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On a sublinear nonlocal fractional problem

Giovanni Molica Bisci [a], **Raffaella Servadei** [b]

[a] Dipartimento di Promozione
delle Scienze Umane e della Qualità della Vita,
Università Telematica San Raffaele Roma,
Via di Val Cannuta 247, 00166 Roma, Italy
giovanni.molicabisci@uniroma5.it

[b] Dipartimento di Scienze Pure e Applicate,
Università degli Studi di Urbino Carlo Bo,
Piazza della Repubblica 13, 61029 Urbino, Italy
raffaella.servadei@uniurb.it

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Abstract. This paper deals with existence results of nonnegative solutions for a one-parameter sublinear elliptic boundary-value problem driven by the classical fractional Laplacian operator. The existence of a weak solution for any parameter λ beyond the first resonance has been proved by using a slight variation of the classical Mountain Pass result due to Ambrosetti and Rabinowitz.

Keywords: variational methods, fractional Laplacian operator, weak solutions, deformation lemma.

1 Introduction

In recent years, nonlocal operators have aroused much interest in the mathematical literature, thanks also to their numerous applications in different fields (physics, mathematical finance, population dynamics, just to name a few). The literature on nonlocal operators and on their applications is very interesting; see, for instance, [6] for an elementary introduction to this topic and for related references.

Nonlocal problems have been studied using different approaches and techniques. Here we want to focus on problems with a variational structure. In this framework a lot of works appeared in the literature about the existence and multiplicity results for nonlocal fractional equations under various assumptions on the nonlinearity. We refer to the recent monograph [14], which is dedicated to the study of fractional nonlocal problems involving superlinear and subcritical nonlinearities, as well as critical nonlinearities, via classical variational methods and other novel approaches.

To complete this analysis, in this paper, our aim is to consider variational nonlocal problems in the presence of sublinear nonlinearities. More precisely, we study the existence of a weak solution for the following nonlocal Dirichlet problem:

$$\begin{aligned} &(-\Delta)^s u + u^\sigma = \lambda u \quad \text{in } \Omega, \\ &u \geqslant 0, \quad u \not\equiv 0 \quad \text{in } \Omega, \\ &u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{$\mathcal{P}_\lambda$}$$

where $s \in]0,1[$ is fixed, and $\Omega \subset \mathbb{R}^n$ ($n > 2s$) is a bounded domain with continuous boundary. Here

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

denotes the classical fractional Laplacian, $\sigma \in]0,1[$, and λ is a real positive parameter.

As for the nonlocal diffusive operator, for the sake of concreteness, we stick here to the prototypical case of the fractional Laplacian, but the arguments that we develop are in fact usable in more general contexts including various interaction kernels of singular type (cfr. Remark 2).

In order to give the weak formulation of problem ($\mathcal{P}_\lambda$), we need to work in a special functional space, which allows us to encode the Dirichlet boundary condition in the variational formulation. More precisely, in the following, $X_0^s(\Omega)$ denotes the Sobolev space of all the functions in the usual fractional Sobolev space $H^s(\mathbb{R}^n)$, which vanish a.e. outside Ω, i.e.,

$$X_0^s(\Omega) := \{ u \in H^s(\mathbb{R}^n) \colon u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.$$

Problem ($\mathcal{P}_\lambda$) has a variational nature since its weak solutions are the critical points of the Euler–Lagrange energy functional $\mathcal{J}_\lambda : X_0^s(\Omega) \to \mathbb{R}$ given by

$$\mathcal{J}_\lambda(u) := \frac{1}{2} \|u\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2} \|u^+\|_2^2 + \frac{1}{\sigma+1} \|u^+\|_{\sigma+1}^{\sigma+1} \tag{1}$$

for every $u \in X_0^s(\Omega)$. Here and in the following, for any $u \in X_0^s(\Omega)$, we denote by u^+ the positive part of u, that is, $u^+ := \max\{0, u\}$. Finally, along the paper, $\lambda_{1,s}$ is the first (positive) eigenvalue of the nonlocal operator $(-\Delta)^s$.

