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# Eigenvalue problems for a $k$-Hessian-type equation*

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**Abstract.** In this work, we focus on the eigenvalue problem for a class of $k$-Hessian-type equations. Under some suitable assumptions, we first determine the intervals of the parameter for the existence of nontrivial radial solutions. To this aim, we apply the eigenvalue theory and Jensen inequality. Finally, the behavior of the solutions with respect to the parameter is analyzed via Guo's fixed point theorem.

**Keywords:** $k$-Hessian equation, existence, dependence on a parameter, fixed point theorem.

## 1 Introduction

The study of the $k$-Hessian-type equations has an important sense in many geometric problems, such as the $k$-Yamabe problems in conformal geometry [19] and conformal invariant elliptic problems [14]. Furthermore, for the case $k = N$, the Monge–Ampère-type equations arise in geometrical optics [12], meteorology [8], and optimal transportation [16].

In this paper, we focus on the following $k$-Hessian-type equation:

$$\begin{aligned} &S_k\bigl(\sigma(D^2u + \alpha|\nabla u|I)\bigr) = \lambda b\bigl(|x|\bigr)\varphi(-u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\alpha \geqslant 0$, $|\nabla u|$ denotes the gradient of $u$, $I$ stands for an identity matrix, $\lambda$ is a positive parameter, $\Omega$ is an open unit ball in $\mathbb{R}^N$ with $N < 2k$ ($k \in \mathbb{N}$), $\varphi$ and $b$ satisfy

(H1) $\varphi \in C([0,\infty),[0,\infty))$;
(H2) $b \in C([0,1],[0,\infty))$, and $b \not\equiv 0$ on any subinterval of $[0,1]$.

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For an arbitrary $N \times N$ real symmetric matrix $M$, $S_k(\sigma(M))$ is the $k$-Hessian-type operator and denotes $k$th elementary symmetric function of eigenvalues of $M$, that is,

$$S_k(\sigma(M)) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant N} \Lambda_{j_1} \cdot \Lambda_{j_2} \cdots \Lambda_{j_k},$$

where $\sigma(M) = (\Lambda_1, \Lambda_2, \ldots, \Lambda_N)$ is the set of eigenvalues $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$. For any $C^2$ function $u(x)$, if $M$ is the Hessian matrix $(D^2u)$ of $u$, then it becomes the so-called $k$-Hessian operator [3].

The main motivation to study (1) comes from the impressive development of theory concerning the nonlinear $k$-Hessian-type equation. Such an equation can be considered as an extension of the Laplace, Monge–Ampère, and $k$-Hessian equation. There are many interesting results on the $k$-Hessian-type equations [1–5, 13, 20, 21, 23, 24, 26–32], starting with the pioneering work of Caffarelli, Nirenberg, and Spruck [3].

### On the $k$-Hessian problems

In 2017, Dai [5] studied the eigenvalue problem for the $k$-Hessian equation

$$\begin{aligned} &S_k\bigl(\sigma\bigl(D^2u\bigr)\bigr) = \lambda^k \varphi(-u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\lambda > 0$, $\Omega$ is an unit ball in $\mathbb{R}^N$ with $N \geqslant 1$, and $\varphi$ satisfies (H1) with $\varphi(s) > 0$ for any $s > 0$ and $\varphi_0, \varphi_\infty \in [0, \infty]$. Here

$$\varphi_0 = \lim_{s \to 0^+} \frac{\varphi(s)}{s^k}, \qquad \varphi_\infty = \lim_{s \to +\infty} \frac{\varphi(s)}{s^k}.$$

He showed the existence, nonexistence, and multiplicity of radial $k$-admissible solutions to (2) for $\lambda$ belonging to different intervals via the bifurcation method.

In 2022, Zhang et al. [31] researched the following eigenvalue problem:

$$\begin{aligned} &(-1)^k S_k^{1/k}\bigl(\sigma\bigl(D^2u\bigr)\bigr) = \lambda \varphi\bigl(|x|, u\bigr) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

where $\lambda > 0$, $\Omega$ is a unit ball in $\mathbb{R}^N$ with $k \leqslant N < 2k$, and $\varphi \in [0,1] \times (0, +\infty) \to (0, +\infty)$ is nonincreasing in $u > 0$ and satisfies

$$0 < \int_0^1 s^{N-1} \varphi^k\bigl(s,\, 1 - s^2\bigr)\, \mathrm{d}s < +\infty.$$

By constructing the upper and lower solutions and combining with Schauder's fixed point theorem, they proved that there exists $\lambda^* > 0$ such that, for any $\lambda \in (\lambda^*, +\infty)$, problem (3) admits at least one radial solution $u$ satisfying

$$1 - |x|^2 \leqslant u(|x|) \leqslant \rho\bigl(1 - |x|^{(2k-N)/k}\bigr) \tag{4}$$

for some positive constant $\rho$.

