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On k-fuzzy metric spaces with applications

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Abstract. With application point of view, Gopal et al. [D. Gopal, W. Sintunavarat, A.S. Ranadive, S. Shukla, The investigation of k-fuzzy metric spaces with the first contraction principle in such spaces, *Soft Comput.*, 27:11081–11089, 2023] generalized the conceptions of a fuzzy metric space and introduced the definition of k-fuzzy metric space. Here a fuzzy set defined in k-fuzzy metric space is a membership function $\mathcal{F}_{\mathcal{Y}} : \mathcal{X} \times \mathcal{X} \times (0, +\infty)^k \to [0, 1]$, that is, the fuzzy distance between two points of the set depends on more than one parameter, and then also introduced first contraction principle in this space. In this sequel, we extend the work on k-fuzzy metric spaces by generalizing Banach contraction principle by introducing various type of inequalities. Here we introduce Tirado-type k-fuzzy contraction condition and prove fixed point theorem for Tirado-type contractive mapping. We also discuss the k-fuzzy ψ-contractive mapping, where $\psi \in \Psi$, and Ψ is a class of mappings defined from $\psi : [0, 1] \to [0, 1]$ that has certain properties, and also obtained fixed point for such class of mappings. Later, we define Ćirić-type contraction inequalities to prove fixed point results by restricting ourselves on l-natural property of the fuzzy space to ensure the existence of fixed point. Between all results, a set of supportive examples are also produced to validate the results. In application section, we discuss the solutions of Volterra-type integral equations and second-order nonlinear ordinary differential equation.

Keywords: fuzzy metric space, k-fuzzy metric space, k-fuzzy Tirado-type contractive mapping, k-fuzzy ψ-contractive mapping, fixed point.

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1 Introduction

The fuzzy set theory was discovered first by Zadeh [19] in 1965. In his seminal paper, Zadeh generalized the crisp sets into fuzzy sets by defining a membership function from a nonempty set $\mathcal{X}$ to $[0, 1]$ called a fuzzy set. Later, in 1975, Kramosil and Michálek [7] launched the fuzzy metric space by replacing the crisp distance between two points with a fuzzy distance between two points. The measurement of the fuzzy distance between two points is defined as the degree of the nearness of points with respect to a parameter $t \in (0, \infty)$ belonging to the $[0, 1]$. Later, George and Veeramani [2] improved the definition of fuzzy metric space given by Kramoshil and Michálek [7] and proved that fuzzy metric space has a Hausdorff topology. In 1988, Grabiec [5] was the first person who introduced fixed point theory involving fuzzy sets and gave the first contraction principle and Edelstein contraction theorem in fuzzy metric space in the sense of George and Veeramani [2]; see also [6,9,12,16–18]. Many author have research in probabilistic metric space like Sherwood et al. [15], Gopal et al. [3], and Sehgal et al. [14]

Very recently, an innovative new space was invented by Gopal et al. [4] from the application point of view known as k-fuzzy metric spaces, where they investigated the first contraction principle for this space. In fuzzy metric space, we prefer the measurement of the fuzzy distance between two points by the degree of the nearness of points with respect to a parameter $t \in (0, +\infty)$. The theory of k-fuzzy metric space is inspired by this concept of measuring the fuzzy distance between two points depending on more than one parameter $t_1, t_2, \dots, t_k \in (0, +\infty)$, those parameters may be time, type of fuel, weather, etc. Gopal et al. [4] defined a fuzzy set from $\mathcal{X}^2 \times (0, +\infty)^k$ to $[0, 1]$ and introduced the definition of a k-fuzzy metric space with supportive examples. They restrict themselves on a mathematical property known as l-natural property of k-fuzzy metric spaces to obtain the fixed point theorem and also discuss some other topological properties of such spaces. They defined an open ball with center at $x \in \mathcal{X}$ and radius $\epsilon \in (0, 1)$ with respect to parameters $t_1, t_2, \dots, t_k \in (0, +\infty)$ and defined open set, closed set depending on the concepts of open balls and also proved that "every open ball in a k-fuzzy metric space is an open set". According to a few more results, k-fuzzy metric spaces have a topology in $\mathcal{X}$ and have a local base at a point $\mathcal{X}$ with the first countable topology. Further, it is proved that "every k-fuzzy metric space is Hausdorff". Later, given the definition of a convergent sequence, Cauchy sequence, and completeness property of the spaces, in the main section, they defined the fuzzy first contraction in k-fuzzy metric space.

2 Preliminaries

In this part, we write some fundamental ideas, which will be essential to accommodate major theorems.

Definition 1. (See [13].) A mapping $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (t-norm) if $\diamond$ satisfies the following assertions:

1. $\diamond$ is commutative and associative, that is, $e \diamond f = f \diamond e$ and $e \diamond (f \diamond g) = (e \diamond f) \diamond g$ for all $e, f, g \in [0, 1]$;

2. $\diamond$ is a continuous map;
3. $1 \diamond e = e$ for all $e \in [0, 1]$;
4. $e \diamond f \leqslant g \diamond h$ whenever $e \leqslant g$ and $f \leqslant h$ for all $e, f, g, h \in [0, 1]$.

Some examples of continuous t-norm are $\diamond_p$ product $e \diamond_{\mathrm{p}} f = e \cdot f$, $\diamond_{\mathrm{m}}$ standard minimum $e \diamond_{\mathrm{m}} f = \min\{e, f\}$, $\diamond_{\mathrm{L}}$ Łukasiewicz maximum $e \diamond_{\mathrm{L}} f = \max\{e + f - 1,\ 0\}$.

Definition 2. (See [2].) A fuzzy metric space is an ordered triple $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$, where $\mathcal{X}$ is a nonempty set, $\diamond$ is a continuous t-norm, and $\mathcal{F}_\mathcal{Y}$ is a fuzzy set defined on $\mathcal{X} \times \mathcal{X} \times (0, +\infty)$ to $[0, 1]$ satisfying the following assertions for all $\eta, \zeta, \kappa \in \mathcal{X}$ and $r, s > 0$:

1. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r) > 0$;
2. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r) = 1$ if and only if $\eta = \zeta$;
3. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r) = \mathcal{F}_\mathcal{Y}(\zeta, \eta, r)$;
4. $\mathcal{F}_\mathcal{Y}(\eta, \kappa,\ r + s) \geqslant \mathcal{F}_\mathcal{Y}(\eta, \zeta, r) \diamond \mathcal{F}_\mathcal{Y}(\zeta, \kappa, s)$;
5. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, \cdot) : (0, \infty) \to [0, 1]$ is continuous mapping.

Definition 3. (See [2].) A space $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is called a natural fuzzy metric space if and only if

$$\lim_{r \to +\infty} \mathcal{F}_\mathcal{Y}(\eta, \zeta, r) = 1$$

for all $\eta, \zeta \in \mathcal{X}$.

The notation of k-fuzzy metric spaces, where $k \in \{1, 2, 3, \dots\}$, is an extension and generalization of the concepts of fuzzy metric space in the sense of George and Veeramani (1994). The membership degree of nearness between two points is depending on the number of parameters k. Now we present introduced idea of k-fuzzy metric by Gopal et al. [4].

Definition 4. (See [4].) Let $\mathcal{X}$ be a nonempty set, $\diamond$ be a continuous t-norm, k be a positive integer, and $\mathcal{F}_\mathcal{Y}$ be a set on $\mathcal{X}^2 \times (0, +\infty)^k$. An ordered triple $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is called a k-fuzzy metric space if the following conditions are satisfied for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$:

1. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, \dots, r_k) > 0$;
2. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, \dots, r_k) = 1$ if and only if $\eta = \zeta$;
3. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, \dots, r_k) = \mathcal{F}_\mathcal{Y}(\zeta, \eta, r_1, r_2, \dots, r_k)$;
4. For any $l \in \{1, 2, 3, \dots, k\}$,

$$\begin{aligned}
&\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, \dots, r_{l-1}, t, r_{l+1}, \dots, r_{k-1}, r_k)\\
&\qquad \diamond \mathcal{F}_\mathcal{Y}(\zeta, \kappa, r_1, r_2, \dots, r_{l-1}, s, r_{l+1}, \dots, r_{k-1}, r_k)\\
&\quad \leqslant \mathcal{F}_\mathcal{Y}(\eta, \kappa, r_1, r_2, \dots, r_{l-1}, t + s, r_{l+1}, \dots, r_{k-1}, r_k);
\end{aligned}$$

5. $\mathcal{F}_\mathcal{Y}(\eta, \zeta, \cdot) : (0, +\infty)^k \to [0, 1]$ is a continuous mapping.

Remark 1. k-fuzzy metric space turns into a fuzzy metric space by putting $k = 1$.

With the application points, Gopal et al. [4] illustrated a few examples of k-fuzzy metric spaces with take care of the physical behaviours of the entities. Let us now discuss some mathematical formulas of k-fuzzy metric as in the form of examples of k-fuzzy metric spaces.