Now we can state the main result of the present paper, which reads as follows.

Theorem 1. *Let $s \in]0,1[$, $n > 2s$, Ω be a bounded domain of $\mathbb{R}^n$ with continuous boundary, and let $\sigma \in]0,1[$.*

Then, for every $\lambda > \lambda_{1,s}$, problem ($\mathcal{P}_\lambda$) admits a weak solution $u_\lambda \in X_0^s(\Omega)$, which is a critical point of Mountain Pass type of the Euler–Lagrange energy functional $\mathcal{J}_\lambda$.

Moreover,

$$\mathcal{J}_\lambda(u_\lambda) \leqslant \frac{1}{2} \frac{1-\sigma}{\sigma+1} \frac{\|v^+\|_{\sigma+1}^{2(\sigma+1)/(1-\sigma)}}{(\lambda\|v^+\|_2^2 - \|v\|^2_{X_0^s(\Omega)})^{(\sigma+1)/(1-\sigma)}} \tag{2}$$

for every $v \in C_\lambda^{(s)}$, where

$$C_\lambda^{(s)} := \{ v \in X_0^s(\Omega) \colon \lambda\|v^+\|_2^2 > \|v\|^2_{X^s(\Omega)} \}.$$

Theorem 1 is proved by using a refined version of [1, Thm. 2.1] applied to the set of the continuous paths in $X_0^s(\Omega)$ given by

$$\Gamma_\lambda^{(s)} := \bigl\{\gamma \in C^0\bigl([0,1], X_0^s(\Omega)\bigr)\colon \exists\, v \in C_\lambda^{(s)} \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = \zeta_v v\bigr\}$$

for every $\lambda > \lambda_{1,s}$, where the constant

$$\zeta_v := \left(\frac{2}{\sigma+1}\right)^{1/(1-\sigma)} \frac{\|v^+\|_{\sigma+1}^{(\sigma+1)/(1-\sigma)}}{(\lambda\|v^+\|_2^2 - \|v\|_{X_0^s(\Omega)}^2)^{1/(1-\sigma)}} \tag{3}$$

is the unique positive real number such that $\mathcal{J}_\lambda(\zeta_v v) = 0$. Subsequently, if we set

$$c := \inf_{\gamma \in \Gamma_\lambda^{(s)}} \sup_{t\in[0,1]} \mathcal{J}_\lambda\bigl(\gamma(t)\bigr), \tag{4}$$

the classical deformation result [1, Lemma 1.3] ensures that the set of critical points

$$K_{\lambda,c}^{(s)} := \bigl\{u \in X_0^s(\Omega)\colon \mathcal{J}_\lambda(u) = c \text{ and } \mathcal{J}_\lambda'(u) = 0\bigr\}$$

is not empty; see Section 3.

Problem $(\mathcal{P}_\lambda)$ is asymptotically linear at infinity, and the Mountain Pass geometry is essentially due to the presence of a sublinear positive perturbation. On the other hand, the main variant of the classical Ambrosetti and Rabinowitz result used here guarantees a precise information of the energy level of the Mountain Pass solutions $u_\lambda \in X_0^s(\Omega)$ for every $\lambda > \lambda_{1,s}$. This estimate is the first step that can be useful in order to potentially prove a bifurcation type result for problem $(\mathcal{P}_\lambda)$ by using suitable recurrence arguments.

Theorem 1 is the nonlocal counterpart of the result got by Porretta in [16]. On one hand, the techniques of the proof follow the ones used in the classical framework of the Laplacian. On the other hand, due to the nonlocality of the problem under consideration, to make these nonlinear methods work, some careful analysis of the fractional spaces involved, as well as their embedding properties, are necessary.

Finally, we notice that Theorem 1 is related to some bifurcation results already present in the current literature in either the classical elliptic or nonlocal setting; see, among others, the papers [7, 8, 13, 16], as well as [9, 10, 15] and the references therein.

The present paper is organized as follows. In Section 2 we recall some results on fractional Sobolev spaces, while Section 3 is devoted to the proof of Theorem 1.