**On the $k$-Hessian-type problems**

In 2022, Zhang et al. [30] extended the results in [31] to the following eigenvalue problem:

$$\begin{aligned} &-S_k^{1/k}\big(\sigma\big(D^2u+\lambda\alpha I\big)\big)=\lambda\varphi\big(|x|,u\big) \quad \text{in } \Omega, \\ &u=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where $\varphi \in [0,1]\times(0,+\infty)\to(0,+\infty)$ is nonincreasing in $u>0$. They proved that there exist two positive constants $\lambda_*$ and $\lambda^*$ such that, for any $\lambda\in(\lambda_*,\lambda^*)$, (5) admits at least one radial solution $u$ satisfying (4).

Recently, in [24], Yang et al. considered the $k$-Hessian-type system

$$\begin{aligned} &S_k\big(\sigma\big(D^2u_i+\alpha|\nabla u_i|I\big)\big)=\varphi_i\big(|x|,-u_1,-u_2,\dots,-u_n\big) \quad \text{in } \Omega, \\ &u_i=0 \quad \text{on } \partial\Omega \end{aligned} \tag{6}$$

and obtained some existence results of radial solutions relying on fixed point index computations.

However, to our knowledge, there are few studies on nonlinear eigenvalue problem for the $k$-Hessian-type equation (1). The main aims of this paper are twofold. One is to derive some new existence results of radial solutions for (1) with the help of the eigenvalue theory. The other is to analyze the behavior of the radial solutions $u_\lambda$ with respect to the parameter $\lambda$ for (1) via Guo's fixed point theorem.

The $k$-Hessian-type equations have important significance in nonlinear science. Many optimal transport problems, geometric optics problems, etc. can be transformed into solving $k$-Hessian differential models, which greatly promotes research and development in related fields of economics and physics. The contributions of our paper are as follows.

1. Compared with the works [5, 30, 31], the augmented gradient term $\alpha|\nabla u|I$ is considered.
2. We supplement the results on the asymptotic behavior of the radial solutions $u_\lambda$ that depend on the parameter $\lambda$, unlike previous works [5, 24, 30, 31], which only considered existence results.
3. The nonlinearity in (1) does not require monotonicity, unlike in [20, 21, 26, 30, 31].
4. Jensen inequality technique is used, which enables us to overcome the difficulties caused by the complexity of the $k$-Hessian-type operator.

The rest of this paper is arranged as follows. In Section 2, we will show some preliminaries. In particular, we introduce the fixed point theory and Jensen inequality. In Section 3, we give the intervals of $\lambda$ for the existence of negative radial solutions for the $k$-Hessian-type equation (1) under some suitable assumptions on $\varphi$. Finally, in Section 4, the behavior of nontrivial radial solution $u_\lambda$ with respect to the parameter $\lambda$ are stated and proved.

**Remark 1.** The general form of the $k$-Hessian-type equation (1) is as follows:

$$S_k\big(\sigma\big(\big(D^2u+M(x,u,Du)\big)\big)\big)=\varphi(x,u,Du) \quad \text{in } \Omega, \tag{7}$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $M$ is a $N \times N$ symmetric matrix function, $\varphi$ is a scalar valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. When $M = 0$, (7) is the standard $k$-Hessian equation. Furthermore, it can be reduced to classical Poisson equation ($k = 1$) and the Monge–Ampère equation ($k = N$). When $M \neq 0$ and $k = N$, (7) is called the Monge–Ampère-type equation.

The domain $\Omega$ plays important roles in many geometric problems. We know that scholars search for suitable geometric forms of the domain $\Omega$ in order to obtain desired results. For example, when $\Omega$ is a star-shaped domain, Tso [18] obtained a nonexistence result of the Dirichlet problem for the case $M = 0$ and $\varphi(x, u, Du) = \varphi(x, -u)$. When $(\Omega, g)$ is a Riemannian manifold with the metric $g$, Guan [10] proved the existence result of the Dirichlet problem for the case $M = \alpha u g$. On a closed Hermitian manifold $(\Omega, g)$, Zhang [25] showed the existence of weak solution for the complex Hessian equation of (7). In [6], Dinew, Pliś and Zhang consider a second-order a priori estimate for solutions of the complex Hessian equations of (7) on a compact Kähler manifold. In [9], Gálvez and Nelli studied the global behavior of solutions for $\det(D^2u) = 0$ in the finitely punctured plane $\Omega \subset \mathbb{R}^2$.

