(w_0^{6q}\sin\frac{6q\pi}{2} + P_5w_0^{5q}\sin\frac{5q\pi}{2} + P_4w_0^{4q}\sin\frac{4q\pi}{2} \\
& + P_3w_0^{3q}\sin\frac{3q\pi}{2} + P_2w_0^{2q}\sin\frac{2q\pi}{2} + P_1w_0^{q}\sin\frac{q\pi}{2} + P_0\bigg)w_0\sin(w_0\xi_0) \\
& + J_0w_0\cos(w_0\xi_0).
\end{aligned}
$$

Lemma 4. *Assume that* $s(\xi) = \alpha(\xi) + \mathrm{i}w(\xi)$ *be the root of Eq.* (15) *at* $\xi = \xi_0$ *that satisfies* $\alpha(\xi_0) = 0$, $w(\xi_0) = w_0$. *Then*

$$
\mathrm{Re}\bigg(\bigg(\frac{\mathrm{d}s}{\mathrm{d}\xi}\bigg)^{-1}\bigg|_{\xi=\xi_0}\bigg) > 0.
$$

Proof. Let $s(\xi) = \alpha(\xi) + \mathrm{i}w(\xi)$ be the root of Eq. (15) near $\xi = \xi_0$ that satisfies $\alpha(\xi_0) = 0$, $w(\xi_0) = w_0$. Next, we verify the condition of transversality. Taking the derivative of Eq. (6) with respect to ξ, we have

$$\begin{aligned}
&\big(6qs^{6q-1} + 5qP_5s^{5q-1} + 4qP_4s^{4q-1} + 3qP_3s^{3q-1} + 2qP_2s^{2q-1} + qP_1s^{q-1}\big)\mathrm{e}^{s\xi}\frac{\mathrm{d}s}{\mathrm{d}\xi} \\
&\quad + \big(s^{6q} + P_5s^{5q} + p_4s^{4q} + P_3s^{3q} + P_2s^{2q} + P_1s^q + P_0\big)\left(s + \xi\frac{\mathrm{d}s}{\mathrm{d}\xi}\right)\mathrm{e}^{s\xi} \\
&\quad - \big(3qQ_3s^{3q-1} + 2qQ_2s^{2q-1} + qQ_1s^{q-1}\big)\frac{\mathrm{d}s}{\mathrm{d}\xi} - J_0\left(s + \xi\frac{\mathrm{d}s}{\mathrm{d}\xi}\right)\mathrm{e}^{-s\xi} = 0. \qquad (18)
\end{aligned}$$

Simplify (18) to get

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\xi}\right)^{-1} = \frac{\kappa(s)}{\lambda(s)} - \frac{\xi}{s}, \qquad (19)$$

where

$$\begin{aligned}
\kappa(s) &= \big(6qs^{6q-1} + 5qP_5s^{5q-1} + 4qP_4s^{4q-1} + 3qP_3s^{3q-1} + 2qP_2s^{2q-1} + qP_1s^{q-1}\big)\mathrm{e}^{s\xi} \\
&\quad - \big(3qQ_3s^{3q-1} + 2qQ_2s^{2q-1} + qQ_1s^{q-1}\big), \\
\lambda(s) &= sJ_0\mathrm{e}^{-s\xi} - s\mathrm{e}^{s\xi}\big(s^{6q} + P_5s^{5q} + p_4s^{4q} + P_3s^{3q} + P_2s^{2q} + P_1s^q + P_0\big).
\end{aligned}$$

Now define the real numbers κ_R, κ_I, λ_R, and λ_I as the real and imaginary parts of $\kappa(s)$ and $\lambda(s)$, namely,

$$\kappa(s) = \kappa_R + \mathrm{i}\kappa_I, \qquad \lambda(s) = \lambda_R + \mathrm{i}\lambda_I, \qquad (20)$$

where κ_R, κ_I, λ_R, and λ_I are as in condition (H4).