2 Preliminaries

In this section, we recall the definition and some properties of fractional Sobolev spaces usefull along the paper.

In the following, $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the Gagliardo norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_2 + \biggl(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\biggr)^{1/2};$$

see [23, Lemma 7]. While $X_0^s(\Omega)$ is the Sobolev space of all the functions in $H^s(\mathbb{R}^n)$, which vanish a.e. outside Ω, i.e.,

$$X_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n):\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

By [18, Lemmas 6 and 7] we can take in $X_0^s(\Omega)$ the norm

$$X_0^s(\Omega) \ni u \mapsto \|u\|_{X_0^s(\Omega)} := \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$

induced by the scalar product defined by

$$\langle u, v\rangle := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

for every $u, v \in X_0^s(\Omega)$; see [18, Lemma 7].

Moreover, along the paper, by $L^\nu(\mathbb{R}^n)$ we denote the classical Lebesgue space endowed with the standard norm $\|\cdot\|_\nu$ for any $\nu \in [1, +\infty]$. By [18, Lemma 8] and [23, Lemma 9] we have the following result; see also [3].

Lemma 1. *Let $s \in]0,1[$, $n > 2s$, and let Ω be a bounded domain of $\mathbb{R}^n$ with continuous boundary. Then the embedding $j : X_0^s(\Omega) \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2_s^*]$, while it is compact whenever $\nu \in [1, 2_s^*[$, where $2_s^* := 2n/(n-2s)$ is the fractional critical Sobolev exponent.*

Finally, $\lambda_{1,s}$ is the first eigenvalue of the nonlocal operator $(-\Delta)^s$, that is, the first eigenvalue of the problem

$$\begin{aligned} &(-\Delta)^s u = \lambda u \quad \text{in } \Omega,\\ &u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned}$$

The variational characterization of $\lambda_{1,s}$ is given by

$$\lambda_{1,s} = \min_{u \in X_0^s(\Omega)\setminus\{0\}} \frac{\|u\|^2_{X_0^s(\Omega)}}{\|u\|_2^2}.$$

For all the properties of the eigenvalues of $(-\Delta)^s$ (and of general nonlocal fractional operators), we refer to [20, Prop. 9 and Appx. A] and [11]. We also recall that a complete study of the spectrum of fractional operators and of their eigenfunctions has been considered in [17, 21, 22].

3 Proof of the main result

In this section, we prove Theorem 1 via variational methods, that is, looking for weak solutions of problem $(\mathcal{P}_\lambda)$ as critical points of the Euler–Lagrange functional $\mathcal{J}_\lambda$ associated with it.

First of all, we show that $u \equiv 0$ is a local minimum point of the energy functional $\mathcal{J}_\lambda$ given in (1). More precisely, we prove the following claim.

Proposition 1. *Let* $\lambda > \lambda_{1,s}$. *Then*

$$\textit{there exists } \varrho > 0 \textit{ such that } \mathcal{J}_\lambda(u) > 0 \textit{ for every } u \in \bar{B}_\varrho \setminus \{0\}, \tag{5}$$

where $\bar{B}_\varrho := \{u \in X_0^s(\Omega)\colon \|u\|_{X_0^s(\Omega)} \leqslant \varrho\}$.

Proof. We argue by contradiction and assume that (5) is not verified. Then there exists a sequence $(u_k)_k \subset X_0^s(\Omega)$ such that

$$\lim_{k\to+\infty} \|u_k\|_{X_0^s(\Omega)} = 0, \quad u_k \neq 0 \quad \text{and} \quad \mathcal{J}_\lambda(u_k) \leqslant 0 \quad \text{for every } k \in \mathbb{N}. \tag{6}$$

By (6), for every $k \in \mathbb{N}$, we know that

$$\frac{1}{2}\|u_k\|^2_{X_0^s(\Omega)} + \frac{1}{\sigma+1}\|u_k^+\|^{\sigma+1}_{\sigma+1} \leqslant \frac{\lambda}{2}\|u_k^+\|_2^2. \tag{7}$$