**Remark 2.** When $k = 1$, the problem in (1) becomes to the following linear Laplace problem:

$$\begin{aligned} &\Delta u + N\alpha|Du| = \lambda b\big(|x|\big)\varphi(-u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{8}$$

where the nonlinearity $\varphi$ satisfies (H1) with $\varphi_0, \varphi_\infty \in [0, \infty]$. Here

$$\varphi_0 = \lim_{s \to 0^+} \frac{\varphi(s)}{s}, \qquad \varphi_\infty = \lim_{s \to +\infty} \frac{\varphi(s)}{s}.$$

At this time, the nonlinearity $\varphi$ can be regarded as a perturbation form of the linear function $f(s) = s$, then the problem in (1) can be reformulated as the perturbation of the linear counterpart problem.

**Remark 3.** The $k$-Hessian-type equations and Monge–Ampère-type equations can be derived from the problems of light reflection and refraction in geometric optics [7, 15, 17, 22]. This type of problem involves the design of the shapes of reflective paraboloids and refractive paraboloids. It can be specifically divided into far-field optics and near-field optics. The description of light reflection in far-field optics can be considered as an optimal transport problem with a cost function; please see [22]. The equation corresponding to its problem has a form of (7). For the situation of light reflection in near-field optics, we take the parallel light reflection problem [17] as an example. The corresponding equation is the following nonlinear equation:

$$\det\left[D^2u - \frac{|Du|^2 - 1}{2u} I\right] = \frac{(1 - |Du|^2)^{N+1}}{(1 + |Du|^2)(2u)^N} f(x, Du), \tag{9}$$

where $u > 0$, $|Du| < 1$. When $|Du| \ll 1$, the right-hand nonlinear term in (9) can be approximated as $1/u^N$.

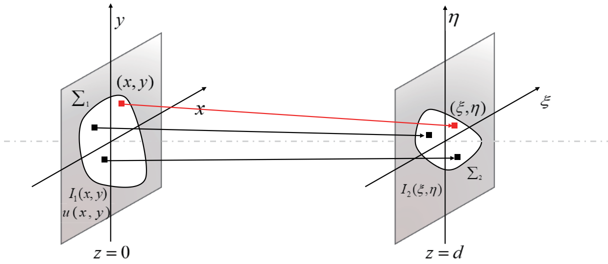


**Figure 1.** Ray mapping based on geometric optics.

Next, we will analyze the following Monge–Ampère model:

$$-\det\bigl(D^2u\bigr) = \gamma\bigl(1-u_y^2\bigr)u_{xx} + \gamma\bigl(1-u_x^2\bigr)u_{yy} + 2\gamma u_x u_y u_{xy} + \gamma^4 d^2\bigl[1 - I_1(x,y)I_2(\xi,\eta)\bigr], \tag{10}$$

where $\gamma = \sqrt{1-|\nabla u|^2}/d$, and $d$ is the distance between two planes. In (10), the nonlinearity $\varphi$ describes the complex physical relationship between phase and light intensity, which cannot be simply represented and controlled by a linear term. This model is used to study phase retrieval problem in geometric optics [15]. As shown in Fig. 1, assuming that there exists a light field with a wavelength of $\lambda$ transmitted along the $z$-axis. The light intensity distribution in the plane $z = 0$ is $I_1(x, y)$, where $(x, y) \in \Sigma_1$. The light intensity distribution at the plane $z = d$ is $I_2(\xi, \eta)$, where $(\xi, \eta) \in \Sigma_2$.

The light satisfies the law of conservation of energy during propagation. From [7]:

$$I_1(x,y) = I_2(\xi,\eta)\bigl|J(x,y)\bigr|, \tag{11}$$

where $|J(x, y)|$ is the Jacobian determinant of coordinate transformation. The relationship between $u(x, y)$ and ray mapping $T(x, y) = (\xi, \eta)$ is

$$T(x,y) = (x,y) + d\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}. \tag{12}$$

The Monge–Ampère model (10) can be derived from Eqs. (11) and (12). Then we solve model (10). The purpose of phase recovery is to calculate the phase distribution $U(x, y)$ at $z = 0$ based on the light intensity distribution of these two planes. Here $U(x, y) = ku(x, y)$, where $k = 2\pi/\lambda$ is the wave number.