Based on (19), (20), and (H4), we have

$$\mathrm{Re}\left(\left(\frac{\mathrm{d}s}{\mathrm{d}\xi}\right)^{-1}\bigg|_{\xi=\xi_0}\right) = \frac{\kappa_R\lambda_R + \kappa_I\lambda_I}{\lambda_{\mathbb{R}}^2 + \lambda_I^2} > 0.$$

The proof is complete. □

Next, we assume that

(H5) The following conditions hold:

$$D_1 = P_5 > 0, \qquad D_2 = \begin{vmatrix} P_5 & 1 \\ P_3 - Q_3 & P_4 \end{vmatrix} > 0,$$

$$D_3 = \begin{vmatrix} P_5 & 1 & 0 \\ P_3 - Q_3 & P_4 & P_5 \\ P_1 - Q_1 & P_2 - Q_2 & P_3 - Q_3 \end{vmatrix} > 0,$$

$$D_4 = \begin{vmatrix} P_5 & 1 & 0 & 0 \\ P_3 - Q_3 & P_4 & P_5 & 1 \\ P_1 - Q_1 & P_2 - Q_2 & P_3 - Q_3 & P_4 \\ 0 & P_0 - Q_0 - J_0 & P_1 - Q_1 & P_2 - Q_2 \end{vmatrix} > 0,$$

$$D_5 = \begin{vmatrix} P_5 & 1 & 0 & 0 & 0 \\ P_3 - Q_3 & P_4 & P_5 & 1 & 0 \\ P_1 - Q_1 & P_2 - Q_2 & P_3 - Q_3 & P_4 & P_5 \\ 0 & P_0 - Q_0 & P_1 - Q_1 & P_2 - Q_2 & P_3 - Q_3 \\ 0 & 0 & 0 & P_0 - Q_0 & P_1 - Q_1 \end{vmatrix} > 0,$$

$$D_6 = (P_0 - Q_0)D_5 > 0.$$

Lemma 5. *If $\xi = 0$ and* (H5) *holds, the zero equilibrium point of system* (2) *is asymptotically stable.*

Proof. When $\xi = 0$, Eq. (6) is

$$\begin{aligned} &s^{6q} + P_5 s^{5q} + P_4 s^{4q} + (P_3 - Q_3)s^{3q} + (P_2 - Q_2)s^{2q} \\ &\quad + (P_1 - Q_1)s^q + (P_0 - Q_0 - J_0) = 0. \end{aligned} \tag{21}$$

If $\lambda = s^q$, the above formula (21) becomes

$$\begin{aligned} &\lambda^6 + P_5\lambda^5 + P_4\lambda^4 + (P_3 - Q_3)\lambda^3 + (P_2 - Q_2)\lambda^2 \\ &\quad + (P_1 - Q_1)\lambda + (P_0 - Q_0 - J_0) = 0. \end{aligned} \tag{22}$$

Equation (22) is a one-variable high-degree equation about λ. According to (H5), all roots λ_i of the characteristic equation (22) satisfy $|\arg(\lambda_i)| > q_i\pi/2$ ($i = 1, 2, \ldots, 6$). By Lemma 1 the zero equilibrium point of system (2) is asymptotically stable. The proof is complete. □

Based on the above analysis results, according to Lemmas 1–5, we have the following theorem.

Theorem 1. *Suppose that hypotheses* (H1)–(H5) *hold. Then the zero equilibrium point of system* (2) *is locally asymptotically stable when $\xi \in [0, \xi_0)$ and a Hopf bifurcation will happen around the equilibrium point for $\xi = \xi_0$.*

4 Numerical simulations

This section will give a numerical simulation example to prove the effectiveness of the conclusions obtained. Take $\varphi_p = 1.2$, $\eta_i = -1$, $h_{2p-1}(x) = l_{2p}(x) = \tanh(x)$, $p = 1, 2, \ldots, 6$, $i = 1, 2, \ldots, 12$, and consider the following system:

$$\begin{aligned} D^{0.9}x_1(t) &= -1.2x_1(t) - \tanh\bigl(x_2(t - \xi_2)\bigr) - \tanh\bigl(x_5(t - \xi_7)\bigr), \\ D^{0.9}x_2(t) &= -1.2x_2(t) - \tanh\bigl(x_3(t - \xi_3)\bigr) - \tanh\bigl(x_6(t - \xi_{10})\bigr), \\ D^{0.9}x_3(t) &= -1.2x_3(t) - \tanh\bigl(x_4(t - \xi_4)\bigr) - \tanh\bigl(x_1(t - \xi_9)\bigr), \\ D^{0.9}x_4(t) &= -1.2x_4(t) - \tanh\bigl(x_5(t - \xi_5)\bigr) - \tanh\bigl(x_2(t - \xi_{12})\bigr), \\ D^{0.9}x_5(t) &= -1.2x_5(t) - \tanh\bigl(x_6(t - \xi_6)\bigr) - \tanh\bigl(x_3(t - \xi_8)\bigr), \\ D^{0.9}x_6(t) &= -1.2x_6(t) - \tanh\bigl(x_1(t - \xi_1)\bigr) - \tanh\bigl(x_4(t - \xi_{11})\bigr). \end{aligned} \tag{23}$$

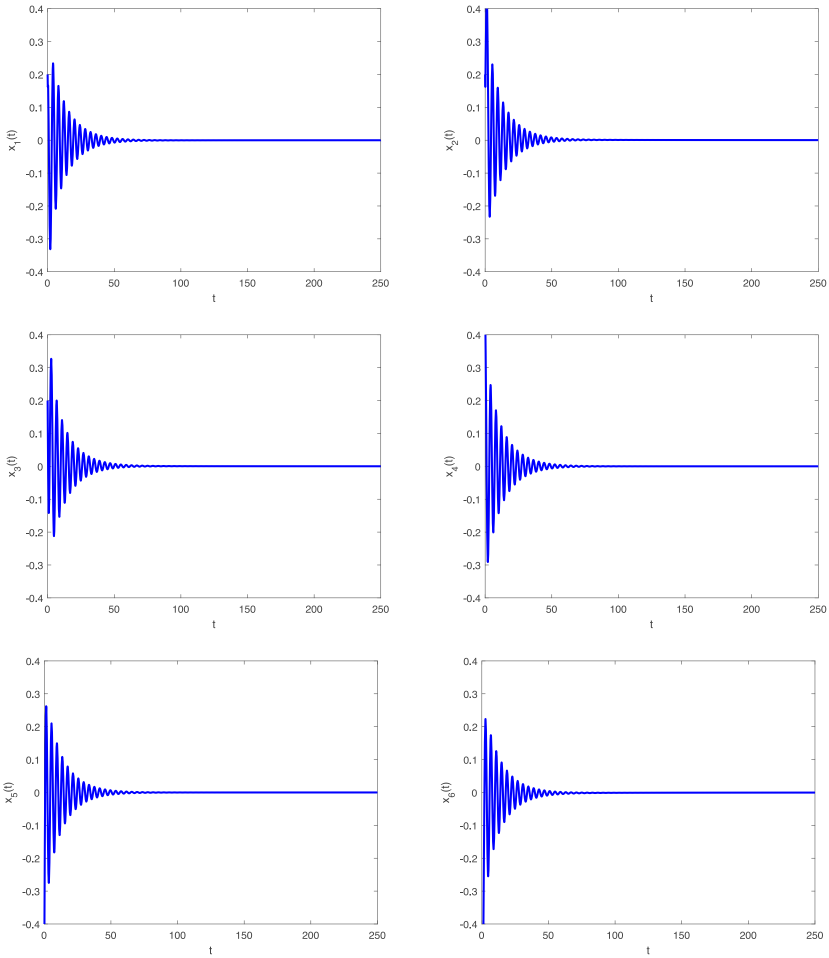

Figure 1. The trajectory of system (23) with respect to time t. When $\tau = 0.42 < \tau_0 = 0.4532$, the zero solution of the model is asymptotic stability.

