Nonlinear Analysis: Modelling and Control, Vol. 29, No. 6, 1167–1185
https://doi.org/10.15388/namc.2024.29.37853

Vilnius University Press

Lie group analysis and its invariants for the class of multidimensional nonlinear wave equations*

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Received: May 15, 2024 / **Revised:** November 5, 2024 / **Published online:** December 1, 2024

Abstract. We systematically classify Lie symmetries of a class of $(2+1)$-dimensional nonlinear wave equations. Our methodology proposes a symmetry classification for Lie generators applicable to four distinct cases inherent within this equation. For each identified category, we comprehensively analyze symmetry reduction and delineate the invariant solutions. Furthermore, we extend our Lie symmetry analysis to encompass reduced $1+1$ partial differential equations (PDEs). Through our investigations, we establish local conservation laws corresponding to each conserved vector, employing the formal Lagrangian approach. Significantly, this classification constitutes a novel contribution to the scientific discourse, as it remains absent from extant literature to date.

Keywords: nonlinear wave equation, Lie symmetries, conservation laws, invariant solutions, symmetry algebra.

1 Introduction

The wave equation, a fundamental mathematical construct, finds widespread utility in elucidating the propagation characteristics of waves across diverse domains of physics, including electromagnetism, fluid mechanics, acoustics, hydrodynamics, general relativity, and quantum mechanics [3, 14]. Consequently, the exploration of wave equations represents a dynamic frontier in applied mathematics research.

*This work was funded by the Deanship of Scientific Research at King Khalid University as part of a large group research project under grant No. RGP.2/16/45.

Tracing its origins, investigations into wave phenomena extend back to antiquity, notably exemplified by Pythagoras' inquiry into the properties of sound waves emanating from vibrating strings in musical instruments during the sixth century BC [8]. After this early exploration, the scientific revolution catalyzed rapid advancements in our understanding of wave dynamics with the contributions of prominent scientists such as Bynam et al. [2] being particularly noteworthy. Furthermore, a transformative paradigm shift in our comprehension of wave phenomena occurred during the nineteenth century, marked by Maxwell's formulation of electromagnetic field theory [22]. This seminal development fundamentally altered the landscape of human perception regarding wave behavior, underscoring the profound impact of scientific inquiry on our understanding of the natural world.

Indeed, mathematicians have directed their efforts towards exploring the genesis and dynamics of waves. Consequently, substantial investigations have been undertaken to derive precise solutions for both linear and nonlinear wave equations. Notably, Cajori and Farlow achieved a significant milestone by obtaining the inaugural exact solution to the linear wave equation [4, 20]. Subsequent endeavors have delved into the realm of nonlinear wave equations, prompting the development of various approximate and numerical methodologies. A plethora of research works, exemplified by references [5, 9, 19, 23, 25], has contributed to this endeavor, further enriching our understanding of wave dynamics.

To comprehend these physical systems effectively, it is imperative to undertake a thorough investigation of wave equations. Numerous authors have delved into various categories of wave equations, either by scrutinizing their solutions or by examining their asymptotic behaviors [7, 21, 24]. Furthermore, certain researchers have opted for a numerical approach to tackle such equations [17]. Additionally, specific instances of wave equations have been subjected to analysis from the perspective of Lie symmetries. For instance, Raza et al. [16] conducted a study on the Lie symmetry analysis of the $1 + 1$ case of the nonlinear wave equation featuring a single arbitrary function. Similarly, Gandarias et al. [6] explored the same analysis for the identical equation but with two arbitrary functions. Ibragimov [10] delved into various forms of unperturbed damped wave equations, delineating their Lie symmetries. Moreover, certain researchers have utilized symmetries of specific classes of damped wave equations to derive analytical solutions [13]. Usamah et al. [1] contributed by classifying multidimensional nonlinear damped wave equations utilizing Lie symmetries. However, despite these endeavors, the investigation of $(2 + 1)$-dimensional wave equations in terms of the Lie point symmetries they accommodate has not received adequate attention in the literature. The focus has primarily been on classifying symmetries of classical wave equations thus far.

Undoubtedly, one of the primary applications of symmetries lies in the formulation of conservation laws for a given system. The pivotal connection between conservation laws and symmetries was initially elucidated by Noether in 1918 [15, 18]. While Noether's theorem offers a potent mechanism for deriving conservation laws, it is constrained by its applicability solely to variational PDEs, necessitating the existence of a Lagrangian. Consequently, to address this limitation, the formal Lagrangian approach has been devised to derive conservation laws for both variational and nonvariational PDEs. Numerous scholarly works expound upon the methodology of obtaining conservation laws via the formal Lagrangian approach, as documented in various references [11, 12].

This study endeavors to explore wave equations through an examination of the $(2+1)$-dimensional nonlinear wave equation delineated as follows:

$$U_{tt} = \operatorname{div}\bigl(F(U)\,\operatorname{grad} U\bigr). \tag{1}$$

The objective is to classify this equation based on the Lie point symmetries it accommodates. These identified symmetries will subsequently inform the execution of various similarity reductions, facilitating the acquisition of exact solutions wherever feasible. Furthermore, the discerned symmetries will be leveraged to formulate conservation laws for select cases of interest, employing the formal Lagrangian approach.

2 Symmetry classification

We shall examine Eq. (1), which can be expressed equivalently in the following manner:

$$U_{tt} = F(U)(U_{xx} + U_{yy}) + F_U\bigl(U_x^2 + U_y^2\bigr). \tag{2}$$

In general, a symmetry of a differential equation refers to a transformation that maintains the invariance of its set of solutions. To derive the symmetry algebra of Eq. (2), we consider the infinitesimal generator of the symmetry algebra in the following form:

$$\mathcal{Z} = \zeta_1\frac{\partial}{\partial x} + \zeta_2\frac{\partial}{\partial y} + \zeta_3\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial U}.$$

To complete our task, we must determine the values of ζ_1, ζ_2, ζ_3, and ρ, while verifying that the operator $\mathcal{Z}$ satisfies the condition for Lie symmetry

$$\mathcal{Z}^{[2]}\bigl(U_{tt} - F(U)(U_{xx} + U_{yy}) - F_U\bigl(U_x^2 + U_y^2\bigr)\bigr)\big|_{(2)} = 0, \tag{3}$$

where $\mathcal{Z}^{[2]}$ is the second extension of $\mathcal{Z}$.