## 2 Preliminaries

In this section, we will present some useful preliminary results.

**Lemma 1.** (*See* [24].) $u(x) = -v(t)$ *is a radial solution for* (1) *if and only if* $v$ *satisfies*

$$\begin{aligned} &\bigl[t^{N-k}\mathrm{e}^{N\alpha t}\bigl(-v'(t)\bigr)^k\bigr]' = \frac{k}{C_{N-1}^{k-1}}\frac{\mathrm{e}^{N\alpha t}t^{N-1}}{(1+\alpha t)^{k-1}}\lambda b(t)\varphi\bigl(v(t)\bigr), \quad 0<t<1, \\ &v'(0) = 0, \qquad v(1) = 0. \end{aligned} \tag{13}$$

Next, we will find a solution of the integral equation for (13)

$$v(t) = \lambda^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau} \tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s, \quad t \in [0,1]. \tag{14}$$

It is well known that $X = C[0,1]$ is a real Banach space endowed with the norm $\|v\| = \max_{t\in[0,1]} |v(t)|$. Define the cone

$$P := \left\{ v\colon\ v \in X,\ v \geqslant 0,\ \min_{t\in[1/4,3/4]} v(t) \geqslant \frac{1}{4}\|v\| \right\} \tag{15}$$

and the operator

$$Tv(t) := \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau} \tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s. \tag{16}$$

**Lemma 2.** *If* (H1) *and* (H2) *hold, then the operator* $T : X \to X$ *is completely continuous and* $T(P) \subset P$.

The proof of Lemma 2 is similar to the proof of the paper [24], which is omitted here.

The following eigenvalue theory will be used to analyze the existence of solutions for (14).

**Lemma 3.** (*See* [11].) *Let* $X$ *be an infinite-dimensional real Banach space and* $P \subset X$ *be a cone. If* $\Omega \subset X$ *is an open subset with* $\theta \in \Omega$ *and* $T : P \cap \Omega \to P$ *is a completely continuous operator with* $T\theta = \theta$ *satisfying*

$$\inf_{v\in P\cap\partial\Omega} Tv > 0,$$

*then* $T$ *admits a proper element in* $P \cap \partial\Omega$ *related to a positive eigenvalue. In other words, there are* $v_0 \in P \cap \partial\Omega$ *and* $\lambda_0$ *so that* $Tv_0 = \lambda_0 v_0$.

The following Guo's fixed point theorem plays a major role in the study of the behavior of the radial solutions $u_\lambda$ with respect to the parameter $\lambda$.

**Lemma 4.** (*See* [11].) *Let* $X$ *be a Banach space and* $P$ *be a cone in* $X$. *Assume that* $\Omega_1$, $\Omega_2$ *are open subsets of* $X$ *with* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, *and* $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *is a completely continuous operator such that one of the following two conditions*

(i) $\|Tv\| \leqslant \|v\|$, $v \in P \cap \partial\Omega_1$ *and* $\|Tv\| \geqslant \|v\|$, $v \in P \cap \partial\Omega_2$;
(ii) $\|Tv\| \geqslant \|v\|$, $v \in P \cap \partial\Omega_1$, *and* $\|Tv\| \leqslant \|v\|$, $v \in P \cap \partial\Omega_2$

*is satisfied. Then* $T$ *admits a fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Jensen integral inequality is presented as follows.

**Lemma 5.** (*See* [24].) *If* $v \in C([\alpha,\beta],[0,\infty))$, *then for* $0 < \gamma \leqslant 1$, *one has*

$$\left( \int_\alpha^\beta v(s)\,\mathrm{d}s \right)^\gamma \geqslant (\beta-\alpha)^{\gamma-1} \int_\alpha^\beta v^\gamma(s)\,\mathrm{d}s.$$

## 3 Existence results

Define

$$d_1 = \int_{1/4}^{3/4} b^{1/k}(\tau)\,\mathrm{d}\tau, \qquad d_2 = \int_0^1 b(\tau)\,\mathrm{d}\tau$$

and

$$L_1 = \left(\frac{k}{\mathrm{e}^{N\alpha}(1+\alpha)^{k-1}C_{N-1}^{k-1}}\right)^{1/k}, \qquad L_2 = \left(\frac{k e^{N\alpha}}{C_{N-1}^{k-1}}\right)^{1/k}.$$

In this section, we will study the existence of the positive solutions for problem (14).