Let $(v_k)_k$ be the sequence in $X_0^s(\Omega)$ defined as

$$v_k := \frac{u_k^+}{\|u_k^+\|_{X_0^s(\Omega)}}.$$

By [19, Lemma 5.2] we know that $v_k^+ \in X_0^s(\Omega)$, while by (7) we deduce that

$$\begin{aligned} \|v_k\|^{\sigma+1}_{\sigma+1} = \frac{\|u_k^+\|^{\sigma+1}_{\sigma+1}}{\|u_k^+\|^{\sigma+1}_{X_0^s(\Omega)}} &\leqslant \frac{\lambda(\sigma+1)}{2} \frac{\|u_k^+\|_2^2}{\|u_k^+\|^{\sigma+1}_{X_0^s(\Omega)}} \\ &\leqslant \frac{\lambda(\sigma+1)}{2}\|u_k^+\|^{1-\sigma}_{X_0^s(\Omega)}\|v_k\|_2^2 \end{aligned} \tag{8}$$

for every $k \in \mathbb{N}$.

Furthermore, since $\|v_k\|_{X_0^s(\Omega)} = 1$ for every $k \in \mathbb{N}$, due to the reflexivity of $X_0^s(\Omega)$, up to a subsequence, still denoted by $(v_k)_k$, there exists $v_\infty \in X_0^s(\Omega)$ such that

$$v_k \rightharpoonup v_\infty \quad \text{weakly in } X_0^s(\Omega) \text{ as } k \to +\infty,$$

and, as a consequence of this and [18, Lemma 8], again up to a subsequence, still denoted by $(v_k)_k$, we have that

$$v_k \to v_\infty \quad \text{in } L^\nu(\Omega) \text{ as } k \to +\infty \text{ for any } \nu \in [1, 2_s^*[. \tag{9}$$

By (6), (8), (9), the fact that $\sigma < 1$ and taking into account that $\|u_k^+\|_{X_0^s(\Omega)} \leqslant \|u_k\|_{X_0^s(\Omega)}$ (see [19, Lemma 5.2]), we deduce that $v_k \to 0$ in $L^{\sigma+1}(\Omega)$ as $k \to +\infty$, which, together with (9), gives that

$$v_\infty \equiv 0. \tag{10}$$

Finally, dividing (7) by $\|u_k^+\|^2_{X_0^s(\Omega)}$, we have that

$$\frac{1}{2} \leqslant \frac{1}{2} + \frac{1}{\sigma+1}\frac{\|u_k^+\|^{\sigma+1}_{\sigma+1}}{\|u_k^+\|^2_{X_0^s(\Omega)}} \leqslant \frac{\lambda}{2}\|v_k\|_2^2,$$

which is a contradiction, thanks to (9) and (10). Then (5) holds true as claimed, and this ends the proof of Proposition 1. □

Now, let us show that the Euler–Lagrange functional $\mathcal{J}_\lambda$ is positive on the boundary of a suitable ball centered on 0. More precisely, we prove the following.

Proposition 2. *Let $\lambda > \lambda_{1,s}$ and $\rho > 0$ be as in Proposition* 1. *Then there exists $\beta \in \mathbb{R}$ such that for every $u \in \partial \bar{B}_\varrho$,*

$$\mathcal{J}_\lambda(u) \geqslant \beta. \tag{11}$$

Proof. We argue by contradiction and we suppose that there exists $(u_k)_k$ in $X_0^s(\Omega)$ such that $\|u_k\|_{X_0^s(\Omega)} = \rho$ and

$$\limsup_{k\to+\infty} \mathcal{J}(u_k) = 0. \tag{12}$$

Since $(u_k)_k$ is bounded in $X_0^s(\Omega)$, there exists $u_\infty \in X_0^s(\Omega)$ such that

$$\begin{aligned} &u_k \rightharpoonup u_\infty \quad \text{weakly in } X_0^s(\Omega),\\ &u_k \to u_\infty \quad \text{in } L^\nu\big(\mathbb{R}^n\big) \end{aligned} \tag{13}$$