Case 1: $F(U)$ is arbitrary function. Solving Eq. (3) provides us with the expressions for infinitesimals ζ_1, ζ_2, ζ_3, and ρ, thus leading to the establishment of five symmetry generators given by

$$\mathcal{Z}_1 = \frac{\partial}{\partial x}, \qquad \mathcal{Z}_2 = \frac{\partial}{\partial y}, \qquad \mathcal{Z}_3 = \frac{\partial}{\partial t},$$
$$\mathcal{Z}_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad \mathcal{Z}_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}.$$

Case 2: $F(U) = aU + b$, where a and b are constants. In this case, the solution of (3) leads to the conclusion that (2) features the generators $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, and $\mathcal{Z}_4$ along with

$$\mathcal{Z}_6 = t\frac{\partial}{\partial t} - \frac{(2aU + 2b)}{a}\frac{\partial}{\partial U}, \qquad \mathcal{Z}_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{(2aU + 2b)}{a}\frac{\partial}{\partial U}.$$

Case 3: $F(U) = c$ (constant). In this case, the solution of (3) leads to the conclusion that (2) features the generators $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, $\mathcal{Z}_4$, and $\mathcal{Z}_5$ along with

$$\begin{aligned}
\mathcal{Z}_8 &= U\frac{\partial}{\partial U},\ \mathcal{Z}_9 = t\frac{\partial}{\partial x} + \frac{x}{c}\frac{\partial}{\partial t}, \qquad \mathcal{Z}_{10} = t\frac{\partial}{\partial y} + \frac{y}{c}\frac{\partial}{\partial t},\\
\mathcal{Z}_{11} &= xy\frac{\partial}{\partial x} + \left(\frac{y^2}{2} + \frac{ct^2}{2} - \frac{x^2}{2}\right)\frac{\partial}{\partial y} + yt\frac{\partial}{\partial t} - \frac{yU}{2}\frac{\partial}{\partial U},\\
\mathcal{Z}_{12} &= xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + \left(\frac{ct^2 + x^2 + y^2}{2c}\right)\frac{\partial}{\partial t} - \frac{tU}{2}\frac{\partial}{\partial U},\\
\mathcal{Z}_{13} &= \left(\frac{y^2}{2} - \frac{ct^2}{2} - \frac{x^2}{2}\right)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - xt\frac{\partial}{\partial t} + \frac{xU}{2}\frac{\partial}{\partial U}.
\end{aligned}$$

Case 4: $F(U) = \mathrm{e}^{\alpha U}$, where α is a constant. In this case, the solution of (3) leads to the conclusion that (2) features the generators $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, and $\mathcal{Z}_4$ along with

$$\mathcal{Z}_{14} = t\frac{\partial}{\partial t} - \frac{2}{\alpha}\frac{\partial}{\partial U}, \qquad \mathcal{Z}_{15} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2}{\alpha}\frac{\partial}{\partial U}.$$

3 Invariant solutions

Similarity reduction for Case 1: $F(U)$ is arbitrary function

(i) For Lie operator $\mathcal{Z}_1 = \partial/\partial x$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}U}{0}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \theta(p, q)$, where $p = y$, $q = t$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$\theta_{qq} - F(\theta)\theta_{pp} - F'(\theta)\theta_p^2 = 0. \tag{4}$$

Let us revisit our method to tackle Eq. (4) with following results:

$$\zeta_p = c_1 p + c_2, \qquad \zeta_q = c_1 q + c_3, \qquad \rho_\theta = 0. \tag{5}$$

Case A. Under the stipulation that $c_2 = 1$ and rest constants are zero, we obtain the characteristic equation from (5) as

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}q}{0} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where $s = q$. By virtue of this transformation we can move on to present the simplified form of Eq. (4) as follows:

$$\tau'' = 0.$$

This gives

$$\tau(s) = c_1 s + c_2,$$

which implies,

$$\theta(p, q) = c_1 q + c_2.$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x, y, t) = c_1 t + c_2.$$

Similarity reduction for Case 2: $F(U) = aU + b$

(i) For Lie operator $\mathcal{Z}_1 + \mathcal{Z}_3 = \partial/\partial x + \partial/\partial t$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}U}{0}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \theta(p, q)$, where $p = y$, $q = -x + t$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$(-a\theta - b + 1)\theta_{qq} + (-a\theta - b)\theta_{pp} - a\big(\theta_p^2 + \theta_q^2\big) = 0. \tag{6}$$

Let us revisit our method to tackle Eq. (6), with following results:

$$\zeta_p = c_1 p + c_2, \qquad \zeta_q = c_1 q + c_3, \qquad \rho_\theta = 0. \tag{7}$$

Case A. Under the stipulation that $c_3 = 1$ and rest constants are zero, we obtain the characteristic equation from (7) as

$$\frac{\mathrm{d}p}{0} = \frac{\mathrm{d}q}{1} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where $s = p$. By virtue of this transformation we can move on to present the simplified form of Eq. (6) as follows:

$$(-a\tau - b)\tau'' - a{\tau'}^2 = 0.$$

This gives

$$\tau(s) = \frac{-b - \sqrt{(2c_1 s + 2c_2)a + b^2}}{a},$$

which implies

$$\theta(p, q) = \frac{-b - \sqrt{(2c_1 p + 2c_2)a + b^2}}{a}.$$

Thus, we can express the solution for Eq. (2) in the initial variables as follows:

$$U(x, y, t) = \frac{-b - \sqrt{(2c_1 y + 2c_2)a + b^2}}{a}.$$

(ii) For Lie operator $\mathcal{Z}_2 + \mathcal{Z}_3 = \partial/\partial y + \partial/\partial t$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{1} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}U}{0}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \theta(p, q)$, where $p = x$, $q = -y + t$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$(-a\theta - b + 1)\theta_{qq} + (-a\theta - b)\theta_{pp} - a\big(\theta_p^2 + \theta_q^2\big) = 0. \tag{8}$$

Let us revisit our method to tackle Eq. (8) with following results:

$$\zeta_p = c_1 p + c_2, \qquad \zeta_q = c_1 q + c_3, \qquad \rho_\theta = 0. \tag{9}$$

Case A. Under the stipulation that $c_2 = 1$, $c_3 = 1$, and rest constants are zero, we obtain the characteristic equation from (9) as