Example 1. (See [4].) Let $(\mathcal{X}, \rho)$ be a metric space, $\diamond$ be the product (minimum) t-norm, $w > 0$ and k be a positive integer. Define a fuzzy set $\mathcal{F}_\mathcal{Y}$ on $\mathcal{X}^2 \times (0, +\infty)^k$ by

$$\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, r_3, \dots, r_k) = \frac{w r_1 r_2 \cdots r_k}{w r_1 r_2 \cdots r_k + \rho(\eta, \zeta)}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$. Then $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is a k-fuzzy metric space.

Example 2. (See [4].) Let $(\mathcal{X}, \rho)$ be a metric space, $\diamond$ be the product (minimum) t-norm, $w > 0$ and k be a positive integer. Define a fuzzy set $\mathcal{F}_\mathcal{Y}$ on $\mathcal{X}^2 \times (0, +\infty)^k$ by

$$\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, r_3, \dots, r_k) = w\left[w + \left(\sum_{i=1}^{k} \frac{1}{r_i}\right)\rho(\eta, \zeta)\right]^{-1}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$. Then $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is a k-fuzzy metric space.

In their research article, Gopal et al. [4] restricted themselves with the l-natural mathematical property like natural fuzzy metric space.

Definition 5. (See [4].) $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is said to be l-natural k-fuzzy metric space if there exists $l \in \{1, 2, \dots, k\}$ such that

$$\lim_{r_l \to +\infty} \mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, \dots, r_l, \dots, r_k) = 1$$

for all $\eta, \zeta \in \mathcal{X}$.

In k-fuzzy metric space, l-naturalness helps for the existence of fixed point in first contraction principle. Assumption of l-natural property of the space $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ cannot be replaced by m-natural property with $m \neq l$. In coming part of this note, we usually write $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1^k)$ in the place of $\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, \dots, r_k)$, where $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$. The following proposition shows the effect of degree of nearness between two points with respect to parameter relation.

Proposition 1. (*See* [4].) *Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a k-fuzzy metric space, $r, r_1, r_2, \dots, r_k > 0$. Suppose that $r_l < r$ for some $l \in \{1, 2, \dots, k\}$. Then*

$$\mathcal{F}_\mathcal{Y}\bigl(\eta, \zeta, r_1^k\bigr) \leqslant \mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, \dots, r_{l-1}, r, r_{l+1}, \dots, r_k)$$

for all $\eta, \zeta \in \mathcal{X}$.

Now we recall the following analytical concepts of the convergent sequence, Cauchy sequence, completeness property of this space.

Definition 6. (See [4].) Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a k-fuzzy metric space. A sequence $\{\beta_n\}$ in $\mathcal{X}$ is said to be convergent and converges to $\beta \in \mathcal{X}$ if and only if for every real $\epsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{F}_\mathcal{Y}\big(\beta_n, \beta, r_1^k\big) > 1 - \epsilon$$

for all $n > n_0$ and $r_1, r_2, \dots, r_k > 0$.

Lemma 1. (*See* [4].) *Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a k-fuzzy metric space. A sequence $\{\beta_n\}$ in $\mathcal{X}$ converges to $\beta \in \mathcal{X}$ if and only if*

$$\lim_{n \to +\infty} \mathcal{F}_\mathcal{Y}\big(\beta_n, \beta, r_1^k\big) = 1$$

for all $r_1, r_2, \dots, r_k > 0$.

In fuzzy metric space, there are two distinct notations of the Cauchy sequences and completeness known as G-Cauchy sequence, G-completeness (Grabic) and M-Cauchy sequence, M-completeness (George and Veeramani). It is well known that definitions of completeness given by George and Veeramani is more appropriate than the Grabic definition. These definitions are also generalized with the following manner.

Definition 7. (See [4].) Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a k-fuzzy metric space, and let $\{\beta_n\}$ be a sequence in $\mathcal{X}$.

1. A sequence $\{\beta_n\}$ is called an M-Cauchy sequence if for every $\epsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$ and $r_1, r_2, \dots, r_k > 0$,
$$\mathcal{F}_\mathcal{Y}\big(\beta_n, \beta_m, r_1^k\big) > 1 - \epsilon.$$
2. A sequence $\{x_n\}$ is called a G-Cauchy sequence if for all $r_1, r_2, \dots, r_k > 0$ and $p > 0$,
$$\lim_{n \to \infty} \mathcal{F}_\mathcal{Y}\big(\beta_n, \beta_{n+p}, r_1^k\big) = 1.$$

Definition 8. (See [4].) Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a k-fuzzy metric space.

1. $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is said to be M-complete if every M-Cauchy sequence in $\mathcal{X}$ converges to some $\beta \in \mathcal{X}$.
2. $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is said to be G-complete if every G-Cauchy sequence in $\mathcal{X}$ converges to some $\beta \in \mathcal{X}$.

Remark 2. (See [4].) For $k = 1$, the M-completeness and G-completeness of a k-fuzzy metric space is equivalent to the M-completeness and G-completeness of a fuzzy metric space as defined by George and Veeramani and Grabic, respectively.

Theorem 1. (*See* [4].) *Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a G-complete k-fuzzy metric space, and let $\mathcal{P}$ be a self-mapping satisfying the following condition*:

$$\mathcal{F}_{\mathcal{Y}_l}^{1/\lambda}\big(\mathcal{P}\eta, \mathcal{P}\zeta, r_1^k\big) \geqslant \mathcal{F}_\mathcal{Y}\big(\eta, \zeta, r_1^k\big)$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$, where $l \in \{1, 2, \dots, k\}$, and $\lambda \in (0,1)$ is a constant. Suppose that $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is an l-natural k-fuzzy metric space. Then $\mathcal{P}$ has a unique fixed point.

3 Main results

In this main part, we are going to extend and establish many contractive inequality in the setting of k-fuzzy metric spaces. For the easiness, $l \in \{1, 2, \dots, k\}$, $a > 0$, $\eta, \zeta \in \mathcal{X}$, and $r_1, r_2, \dots, r_k > 0$, we write ${\mathcal{F_Y}}_l^a(\eta, \zeta, r_1^k)$ in the place of $\mathcal{F_Y}(\eta, \zeta, r_1, \dots, r_{l-1}, r_l/a, r_{l+1}, \dots, r_k)$, where $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ is a k-fuzzy metric space. With the inspiration of this [4], we extend this idea of l-naturalness, which is depending on more then one parameter in the following manners called as generalized natural property of a k-fuzzy metric space.

Definition 9. $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ is said to be generalized natural property of a k-fuzzy metric space if there exist one or more than one parameter(s) $l_i \in \{1, 2, \dots, k\}$, where $i = 1, 2, \dots, m$, $m \leqslant k$, such that for all $\eta, \zeta \in \mathcal{X}$,

$$\lim_{r_{l_1}, r_{l_2}, \dots, r_{l_m} \to +\infty} \mathcal{F_Y}(\eta, \zeta, r_1, r_2, \dots, r_{l_i}, \dots, r_k) = 1.$$

We can see an example of generalized naturalness of k-fuzzy metric space.

Example 3. (See [4].) Let $\mathcal{X} = \mathbb{R}^k$, where k is a positive integer, $w > 0$, and let $\diamond$ be the product t-norm. Define a fuzzy set $\mathcal{F_Y}$ on $\mathcal{X}^2 \times (0, +\infty)^k$ by

$$\begin{aligned}
&\lim_{r_1, r_2, \dots, r_{k-1} \to +\infty} \mathcal{F_Y}(\eta, \zeta, r_1, r_2, r_3, \dots, r_k) \\
&\quad = \lim_{r_1, r_2, \dots, r_{k-1} \to +\infty} w \left[w + \sum_{i=1}^{k-1} \frac{|\eta_i - \zeta_i|}{r_i} \right]^{-1} = 1
\end{aligned}$$

for all $\eta = (\eta_1, \eta_2, \dots, \eta_{k-1}), \zeta = (\zeta_1, \zeta_2, \dots, \zeta_{k-1}) \in \mathcal{X}$. Then $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ has a generalized natural property of k-fuzzy metric space.

Remark 3. If we put $m = 1$ in Definition 9, the space reduces into 1-natural k-fuzzy metric space.

We start generalizing Tirado-type contraction principle in this space.

Definition 10. Let $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ be a k-fuzzy metric space. A mapping $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ is called a Tirado-type k-fuzzy contraction mapping if

$$1 - \mathcal{F_Y}\big(\mathcal{P}\eta, \mathcal{P}\zeta, r_1^k\big) \leqslant \alpha\big(1 - \mathcal{F_Y}\big(\eta, \zeta, r_1^k\big)\big)$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$, where $\eta \neq \zeta$, $\alpha \in (0, 1)$ is a constant.

