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Comparative approaches to solving the $(2+1)$-dimensional generalized coupled nonlinear Schrödinger equations with four-wave mixing

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Abstract. This paper extensively studies the propagation of optical solitons within the framework of $(2+1)$-dimensional generalized coupled nonlinear Schrödinger equations. The investigation employs three worldly integration techniques: the enhanced direct algebraic method, the enhanced Kudryashov method, and the new projective Riccati equation method. Through the application of these methods, a broad spectrum of soliton solutions has been uncovered, including bright, dark, singular, and straddled solitons. Additionally, this study reveals solutions characterized by Jacobi and Weierstrass elliptic functions, enriching the understanding of the dynamics underpinning optical solitons in complex systems. The diversity of the soliton solutions obtained demonstrates the versatility and efficacy of the employed integration techniques and contributes significantly to the theoretical and practical knowledge of nonlinear optical systems.

Keywords: optical solitons, generalized coupled nonlinear Schrödinger equations, enhanced direct algebraic method, enhanced Kudryashov method, new projective Riccati equation method.

1 Introduction

Optical solitons are fundamental to advancing nonlinear optical systems and have become a focal point of scientific research due to their unique characteristics and potential applications [14,27]. These solitary wave packets, remarkable for their ability to maintain their shape over long distances, play a crucial role in transmitting information through optical fibers. Their stability and self-reinforcing nature make them ideal for optical communication and fiber optics technology applications, where data integrity and efficiency are paramount. Unlike traditional waveforms that may dissipate and lose information during transmission, solitons preserve their form and speed, making them highly effective for long-range data transfer. This property enhances the capacity and reliability of communication networks and enables higher data rates and more robust signal integrity against noise and physical impairments in optical channels. The study of optical solitons opens new avenues for technological innovations and enhances our understanding of nonlinear phenomena in optical systems [8,15,21,23]. Furthermore, exploring the dynamics of these solitons has implications across various fields, including telecommunications, medical imaging, and even quantum computing, as they allow for novel ways to manipulate light that can lead to breakthroughs in signal processing and transmission technology. As research continues to delve deeper into the behavior and interactions of optical solitons, we can expect to see transformative advancements in how information is transmitted, processed, and utilized in an increasingly data-driven world. Ultimately, the ongoing exploration of these fascinating wave phenomena represents a critical intersection of theoretical inquiry and practical application, promising to reshape the landscape of modern optical technologies [1, 17, 22, 25, 30, 31].

The nonlinear Schrödinger equation (NLSE) is a fundamental equation that plays a pivotal role in various fields of physics, including optics, quantum mechanics, and wave dynamics. Different types of analytical methods were introduced in literature to find exact solutions of differential equations with real-world applications [13, 19, 20]. The NLSE has been the subject of extensive research with various methods employed to explore its solutions. Techniques such as the modified simple equation method [7], Kudryashov method [24], the new Kudryashov method [29] improved modified Sardar subequation method [16], Sine-Gordon expansion method [18], generalized projective Riccati equation method [2], and Lie symmetry analysis [10–12] have been pivotal in uncovering a diverse range of soliton solutions. Moreover, Hashemi in [9] introduced a novel technique base on the variable coefficient third-degree generalized Abel equation method for solving the stochastic Shrödinger–Hirota equation. These methodologies have enabled researchers to systematically analyze and categorize solitons, shedding light on their properties and behaviors in nonlinear optical systems. The field has witnessed significant advancements through these worldly integration techniques, enriching our theoretical and practical understanding of optical solitons. This body of work demonstrates the versatility and efficacy of these methods. It underscores the continued importance of exploring the NLSE to advance optical soliton theory and its applications in emerging technologies [5, 26].

Recently, the $(2+1)$-dimensional generalized coupled NLSEs, which will extract more abundant physical phenomena, described below [28]:

$$\begin{aligned} &\mathrm{i}U_t + a(U_{xx} + U_{yy}) + 2\bigl(b|U^2| + c|V^2| + dUV^* + d^*U^*V\bigr)U = 0\\ &\mathrm{i}V_t + a(V_{xx} + V_{yy}) + 2\bigl(b|U^2| + c|V^2| + dUV^* + d^*U^*V\bigr)V = 0. \end{aligned} \tag{1}$$

The functions U and V in this equation represent the envelopes of two circularly polarized waves, expressed in terms of x, y, and t. The parameter a represents group velocity dispersion, while b and c are real constants that account for self-phase modulation and cross-phase modulation, respectively. Additionally, d is a complex constant that reflects the effects of four-wave mixing, and the symbol "$*$" signifies a complex conjugate. In this study, the coupled NLSEs with cross-spatial dispersion are first reported using three distinct techniques: the enhanced direct algebraic method, the enhanced Kudryashov's, and the projective Riccati equations methods. These techniques can provide different forms of solitons, including bright solitons, dark solitons, singular solitons, and straddled solitons, along with other different types of solutions involving the Jacobi and Weierstrass elliptic function solutions.