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}q}{1} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where $s = -p + q$. By virtue of this transformation we can move on to present the simplified form of Eq. (8) as follows:

$$(-2a\tau - 2b + 1)\tau'' - 2a{\tau'}^2 = 0.$$

This gives

$$\tau(s) = \frac{-2b + 1 + \sqrt{(4sc_1 + 4c_2)a + 4(b - 1/2)^2}}{2a},$$

which implies

$$\theta(p, q) = \frac{-2b + 1 + \sqrt{(4(-p + q)c_1 + 4c_2)a + 4(b - 1/2)^2}}{2a}.$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x, y, t) = \frac{-2b + 1 + \sqrt{(4(-x - y + t)c_1 + 4c_2)a + 4(b - 1/2)^2}}{2a}.$$

(iii) For Lie operator $\mathcal{Z}_3 = \partial/\partial t$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}U}{0}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \theta(p, q)$, where $p = x$, $q = y$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$(-a\theta - b)\theta_{qq} + (-a\theta - b)\theta_{pp} - a\big(\theta_p^2 + \theta_q^2\big) = 0. \tag{10}$$

Let us revisit our method to tackle Eq. (10) with following results:

$$\begin{aligned}
\zeta_p &= \mathrm{i} f_3(q - \mathrm{i}p) - \mathrm{i} f_4(q + \mathrm{i}p) + c_2,\\
\zeta_q &= f_3(q - \mathrm{i}p) + f_4(q + \mathrm{i}p),\\
\rho_\theta &= \frac{f_1(q - \mathrm{i}p) + f_2(q + \mathrm{i}p)}{\theta + b/a} + c_1\left(\theta + \frac{b}{a}\right).
\end{aligned} \tag{11}$$

Case A. Under the stipulation that $c_1 = 1$, $c_2 = 1$, and rest constants are zero, we obtain the characteristic equation from (11) as

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}q}{0} = \frac{\mathrm{d}\theta}{\theta + b/a}.$$

From this we obtain the similarity variables $\theta(p, q) = \mathrm{e}^p \tau(s) - b/a$, where $s = q$. By virtue of this transformation we can move on to present the simplified form of Eq. (10) as follows:

$$-2a\tau^2 - a\tau\tau'' - a\tau'^2 = 0.$$

This gives

$$\tau(s) = \sqrt{-c_1 \sin(2s) + c_2 \cos(2s)},$$

which implies

$$\theta(p, q) = \mathrm{e}^p \sqrt{-c_1 \sin(2q) + c_2 \cos(2q)} - \frac{b}{a}.$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x, y, t) = \mathrm{e}^x \sqrt{-c_1 \sin(2y) + c_2 \cos(2y)} - \frac{b}{a}.$$

Similarity reductions for Case 3: $F(U) = c$

(i) For Lie operator $\mathcal{Z}_3 + \mathcal{Z}_8 = \partial/\partial t + U\partial/\partial U$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}U}{U}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \mathrm{e}^t \theta(p, q)$, where $p = x$, $q = y$. By means of this transformation mechanism, Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$-c(\theta_{pp} + \theta_{qq}) + \theta = 0. \tag{12}$$

Let us revisit our method to tackle Eq. (12) with following results:

$$\begin{aligned}
\zeta_p &= -c_6 q + c_8, \qquad \zeta_q = c_6 p + c_7,\\
\rho_\theta &= \frac{1}{\mathrm{e}^{\sqrt{a_1 p}}}\biggl(c_5\bigl(\bigl(\mathrm{e}^{\sqrt{a_1 p}}\bigr)^2 c_2 + c_3\bigr) \cos \frac{\sqrt{ca_1 - 1}q}{\sqrt{c}}\\
&\quad + c_4\bigl(\bigl(\mathrm{e}^{\sqrt{a_1 p}}\bigr)^2 c_2 + c_3\bigr) \sin \frac{\sqrt{a_1 c - 1}q}{\sqrt{c}} + c_1 \theta \mathrm{e}^{\sqrt{a_1 p}}\biggr).
\end{aligned} \tag{13}$$

Case A. Under the stipulation that $c_6 = 1$ and rest constants are zero, we obtain the characteristic equation from (13) as

$$\frac{\mathrm{d}p}{-q} = \frac{\mathrm{d}q}{p} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p,q) = \tau(s)$, where $s = p^2 + q^2$. By virtue of this transformation we can move on to present the simplified form of Eq. (12) as follows:

$$-4cs\tau'' - 4c\tau' + \tau = 0.$$

This gives

$$\tau(s) = c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{s}}{\sqrt{c}}\right) + c_2 \operatorname{BesselY}\left(0, \mathrm{i}\frac{\sqrt{s}}{\sqrt{c}}\right),$$

which implies

$$\theta(p,q) = c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{p^2+q^2}}{\sqrt{c}}\right) + c_2 \operatorname{BesselY}\left(0, \mathrm{i}\frac{\sqrt{p^2+q^2}}{\sqrt{c}}\right).$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x,y,t) = \mathrm{e}^t\left(c_1 \operatorname{BesselI}\left(0, \frac{\sqrt{x^2+y^2}}{\sqrt{c}}\right) + c_2 \operatorname{BesselY}\left(0, \mathrm{i}\frac{\sqrt{x^2+y^2}}{\sqrt{c}}\right)\right).$$

Case B. Under the stipulation that $c_1 = 1$, $c_7 = 1$, $c_8 = 1$, and rest constants are zero, we obtain the characteristic equation from (13) as

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}q}{1} = \frac{\mathrm{d}\theta}{\theta}.$$

From this we obtain the similarity variables $\theta(p,q) = \mathrm{e}^p\tau(s)$, where $s = -p + q$. By virtue of this transformation we can move on to present the simplified form of Eq. (12) as follows:

$$-\frac{2c}{\mathrm{e}^s}\tau'' + \frac{2c}{\mathrm{e}^s}\tau' - \frac{c}{\mathrm{e}^s}\tau = 0.$$

This gives

$$\tau(s) = c_1\mathrm{e}^{(c+\mathrm{i}\sqrt{c}\sqrt{c-2})s/(2c)} + c_2\mathrm{e}^{(c-\mathrm{i}\sqrt{c}\sqrt{c-2})s/(2c)},$$

which implies,

$$\theta(p,q) = \mathrm{e}^p\left(c_1\mathrm{e}^{(c+\mathrm{i}\sqrt{c}\sqrt{c-2})(-p+q)/(2c)} + c_2\mathrm{e}^{(c-\mathrm{i}\sqrt{c}\sqrt{c-2})(-p+q)/(2c)}\right).$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x,y,t) = \mathrm{e}^{x+t}\left(c_1\mathrm{e}^{(c+\mathrm{i}\sqrt{c}\sqrt{c-2})(-x+y)/(2c)} + c_2\mathrm{e}^{(c-\mathrm{i}\sqrt{c}\sqrt{c-2})(-x+y)/(2c)}\right).$$