It is obvious that system (23) has an equilibrium point. We take $q = 0.9$ and can find out by numerical calculation $D_1 = 7.2000 > 0$, $D_2 = 112.9600 > 0$, $D_3 = 2092.4057 > 0$, $D_4 = 27206.6148 > 0$, $D_5 = 613226.6916 > 0$, $D_6 = 10308330.8754 > 0$, so (H5) is satisfied. It is calculated that Eq. (15) has two positive real roots, and $\omega_0 = 1.56974$, $\xi_0 = 0.4532$, $(\kappa_R\lambda_R + \kappa_I\lambda_I)/(\lambda_R^2 + \lambda_I^2) = 0.8092 > 0$, which show that (H3) and (H4) also hold. Therefore, the value satisfies the requirements of Theorem 1. Now, let $\xi_i = 0.14$ ($i = 1, 2, \dots, 12$), that is, $\xi = 0.42 < 0.4532$. Figures 1–3 show that system (23) is locally asymptotically stable at the zero equilibrium point. Let $\xi_i = 0.2$ ($i = 1, 2, \dots, 12$), that is, $\xi = 0.60 > 0.4532$. Figures 4–6 show that system (23) loses its stability near the zero equilibrium point, and Hopf bifurcation phenomenon appears.

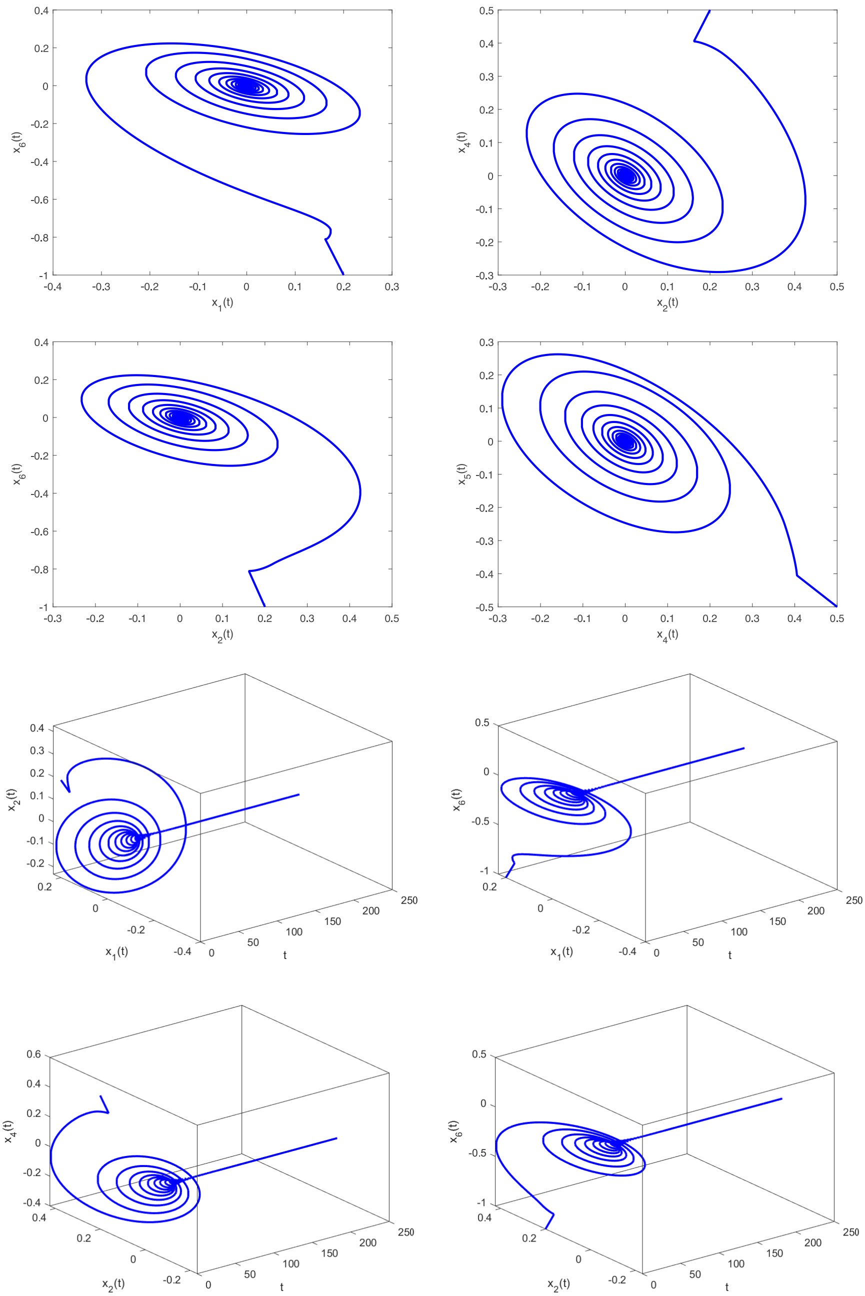

Figure 2. Part of the phase diagram of system (23) with respect to time t. When $\tau = 0.42 < \tau_0 = 0.4532$, the zero solution of the model is asymptotic stability.

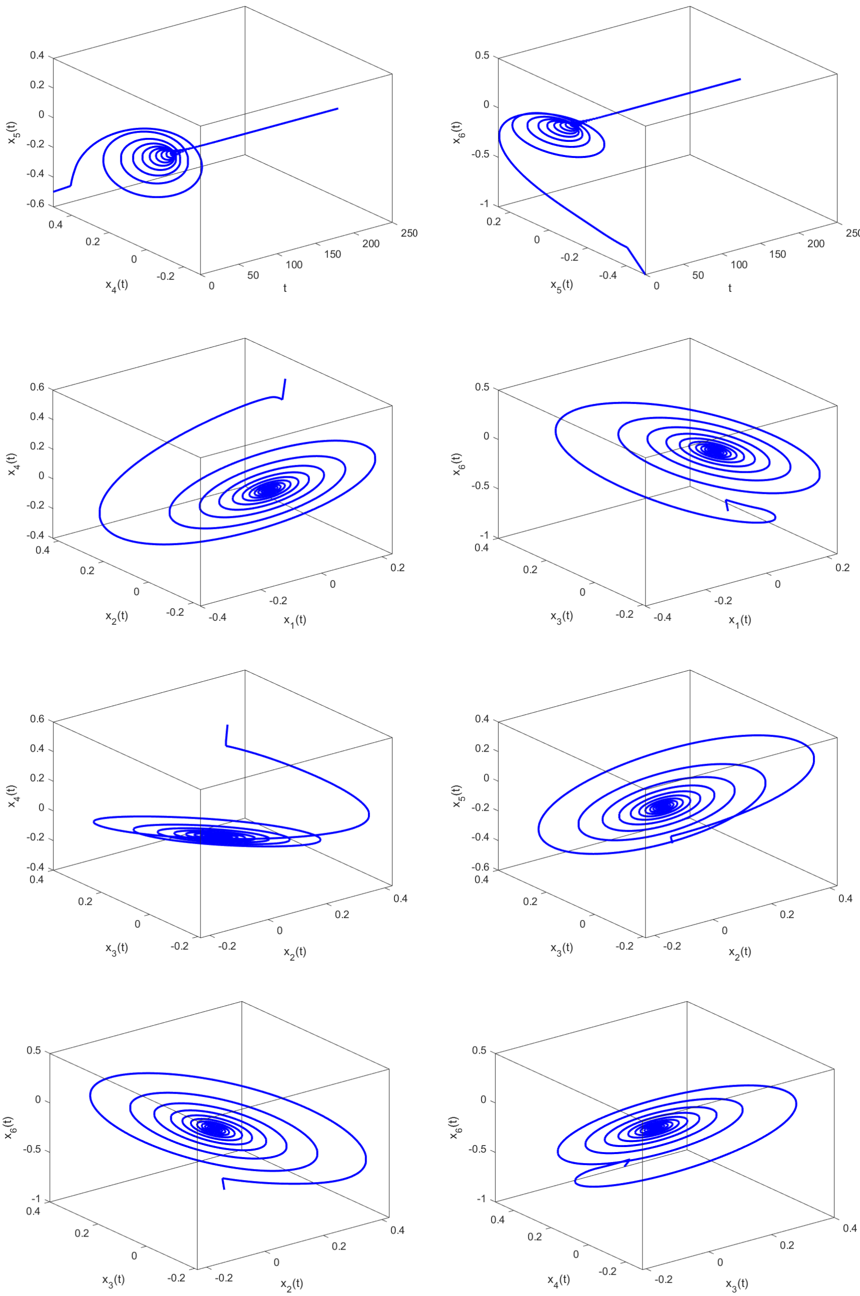

Figure 3. Part of the phase diagram of system (23) with respect to time t. When $\tau = 0.42 < \tau_0 = 0.4532$, the zero solution of the model is asymptotic stability.