Assume that

$$U(x,y,t) = Q_1(\eta)\mathrm{e}^{\mathrm{i}\phi(x,y,t)}, \qquad V(x,y,t) = Q_2(\eta)\mathrm{e}^{\mathrm{i}\phi(x,y,t)}.$$

In the expressions $\eta = k(x+y-\nu t)$ and $\phi(x,y,t) = -\varpi_1 x - \varpi_2 y + \omega t + \theta_0$, where v, ω, κ, and θ_0 represent velocity, frequency, wave number, and phase constants, respectively, let us utilize these transformations to decompose the system into the ensuing real and imaginary systems. The real parts are as follows:

$$\begin{aligned} &Q_1\big(a\big(\varpi_1^2+\varpi_2^2\big) - 2cQ_2^2 + \omega\big) - 2ak^2Q_1'' - 2bQ_1^3 - 2(d+d^*)Q_2Q_1^2 = 0, \\ &Q_2\big(a\big(\varpi_1^2+\varpi_2^2\big) - 2bQ_1^2 + \omega\big) - 2ak^2Q_2'' - 2cQ_2^3 - 2(d+d^*)Q_1Q_2^2 = 0. \end{aligned}$$

While the imaginary parts are

$$Q_1'\big(2a(\varpi_1+\varpi_2) + v\big) = 0, \qquad Q_2'\big(2a(\varpi_1+\varpi_2) + v\big) = 0.$$

After substituting each imaginary part into its corresponding real part and applying the transformation $Q_2 = \mu Q_1$, the system of real parts simplifies to the identical equation, namely,

$$-2ak^2Q_1'' + Q_1\big(a\big(\varpi_1^2+\varpi_2^2\big) + \omega\big) - 2Q_1^3(b+c+d+d^*). \tag{2}$$

When the homogeneous balance implemented between Q_1'' and Q_1^3 in Eq. (2), it implies $N=1$.

2 Overview of the integration algorithms

Suppose that we have a nonlinear evolution equation in the form

$$F(U, U_t, U_x, U_y, U_{xx}, U_{yy}, U_{xt}, U_{yt}, \dots) = 0 \tag{3}$$

with $U(X) = U(x,y,t)$. Use the following traveling wave transformation:

$$U(X) = Q_1(\eta), \quad \eta = k(x+y-\nu t), \quad k \neq 0. \tag{4}$$

Here v symbolizes the speed of the wave. Consequently, Eq. (4) can be converted into the subsequent nonlinear ordinary differential equation

$$F(Q_1, {Q_1}', {Q_1}'', {Q_1}''', \dots) = 0. \tag{5}$$

2.1 The enhanced direct algebraic method

The enhanced direct algebraic technique possesses the capability to derive several forms of closed-form exact solutions, encompassing bright, dark, single, and straddled solitons, as well as diverse solutions such as Jacobi and Weierstrass elliptic function solutions [4]. Nonetheless, the diversity of its solutions might lose certain qualities if the governing model fails to restore either a bright or dark soliton.

Step 1. Let us imagine that the solution to Eq. (5) can be represented as

$$Q_1(\eta) = \alpha_0 + \sum_{i=1}^{N} \left[\alpha_i \theta(\eta)^i + \beta_i \theta(\eta)^{-i}\right], \tag{6}$$

where

$$\theta'(\eta)^2 = \sum_{l=0}^{4} \tau_1 \theta(\eta)^l. \tag{7}$$

Furthermore, τ_l, $l = 0, \ldots, 4$, represent constants with the condition that $\tau_4 \neq 0$. Eq. (7) presents various solutions of diverse natures as documented in [4].