(ii) For Lie operator $\mathcal{Z}_1 + \mathcal{Z}_8 = \partial/\partial x + U\partial/\partial U$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}U}{U}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \mathrm{e}^x\theta(p, q)$, where $p = t$, $q = y$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$-c(\theta_{qq} + \theta_{pp}) - c\theta = 0. \tag{14}$$

Let us revisit our method to tackle Eq. (14) with following results:

$$\begin{aligned} &\zeta_p = c_4 q + c_5, \qquad \zeta_q = c_4 cp + c_6, \\ &\rho_\theta = c_1\theta + c_2\mathrm{e}^{a_1(p\sqrt{c}+q)}c_3\mathrm{e}^{\sqrt{c}(-q/\sqrt{c}+p)/(4a_1)}. \end{aligned} \tag{15}$$

Case A. Under the stipulation that $c_4 = 1$, $c_6 = 1$, and rest constants are zero, we obtain the characteristic equation from (15) as

$$\frac{\mathrm{d}p}{q} = \frac{\mathrm{d}q}{cp + 1} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where $s = -cp^2 + q^2 - 2p$. By virtue of this transformation we can move on to present the simplified form of Eq. (14) as follows:

$$(-4cs + 4)\tau'' - c(4\tau' + \tau) = 0.$$

This gives

$$\tau(s) = c_1 \operatorname{BesselJ}\left(0, \frac{\sqrt{-1 + cs}}{\sqrt{c}}\right) + c_2 \operatorname{BesselY}\left(0, \frac{\sqrt{-1 + cs}}{\sqrt{c}}\right),$$

which implies

$$\begin{aligned} \theta(p, q) = c_1 \operatorname{BesselJ}&\left(0, \frac{\sqrt{-1 + c(-cp^2 + q^2 - 2p)}}{\sqrt{c}}\right) \\ &+ c_2 \operatorname{BesselY}\left(0, \frac{\sqrt{-1 + c(-cp^2 + q^2 - 2p)}}{\sqrt{c}}\right). \end{aligned}$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$\begin{aligned} U(x, y, t) = \mathrm{e}^x\biggl(c_1 \operatorname{BesselJ}&\left(0, \frac{\sqrt{-1 + c(-ct^2 + y^2 - 2t)}}{\sqrt{c}}\right) \\ &+ c_2 \operatorname{BesselY}\left(0, \frac{\sqrt{-1 + c(-ct^2 + y^2 - 2t)}}{\sqrt{c}}\right)\biggr). \end{aligned}$$

(iii) For Lie operator $\mathcal{Z}_9 = t\partial/\partial x + (x/c)\partial/\partial t$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{t} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{\frac{x}{c}} = \frac{\mathrm{d}U}{0}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x,y,t) = \theta(p,q)$, where $p = y$, $q = t^2 - x^2/c$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1+1)$ PDE outlined as follows:

$$4q\theta_{qq} - c\theta_{pp} + 4\theta_q = 0. \tag{16}$$

Let us revisit our method to tackle Eq. (16) with following results:

$$\begin{aligned} &\zeta_p = c_1 q + \frac{c_1}{c}p^2 + c_2 p + c_3, \qquad \zeta_q = 2q\left(\frac{2p}{c}c_1 + c_2\right), \\ &\rho_\theta = \frac{1}{c\mathrm{e}^{\sqrt{a_1}p}}\Big(\big(\mathrm{e}^{\sqrt{a_1}p}\theta c_4 + \big(c_7 \operatorname{BesselJ}(0, \sqrt{-ca_1}\sqrt{q}) \\ &\quad + c_8 \operatorname{BesselY}(0, \sqrt{-ca_1}\sqrt{q})\big)\big(\big(\mathrm{e}^{\sqrt{a_1}p}\big)^2 c_5 + c_6\big)\big)c - c_1\mathrm{e}^{\sqrt{a_1}p}p\theta\Big). \end{aligned} \tag{17}$$

Case A. Under the stipulation that $c_2 = 1$ and rest constants are zero, we obtain the characteristic equation from (17) as

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}q}{2q} = \frac{\mathrm{d}\theta}{0}.$$

From this, we obtain the similarity variables $\theta(p,q) = \tau(s)$ where, $s = q/p^2$. By virtue of this transformation we can move on to present the simplified form of Eq. (16) as follows:

$$-4cs^2\tau'' - 6cs\tau' + 4s\tau'' + 4\tau' = 0.$$

This gives

$$\tau(s) = c_1 + c_2 \arctan(\sqrt{cs - 1}),$$

which implies

$$\theta(p,q) = c_1 + c_2 \arctan\left(\sqrt{c\frac{q}{p^2} - 1}\right).$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x,y,t) = c_1 + c_2 \arctan\left(\sqrt{\frac{ct^2 - x^2}{y^2} - 1}\right).$$

(iv) For Lie operator $\mathcal{Z}_8 + \mathcal{Z}_9 = U\partial/\partial U + t\partial/\partial x + (x/c)\partial/\partial t$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{t} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{x/c} = \frac{\mathrm{d}U}{U}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = (\sqrt{c}x + ct)^{\sqrt{c}}\theta(p, q)$, where $p = y$, $q = t^2 - x^2/c$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$- c^{3/2}\theta_{pp} + 4\sqrt{c}\, q\theta_{qq} + 4\sqrt{c}\theta_q + 4c\theta_q = 0. \tag{18}$$