as $k \to +\infty$ for any $\nu \in [1, 2_s^*[$. By (12), (13), and the weak lower semicontinuity of the norm we get that

$$\|u_\infty\|_{X_0^s(\Omega)} \leqslant \rho \tag{14}$$

and

$$\begin{aligned} 0 &\geqslant \liminf_{k\to+\infty} \mathcal{J}_\lambda(u_k) \geqslant \frac{1}{2}\|u_\infty\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2}\|u_\infty\|_2^2 + \frac{1}{\sigma+1}\|u_\infty\|_{\sigma+1}^{\sigma+1}\\ &= \mathcal{J}_\lambda(u_\infty). \end{aligned} \tag{15}$$

By (5), (14), and (15) we deduce that

$$u_\infty \equiv 0. \tag{16}$$

Moreover, by definition of $\mathcal{J}_\lambda$ we have that

$$0 \leqslant \frac{1}{2}\|u_k\|^2_{X_0^s(\Omega)} = \mathcal{J}_\lambda(u_k) + \frac{\lambda}{2}\|u_k^+\|^2_{L^2(\Omega)} - \frac{1}{\sigma+1}\|u_k^+\|^{\sigma+1}_{L^{\sigma+1}(\Omega)}. \tag{17}$$

Taking into account (12) and the fact that $\sigma > 0$, by (13), (16), and (17) we obtain that

$$\rho = \|u_k\|_{X_0^s(\Omega)} \to 0$$

as $k \to +\infty$, which is a contradiction. Hence, (11) holds true, and this concludes the proof of Proposition 2. □

Now, let us show that the functional $\mathcal{J}_\lambda$ satisfies the Palais–Smale condition at any level $\mu \in \mathbb{R}$. Aiming at this purpose, we prove the following result.

Proposition 3. *Let $\mu \in \mathbb{R}$, and let $(u_k)_k$ be a sequence in $X_0^s(\Omega)$ such that $\mathcal{J}_\lambda(u_k) \to \mu$ and*

$$\sup\big\{\big|\langle \mathcal{J}_\lambda'(u_k), \varphi\rangle\big|\colon \varphi \in X_0^s(\Omega),\ \|\varphi\|_{X_0^s(\Omega)} = 1\big\} \to 0 \tag{18}$$

as $k \to +\infty$. Then, up to a subsequence, $(u_k)_k$ converges in $X_0^s(\Omega)$ as $k \to +\infty$.

Proof. First of all, we show that $(u_k)_k$ is bounded in $X^s_0(\Omega)$. To this aim, we argue by contradiction and suppose that as $k \to +\infty$,

$$\|u_k\|_{X^s_0(\Omega)} \to +\infty. \tag{19}$$

Setting

$$v_k := \frac{u_k}{\|u_k\|_{X^s_0(\Omega)}},$$

we have that $(v_k)_k$ is bounded in $X^s_0(\Omega)$, and, by the embedding properties of $X^s_0(\Omega)$ into the Lebesgue spaces, up to a subsequence, still denoted by v_k, we may assume that there exists $v_\infty \in X^s_0(\Omega)$ such that

$$\begin{aligned} &v_k \rightharpoonup v_\infty \quad \text{weakly in } X^s_0(\Omega),\\ &v_k \to v_\infty \quad \text{in } L^\nu\big(\mathbb{R}^n\big) \end{aligned} \tag{20}$$

as $k \to +\infty$ and for any $\nu \in [1, 2^*_s[$; see, for instance, [2, Thm. IV.9].