Now we establish fixed point result using Tirado-type k-fuzzy contraction principle.

Theorem 2. *Let $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ be a Tirado-type k-fuzzy contraction mapping and $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ be a G-complete k-fuzzy metric space. Suppose that $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ is a k-fuzzy metric space. Then $\mathcal{P}$ has a unique fixed point.*

Proof. Let $\eta_0 \in \mathcal{X}$ be any arbitrary point. Define a sequence $\{\eta_n\}$ by Picard iteration method $\eta_n = \mathcal{P}\eta_{n-1}$ for all $n \in \mathbb{N} \cup \{0\}$. We must show that this sequence is a G-Cauchy sequence. For any $n \in \mathbb{N} \cup \{0\}$, we have

$$1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big) \leqslant \alpha\big(1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_{n-1}, \eta_n, r_1^k\big)\big).$$

By repeating in this manner, we obtain

$$1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big) \leqslant \alpha^n\big(1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_0, \eta_1, r_1^k\big)\big) \tag{1}$$

for all $n \in \mathbb{N}$. Taking limit as n tends to $+\infty$ and since $\alpha \in (0, 1)$, we conclude by (1),

$$\lim_{n\to+\infty} \big(1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big)\big) \leqslant \lim_{n\to+\infty} \alpha^n\big(1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_0, \eta_1, r_1^k\big)\big),$$

that is,

$$\lim_{n\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big) = 1 \tag{2}$$

for all $r_1, r_2, r_3, \dots, r_k > 0$. For each $n \in \mathbb{N}$ and $p > 0$,

$$\begin{aligned}
\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+p}, r_1^k\big) &\geqslant \mathcal{F}_{\mathcal{Y}}\left(\eta_n, \eta_{n+1}, r_1, r_2, \dots, \frac{r_l}{2}, \dots, r_k\right) \\
&\quad \diamond \mathcal{F}_{\mathcal{Y}}\left(\eta_{n+1}, \eta_{n+p}, r_1, r_2, \dots, \frac{r_l}{2}, \dots, r_k\right), \\
\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+p}, r_1^k\big) &\geqslant {\mathcal{F}_{\mathcal{Y}}}_l^2\big(\eta_n, \eta_{n+1}, r_1^k\big) \diamond {\mathcal{F}_{\mathcal{Y}}}_l^{2^2}\big(\eta_{n+1}, \eta_{n+2}, r_1^k\big) \\
&\quad \diamond \cdots \diamond {\mathcal{F}_{\mathcal{Y}}}_l^{2^{p-1}}\big(\eta_{n+p-2}, \eta_{n+p-1}, r_1^k\big) \\
&\quad \diamond {\mathcal{F}_{\mathcal{Y}}}_l^{2^p}\big(\eta_{n+p-1}, \eta_{n+p}, r_1^k\big)
\end{aligned} \tag{3}$$

for all $r_1, r_2, \dots, r_k > 0$. Letting limit as $n \to +\infty$ and by using (2), we have

$$\lim_{n\to+\infty} {\mathcal{F}_{\mathcal{Y}}}_l^a\big(\eta_n, \eta_{n+1}, r_1^k\big) = 1$$

for all $r_1, r_2, \dots, r_k > 0$ and $a > 0$. Inequality (3) yields

$$\lim_{n\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+p}, r_1^k\big) \geqslant 1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1 = 1$$

for all $r_1, r_2, \dots, r_k > 0$ and $p > 0$. Thus, the sequence $\{\eta_n\}$ is a G-Cauchy sequence in $\mathcal{X}$. Since the space $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is a G-complete, there exists $u \in \mathcal{X}$ such that the sequence $\{\eta_n\}$ converges to u, that is,

$$\lim_{n\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_n, u, r_1^k\big) = 1 \tag{4}$$

for all $r_1, r_2, \dots, r_k > 0$. Now we must prove that u is a fixed point for a self-map $\mathcal{P}$.

$$1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_{n+1}, \mathcal{P}u, r_1^k\big) = 1 - \mathcal{F}_{\mathcal{Y}}\big(\mathcal{P}\eta_n, \mathcal{P}u, r_1^k\big) \leqslant \alpha\big(1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_n, u, r_1^k\big)\big).$$

Letting limit as $n \to +\infty$ and by using (4),

$$\lim_{n\to+\infty} \big(1 - \mathcal{F}_{\mathcal{Y}}\big(\eta_{n+1}, \mathcal{P}u, r_1^k\big)\big) = 0,$$

that is,

$$\mathcal{F}_{\mathcal{Y}}\big(\eta_{n+1}, \mathcal{P}u, r_1^k\big) = 1 \tag{5}$$

for all $r_1, r_2, \dots, r_k > 0$. For any $n \in \mathbb{N}$, we have

$$\mathcal{F}_{\mathcal{Y}}\big(u, \mathcal{P}u, r_1^k\big) \geqslant \mathcal{F}_{\mathcal{Y}}{}_l^2\big(u, \eta_{n+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{n+1}, \mathcal{P}u, r_1^k\big).$$

Letting limit as $n \to +\infty$ and using together (4) and (5), we have

$$\mathcal{F}_{\mathcal{Y}}\big(u, \mathcal{P}u, r_1^k\big) = 1$$

for all $r_1, r_2, \dots, r_k > 0$. This implies $\mathcal{P}u = u$, that is, u is a fixed point of $\mathcal{P}$.

For the uniqueness, suppose that v is another fixed point of $\mathcal{P}$ such that $u \neq v$. Then there exist $r_1, r_2, \dots, r_k > 0$ such that

$$\mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big) < 1,$$

that is,

$$1 - \mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big) > 0.$$

Now

$$k\big(1 - \mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big)\big) \geqslant 1 - \mathcal{F}_{\mathcal{Y}}\big(\mathcal{P}u, \mathcal{P}v, r_1^k\big) = 1 - \mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big),$$

which implies that $k \geqslant 1$, which is a contradiction. Therefore, we must have $u = v$, that is, fixed point of $\mathcal{P}$ is unique. □

For the existence of the fixed point, we are going to discuss the following example with generalized natural property of space for Tirado-type k-fuzzy contraction principle.

Example 4. Consider $\mathcal{X} = [0,1]^2$ with a usual metric $(\mathcal{X}, \rho)$. $\diamond$ is the product t-norm, $w > 0$, and $k \in \mathbb{Z}^+$. Define a membership function $\mathcal{F}_{\mathcal{Y}} : \mathcal{X}^2 \times (0, +\infty)^k \to [0,1]$ by

$$\mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3) = \frac{w}{w + \rho(\eta, \zeta)(\frac{1}{r_1} + \frac{1}{r_2})}$$

for all $\eta = (\eta_1, \eta_2), \zeta = (\zeta_1, \zeta_2) \in \mathcal{X}$ and $r_1, r_2, r_3 \in (0, +\infty)$. Then $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is a G-complete 3-fuzzy metric space.

In addition,

$$\lim_{r_1, r_2 \to +\infty} \mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3) = 1$$

for all $\eta, \zeta \in \mathcal{X}$, $r_3 > 0$, and the space $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is a generalized natural 3-fuzzy metric space.

Define a mapping $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{P}(\eta_1, \eta_2) = \left(\frac{\eta_1}{2}, \frac{\eta_2}{2}\right)$$

for all $(\eta_1, \eta_2) \in \mathcal{X}$.

Consider $\eta = (\eta_1, \eta_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathcal{X}$ and $r_1, r_2 > 0$. We get

$$\begin{aligned}
&\alpha\bigl(1 - \mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3)\bigr)\\
&\quad = \alpha\left(1 - \frac{w}{w + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(\frac{1}{r_1} + \frac{1}{r_2})}\right)\\
&\quad = \alpha\left(1 - \frac{w r_1 r_2}{w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\right)\\
&\quad = \alpha\left(\frac{(|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}{w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\right).
\end{aligned}$$

If

$$\alpha \in \left(\frac{w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}{2 w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}, 1\right),$$

$$\begin{aligned}
&\alpha\bigl(1 - \mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3)\bigr)\\
&\quad \geqslant \left(\frac{w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}{2 w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\right)\\
&\qquad \times \left(\frac{(|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}{w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\right)\\
&\quad = \frac{(|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}{2 w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\\
&\quad = \frac{2 w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2) - 2 w r_1 r_2}{2 w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\\
&\quad = 1 - \frac{2 w r_1 r_2}{2 w r_1 r_2 + (|\eta_1 - \zeta_1| + |\eta_2 - \zeta_2|)(r_1 + r_2)}\\
&\quad = 1 - \frac{w}{w + (\frac{|\eta_1 - \zeta_1|}{2} + \frac{|\eta_2 - \zeta_2|}{2})(\frac{1}{r_1} + \frac{1}{r_2})}\\
&\quad = 1 - \mathcal{F}_{\mathcal{Y}}(\mathcal{P}\eta, \mathcal{P}\zeta, r_1, r_2, r_3).
\end{aligned}$$

Thus, $\mathcal{P}$ is a Tirado-type k-fuzzy contraction mapping. Therefore, $(0, 0)$ is a unique fixed point of self-map $\mathcal{P}$.