Let us revisit our method to tackle Eq. (18) with following results:

$$\begin{aligned}
&\zeta_p = c_1 q + \frac{c_1}{c}p^2 + c_2 p + c_3, \qquad \zeta_q = 2q\left(\frac{2p}{c}c_1 + c_2\right),\\
&\rho_\theta = \frac{1}{c^{3/2}\mathrm{e}^{\sqrt{a_1}p}}\Big(\big(\big(\mathrm{e}^{\sqrt{a_1}p}\big)^2 c_5 + c_6\big)\big(\operatorname{BesselJ}(\sqrt{c}, \sqrt{-ca_1}\sqrt{q})c_7\\
&\quad + \operatorname{BesselY}(\sqrt{c}, \sqrt{ca_1}\sqrt{q})c_8\big)c^{3/2}q^{-\sqrt{c}/2}\\
&\quad + \mathrm{e}^{\sqrt{a_1}p}\big(-c_1 p\sqrt{c} + c_4 c^{3/2} - 2c_1 pc\big)\theta\Big).
\end{aligned} \tag{19}$$

Case A. Under the stipulation that $c_2 = 1$, $c_3 = 1$, and rest constants are zero, we obtain the characteristic equation from (19) as

$$\frac{\mathrm{d}p}{p+1} = \frac{\mathrm{d}q}{2q} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where, $s = q/((p + 1)^2)$. By virtue of this transformation we can move on to present the simplified form of Eq. (18) as follows:

$$\big(-4s^2\tau'' - 6s\tau'\big)c^{3/2} + 4\sqrt{c}s\tau'' + 4(c + \sqrt{c}\,)\tau' = 0.$$

This gives

$$\tau(s) = c_1 + \operatorname{hypergeom}\left(\left[-\sqrt{c}, \frac{1}{2}, -\sqrt{c}\right], [1 - \sqrt{c}\,], cs\right)s^{-\sqrt{c}}c_2,$$

which implies

$$\begin{aligned}
\theta(p, q) = c_1 + \operatorname{hypergeom}&\left(\left[-\sqrt{c}, \frac{1}{2}, -\sqrt{c}\right], [1 - \sqrt{c}\,], \frac{cq}{(p+1)^2}\right)\\
&\times q^{-\sqrt{c}}(p+1)^{2\sqrt{c}}c_2.
\end{aligned}$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$\begin{aligned}
U(x, y, t) = \Bigg(c_1 + \operatorname{hypergeom}&\left(\left[-\sqrt{c}, \frac{1}{2}, -\sqrt{c}\right], [1 - \sqrt{c}\,], \frac{c(t^2 - x^2/c)}{(y+1)^2}\right)\\
&\times \left(t^2 - \frac{x^2}{c}\right)^{-\sqrt{c}}(y+1)^{2\sqrt{c}}c_2\Bigg)(\sqrt{c}x + ct)^{\sqrt{c}}.
\end{aligned}$$

Similarity reductions for Case 4: $F(U) = \mathrm{e}^{\alpha U}$

(i) For Lie operator $\mathcal{Z}_1 = \partial/\partial x$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}U}{0}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = \theta(p, q)$, where $p = t$, $q = y$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1 + 1)$ PDE outlined as follows:

$$\left(-\alpha\theta_q^2 - \theta_{qq}\right)\mathrm{e}^{\alpha\theta} + \theta_{pp} = 0. \tag{20}$$

Let us revisit our method to tackle Eq. (20) with following results:

$$\zeta_p = c_1 p + c_2, \qquad \zeta_q = c_3 q + c_4, \qquad \rho_\theta = \frac{-2c_1 + 2c_3}{\alpha}. \tag{21}$$

Case A. Under the stipulation that $c_2 = 1$, $c_4 = 1$, and rest constants are zero, we obtain the characteristic equation from (21) as

$$\frac{\mathrm{d}p}{1} = \frac{\mathrm{d}q}{1} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where $s = -p + q$. By virtue of this transformation we can move on to present the simplified form of Eq. (20) as follows:

$$\left(-\alpha\tau'^2 - \tau''\right)\mathrm{e}^{\alpha\tau} + \tau'' = 0.$$

This gives

$$\tau(s) = \frac{-\operatorname{LambertW}\left(-\mathrm{e}^{\alpha(-c_1 s - c_2)}\right)}{\alpha} - c_1 s - c_2,$$

which implies

$$\theta(p, q) = \frac{-\operatorname{LambertW}\left(-\mathrm{e}^{\alpha(-c_1(-p+q) - c_2)}\right)}{\alpha} - c_1(-p + q) - c_2.$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x, y, t) = \frac{-\operatorname{LambertW}\left(-\mathrm{e}^{\alpha(-c_1(-t+y) - c_2)}\right)}{\alpha} - c_1(-t + y) - c_2.$$

(ii) For Lie operator $\mathcal{Z}_{14} = t\partial/\partial t - (2/\alpha)\partial/\partial U$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}U}{-2/\alpha}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = -(2/\alpha)\ln(t) + \theta(p, q)$, where $p = x$, $q = y$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1+1)$ PDE outlined as follows:

$$2 - \alpha\big(\alpha\theta_p^2 + \alpha\theta_q^2 + \theta_{pp} + \theta_{qq}\big)\mathrm{e}^{\alpha\theta} = 0. \tag{22}$$

Let us revisit our method to tackle Eq. (22) with following results:

$$\begin{aligned}
\zeta_p &= \frac{\alpha(f_4(q - \mathrm{i}p) + f_6(q + \mathrm{i}p))}{2} - c_2 q + \frac{c_1}{2}\alpha p + c_4,\\
\zeta_q &= -\frac{\mathrm{i}\alpha}{2}\big(-f_6(q + \mathrm{i}p) + f_4(q - \mathrm{i}p)\big) + c_2 p + \frac{c_1}{2}\alpha q + c_3,\\
\rho_\theta &= c_1 + \big(f_3(q - \mathrm{i}p) + p f_4(q - \mathrm{i}p) + f_5(q + \mathrm{i}p) + f_6(q + \mathrm{i}p)p\big)\mathrm{e}^{-\alpha\theta}.
\end{aligned} \tag{23}$$

Case A. Under the stipulation that $c_2 = 1$ and rest constants are zero, we obtain the characteristic equation from (23) as

$$\frac{\mathrm{d}p}{-q} = \frac{\mathrm{d}q}{p} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p, q) = \tau(s)$, where $s = p^2 + q^2$. By virtue of this transformation we can move on to present the simplified form of Eq. (22) as follows:

$$2 - 4\alpha\big(\alpha s {\tau'}^2 + s\tau'' + \tau'\big)\mathrm{e}^{\alpha\tau} = 0.$$