By (18) and (19) we get that

$$\frac{\langle \mathcal{J}'_\lambda(u_k), \varphi\rangle}{\|u_k\|_{X^s_0(\Omega)}} \to 0$$

as $k \to +\infty$ for any $\varphi \in X^s_0(\Omega)$. By this, taking into account (19) and (20), we obtain that

$$\begin{aligned} &\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \big(v_\infty(x) - v_\infty(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad = \lambda \int\limits_{\Omega} v^+_\infty(x)\varphi(x)\,\mathrm{d}x \end{aligned}$$

for any $\varphi \in X^s_0(\Omega)$. This means that v_∞ is a nonnegative eigenfunction of $(-\Delta^s)$ whose corresponding eigenvalue is λ, provided $v_\infty \not\equiv 0$. Since $\lambda > \lambda_{1,s}$, this cannot occur. Then $v_\infty \equiv 0$ in Ω.

Finally, testing (18) with $\varphi = u_k/\|u_k\|^2_{X^s_0(\Omega)}$, thanks to (19), (20), and the fact that $v_\infty \equiv 0$ and $\sigma < 1$, we deduce

$$0 = \lim_{k\to+\infty} \frac{\langle \mathcal{J}'_\lambda(u_k), u_k\rangle}{\|u_k\|^2_{X^s_0(\Omega)}} = \lim_{k\to+\infty}\left(1 - \lambda\|v^+_k\|^2_2 + \frac{\|v^+_k\|^{\sigma+1}_{\sigma+1}}{\|u_k\|^{1-\sigma}_{X^s_0(\Omega)}}\right) = 1,$$

which is a contradiction. Thus, $(u_k)_k$ is bounded in $X^s_0(\Omega)$.

As a consequence of this, we get that there exists $u_\infty \in X^s_0(\Omega)$ such that, up to a subsequence, still denoted by u_k,

$$\begin{aligned} &u_k \rightharpoonup u_\infty \quad \text{weakly in } X^s_0(\Omega),\\ &u_k \to u_\infty \quad \text{in } L^\nu\big(\mathbb{R}^n\big) \end{aligned} \tag{21}$$

as $k \to +\infty$ and for any $\nu \in [1, 2^*_s[$.

Since $(u_k)_k$ is a Palais–Smale sequence for $\mathcal{J}_\lambda$, we know that

$$\langle \mathcal{J}_\lambda'(u_k), u_k\rangle_{X_0^s(\Omega)} \to 0, \qquad \langle \mathcal{J}_\lambda'(u_k), u_\infty\rangle_{X_0^s(\Omega)} \to 0$$

as $k \to +\infty$, which implies

$$\|u_k\|_{X_0^s(\Omega)} \to \|u_\infty\|_{X_0^s(\Omega)}$$

as $k \to +\infty$. This and the first convergence in (21) give that $u_k \to u$ in $X_0^s(\Omega)$ as $k \to +\infty$. Hence, $\mathcal{J}_\lambda$ satisfies the Palais–Smale compactness condition. This concludes the proof of Proposition 3. □

Now, we are in position to prove our main existence result stated in Theorem 1.

3.1 Proof of Theorem 1.

Thanks to Propositions 1, 2, and 3, now we can show that the value c given in (4) is critical for the energy functional $\mathcal{J}_\lambda$.

To this aim, let ϱ and β be as in (11). Thanks to (5), the definition of ζ_v given in (3) immediately yields

$$\|\zeta_v v\|_{X_0^s(\Omega)} > \varrho.$$

Thus, for every path $\gamma \in \Gamma_\lambda^{(s)}$, there exists $t_\gamma \in [0,1]$ such that $\gamma(t_\gamma) \in \partial\bar{B}_\varrho$. Hence, by (11)

$$\sup_{t\in[0,1]} \mathcal{J}_\lambda\big(\gamma(t)\big) \geqslant \mathcal{J}_\lambda\big(\gamma(t_\gamma)\big) \geqslant \beta,$$

so that $c \geqslant \beta$.