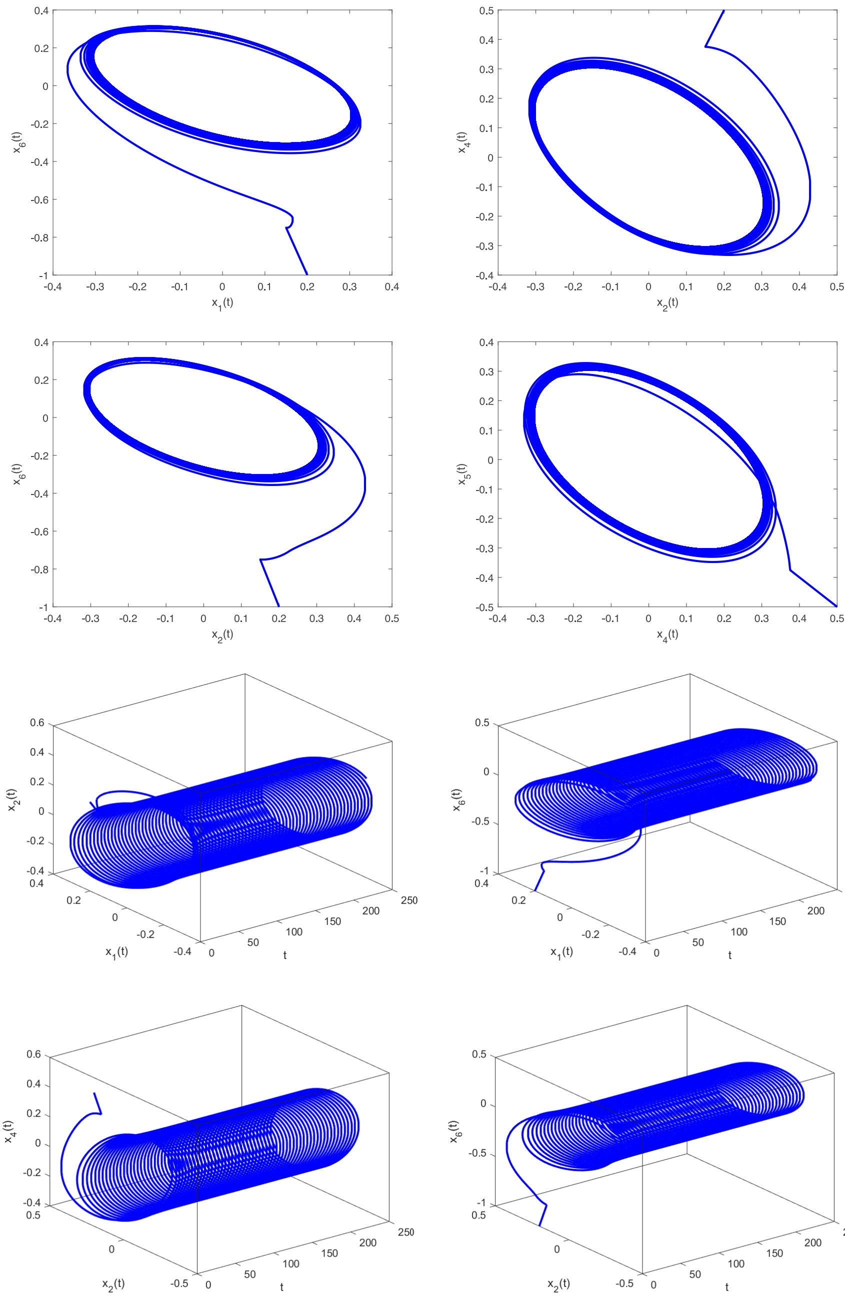

Figure 4. Part of the phase diagram of system (23) with respect to time t. When $\tau = 0.60 > \tau_0 = 0.4532$, the model loses stability and presents a Hopf bifurcation at the equilibrium point.