Example 5. Consider $\mathcal{X} = [0, 1]$ with a usual metric $(\mathcal{X}, \rho)$. $\diamond$ is the product t-norm, $w > 1$, and $k \in \mathbb{Z}^+$. Define a membership function $\mathcal{F}_{\mathcal{Y}} : \mathcal{X}^2 \times (0, +\infty)^k \to [0, 1]$ by

$$\mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3) = \frac{w r_1 r_2}{w r_1 r_2 + \rho(\eta, \zeta)}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, r_3 > 0$. Then $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is a G-complete 3-fuzzy metric space.

In addition,

$$\lim_{r_1, r_2 \to +\infty} \mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3) = 1$$

for all $\eta, \zeta \in \mathcal{X}$, and the space $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is a generalized natural 3-fuzzy metric space.

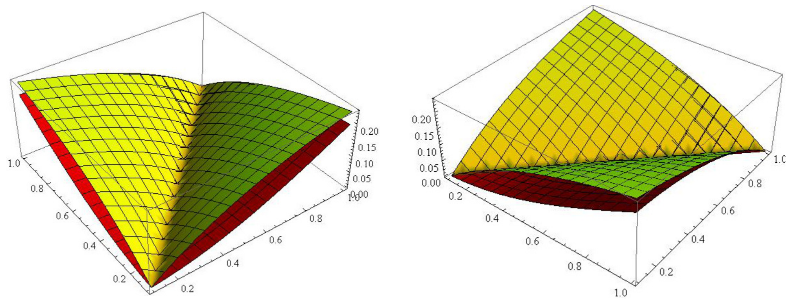

Figure 1. Red colour defined by L.H.S., and yellow colour defined by R.H.S. in Eq. (6).

Define a mapping $\mathcal{P}:\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{P}(\eta)=\begin{cases}\frac{\eta^2}{4} & \text{if } \eta\neq 0,\\ 0 & \text{if } \eta=0\end{cases}$$

for all $\eta,\zeta\in\mathcal{X}$. Here $\mathcal{P}$ is not continuous. We consider two cases:

1. If $\eta\neq 0$ and $\zeta\neq 0$, then for $\alpha=1/2$,

$$\begin{aligned}&\big[1-\mathcal{F}_{\mathcal{Y}}(\mathcal{P}\eta,\mathcal{P}\zeta,r_1,r_2)\big]\\ &\quad=\left[1-\mathcal{F}_{\mathcal{Y}}\left(\frac{\eta^2}{4},\frac{\zeta^2}{4},r_1,r_2\right)\right]=\left[1-\frac{wr_1r_2}{wr_1r_2+|\frac{\eta^2}{4}-\frac{\zeta^2}{4}|}\right]\\ &\quad=\left[1-\frac{wr_1r_2}{wr_1r_2+\frac{1}{4}|\eta-\zeta|}\right]\leqslant\frac{1}{2}\left[1-\frac{wr_1r_2}{wr_1r_2+|\eta-\zeta|}\right]\\ &\quad=\alpha\big[1-\mathcal{F}_{\mathcal{Y}}(\eta,\zeta,r_1,r_2)\big].\end{aligned}\tag{6}$$

2. If $\eta\neq 0$ and $\zeta=0$, then for $\alpha=1/2$,

$$\begin{aligned}&\big[1-\mathcal{F}_{\mathcal{Y}}(\mathcal{P}\eta,\mathcal{P}\zeta,r_1,r_2)\big]\\ &\quad=\left[1-\mathcal{F}_{\mathcal{Y}}\left(\frac{\eta^2}{4},0,r_1,r_2\right)\right]=\left[1-\frac{wr_1r_2}{wr_1r_2+|\frac{\eta^2}{4}-0|}\right]\\ &\quad\leqslant\frac{1}{2}\left[1-\frac{wr_1r_2}{wr_1r_2+|\eta-0|}\right]=\alpha\big[1-\mathcal{F}_{\mathcal{Y}}(\eta,\zeta,r_1,r_2)\big].\end{aligned}\tag{7}$$

Thus, $\mathcal{P}$ is a Tirado-type k-fuzzy contraction mapping for $w\cdot r_1\cdot r_2\geqslant 1$. Hence 0 is a unique fixed point of self-map $\mathcal{P}$.

Let Ψ be the class of all mapping $\psi:[0,1]\to[0,1]$ such that ψ is continuous, nondecreasing and $\psi(t)>t$ for all $t\in(0,1)$. Using this class of mapping, Mihet [8] defined a fuzzy ψ-contractive mapping. Now we extend this idea in k-fuzzy ψ-contractive mapping in this space and state fixed point theorem without using natural property in the following manner:

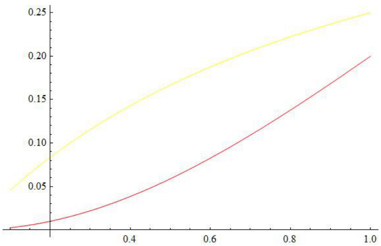

Figure 2. Red colour defined by L.H.S., and yellow colour defined by R.H.S. in Eq. (7).

Definition 11. A self-map $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ is called a k-fuzzy ψ-contractive mapping if $\mathcal{P}$ satisfies the following inequality:

$$\mathcal{F_Y}\big(\eta, \zeta, r_1^k\big) > 0 \implies \mathcal{F_Y}\big(\mathcal{P}\eta, \mathcal{P}\zeta, r_1^k\big) \geqslant \psi\big(\mathcal{F_Y}\big(\eta, \zeta, r_1^k\big)\big)$$

for all $r_1, r_2, \ldots, r_k > 0$ and $\psi \in \Psi$.

Theorem 3. *Let $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ be an G-complete k-fuzzy metric space and $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ be a k-fuzzy ψ-contractive mapping. If there exists $\eta_0 \in \mathcal{X}$ such that $\mathcal{F_Y}(\eta_0, \mathcal{P}\eta_0, r_1^k) > 0$ for all $r_1, r_2, \ldots, r_k > 0$, then $\mathcal{P}$ has a fixed point.*

Proof. Let $\eta_0 \in \mathcal{X}$ be any arbitrary point. Define a sequence $\{\eta_n\}$ by Picard iteration method $\eta_n = \mathcal{P}\eta_{n-1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then for any $\eta_0 \in \mathcal{X}$ such that $\mathcal{F_Y}(\eta_0, \mathcal{P}\eta_0, r_1^k) > 0$ for all $r_1, r_2, \ldots, r_k > 0$, we have

$$\mathcal{F_Y}\big(\eta_1, \eta_2, r_1^k\big) = \mathcal{F_Y}\big(\mathcal{P}\eta_0, \mathcal{P}\eta_1, r_1^k\big) \geqslant \psi\big(\mathcal{F_Y}\big(\eta_0, \eta_1, r_1^k\big)\big) > \mathcal{F_Y}\big(\eta_0, \eta_1, r_1^k\big)$$

for all $r_1, r_2, \ldots, r_k > 0$ and

$$\mathcal{F_Y}\big(\eta_2, \eta_3, r_1^k\big) = \mathcal{F_Y}\big(\mathcal{P}\eta_1, \mathcal{P}\eta_2, r_1^k\big) \geqslant \psi\big(\mathcal{F_Y}\big(\eta_1, \eta_2, r_1^k\big)\big) > \mathcal{F_Y}\big(\eta_1, \eta_2, r_1^k\big)$$

for all $r_1, r_2, \ldots, r_k > 0$. Inductively, we obtain

$$\mathcal{F_Y}\big(\eta_{n+1}, \eta_{n+2}, r_1^k\big) \geqslant \mathcal{F_Y}\big(\eta_n, \eta_{n+1}, r_1^k\big)$$

for all $r_1, r_2, \ldots, r_k > 0$. Thus, sequence $\{\mathcal{F_Y}(\eta_n, \eta_{n+1}, r_1^k)\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of number in $(0, 1]$. Consider

$$\lim_{n\to+\infty} \mathcal{F_Y}\big(\eta_n, \eta_{n+1}, r_1^k\big) = l,$$

where $l \in (0, 1]$, for any fixed $t_l > 0$, $r_1, r_2, \ldots, r_k > 0$.

Now again starting from $\mathcal{F_Y}(\eta_0, \mathcal{P}\eta_0, r_1^k) > 0$,

$$\mathcal{F_Y}\big(\eta_n, \eta_{n+1}, r_1^k\big) \geqslant \psi\big(\mathcal{F_Y}\big(\eta_{n-1}, \eta_n, r_1^k\big)\big) > \mathcal{F_Y}\big(\eta_{n-1}, \eta_n, r_1^k\big)$$

for all $r_1, r_2, \ldots, r_k > 0$. Letting limit as $n \to +\infty$ and since ψ is continuous, we get $l \geqslant \psi(l) > l$, which is a contradiction. This implies $l = 1$, and thus

$$\lim_{n\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big) = 1 \tag{8}$$

for all $r_1, r_2, \ldots, r_k > 0$.