Now, we prove that $K_{\lambda,c}^{(s)} \neq \emptyset$. To get this goal, we argue again by contradiction and assume that $K_{\lambda,c}^{(s)} = \emptyset$. In such a case, by [1, Lemma 1.3 (2) and (6)] there exist a continuous deformation $\eta : [0,1] \times X_0^s(\Omega) \to X_0^s(\Omega)$ and $\varepsilon > 0$ such that

(i) $\eta(t,u) = u$ for every $u \notin \mathcal{J}_\lambda^{-1}([c-\beta/2,\, c+\beta/2])$;
(ii) $\mathcal{J}_\lambda(\eta(1,u)) \leqslant c - \varepsilon$ for every $u \in X_0^s(\Omega)$ such that $\mathcal{J}_\lambda(u) \leqslant c + \varepsilon$.

Hence, if $\gamma_\varepsilon \in \Gamma_\lambda^{(s)}$ and $\sup_{t\in[0,1]} \mathcal{J}_\lambda(\gamma_\varepsilon(t)) \leqslant c + \varepsilon$, we have that

$$\eta(1,\gamma_\varepsilon(t)) \in \Gamma_\lambda^{(s)} \quad \text{and} \quad \sup_{t\in[0,1]} \big(\eta\big(1,\gamma_\varepsilon(t)\big)\big) \leqslant c - \varepsilon,$$

which contradicts the definition of c. Thus, $K_{\lambda,c}^{(s)} \neq \emptyset$.

As a consequence of this, there exists $u_\lambda \in X_0^s(\Omega)$ such that

$$\mathcal{J}_\lambda(u_\lambda) = c \quad \text{and} \quad \mathcal{J}_\lambda'(u_\lambda) = 0.$$

Clearly, u_λ is a weak solution of problem $(\mathcal{P}_\lambda)$, and, since the path defined by $\gamma_v(t) := t\zeta_v v$ for every $t \in [0,1]$ belongs to $\Gamma_\lambda^{(s)}$ for every $v \in C_\lambda^{(s)}$, we also have

$$\mathcal{J}_\lambda(u_\lambda) = c = \inf_{\gamma\in\Gamma_\lambda^{(s)}} \sup_{t\in[0,1]} \mathcal{J}_\lambda\big(\gamma(t)\big) \leqslant \sup_{t\in[0,1]} \mathcal{J}_\lambda\big(\gamma_v(t)\big). \tag{22}$$

Inequality (2) immediately follows from (22) observing that, for fixed $v \in C_\lambda^{(s)}$, one has

$$\sup_{t\in[0,1]} \mathcal{J}_\lambda\bigl(\gamma_v(t)\bigr) = \mathcal{J}_\lambda\left(\left(\frac{2}{\sigma+1}\right)^{1/(\sigma-1)} \zeta_v v\right)$$
$$= \frac{1}{2}\,\frac{1-\sigma}{\sigma+1}\,\frac{\|v^+\|_{\sigma+1}^{2(\sigma+1)/(1-\sigma)}}{(\lambda\|v^+\|_2^2 - \|v\|_{X_0^s(\Omega)}^2)^{(\sigma+1)/(1-\sigma)}}.$$

The proof of Theorem 1 is now complete.

3.2 Final comments

We end this paper with some remarks.

Remark 1. We notice that the above proof is patterned after that of [16, Thm. 1.1]. Furthermore, the extension of the above Theorem 1 to nonlocal equations involving the fractional p-Laplacian operator should be investigated by using [12, Prop. 2.3]. We refer to the book [14] and to the quoted paper [6] as general references on the subject treated here.

Remark 2. Theorem 1 continues to hold also if we replace the fractional operator $(-\Delta)^s$ with a more general integrodifferential operator like $\mathcal{L}_K$ defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \bigl(u(x+y) + u(x-y) - 2u(x)\bigr) K(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is such that

(i) $mK \in L^1(\mathbb{R}^n)$ with $m(x) = \min\{|x|^2, 1\}$;
(ii) there exists $\theta > 0$ such that $K(x) \geqslant \theta |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

For the properties of the fractional Sobolev spaces associated with $\mathcal{L}_K$, as well as for its eigenvalue Dirichlet problem, we refer to [17, 20, 22, 23].

For the (s,p)-Laplacian operator, possibly with measurable coefficients, the readers could also refer to the first approach in [4, 5].

Conflicts of interest. The authors declare no conflicts of interest.

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