Case 1. If $\tau_0 = \tau_1 = \tau_3 = 0$, we get bell-shaped soliton with $\tau_2 > 0$, $\tau_4 < 0$ and singular soliton with $\tau_2 > 0$, $\tau_4 > 0$:

$$\theta(\eta) = \sqrt{-\frac{\tau_2}{\tau_4}} \operatorname{sech}\left[\sqrt{\tau_2}\eta\right], \quad \tau_2 > 0,\ \tau_4 < 0,$$
$$\theta(\eta) = \sqrt{\frac{\tau_2}{\tau_4}} \operatorname{csch}\left[\sqrt{\tau_2}\eta\right], \quad \tau_2 > 0,\ \tau_4 > 0.$$

Case 2. For $\tau_0 = \tau_2^2/(4\tau_4)$, $\tau_1 = \tau_3 = 0$, we obtain kink-shaped and singular solitons for $\tau_2 < 0$, $\tau_4 > 0$:

$$\theta(\eta) = \sqrt{-\frac{\tau_2}{2\tau_4}} \tanh\left[\sqrt{\frac{-\tau_2}{2}}\eta\right], \quad \tau_2 < 0,\ \tau_4 > 0,$$
$$\theta(\eta) = \sqrt{-\frac{\tau_2}{2\tau_4}} \coth\left[\sqrt{\frac{-\tau_2}{2}}\eta\right], \quad \tau_2 < 0,\ \tau_4 > 0.$$

Case 3. For $\tau_1 = \tau_3 = 0$, we have Jacobi elliptic doubly periodic type (JEDPT) solution for various choices of τ_0 as follows:

$$\theta(\eta) \pm \sqrt{-\frac{m^2\tau_2}{(2m^2-1)\tau_4}} \operatorname{cn}\left(\sqrt{\frac{\tau_2}{(2m^2-1)}}\eta \,\Big|\, m\right), \quad \tau_0 = \frac{m^2(1-m^2)\tau_2^2}{(2m^2-1)^2\tau_4},$$
$$\theta(\eta) \pm \sqrt{-\frac{m^2\tau_2}{(2-m^2)\tau_4}} \operatorname{dn}\left(\sqrt{\frac{\tau_2}{(2-m^2)}}\eta \,\Big|\, m\right), \quad \tau_0 = \frac{(1-m^2)\tau_2^2}{(2-m^2)^2\tau_4},$$
$$\theta(\eta) \pm \sqrt{-\frac{m^2\tau_2}{(m^2+1)\tau_4}} \operatorname{sn}\left(\sqrt{-\frac{\tau_2}{(m^2+1)}}\eta \,\Big|\, m\right), \quad \tau_0 = \frac{m^2\tau_2^2}{(m^2+1)^2\tau_4}.$$

Case 4. If we set $\tau_1 = \tau_3 = 0$, we obtain Weierstrass elliptic doubly periodic type solutions

$$\theta(\eta) = \frac{3\wp'(\eta; g_2, g_3)}{\sqrt{\tau_4}[6\wp(\eta; g_2, g_3) + \tau_2]}, \quad \tau_4 > 0,$$
$$\theta(\eta) = \frac{\sqrt{\tau_0}[6\wp(\eta; g_2, g_3) + \tau_2]}{3\wp'(\eta; g_2, g_3)}, \quad \tau_0 > 0,$$

where $g_2 = \tau_2^2/12 + \tau_0\tau_4$ and $g_3 = (\tau_2/216)(36\tau_0\tau_4 - \tau_2^2)$.

Case 5. When $\tau_0 = \tau_1 = 0$, we obtain straddled soliton solutions, where $\tau_2 > 0$, outlined as

$$\theta(\eta) = \frac{-\tau_2 \operatorname{sech}^2[\frac{1}{2}\sqrt{\tau_2}\eta]}{\pm 2\sqrt{\tau_2\tau_4}\tanh[\frac{1}{2}\sqrt{\tau_2}\eta] + \tau_3}, \qquad \tau_4 > 0,$$
$$\theta(\eta) = \frac{\tau_2 \operatorname{csch}^2[\frac{1}{2}\sqrt{\tau_2}\eta]}{\pm 2\sqrt{\tau_2\tau_4}\coth[\frac{1}{2}\sqrt{\tau_2}\eta] + \tau_3}, \qquad \tau_4 > 0,$$
$$\theta(\eta) = \frac{-\tau_2\tau_3 \operatorname{sech}^2[\frac{1}{2}\sqrt{\tau_2}\eta]}{\tau_3^2 - \tau_2\tau_4(1 - \tanh[\frac{1}{2}\sqrt{\tau_2}\eta])^2}, \quad \tau_3 \neq 0,$$
$$\theta(\eta) = \frac{\tau_2\tau_3 \operatorname{csch}^2[\frac{1}{2}\sqrt{\tau_2}\eta]}{\tau_3^2 - \tau_2\tau_4(1 - \coth[\frac{1}{2}\sqrt{\tau_2}\eta])^2}, \quad \tau_3 \neq 0.$$

Step 2. Determine the balance number N in Eqs. (6) and (5).