Next, we must show that $\{\eta_n\}$ is a G-Cauchy sequence. Take on the a contrary that $\{\eta_n\}$ is not a G-Cauchy sequence. Then there is $\epsilon \in (0, 1)$ such that for each $s \in \mathbb{N}$, there exist $m(s), n(s) \in \mathbb{N}$ with $m(s) > n(s) \geqslant s$ and

$$\mathcal{F}_{\mathcal{Y}}\big(\eta_{m(s)}, \eta_{n(s)}, r_1^k\big) \leqslant 1 - \epsilon \tag{9}$$

for all $r_1, r_2, \ldots, r_k > 0$. Let, for each s, $m(s)$ be the least integer exceeding $n(s)$ satisfying the above property, that is,

$$\mathcal{F}_{\mathcal{Y}}\big(\eta_{m(s)-1}, \eta_{n(s)}, r_1^k\big) > 1 - \epsilon. \tag{10}$$

Using (9), then applying property of triangular inequality and using (10),

$$\begin{aligned} 1 - \epsilon &\geqslant \mathcal{F}_{\mathcal{Y}}\big(\eta_{m(s)}, \eta_{n(s)}, r_1^k\big) \\ &\geqslant \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)-1}, \eta_{n(s)}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)-1}, \eta_{m(s)}, r_1^k\big) \\ &\geqslant (1-\epsilon) \diamond \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)-1}, \eta_{m(s)}, r_1^k\big). \end{aligned}$$

Taking limit as $s \to +\infty$, it follows that

$$\lim_{s\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_{m(s)}, \eta_{n(s)}, r_1^k\big) = 1 - \epsilon \tag{11}$$

for all $r_1, r_2, \ldots, r_k > 0$. Now

$$\begin{aligned} &\mathcal{F}_{\mathcal{Y}}\big(\eta_{m(s)}, \eta_{n(s)}, r_1^k\big) \\ &\quad\geqslant \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)}, \eta_{m(s)+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)+1}, \eta_{n(s)}, r_1^k\big) \\ &\quad\geqslant \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)}, \eta_{m(s)+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}}{}_l^{2^2}\big(\eta_{m(s)+1}, \eta_{n(s)+1}, r_1^k\big) \\ &\qquad \diamond \mathcal{F}_{\mathcal{Y}}{}_l^{2^2}\big(\eta_{n(s)+1}, \eta_{n(s)}, r_1^k\big) \\ &\quad\geqslant \mathcal{F}_{\mathcal{Y}}{}_l^2\big(\eta_{m(s)}, \eta_{m(s)+1}, r_1^k\big) \diamond \psi\big(\mathcal{F}_{\mathcal{Y}}{}_l^{2^2}\big(\eta_{m(s)}, \eta_{n(s)}, r_1^k\big)\big) \\ &\qquad \diamond \mathcal{F}_{\mathcal{Y}}{}_l^{2^2}\big(\eta_{n(s)+1}, \eta_{n(s)}, r_1^k\big). \end{aligned}$$

Letting limit as $s \to +\infty$ and using (8), (11), we get

$$1 - \epsilon \geqslant 1 \diamond \psi(1-\epsilon) \diamond 1 > 1 - \epsilon,$$

which is a contradiction. Thus, $\{\eta_n\}$ is a Cauchy sequence. Therefore,

$$\lim_{s\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_{m(s)}, \eta_{n(s)}, r_1^k\big) = 1$$

for all $r_1, r_2, \ldots, r_k > 0$. Since the k-fuzzy metric space $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is complete, so that Cauchy sequence $\{\eta_n\}$ is convergent at u in $\mathcal{X}$ such that

$$\lim_{n\to+\infty} \mathcal{F}_{\mathcal{Y}}\big(\eta_n, u, r_1^k\big) = 1$$

for all $r_1, r_2, \ldots, r_k > 0$. Next, we must prove that $\mathcal{P}$ has fixed point $u \in \mathcal{X}$. Suppose that $\mathcal{P}u \neq u$, so that $\mathcal{F}_{\mathcal{Y}}(u, \mathcal{P}u, r_1^k) > 0$ for all $r_1, r_2, \ldots, r_k > 0$. Without loss of generality, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \mathcal{P}u, r_1^k\big) > 0 \\
&\quad\Longrightarrow\quad \mathcal{F}_{\mathcal{Y}}\big(\mathcal{P}\eta_n, \mathcal{P}u, r_1^k\big) \geqslant \psi\big(\mathcal{F}_{\mathcal{Y}}\big(\eta_n, u, r_1^k\big)\big) > \mathcal{F}_{\mathcal{Y}}\big(\eta_n, u, r_1^k\big).
\end{aligned}$$

Letting limit as $n \to +\infty$, we get $\mathcal{F}_{\mathcal{Y}}(u, \mathcal{P}u, r_1^k) > 1$, which is a contradiction. Therefore, $\mathcal{F}_{\mathcal{Y}}(u, \mathcal{P}u, r_1^k) = 1$ for all $r_1, r_2, \ldots, r_k \in (0, 1)$. Thus, u is a fixed point for a self-map $\mathcal{P}$. Next, we must show that fixed point is unique. Suppose v is another fixed point of self-map $\mathcal{Q}$ such that $u \neq v$, that is, $\mathcal{F}_{\mathcal{Y}}(u, v, r_1^k) > 0$. This implies that

$$\mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big) = \mathcal{F}_{\mathcal{Y}}\big(\mathcal{P}u, \mathcal{P}v, r_1^k\big) \geqslant \psi\big(\mathcal{F}_{\mathcal{Y}}(u, v, r_1^k)\big) > \mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big)$$

for all $r_1, r_2, \ldots, r_k \in (0, 1)$, which is a contradiction. Therefore, fixed point of Q is unique. □

The following example satisfies the above theorem.

Example 6. Let $\mathcal{X} = [0, 1]$ and $(\mathcal{X}, \rho)$ be a usual fuzzy metric space, and let $\diamond$ be the product t-norm, $w > 0$, and $k = 2$. Define the fuzzy set $\mathcal{F}_{\mathcal{Y}}$ on $\mathcal{X}^2 \times (0, +\infty)^k$ by

$$\mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3) = \exp\left\{-2\rho(\eta, \zeta)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, r_3 > 0$. Then $(\mathcal{X}, \mathcal{F}_{\mathcal{Y}}, \diamond)$ is a G-complete 2-fuzzy metric space. Define a self-map $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ by $\mathcal{P}\eta = \eta/4$ for all $\eta \in \mathcal{X}$. Take $\psi(t) = \sqrt{t} > t$ for all $t \in (0, 1)$.

$$\begin{aligned}
&\mathcal{F}_{\mathcal{Y}}(\mathcal{P}\eta, \mathcal{P}\zeta, r_1, r_2, r_3) \\
&\quad= \mathcal{F}_{\mathcal{Y}}\left(\frac{\eta}{4}, \frac{\zeta}{4}, r_1, r_2, r_3\right) = \exp\left\{-2\rho\left(\frac{\eta}{4}, \frac{\zeta}{4}\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\} \\
&\quad= \exp\left\{-\frac{|\eta - \zeta|}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\} \geqslant \exp\left\{-|\eta - \zeta|\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\} \\
&\quad= \left[\exp\left\{-2|\eta - \zeta|\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\}\right]^{1/2} = \big[\mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3)\big]^{1/2} \\
&\quad= \psi\big(\mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3)\big).
\end{aligned}$$

Thus, $\mathcal{Q}$ is a 2-fuzzy ψ-contractive mapping with respect to $\psi(t) = \sqrt{t}$. The point $\eta = 0$ is the fixed point for self-map $\mathcal{P}$.

The following example is a discontinuous function.

Example 7. Let $\mathcal{X} = [0,1]$ and $(\mathcal{X}, \rho)$ be a usual fuzzy metric space, $\diamond$ be the product t-norm, $w > 0$, and $k = 2$. Define the fuzzy set $\mathcal{F}_\mathcal{Y}$ on $\mathcal{X}^2 \times (0, +\infty)^k$ by

$$\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, r_3) = \frac{w}{w + \rho(\eta, \zeta)(\frac{1}{r_1} + \frac{1}{r_2})}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, r_3 > 0$. Then $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is a G-complete 2-fuzzy metric space. Define a self-map $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{P}\eta = \begin{cases} \frac{1}{4}\eta & \text{if } \eta \in [0,1), \\ \frac{1}{5} & \text{if } \eta = 1 \end{cases}$$

for all $\eta \in \mathcal{X}$. Here $\mathcal{P}$ is not continuous at the point $\eta = 1$. Take $\psi(t) = \sqrt{t} > t$ for all $t \in (0,1)$.