This gives

$$\tau(s) = \frac{-\ln 2 + \ln(-2c_2\alpha\ln(s) + 2c_1\alpha + s)}{\alpha},$$

which implies

$$\theta(p, q) = \frac{-\ln 2 + \ln(-2c_2\alpha\ln(p^2 + q^2) + 2c_1\alpha + p^2 + q^2)}{\alpha}.$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$U(x, y, t) = -\frac{2}{\alpha}\ln t + \frac{-\ln 2 + \ln(-2c_2\alpha\ln(x^2 + y^2) + 2c_1\alpha + x^2 + y^2)}{\alpha}.$$

(iii) For Lie operator $\mathcal{Z}_3 + \mathcal{Z}_{14} = \partial/\partial t + t\partial/\partial t - (2/\alpha)\partial/\partial U$, the formulation of the Lagrange equation can be accomplished as follows:

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}t}{1+t} = \frac{\mathrm{d}U}{-2/\alpha}.$$

Upon solving the Lagrange equation stated above, we attain the solution $U(x, y, t) = -(2/\alpha)\ln(1+t) + \theta(p, q)$, where $p = x$, $q = y$. By means of this transformation mechanism Eq. (2) undergoes conversion into the $(1+1)$ PDE outlined as follows:

$$2 - \alpha\big(\alpha\theta_p^2 + \alpha\theta_q^2 + \theta_{pp} + \theta_{qq}\big)\mathrm{e}^{\alpha\theta} = 0. \tag{24}$$

Let us revisit our method to tackle Eq. (24) with following results:

$$\begin{aligned}\zeta_p &= \frac{\alpha(f_4(q-\mathrm{i}p)+f_6(q+\mathrm{i}p)}{2} - c_2 q + \frac{c_1}{2}\alpha p + c_4,\\ \zeta_q &= -\frac{\mathrm{i}\alpha}{2}\bigl(-f_6(q+\mathrm{i}p)+f_4(q-\mathrm{i}p)\bigr) + c_2 p + \frac{c_1}{2}\alpha q + c_3,\\ \rho_\theta &= c_1 + \bigl(f_3(q-\mathrm{i}p)+pf_4(q-\mathrm{i}p)+f_5(q+\mathrm{i}p)+f_6(q+\mathrm{i}p)p\bigr)\mathrm{e}^{-\alpha\theta}.\end{aligned} \tag{25}$$

Case A. Under the stipulation that $c_2 = 1$, $c_3 = 1$, $c_4 = 1$, and rest constants are zero, we obtain the characteristic equation from (25) as

$$\frac{\mathrm{d}p}{-q+1} = \frac{\mathrm{d}q}{p+1} = \frac{\mathrm{d}\theta}{0}.$$

From this we obtain the similarity variables $\theta(p,q) = \tau(s)$, where $s = -q^2/2 - p^2/2 + q - p$. By virtue of this transformation we can move on to present the simplified form of Eq. (24) as follows:

$$2 + 2\alpha\bigl((s-1)\tau'' + \tau'\bigl(1+\alpha(s-1)\tau'\bigr)\bigr)\mathrm{e}^{\alpha\tau} = 0.$$

This gives

$$\tau(s) = \frac{\ln\left(-c_2\alpha\ln\left(s-1\right) + c_1\alpha - \ln\left(s-1\right) - s + 1\right)}{\alpha},$$

which implies

$$\begin{aligned}\theta(p,q) = \frac{1}{\alpha}\bigl(&-\ln 2 + \ln\bigl(-2c_2\alpha\ln\bigl(-p^2-q^2-2p+2q-2\bigr) + 2c_2\alpha\ln 2 + 2c_1\alpha\\ &+ p^2 + q^2 - 2\ln\bigl(-p^2-q^2-2p+2q-2\bigr) + 2\ln 2 + 2p - 2q + 2\bigr)\bigr).\end{aligned}$$

Thus, we can express the solution for Eq. (2) in the initial variables as

$$\begin{aligned}U(x,y,t) = -\frac{2}{\alpha}\ln(1+t) + \frac{1}{\alpha}\bigl(&-\ln 2 + \ln\bigl(-2c_2\alpha\ln\bigl(-x^2-y^2-2x+2y-2\bigr)\\ &+ 2c_2\alpha\ln 2 + 2c_1\alpha + x^2 + y^2 - 2\ln\bigl(-x^2-y^2-2x+2y-2\bigr)\\ &+ 2\ln 2 + 2x - 2y + 2\bigr)\bigr).\end{aligned}$$

4 Conservation laws for Eq. (2)

In the cited literature [11,12], Ibragimov's seminal theorem focuses on the preserved flow of differential equations. This theorem showcases notable versatility in scenarios, where the count of equations corresponds to the number of dependent variables in the system. We examine a PDE of kth order represented as

$$\mathcal{H} = \mathcal{H}(\mathbf{x}, U, U_1, U_2, \dots, U_p). \tag{26}$$

In the provided context, where $U = U(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(x_1, x_2, \dots, x_m)$, the formal Lagrangian for Eq. (26) being given, we can derive an adjoint equation as follows:

$$\mathcal{H}^\star \equiv \frac{\delta}{\delta U}(w\mathcal{H}), \tag{27}$$

where the operator $\delta/\delta U$ is defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} + \sum_{i=1}^{\infty}(-1)^s \mathcal{D}_{i_1}\cdots\mathcal{D}_{i_s}\frac{\partial}{\partial U_{i_1\cdots i_s}},$$

and the total derivative operator $\mathcal{D}_i$ is given by

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + U_i\frac{\partial}{\partial U} + U_{ij}\frac{\partial}{\partial U_j} + \cdots.$$

Theorem 1. *For any Lie point, Lie–Bäcklund, or nonlocal symmetry of Eq.* (26) *as specified by*

$$\mathcal{Z} = \zeta^i\frac{\partial}{\partial x^i} + \rho\frac{\partial}{\partial U},$$

where $\mathcal{L}$ serves as a formal Lagrangian, the conserved vectors for Eq. (27) *can be formally defined as*

$$\begin{aligned}\mathcal{F}^i = \zeta^i\mathcal{L} + P\biggl[\frac{\partial\mathcal{L}}{\partial U_i} - \mathcal{D}_j\biggl(\frac{\partial\mathcal{L}}{\partial U_{ij}}\biggr) + \mathcal{D}_j\mathcal{D}_k\biggl(\frac{\partial\mathcal{L}}{\partial U_{ijk}}\biggr) + \cdots\biggr]\\ + \mathcal{D}_j(P)\biggl[\frac{\partial\mathcal{L}}{\partial U_{ij}} - \mathcal{D}_k\biggl(\frac{\partial\mathcal{L}}{\partial U_{ijk}}\biggr) + \cdots\biggr] + \mathcal{D}_j\mathcal{D}_k(P)\biggl[\frac{\partial\mathcal{L}}{\partial U_{ijk}} + \cdots\biggr]\cdots.\end{aligned}$$

In this context, P is defined as

$$P = \rho - \zeta^i U_i,$$

where $\mathcal{D}_i(\mathcal{F}^i) = 0$.