**Theorem 1.** *If* (H1) *and* (H2) *are satisfied and $\varphi_\infty \in (0, +\infty)$, then there exist three positive constants $\beta_1$, $\lambda_1$, and $\lambda_2$ such that, for any $R > \beta_1$, Eq.* (14) *has at least one solution $v_R$ satisfying $\|v_R\| = R$ in $P$ for some $\lambda$ belonging to $[\lambda_1, \lambda_2]$.*

*Proof.* In view of (14) and (16), we know that (14) admits a solution $v_R(t)$, which corresponds to $\lambda > 0$ if and only if $T$ admits a proper element $v_R$, which corresponds to the eigenvalue $1/\lambda^k$. For $\varphi_\infty \in (0, +\infty)$, there exist $l_2 > l_1 > 0$ and $J_1 > 0$ so that $l_1 s^k < \varphi(s) < l_2 s^k$ for $s \geqslant J_1$. Let $\beta_1 = 4J_1$ and

$$\Omega_R = \{v\colon\ v \in X,\ \|v\| < R\}.$$

Because $R > \beta_1$, for any $v \in P \cap \partial\Omega_R$, we can see that

$$\min_{t\in[1/4,\,3/4]} v(t) \geqslant \frac{1}{4}\|v\| = \frac{1}{4}R > \frac{1}{4}\beta_1 = J_1.$$

Thus, for any $v \in P \cap \partial\Omega_R$ and $t \in [0, 1]$, we derive from Lemma 5 that

$$\begin{aligned}
Tv(t) &= \int_t^1 \left(\frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\big(v(\tau)\big)\,\mathrm{d}\tau\right)^{1/k} \mathrm{d}s \\
&\geqslant \left(\frac{k}{C_{N-1}^{k-1}}\right)^{1/k} \\
&\quad \times \int_t^1 \left(\frac{s^{k-N}}{\mathrm{e}^{N\alpha s}}\right)^{1/k} s^{1/k-1} \int_0^s \left(\frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}}\right)^{1/k} b^{1/k}(\tau)\varphi^{1/k}\big(v(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant \left(\frac{k}{\mathrm{e}^{N\alpha}(1+\alpha)^{k-1}C_{N-1}^{k-1}}\right)^{1/k} \int_t^1\int_0^s \tau^{(N-1)/k} b^{1/k}(\tau)\varphi^{1/k}\big(v(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= L_1 \int_t^1\int_0^s \tau^{(N-1)/k} b^{1/k}(\tau)\varphi^{1/k}\big(v(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\geqslant L_1 \int_{3/4}^{1} \int_{1/4}^{3/4} \tau^{(N-1)/k} b^{1/k}(\tau) l_1^{1/k} v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant L_1 \int_{3/4}^{1} \int_{1/4}^{3/4} \tau^{(N-1)/k} b^{1/k}(\tau) l_1^{1/k} \left(\frac{1}{4}\|v\|\right) \mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant L_1 \left(\frac{1}{4}\right)^{(N-1)/k} l_1^{1/k} \frac{1}{16}\|v\| \int_{1/4}^{3/4} b^{1/k}(\tau)\,\mathrm{d}\tau \\
&= L_1 \left(\frac{1}{4}\right)^{(N+2k-1)/k} l_1^{1/k} d_1 \|v\|,
\end{aligned} \tag{17}$$

which means that

$$\inf_{v \in H \cap \partial\Omega_R} Tv \geqslant L_1 \left(\frac{1}{4}\right)^{(N+2k-1)/k} l_1^{1/k} d_1 R > 0.$$

For any $R > \beta_1$, by Lemma 3, we can know that $T$ possesses a proper element $v_R \in P$, which corresponds to the eigenvalue $\mu_R > 0$. Moreover, one can see that $v_R$ satisfies $\|v_R\| = R$. Let $\lambda_R = 1/\mu_R^k$. Therefore, we immediately have

$$Tv_R = \mu_R v_R = \lambda_R^{-1/k} v_R. \tag{18}$$

Based on the above proof, for any $R > \beta_1$, problem (14) has a positive solution $v_R$ with $v_R \in P \cap \partial\Omega_R$ corresponding to $\lambda = \lambda_R > 0$. By (18), one has $v_R = \lambda_R^{1/k} T v_R$, that is,

$$v_R(t) = \lambda_R^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau} \tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau) \varphi\bigl(v_R(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s$$