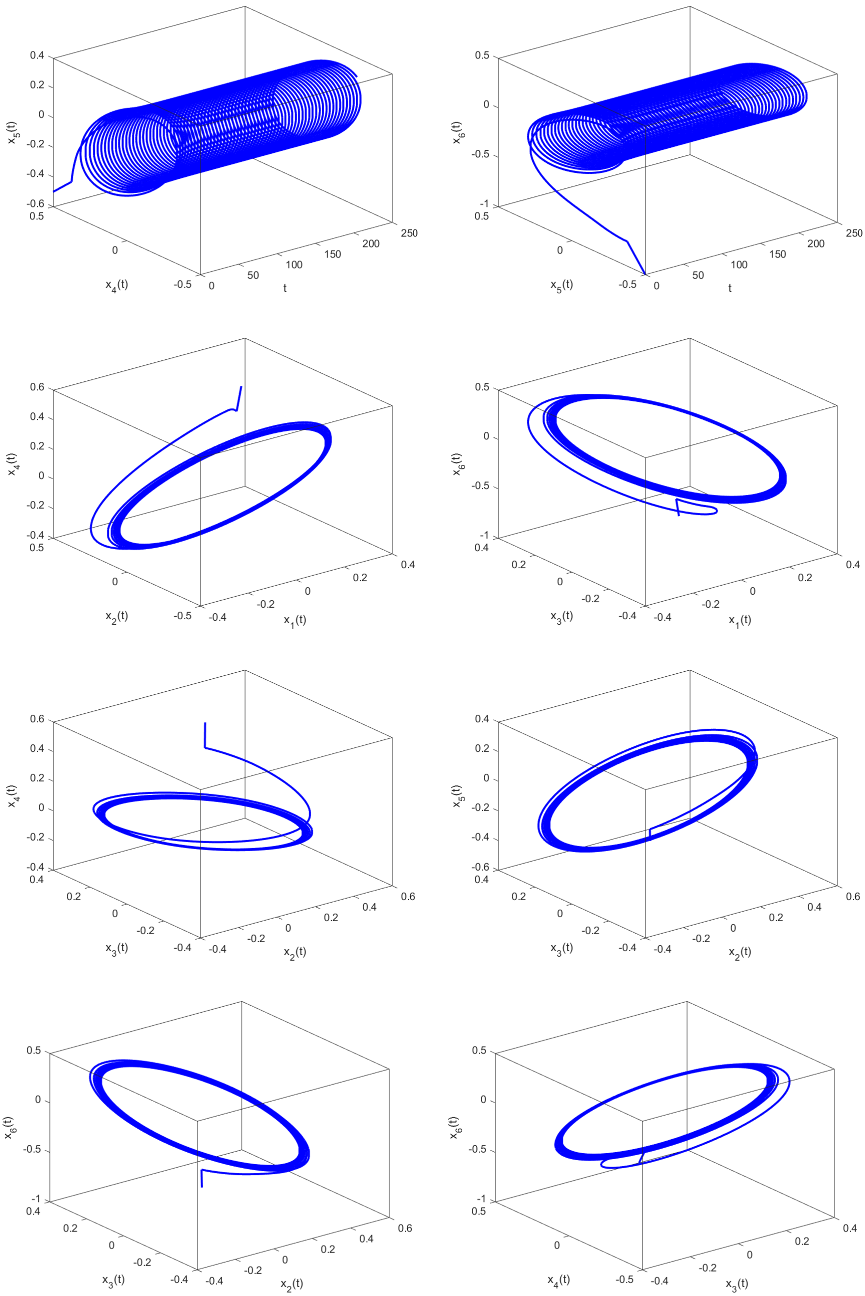

Figure 5. Part of the phase diagram of system (23) with respect to time t. When $\tau = 0.60 > \tau_0 = 0.4532$, the model loses stability and presents a Hopf bifurcation at the equilibrium point.