Theorem 2. *The adjoint equation for Eq.* (2) *when $F(U)$ is arbitrary function, expressed as follows*:

$$\mathcal{H}^\star = U_{tt} - \mathrm{e}^{\alpha U}(U_{xx} + U_{yy}) - \alpha\mathrm{e}^{\alpha U}\bigl(U_x^2 + U_y^2\bigr) = 0,$$

where

$$\mathcal{L} = w(x,y,t)\bigl(U_{tt} - F(U)(U_{xx} + U_{yy}) - F_U\bigl(U_x^2 + U_y^2\bigr)\bigr).$$

Under the principles delineated in the Ibragimov theorem, each symmetry generator corresponds to a conserved vector. Accordingly, we advance to compute the conserved vectors using Theorem 1.

(I) When examining the vector field $\mathcal{Z}_1 = \partial/\partial x$ and $P = -U_x$, the ensuing conserved vectors are as follows:

$$\begin{aligned}
\mathcal{F}_1^t &= w(U_x - U_{xx}),\\
\mathcal{F}_1^x &= w\big(U_{tt} - F(U)(U_{xx} + U_{yy}) - F_U\big(U_x^2 + U_y^2\big)\big)\\
&\quad - U_x(-2F_U w U_x + w_x F(U) + wU_x F_U) + wU_{xx}F(U),\\
\mathcal{F}_1^y &= -U_x\big(-2wU_y F_U + w_x F(U) + ww_x F_U\big) + wU_{xy}F(U).
\end{aligned}$$

Similarly, we can compute the conserved vectors for remaining symmetry generators.

Now we consider Case 2 when $F(U) = aU + b$, where a and b are constants.

Theorem 3. *The adjoint equation for Eq.* (2) *when $F(U) = aU + b$, where a and b are constants, expressed as follows*:

$$\mathcal{H}^\star = w_{tt} - aw_{xx}U - aUw_{yy} - bw_{xx} - bw_{yy} = 0,$$

where

$$\mathcal{L} = w(x, y, t)\big(U_{tt} - (aU + b)(U_{xx} + U_{yy}) - a(U_x^2 + U_y^2)\big).$$

Based on Ibragimov's theorem (Theorem 1), we advance towards computing conserved vectors in the following manner.

(II) When examining the vector field $\mathcal{Z}_1 = \partial/\partial x$ and $P = -U_x$, the ensuing conserved vectors are as follows:

$$\begin{aligned}
\mathcal{F}_2^t &= w(U_x - U_{xx}),\\
\mathcal{F}_2^x &= w\big(U_{tt} - (aU + b)(U_{xx} + U_{yy}) - a\big(U_x^2 + U_y^2\big)\big)\\
&\quad - U_x(-2awU_x + w_x(aU + b) + awU_x) + wU_{xx}(aU + b)\\
&\quad - U_x\big(-2F_U wU_x + w_x F(U) + wU_x F_U\big) + wU_{xx}F(U),\\
\mathcal{F}_2^y &= -U_x\big(-2awU_y + w_x(aU + b) + awU\big) + wU_{xy}(aU + b).
\end{aligned}$$

Similarly, we can compute the conserved vectors for remaining symmetry generators. Now we consider Case 3 when $F(U) = c$ (constant).

Theorem 4. *The adjoint equation for Eq.* (2) *when $F(U) = c$, where c is a constant, expressed as follows*:

$$\mathcal{H}^\star = U_{tt} - cw_{xx} - cw_{yy} = 0,$$

where

$$\mathcal{L} = w(x, y, t)\big(U_{tt} - c(U_{xx} + U_{yy})\big).$$

Based on Ibragimov's theorem (Theorem 1), we advance towards computing conserved vectors in the following manner.

(III) When examining the vector field $\mathcal{Z}_1 = \partial/\partial x$ and $P = -U_x$, the ensuing conserved vectors are as follows:

$$\begin{aligned}
\mathcal{F}_3^t &= w(U_x - U_{xx}),\\
\mathcal{F}_3^x &= w\big(U_{tt} - c(U_{xx} + U_{yy})\big) - cw_xU_x + cwU_{xx},\\
\mathcal{F}_3^y &= -c(w_xU_x - wU_{xy}).
\end{aligned}$$

Similarly, we can compute the conserved vectors for remaining symmetry generators. Now we consider Case 4 when $F(U) = \mathrm{e}^{\alpha U}$, where α is a constant.

Theorem 5. *The adjoint equation for Eq.* (2) *when* $\mathbf{F}(\mathbf{U}) = \mathrm{e}^{\alpha \mathbf{U}}$, *where* α *is a constant, expressed as follows*:

$$\mathcal{H}^\star = w_{tt} - \mathrm{e}^{\alpha U} w_{xx} - \mathrm{e}^{\alpha U} w_{yy} = 0,$$

where

$$\mathcal{L} = w(x,y,t)\big(U_{tt} - \mathrm{e}^{\alpha U}(U_{xx} + U_{yy}) - \alpha \mathrm{e}^{\alpha U}\big(U_x^2 + U_y^2\big)\big).$$

Based on Ibragimov's theorem (Theorem 1), we advance towards computing conserved vectors in the following manner.