1. If $\eta \in [0,1)$ and $\zeta \in [0,1)$, then

$$\begin{aligned} &\mathcal{F}_\mathcal{Y}(\mathcal{P}\eta, \mathcal{P}\zeta, r_1, r_2, r_3) \\ &\quad = \frac{w}{w + |\frac{1}{4}\eta - \frac{1}{4}\zeta|(\frac{1}{r_1} + \frac{1}{r_2})} \geqslant \psi\biggl(\frac{w}{w + |\eta - \zeta|(\frac{1}{r_1} + \frac{1}{r_2})}\biggr) \\ &\quad = \psi\bigl(\mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, r_3)\bigr). \end{aligned} \tag{12}$$

2. If $\eta \in [0,1)$ and $\zeta = 1$, then

$$\begin{aligned} &\mathcal{F}_\mathcal{Y}(\mathcal{P}\eta, \mathcal{P}\zeta, r_1, r_2, r_3) \\ &\quad = \frac{w}{w + |\frac{1}{4}\eta - \frac{1}{5}|(\frac{1}{r_1} + \frac{1}{r_2})} \geqslant \psi\biggl(\frac{w}{w + |\eta - 1|(\frac{1}{r_1} + \frac{1}{r_2})}\biggr) \\ &\quad = \mathcal{F}_\mathcal{Y}(\eta, \zeta, r_1, r_2, r_3). \end{aligned} \tag{13}$$

3. If $\eta = 1$ and $\zeta = 1$, then the contraction condition is obviously satisfied. Hence, $\mathcal{P}$ is a 2-fuzzy ψ-contractive mapping with respect to $\psi(t) = \sqrt{t} > t$, and the point $\eta = 0$ is the fixed point for the self-map.

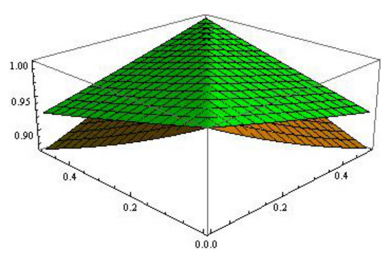

Figure 3. Red colour defined by L.H.S., and yellow colour defined by R.H.S. in Eq. (12).

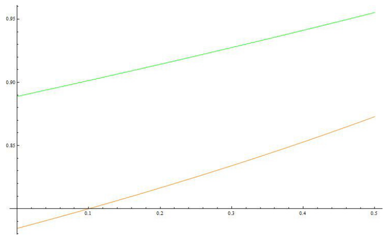

Figure 4. Red colour defined by L.H.S., and yellow colour defined by R.H.S. in Eq. (13).

Theorem 4. *Let $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ be a G-complete l-natural k-fuzzy metric space with $\diamond$ as minimum t-norm. Let $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ be a self-mapping satisfying*

$$\begin{aligned} {\mathcal{F_Y}}_l^{1/\lambda}\big(\mathcal{P}\eta, \mathcal{P}\zeta, r_1^k\big) \geqslant \min\big\{&\mathcal{F_Y}\big(\eta, \zeta, r_1^k\big), \mathcal{F_Y}\big(\eta, \mathcal{P}\eta, r_1^k\big), \mathcal{F_Y}\big(\zeta, \mathcal{P}\zeta, r_1^k\big), \\ &{\mathcal{F_Y}}_l^{1/2}\big(\eta, \mathcal{P}\zeta, r_1^k\big) \diamond {\mathcal{F_Y}}_l^{1/2}\big(\zeta, \mathcal{P}\eta, r_1^k\big)\big\} \end{aligned} \tag{14}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \ldots, r_k > 0$, where $l \in \{1, 2, \ldots, k\}$, and $\lambda \in (0, 1)$ is a constant. Suppose that $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ is an l-natural k-fuzzy metric space. Then $\mathcal{P}$ has a unique fixed point.

Proof. First, we will show that if a fixed point of $\mathcal{P}$ exists, then it is unique. Suppose that u and v are fixed points of $\mathcal{P}$. Using (14), we have

$$\begin{aligned} &\mathcal{F_Y}\big(u, v, r_1^k\big) \\ &\quad= \mathcal{F_Y}\big(\mathcal{P}u, \mathcal{P}v, r_1^k\big) \\ &\quad\geqslant \min\big\{{\mathcal{F_Y}}_l^{\lambda}\big(u, v, r_1^k\big), {\mathcal{F_Y}}_l^{\lambda}\big(u, \mathcal{P}v, r_1^k\big), {\mathcal{F_Y}}_l^{\lambda}\big(v, \mathcal{P}v, r_1^k\big), \\ &\qquad\qquad {\mathcal{F_Y}}_l^{\lambda/2}\big(u, \mathcal{P}v, r_1^k\big) \diamond {\mathcal{F_Y}}_l^{\lambda/2}\big(v, \mathcal{P}u, r_1^k\big)\big\} \\ &\quad= \min\big\{{\mathcal{F_Y}}_l^{\lambda}\big(u, v, r_1^k\big), {\mathcal{F_Y}}_l^{\lambda}\big(u, v, r_1^k\big), {\mathcal{F_Y}}_l^{\lambda}\big(v, v, r_1^k\big), \\ &\qquad\qquad {\mathcal{F_Y}}_l^{\lambda/2}\big(u, v, r_1^k\big) \diamond {\mathcal{F_Y}}_l^{\lambda/2}\big(v, u, r_1^k\big)\big\} \\ &\quad= \min\big\{{\mathcal{F_Y}}_l^{\lambda}\big(u, v, r_1^k\big), 1, 1, \\ &\qquad\qquad \big({\mathcal{F_Y}}_l^{\lambda}\big(u, u, r_1^k\big) \diamond {\mathcal{F_Y}}_l^{\lambda}\big(u, v, r_1^k\big)\big) \diamond {\mathcal{F_Y}}_l^{\lambda}\big(v, v, r_1^k\big) \diamond {\mathcal{F_Y}}_l^{\lambda}\big(v, u, r_1^k\big)\big\} \\ &\quad= {\mathcal{F_Y}}_l^{\lambda}\big(u, v, r_1^k\big). \end{aligned}$$

By repeating this process, we obtain

$$\mathcal{F_Y}\big(u, v, r_1^k\big) \geqslant {\mathcal{F_Y}}_l^{\lambda^n}\big(u, v, r_1^k\big)$$

for all $n \in \mathbb{N}$. Note that, if $\{a_n\}$ be any sequence such that $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$, then since $(\mathcal{X}, \mathcal{F_Y}, \diamond)$ is l-natural, we have

$$\lim_{n\to+\infty} {\mathcal{F_Y}}_l^{a_n}\big(u, v, r_1^k\big) = 1 \tag{15}$$

for all $r_1, r_2, \ldots, r_k > 0$. By using (15), we obtain

$$\mathcal{F}_{\mathcal{Y}}\big(u, v, r_1^k\big) = 1$$

for all $r_1, r_2, \ldots, r_k > 0$, that is, $u = v$. Therefore, the fixed point of $\mathcal{P}$ is unique. To obtain a fixed point for $\mathcal{P}$, we choose any arbitrary point $\eta_0 \in \mathcal{X}$ and construct an iterative Picard sequence $\{\eta_n\}$ by $\eta_n = \mathcal{P}\eta_{n-1}$ for all $n \in \mathbb{N}_0$:

$$\begin{aligned}
&\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big)\\
&\quad = \mathcal{F}_{\mathcal{Y}}\big(\mathcal{P}\eta_{n-1}, \mathcal{P}\eta_n, r_1^k\big)\\
&\quad \geqslant \min\big\{\mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big), \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big),\\
&\qquad\qquad \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big), \mathcal{F}_{\mathcal{Y}_l}^{\lambda/2}\big(\eta_{n-1}, \eta_{n+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{\lambda/2}\big(\eta_n, \eta_n, r_1^k\big)\big\}\\
&\quad = \min\big\{\mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big), \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big), \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big),\\
&\qquad\qquad \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_n, \eta_{n+1}, r_1^k\big) \diamond 1\big\}.
\end{aligned}$$

If

$$\min\big\{\mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big), \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_n, \eta_{n+1}, r_1^k\big)\big\} = \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_n, \eta_{n+1}, r_1^k\big),$$

nothing will be proved, so that

$$\min\big(\mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big), \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_n, \eta_{n+1}, r_1^k\big)\big) = \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big).$$

Therefore,

$$\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big) \geqslant \mathcal{F}_{\mathcal{Y}_l}^{\lambda}\big(\eta_{n-1}, \eta_n, r_1^k\big).$$