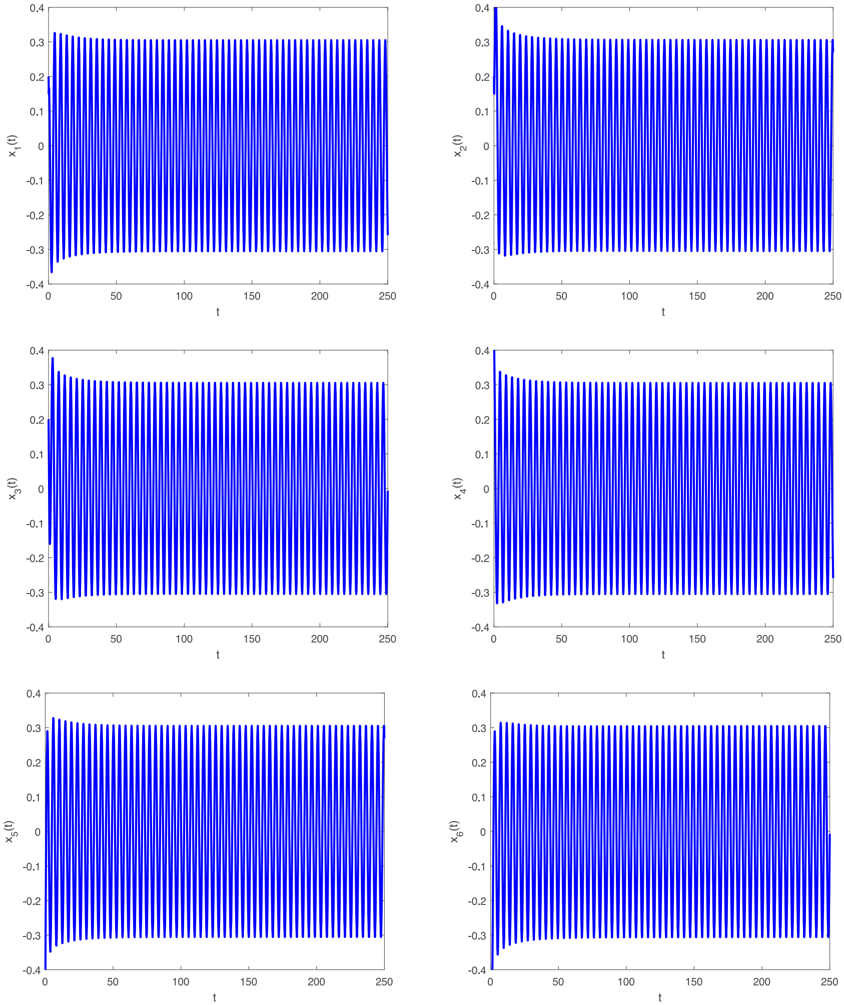

Figure 6. The trajectory of system (23) with respect to time t. When $\tau = 0.60 > \tau_0 = 0.4532$, the model loses stability and presents a Hopf bifurcation at the equilibrium point.

5 Conclusions

As is well known, the study for dynamics of fractional neural network systems with time delay has become a hot topic in the field of nonlinear research. However, most current neural network systems only involve the delay between adjacent nodes, and there is a lack of research on the delay between nonadjacent nodes. In contrast, the nearest-neighbor coupled neural network system studied in this paper connects each node to its surrounding adjacent or nonadjacent nodes, resulting in a more complex topology that is better suited to practical applications. This paper analyzes the characteristic equation of a particular class of multidelay fractional nearest-neighbor coupled neural network model and selects the sum of time delays as branch parameters to obtain the stability of the system at the equilibrium point, as well as the sufficient conditions for Hopf bifurcation. The research results indicate that time delay has a significant impact on the stability of the system. When the time delay exceeds a critical value, the system loses stability and produces Hopf bifurcation. The results of this study provide theoretical guidance for the practical application of neural network systems and open up possibilities for neural network systems to better serve human society.

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