(IV) When examining the vector field $\mathcal{Z}_1 = \partial/\partial x$ and $P = -U_x$, the ensuing conserved vectors are as follows:

$$\begin{aligned}
\mathcal{F}_4^t &= w(U_x - U_{xx}),\\
\mathcal{F}_4^x &= \mathrm{e}^{\alpha U}\big(-\alpha wU_y^2 + \alpha wU_x^2 - \alpha wU_x^2 - wU_{yy} - w_xU_x + wU_{tt}\big),\\
\mathcal{F}_4^y &= \mathrm{e}^{\alpha U}\big(2\alpha wU_xU_y - \alpha wU_x^2 + wU_{xy} - w_xU_x\big).
\end{aligned}$$

5 Discussion and conclusion

This paper undertook a comprehensive symmetry classification of the $(2+1)$-dimensional nonlinear wave equation. It has been demonstrated that the equation admits a minimal subalgebra of five dimensions. In certain intriguing scenarios, this algebra can be expanded to encompass ten additional symmetries. Additionally, reductions of the analyzed equation have been conducted, leading to the derivation of several exact solutions. Moreover, the formal Lagrangian has been provided for the majority of cases discussed in the paper, while conservation laws have been formulated for select cases of interest. The investigation presented herein sets the stage for further exploration into nonlinear wave equations prevalent in mathematical physics and other scientific domains. Specifically, the exploration of the $(3+1)$ nonlinear wave equation could be a promising avenue for future research.

Author contributions. All authors (A.H., M.U., F.D.Z., and A.M.Z.) have contributed as follows: methodology, A.H., M.U.; formal analysis, F.D.Z. and A.M.Z.; software, A.H., M.U., and F.D.Z.; validation, F.D.Z., and A.M.Z.; writing – original draft preparation, A.H., and M.U.; writing – review and editing, A.H., and M.U. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

References

1. U.S. Al-Ali, A.H. Bokhari, A.H. Kara, F.D. Zaman, On the symmetries and conservation laws of the multidimensional nonlinear damped wave equations, *Adv. Math. Phys.*, **2017**(1):9401205, 2017, https://doi.org/10.1155/2017/9401205.
2. W.F. Bynum, E.J. Browne, R. Porter (Eds.), *Dictionary of the History of Science*, Princeton Univ. Press, Princeton, 1981.
3. R.G. Dean, R.A. Dalrymple, *Water Wave Mechanics for Engineers and Scientists*, World Scientific, New Jersey, 1991, https://doi.org/10.1142/1232.
4. S.J. Farlow, *Partial Differential Equations for Scientists and Engineers*, Dover Books Math., Dover Publications, Mineola, NY, 1993.
5. B. Fornberg, G.B. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, *Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **289**(1361):373–404, 1978, https://doi.org/10.1098/rsta.1978.0064.
6. M.L. Gandarias, M. Torrisi, A. Valenti, Symmetry classification and optimal systems of a non-linear wave equation, *Int. J. Non-Linear Mech.*, **39**(3):389–398, 2004, https://doi.org/10.1016/S0020-7462(02)00195-6.
7. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differ. Equations*, **109**(2):295–308, 1994, https://doi.org/10.1006/jdeq.1994.1051.
8. G.W. Griffiths, W.E. Schiesser, Linear and nonlinear waves, http://www.scholarpedia.org/, 2009.
9. A. Hussain, M. Usman, B.R. Al-Sinan, W.M. Osman, T.F. Ibrahim, Symmetry analysis and closed-form invariant solutions of the non-linear wave equations in elasticity using optimal system of Lie subalgebra, *Chin. J. Phys.*, **83**(1):1–13, 2023, https://doi.org/10.1016/j.cjph.2023.02.011.
10. N.H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, CRC, Boca Raton, FL, 1995.
11. N.H. Ibragimov, A new conservation theorem, *J. Math. Anal. Appl.*, **333**(1):311–328, 2007, https://doi.org/10.1016/j.jmaa.2006.10.078.
12. N.H. Ibragimov, Nonlinear self-adjointness and conservation laws, *J. Phys. A, Math. Theor.*, **44**(43):432002, 2011, https://doi.org/10.1088/1751-8113/44/43/432002.
13. A.H. Kara, A.H. Bokhari, F.D. Zaman, On the exact solutions of the nonlinear wave and ϕ^4-model equations, *J. Nonlinear Math. Phys.*, **15**(1):105–11, 2008, https://doi.org/10.2991/jnmp.2008.15.s1.9.
14. C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W.H. Freeman, San Francisco, 1973.
15. E. Noether, Invariante variationsprobleme, *Gött. Nachr.*, **1918**:235–257, 1918.
16. A. Raza, F.M. Mahomed, F.D. Zaman, A.H. Kara, Optimal system and classification of invariant solutions of nonlinear class of wave equations and their conservation laws, *J. Math. Anal. Appl.*, **505**(1):125615, 2022, https://doi.org/10.1016/j.jmaa.2021.125615.

17. M.A. Rincon, M.I. Copetti, Numerical analysis for a locally damped wave equation, *J. Appl. Anal. Comput.*, **3**:169–182, 2013, https://doi.org/10.11948/2013013.

18. P. Scherrer, Bestimmung der größe und der inneren struktur von kolloidteilchen mittels röntgenstrahlen, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, **1918**:98–100, 1918.

19. V.K. Sharma, Chirped soliton-like solutions of generalized nonlinear Schrödinger equation for pulse propagation in negative index material embedded into a Kerr medium., *Indian J. Phys.*, **90**(11):1271–1276, 2016, https://doi.org/10.1007/s12648-016-0840-y.

20. C. Smoryński, *History of Mathematics: A Supplement*, Springer, New York, 2007, https://doi.org/10.1007/978-0-387-75481-9.

21. J.A. Stratton, *Electromagnetic Theory*, IEEE Press, Piscatawey NJ / John Wiley & Sons, Hoboken, NJ, 2007.

22. B. Thidé, *Electromagnetic Field Theory*, Upsilon Books, Uppsala, 2004.

23. M. Usman, F.D. Zaman, Lie symmetry analysis and conservation laws of non-linear $(2+1)$ elastic wave equation, *Arab. J. Math.*, **12**:265–276, 2022, https://doi.org/10.1007/s40065-022-00392-y.

24. B. Vick, R.L. West, Analysis of damped waves using the boundary element method, in J.I. Frankel, C.A. Brebbia, M.A.H. Aliabadi (Eds.), *Boundary Element Technology XII. Proceedings of the 12th International Conference, BETECH XII, Knoxville, TN, 9–11 April, 1997*, WIT Trans. Model. Simulation, WIT Press / Computational Mechanics Publications, Southampton, 1997, pp. 265–278.

25. S.B. Wineberg, J.F. McGrath, E.F. Gabl, L.R. Scott, C.E. Southwell, Implicit spectral methods for wave propagation problems, *J. Comput. Phys.*, **97**(2):311–336, 1991, https://doi.org/10.1016/0021-9991(91)90002-3.