with $\|v_R\| = R$. On the one hand, we derive

$$\begin{aligned}
v_R(t) &= \lambda_R^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau} \tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau) \varphi\bigl(v_R(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&\leqslant \lambda_R^{1/k} \int_0^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^1 \frac{\mathrm{e}^{N\alpha\tau} \tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau) \varphi\bigl(v_R(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&\leqslant \lambda_R^{1/k} \left( \frac{k\mathrm{e}^{N\alpha}}{C_{N-1}^{k-1}} \right)^{1/k} \int_0^1 \left( s^{k-N} \int_0^1 b(\tau) \varphi\bigl(v_R(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_R^{1/k} L_2 \int_0^1 s^{(k-N)/k}\,\mathrm{d}s \left( \int_0^1 b(\tau)\varphi\bigl(v_R(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \\
&\leqslant \lambda_R^{1/k} L_2 \frac{k}{2k-N} \left( \int_0^1 b(\tau) l_2 v_R^k(\tau)\,\mathrm{d}\tau \right)^{1/k} \\
&\leqslant \lambda_R^{1/k} L_2 \frac{k}{2k-N} \left( \int_0^1 b(\tau) l_2 \|v_R\|^k\,\mathrm{d}\tau \right)^{1/k} \\
&\leqslant (\lambda_R l_2)^{1/k} L_2 \frac{k}{2k-N} \|v_R\| \left( \int_0^1 b(\tau)\,\mathrm{d}\tau \right)^{1/k} \\
&= (\lambda_R l_2 d_2)^{1/k} L_2 \frac{k}{2k-N} \|v_R\| \quad \forall t \in [0,1], \qquad (19)
\end{aligned}
$$

which means that

$$
\|v_R\| = R \leqslant (\lambda_R l_2 d_2)^{1/k} L_2 \frac{k}{2k-N} \|v_R\|.
$$

Thus,

$$
\lambda_R \geqslant \frac{1}{l_2 d_2} \left( \frac{2k-N}{kL_2} \right)^k = \lambda_1.
$$

On the other hand, we have

$$
\begin{aligned}
v_R(t) &= \lambda_R^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v_R(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&\geqslant \lambda_R^{1/k} L_1 \left( \frac{1}{4} \right)^{(N+2k-1)/k} l_1^{1/k} d_1 \|v_R\|, \qquad (20)
\end{aligned}
$$

which shows that

$$
\|v_R\| = R \geqslant \lambda_R^{1/k} L_1 \left( \frac{1}{4} \right)^{(N+2k-1)/k} l_1^{1/k} d_1 \|v_R\|.
$$

Consequently,

$$
\lambda_R \leqslant \frac{4^{N+2k-1}}{l_1 d_1^k L_1^k} = \lambda_2.
$$

Thus, we have $\lambda_R \in [\lambda_1, \lambda_2]$. So, we complete the proof of Theorem 1. □

Based on a similar analysis of Theorem 1, one has the following result.

**Theorem 2.** *If* (H1) *and* (H2) *are satisfied and* $\varphi_0 \in (0, +\infty)$, *then there exist three positive constants* $\widehat{\beta}_1$, $\lambda_3$, *and* $\lambda_4$ *such that, for any* $0 < r < \widehat{\beta}_1$, *Eq.* (14) *has at least one solution* $v_r$ *satisfying* $\|v_r\| = r$ *in* $P$ *for some* $\lambda$ *belonging to* $[\lambda_3, \lambda_4]$.

## 4 The dependence results

In this section, we consider the behavior of the solutions $v_\lambda$ with respect to the parameter $\lambda$.

For $v \in P$, let us define $\widehat{T} : P \to X$ as

$$\widehat{T}v(t) = \lambda^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s, \tag{21}$$

where $t \in [0,1]$, and $P$ is defined in (15). It is a standard matter that $\widehat{T}$ is completely continuous.

**Theorem 3.** *If* (H1) *and* (H2) *are satisfied and* $\varphi_0 = 0$, $\varphi_\infty = \infty$, *then for every* $\lambda > 0$, *Eq.* (14) *has at least one solution* $v_\lambda$ *satisfying* $\lim_{\lambda\to 0^+} \|v_\lambda\| = \infty$.