By repeating this process, we obtain

$$\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+1}, r_1^k\big) \geqslant \mathcal{F}_{\mathcal{Y}_l}^{\lambda^n}\big(\eta_0, \eta_1, r_1^k\big) \tag{16}$$

for all $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, $p > 0$, and $r_1, r_2, \ldots, r_k > 0$, we have

$$\begin{aligned}
&\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+p}, r_1^k\big)\\
&\quad \geqslant \mathcal{F}_{\mathcal{Y}_l}^{2}\big(\eta_n, \eta_{n+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2}\big(\eta_{n+1}, \eta_{n+p}, r_1^k\big)\\
&\quad \geqslant \mathcal{F}_{\mathcal{Y}_l}^{2}\big(\eta_n, \eta_{n+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^2}\big(\eta_{n+1}, \eta_{n+2}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^2}\big(\eta_{n+2}, \eta_{n+p}, r_1^k\big)\\
&\quad \geqslant \mathcal{F}_{\mathcal{Y}_l}^{2}\big(\eta_n, \eta_{n+1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^2}\big(\eta_{n+1}, \eta_{n+2}, r_1^k\big)\\
&\qquad \diamond \cdots \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^{p-1}}\big(\eta_{n+p-2}, \eta_{n+p-1}, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^p}\big(\eta_{n+p-1}, \eta_{n+p}, r_1^k\big).
\end{aligned}$$

By using (16), we obtain

$$\begin{aligned}
\mathcal{F}_{\mathcal{Y}}\big(\eta_n, \eta_{n+p}, r_1^k\big) &\geqslant \mathcal{F}_{\mathcal{Y}_l}^{2\lambda^n}\big(\eta_0, \eta_1 r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^2\lambda^{n+1}}\big(\eta_0, \eta_1, r_1^k\big)\\
&\quad \diamond \cdots \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^{p-1}\lambda^{n+p-2}}\big(\eta_0, \eta_1, r_1^k\big) \diamond \mathcal{F}_{\mathcal{Y}_l}^{2^p\lambda^{n+p-1}}\big(\eta_0, \eta_1, r_1^k\big).
\end{aligned}$$

Since $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is l-natural, it follows from the above inequality that

$$\lim_{n\to+\infty} \mathcal{F}_\mathcal{Y}\big(\eta_n, \eta_{n+1}, r_1^k\big) = 1.$$

Therefore, $\{\eta_n\}$ is a G-Cauchy sequence. By the G-completeness of $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ there exists $u \in \mathcal{X}$ such that

$$\lim_{n\to+\infty} \mathcal{F}_\mathcal{Y}\big(\eta_n, u, r_1^k\big) = 1 \tag{17}$$

for all $r_1, r_2, \dots, r_k > 0$.

Now, we must shows that u is a fixed point of $\mathcal{P}$. Since $\mathcal{P}$ is continuous map, we have

$$\begin{aligned} \mathcal{F}_\mathcal{Y}\big(u, \mathcal{P}u, r_1^k\big) &\geqslant {\mathcal{F}_\mathcal{Y}}_l^2\big(u, \eta_n, r_1^k\big) \diamond {\mathcal{F}_\mathcal{Y}}_l^2\big(\eta_n, \mathcal{P}u, r_1^k\big) \\ &= {\mathcal{F}_\mathcal{Y}}_l^2\big(u, \eta_n, r_1^k\big) \diamond {\mathcal{F}_\mathcal{Y}}_l^2\big(\mathcal{P}\eta_{n-1}, \mathcal{P}u, r_1^k\big) \\ &\geqslant {\mathcal{F}_\mathcal{Y}}_l^2\big(u, \eta_n, r_1^k\big) \diamond {\mathcal{F}_\mathcal{Y}}_l^{2\lambda}\big(\eta_{n-1}, u, r_1^k\big). \end{aligned}$$

By using (17) in above inequality, we obtain

$$\mathcal{F}_\mathcal{Y}\big(u, \mathcal{P}u, r_1^k\big) = 1$$

for all $r_1, r_2, \dots, r_k > 0$, that is, $\mathcal{P}u = u$. Thus, u is the unique fixed point of $\mathcal{P}$. □

Theorem 5. *Let $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ be a G-complete l-natural k-fuzzy metric space with $\diamond$ as minimum t-norm. Let $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ be a self-mapping satisfying*

$${\mathcal{F}_\mathcal{Y}}_l^{1/\lambda}\big(\mathcal{P}\eta, \mathcal{P}\zeta, r_1^k\big) \geqslant \min\big\{\mathcal{F}_\mathcal{Y}\big(\eta, \zeta, r_1^k\big), \mathcal{F}_\mathcal{Y}\big(\eta, \mathcal{P}\eta, r_1^k\big), \mathcal{F}_\mathcal{Y}\big(\zeta, \mathcal{P}\zeta, r_1^k\big)\big\}$$

for all $\eta, \zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$, where $l \in \{1, 2, \dots, k\}$, and $\lambda \in (0, 1)$ is a constant. Suppose that $(\mathcal{X}, \mathcal{F}_\mathcal{Y}, \diamond)$ is an l-natural k-fuzzy metric space. Then $\mathcal{P}$ has a unique fixed point.

Proof. We can follow the procedure similar to that of Theorem 4. □

4 Applications

4.1 Application to nonlinear Volterra integral equations

We will prove the existence and uniqueness of a solution for the given integral equation of Volterra kind

$$\kappa(t) = g(t) + \int_0^a h(t, s) \cdot f\big(t, s, \kappa(s)\big)\,\mathrm{d}s, \quad t \in [0, a], \tag{18}$$

where $a > 0$, $f : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$, and $h : [0, a] \times [0, a] \to \mathbb{R}$.

Let $\mathcal{X} = \mathrm{C}([0,a],\mathbb{R})$ be the set of all continuous functions $g : [0,a] \to \mathbb{R}$. It is well known that $\mathcal{X}$, equipped with Bioielecki's norm

$$\|\kappa\| = \sup_{t\in[0,a]} \mathrm{e}^{-t}\bigl|\kappa(t)\bigr|,$$

is a Banach space. Thus, $\mathcal{X}$, endowed with the distance associated with Bielecki's norm

$$\rho(\kappa,\gamma) = \sup_{t\in[0,a]} \mathrm{e}^{-t}\bigl|\kappa(t)-\gamma(t)\bigr|$$

for all $\kappa,\gamma \in \mathcal{X}$, is a complete metric space.

Define G-complete k-fuzzy metric space $\mathcal{F}_{\mathcal{Y}} : \mathcal{X}\times\mathcal{X}\times[0,+\infty] \to [0,1]$ by

$$\mathcal{F}_{\mathcal{Y}}\bigl(\eta,\zeta,r_1^k\bigr) = \mathrm{e}^{-\rho(\eta,\zeta)}\sum_{i=1}^{k}\frac{1}{r_i}$$

for all $\eta,\zeta \in \mathcal{X}$ and $r_1, r_2, \dots, r_k > 0$.

Theorem 6. *Suppose that the following conditions hold*:

(i) *For all* $t, s \in [0,a]$, $\kappa,\gamma \in \mathcal{X}$,

$$\bigl|f\bigl(t,s,\kappa(s)\bigr) - f\bigl(t,s,\gamma(s)\bigr)\bigr| < \frac{|\kappa(s)-\gamma(s)|}{2}.$$

(ii) *For all* $t, s \in [0,a]$,

$$\sup_{t\in[0,a]}\int_0^a \bigl|h(t,s)\bigr|\,\mathrm{d}s \leqslant \frac{1}{a}.$$

Then the integral equation (18) *has a unique solution in* $\mathcal{X}$.

Proof. Let the mapping $\mathfrak{L} : \mathcal{X} \to \mathcal{X}$ be the integral operator defined by

$$\mathfrak{L}\bigl(\kappa(t)\bigr) = g(t) + \int_0^a h(t,s)\cdot f\bigl(t,s,\kappa(s)\bigr)\,\mathrm{d}s,$$

for $\kappa \in \mathcal{X}$ and $t, s \in [0,a]$.