*Proof.* For $\varphi_0 = 0$, there exists $r > 0$ so that

$$\varphi(s) \leqslant \frac{1}{\lambda d_2}\left(\frac{2k-N}{kL_2}\right)^k s^k \quad \forall\, 0 \leqslant s \leqslant r.$$

Therefore, for any $v \in P \cap \partial\Omega_r$ and $t \in [0,1]$, one has

$$\begin{aligned}
\widehat{T}v(t) &= \lambda^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&\leqslant \lambda^{1/k} \int_0^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^1 \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&\leqslant \lambda^{1/k} \left(\frac{k\mathrm{e}^{N\alpha}}{C_{N-1}^{k-1}}\right)^{1/k} \int_0^1 \left( s^{k-N} \int_0^1 b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&= \lambda^{1/k} L_2 \int_0^1 s^{(k-N)/k}\,\mathrm{d}s \left( \int_0^1 b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \\
&\leqslant \lambda^{1/k} L_2 \frac{k}{2k-N} \left( \int_0^1 b(\tau) \frac{1}{\lambda d_2}\left(\frac{2k-N}{kL_2}\right)^k v^k(\tau)\,\mathrm{d}\tau \right)^{1/k} \\
&\leqslant \lambda^{1/k} L_2 \frac{k}{2k-N} \left( \int_0^1 b(\tau) \frac{1}{\lambda d_2}\left(\frac{2k-N}{kL_2}\right)^k \|v\|^k\,\mathrm{d}\tau \right)^{1/k} \leqslant \|v\|.
\end{aligned}$$

For $\varphi_\infty = \infty$, there exists $\widehat{R} > 0$ so that $\varphi(s) \geqslant \varepsilon s^k$ for all $s \geqslant \widehat{R}$, where $\varepsilon$ satisfies

$$\lambda^{1/k} L_1 \left(\frac{1}{4}\right)^{(N+2k-1)/k} \varepsilon^{1/k} d_1 \geqslant 1.$$

Set $R > \max\{r, \widehat{R}\}$. Thereby, for any $v \in P \cap \partial\Omega_R$, one deduces that

$$\min_{t\in[1/4,3/4]} v(t) \geqslant \frac{1}{4}\|v\| \geqslant \widehat{R}.$$

Since $0 < 1/k \leqslant 1$, we obtain from Lemma 5 that

$$\begin{aligned}
\widehat{T}v(t) &= \lambda^{1/k} \int_t^1 \left( \frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi\bigl(v(\tau)\bigr)\,\mathrm{d}\tau \right)^{1/k} \mathrm{d}s \\
&\geqslant \lambda^{1/k} \left( \frac{k}{C_{N-1}^{k-1}} \right)^{1/k} \\
&\quad \times \int_t^1 \left( \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \right)^{1/k} s^{1/k-1} \int_0^s \left( \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} \right)^{1/k} b^{1/k}(\tau)\varphi^{1/k}\bigl(v(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant \lambda^{1/k} \left( \frac{k}{\mathrm{e}^{N\alpha}(1+\alpha)^{k-1}C_{N-1}^{k-1}} \right)^{1/k} \int_t^1 \int_0^s \tau^{(N-1)/k} b^{1/k}(\tau)\varphi^{1/k}\bigl(v(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \lambda^{1/k} L_1 \int_t^1 \int_0^s \tau^{(N-1)/k} b^{1/k}(\tau)\varphi^{1/k}\bigl(v(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant \lambda^{1/k} L_1 \int_{3/4}^1 \int_{1/4}^{3/4} \tau^{(N-1)/k} b^{1/k}(\tau)\varepsilon^{1/k} v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant \lambda^{1/k} L_1 \int_{3/4}^1 \int_{1/4}^{3/4} \tau^{(N-1)/k} b^{1/k}(\tau)\varepsilon^{1/k} \frac{\|v\|}{4}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\geqslant \lambda^{1/k} L_1 \left(\frac{1}{4}\right)^{(N-1)/k} \varepsilon^{1/k} \frac{\|v\|}{16} \int_{1/4}^{3/4} b^{1/k}(\tau)\,\mathrm{d}\tau \\
&= \lambda^{1/k} L_1 \left(\frac{1}{4}\right)^{(N+2k-1)/k} \varepsilon^{1/k} d_1 \|v\| \geqslant \|v\|.
\end{aligned}$$

By (i) of Lemma 4, $\widehat{T}$ has a fixed point $v(t) \in P \cap (\overline{\Omega}_R \setminus \Omega_r)$. Thus, problem (14) has a positive solution.

Here we claim that $\lim_{\lambda\to 0^+} \|v_\lambda\| = \infty$. In fact, assuming otherwise, there are a constant $\xi > 0$ and a sequence $\lambda_m \to 0^+$ such that

$$\|v_{\lambda_m}\| \leqslant \xi \quad (m = 1, 2, 3, \dots).$$

Moreover, the sequence $\{\|v_{\lambda_m}\|\}$ has a subsequence converging to a constant $\zeta$ ($0 \leqslant \zeta \leqslant \xi$). For convenience, let us suppose that $\{\|v_{\lambda_m}\|\}$ itself converges to $\zeta$. When $\zeta > 0$, for sufficiently large $m$, one has $\|v_{\lambda_m}\| > \zeta/2$. Hence

$$\begin{aligned}\frac{1}{\lambda_m^{1/k}} &= \frac{\| \int_t^1 (\frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi(v(\tau))\,\mathrm{d}\tau)^{1/k}\,\mathrm{d}s\|}{\|v_{\lambda_m}\|} \\ &\leqslant \frac{\| \int_0^1 (\frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^1 \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi(v(\tau))\,\mathrm{d}\tau)^{1/k}\,\mathrm{d}s\|}{\|v_{\lambda_m}\|} \\ &\leqslant \frac{(M_1 d_2)^{1/k} L_2 \frac{k}{2k-N}}{\|v_{\lambda_m}\|} < \frac{2(M_1 d_2)^{1/k} L_2 \frac{k}{2k-N}}{\zeta},\end{aligned}$$

where $M_1 = \max\{\varphi(v),\ r < \|v\| < R\}$, which contradicts $\lambda_m \to 0^+$. When $\zeta = 0$, for sufficiently large $m$, one gets $\|v_{\lambda_m}\| \to 0$, and thus it follows from $\varphi_0 = 0$ that there exists $\delta > 0$ so that for any $\varepsilon$,

$$\varphi(v_{\lambda_m}) \leqslant \varepsilon v_{\lambda_m}^k \quad \forall\, 0 \leqslant v_{\lambda_m} \leqslant \delta.$$

Therefore, for $v_{\lambda_m} \in P \cap \partial\Omega_\delta$ and $\|v_{\lambda_m}\| = \delta$, we deduce that

$$\begin{aligned}\frac{1}{\lambda_m^{1/k}} &= \frac{\| \int_t^1 (\frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^s \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi(v(\tau))\,\mathrm{d}\tau)^{1/k}\,\mathrm{d}s\|}{\|v_{\lambda_m}\|} \\ &\leqslant \frac{\| \int_0^1 (\frac{k}{C_{N-1}^{k-1}} \frac{s^{k-N}}{\mathrm{e}^{N\alpha s}} \int_0^1 \frac{\mathrm{e}^{N\alpha\tau}\tau^{N-1}}{(1+\alpha\tau)^{k-1}} b(\tau)\varphi(v(\tau))\,\mathrm{d}\tau)^{1/k}\,\mathrm{d}s\|}{\|v_{\lambda_m}\|} \\ &\leqslant \frac{(\varepsilon d_2)^{1/k} L_2 \frac{k}{2k-N} \|v_{\lambda_m}\|}{\|v_{\lambda_m}\|} = (\varepsilon d_2)^{1/k} L_2 \frac{k}{2k-N}.\end{aligned}$$

Due to the arbitrariness of $\varepsilon$, we can see that $\lim_{m\to+\infty} \lambda_m = \infty$, but this contradicts $\lambda_m \to 0^+$.

Thus, one has $\lim_{\lambda\to 0^+} \|v_\lambda\| = \infty$. So, we complete the proof of Theorem 3. □

Based on a similar analysis of Theorem 3, we can deduce the following result.

**Theorem 4.** *If* (H1) *and* (H2) *are satisfied and* $\varphi_0 = \infty$, $\varphi_\infty = 0$, *then for every* $\lambda > 0$, *Eq.* (14) *has at least one solution* $v_\lambda$ *satisfying* $\lim_{\lambda\to 0^+} \|v_\lambda\| = 0$.

**Remark 4.** The dependence here refers to the dependence of the asymptotic behavior of the solutions $v_\lambda$ on the parameter $\lambda$. Now we consider the nonlinearity $\varphi(s) = s^\gamma$, $\gamma > 0$. If $\gamma > k$, then we know from Theorem 3 that the solution $v_\lambda$ blows up as $\lambda \to 0^+$. On the other hand, when $\gamma < k$, it follows from Theorem 4 that the solution $v_\lambda$ vanishes as $\lambda \to 0^+$.

**Conflicts of interest.** The authors declare no conflicts of interest.

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