Consider $\psi(t) = \sqrt{t} > t$ for all $t \in (0,1)$ such that $\psi \in \Psi$. Now

$$\begin{aligned}
&\mathcal{F}_{\mathcal{Y}}\bigl(\mathfrak{L}\kappa,\mathfrak{L}\gamma,r_1^k\bigr)\\
&\quad= \mathcal{F}_{\mathcal{Y}}\bigl(\mathfrak{L}\kappa,\mathfrak{L}\gamma,r_1,r_2,\dots,r_1,\dots,r_k\bigr) = \exp\left\{-\rho(\mathfrak{L}\kappa,\mathfrak{L}\gamma)\sum_{i=1}^{k}\left(\frac{1}{r_i}\right)\right\}\\
&\quad= \exp\left\{-\sup_{t\in[0,a]}\left\{\left|\int_0^a h(t,s)\cdot f\bigl(t,s,\kappa(s)\bigr)\,\mathrm{d}s - \int_0^a h(t,s)\cdot f\bigl(t,s,\gamma(s)\bigr)\,\mathrm{d}s\right|\right\}\right.\\
&\qquad\qquad \left.\times\sum_{i=1}^{k}\left(\frac{1}{r_i}\right)\right\}
\end{aligned}$$

$$= \exp\left\{-\sup_{t\in[0,a]}\left\{\int_0^a \left|h(t,s)\right| \mathrm{d}s \int_0^a \left|f\big(t,s,\kappa(s)\big) - f\big(t,s,\gamma(s)\big)\right| \mathrm{d}s\right\} \sum_{i=1}^{k}\left(\frac{1}{r_i}\right)\right\}$$
$$\geqslant \exp\left\{-\sup_{t\in[0,a]}\left\{\frac{1}{a}\cdot\int_0^a \frac{|\kappa(s)-\gamma(s)|}{2}\,\mathrm{d}s\right\} \sum_{i=1}^{k}\left(\frac{1}{r_i}\right)\right\}$$
$$= \left(\exp\left\{-\sup_{t\in[0,a]}\left|\kappa(s)-\gamma(s)\right| \sum_{i=1}^{k}\left(\frac{1}{r_i}\right)\right\}\right)^{1/2}$$
$$= \big(\mathcal{F}_{\mathcal{Y}}\big(\kappa,\gamma,r_1^k\big)\big)^{1/2} = \psi\big(\mathcal{F}_{\mathcal{Y}}\big(\kappa,\gamma,r_1^k\big)\big).$$

Since all assertions of Theorem 3 are satisfied, the map $\mathfrak{L}$ has unique fixed point. This means that the integral equation (18) has a solution. □

4.2 Application to second-order differential equations

Now we discuss the existence and uniqueness of solutions of the boundary value problem for the second-order differential equation

$$-\frac{\mathrm{d}^2\kappa}{\mathrm{d}t^2} = f\big(t,\kappa(t)\big), \quad t\in\mathbb{I}, \tag{19}$$

with boundary conditions $\kappa(0)=\kappa(1)=0$, where $\mathbb{I}=[0,1]$ and $f:\mathbb{I}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Let $\mathcal{X} = \mathrm{C}(\mathbb{I},\mathbb{R})$ be the space of all continuous function $\kappa:\mathbb{I}\to\mathbb{R}$. It is well known that $\mathcal{X}$, endowed with

$$\sigma_\infty(\eta,\zeta) = \sup_{t\in[0,1]}\left|\eta(t)-\zeta(t)\right|$$

for all $\eta,\zeta\in\mathcal{X}$, is a complete metric space. Let us define fuzzy set $\mathcal{F}_{\mathcal{Y}}$ on $\mathcal{X}^2\times(0,+\infty)^2$ by

$$\mathcal{F}_{\mathcal{Y}}(\eta,\zeta,r_1,r_2) = \exp\frac{\sigma_\infty(\eta,\zeta)}{r_1}$$

for all $\eta,\zeta\in\mathcal{X}$ and $r_1,r_2>0$. Then $(\mathcal{X},\mathcal{F}_{\mathcal{Y}},\diamond)$ is a G-complete 2-fuzzy metric space ($k=2$). Moreover,

$$\lim_{r_1\to+\infty}\mathcal{F}_{\mathcal{Y}}(\eta,\zeta,r_1,r_2) = 1$$

for all $\eta,\zeta\in\mathcal{X}$, that is, $(\mathcal{X},\mathcal{F}_{\mathcal{Y}},\diamond)$ is a l-natural 2-fuzzy metric space.

Theorem 7. *Suppose that the following condition holds for all $z,w\in\mathbb{R}$ and for all $r\in\mathbb{I}$:*

$$\left|f(r,z)-f(r,w)\right| < |z-w|.$$

Then problem (19) *has a unique solution in* $\eta^*\in\mathrm{C}(\mathbb{I},\mathbb{R})$.

Proof. It is well known that problem (19) is equivalent to the following integral equation:

$$\eta(t) = \int_0^1 \mathcal{G}(t,r) f\big(r,\eta(r)\big)\,\mathrm{d}r$$

for all $t \in \mathbb{I}$, where $\mathcal{G}$ is Green function associated to problem (19) and given by

$$\mathcal{G}(t,r) = \begin{cases} t(1-r) & \text{for } 0 \leqslant t \leqslant r \leqslant 1, \\ r(1-t) & \text{for } 0 \leqslant r \leqslant t \leqslant 1. \end{cases}$$

The $\eta \in \mathrm{C}(\mathbb{I},\mathbb{R})$ is a solution of equation (19) if and only if $\eta \in \mathrm{C}(\mathbb{I},\mathbb{R})$ is a solution of the integral equation (19). Now we can define a mapping $\mathfrak{L} : \mathcal{X} \to \mathcal{X}$ as follows:

$$\mathfrak{L}\eta(t) = \int_0^1 \mathcal{G}(t,r) f\big(r,\eta(r)\big)\,\mathrm{d}r$$

for all $t \in \mathbb{I}$ and $\eta \in \mathcal{X}$. Let $\eta, \zeta \in \mathcal{X}$ such that $\mathfrak{L}\eta \neq \mathfrak{L}\zeta$,

$$\begin{aligned} \big|\mathfrak{L}\eta(t) - \mathfrak{L}\zeta(t)\big| &= \left| \int_0^t \mathcal{G}(t,r) f\big(r,\eta(r)\big)\,\mathrm{d}r - \int_0^t \mathcal{G}(t,r) f\big(r,\zeta(r)\big)\,\mathrm{d}r \right| \\ &= \int_0^t \mathcal{G}(t,r) \big| f\big(r,\eta(r)\big) - f\big(r,\zeta(r)\big)\big|\,\mathrm{d}r \\ &\leqslant \big|\eta(r) - \zeta(r)\big| \int_0^1 \mathcal{G}(t,r)\,\mathrm{d}r. \end{aligned}$$

Since $\sup_{t\in[0,1]} \int_0^1 \mathcal{G}(t,r)\,\mathrm{d}r = 1/8$,

$$\begin{aligned} &\sup_{t\in[0,1]} \big|\mathfrak{L}\eta(t) - \mathfrak{L}\zeta(t)\big| \leqslant \frac{1}{8} \sup_{t\in[0,1]} \big|\eta(r) - \zeta(r)\big| \\ &\implies \sigma_\infty(\mathfrak{L}\eta, \mathfrak{L}\zeta) \leqslant \frac{1}{8}\sigma_\infty(\eta,\zeta) \implies -\frac{\sigma_\infty(\eta,\zeta)}{r_1} \leqslant -\frac{\sigma_\infty(\mathfrak{L}\eta,\mathfrak{L}\zeta)}{\frac{1}{8}r_1} \\ &\implies \exp\left\{-\left(\frac{\sigma_\infty(\eta,\zeta)}{r_1}\right)\right\} \leqslant \exp\left\{-\left(\frac{\sigma_\infty(\mathfrak{L}\eta,\mathfrak{L}\zeta)}{\frac{1}{8}r_1}\right)\right\}. \end{aligned}$$

Choose $\lambda = 1/8 \in (0,1)$, then

$$\begin{aligned} &\exp\left\{-\left(\frac{\sigma_\infty(\eta,\zeta)}{r_1}\right)\right\} \leqslant \exp\left\{-\left(\frac{\sigma_\infty(\mathfrak{L}\eta,\mathfrak{L}\zeta)}{\lambda r_1}\right)\right\} \\ &\implies \mathcal{F}y(\eta,\zeta,r_1,r_2) \leqslant \mathcal{F}y_1^{1/\lambda}(\mathfrak{L}\eta,\mathfrak{L}\zeta,r_1,r_2). \end{aligned}$$

Therefore, $\mathfrak{L}$ has unique fixed point $\eta^* \in \mathcal{X}$. Thus, problem (19) has a unique solution $\eta^* \in \mathrm{C}(\mathbb{I},\mathbb{R})$. □

5 Conclusion and future scope

The motivational new fuzzy metric space invented by Gopal [4], they have proved Banach contraction principle with the concern of k-fuzzy metric space with the restricted property l-naturalness of the space. In this article, we have generalized some more famed contraction like Tirado-type, Mihet-type, and few more in terms of k-fuzzy metric spaces. While preparing the example of such theorems, we saw that the space is not natural, we need to consider more than one parameter tending to infinity, only then we will get $\mathcal{F}_{\mathcal{Y}}(\eta, \zeta, r_1, r_2, r_3)$ tends to 1.

Further, researchers may go with this direction to define some other concepts of k-fuzzy metric space or may define new definition of fuzzy spaces. Moreover, we can prepare more theorems in Suzuki-type contraction like Chandra et al. [1], Patel et al. [10], and also calculate best proximity point for this space like Patel et al. [11].

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