Step 3. By inserting Eqs. (6) and (7) into Eq. (5), we generate a polynomial denoted as $\theta(\eta)$. This polynomial manipulation entails grouping terms with comparable powers and equating the resulting expression to zero. This process leads to an overdetermined set of algebraic equations, solvable using Mathematica to ascertain the unidentified parameters in Eqs. (4) and (6). Consequently, we attain the precise solutions for Eq. (3).

2.2 The projective Riccati equations method

The projective Riccati equations method is a powerful and reliable approach for extracting straddled solitons, which are hybrids of various soliton types, such as bright-dark, bright-singular, and dark-singular solitons [6]. This versatility aligns well with the phenomena associated with both bright and dark solitons.

Step 1. Let us say that the answer to Eq. (5) can be formulated as

$$Q_1(\eta) = \alpha_0 + \sum_{i=1}^{N} \theta(\eta)^{i-1}\big[\alpha_i\theta(\eta) + \beta_i\phi(\eta)\big],$$

where $\theta(\eta)$ and $\phi(\eta)$ satisfy

$$\theta'(\eta) = -\theta(\eta)\phi(\eta), \qquad \phi'(\eta) = 1 - \phi^2(\eta) - r\theta(\eta), \tag{8}$$
$$\phi^2(\eta) = 1 - 2r\theta(\eta) + R(r)\theta^2(\eta), \tag{9}$$

where $r \neq 0$, and α_0, α_i, and β_i, $i = 0, 1, 2, \dots, N$, are constants.

Step 2. The solutions to Eq. (8) are outlined as follows [6].

Case 1. $R(r) = 0$.

$$\theta(\eta) = \frac{1}{2r}\operatorname{sech}^2\left[\frac{\eta}{2}\right] \quad \text{and} \quad \phi(\eta) = \tanh\left[\frac{\eta}{2}\right]$$

or

$$\theta(\eta) = -\frac{1}{2r}\operatorname{csch}^2\left[\frac{\eta}{2}\right] \quad \text{and} \quad \phi(\eta) = \coth\left[\frac{\eta}{2}\right].$$

Case 2. $R(r) = (24/25)r^2$.

$$\theta(\eta) = \frac{1}{r}\frac{5\operatorname{sech}[\eta]}{5\operatorname{sech}[\eta] \pm 1} \quad \text{and} \quad \phi(\eta) = \frac{\tanh[\eta]}{1 \pm 5\operatorname{sech}[\eta]}.$$

Case 3. $R(r) = (5/9)r^2$.

$$\theta(\eta) = \frac{1}{r}\frac{3\operatorname{sech}[\eta]}{3\operatorname{sech}[\eta] \pm 2} \quad \text{and} \quad \phi(\eta) = \frac{2}{2\coth[\eta] \pm 3\operatorname{csch}[\eta]}.$$

Case 4. $R(r) = r^2 - 1$.

$$\theta(\eta) = \frac{4\operatorname{sech}[\eta]}{3\tanh[\eta] + 4r\operatorname{sech}[\eta] + 5} \quad \text{and} \quad \phi(\eta) = \frac{5\tanh[\eta] + 3}{3\tanh[\eta] + 4r\operatorname{sech}[\eta] + 5}$$

or

$$\theta(\eta) = \frac{\operatorname{sech}[\eta]}{r\operatorname{sech}[\eta] + 1} \quad \text{and} \quad \phi(\eta) = \frac{\tanh[\eta]}{r\operatorname{sech}[\eta] + 1}.$$

Case 5. $R(r) = r^2 + 1$.

$$\theta(\eta) = \frac{\operatorname{csch}[\eta]}{r\operatorname{csch}[\eta] + 1} \quad \text{and} \quad \phi(\eta) = \frac{\coth[\eta]}{r\operatorname{csch}[\eta] + 1}.$$

2.3 The enhanced Kudryashov method

The enhanced Kudryashov method is the most simple and robust method to recover the significant solitons, including the bright, dark, and singular solitons; it is applicable to all nonlinear phenomena [3].

Step 1. Suppose that the solution of Eq. (5) can be expressed in the form

$$Q_1(\eta) = \alpha_0 + \sum_{i=1}^{N}\left[\alpha_i\theta(\eta)^i + \beta_i\frac{\theta'(\eta)}{\theta(\eta)^i}\right], \qquad \theta'^2(\eta) = \theta^2(\eta)\bigl(1 - \chi\theta^2(\eta)\bigr), \tag{10}$$

where α_0, χ, α_i, β_i, $i = 1, 2, \ldots, N$, are constants.

Step 2. Eq. (10) gives the soliton waves

$$\theta(\eta) = \frac{4d}{4d^2\mathrm{e}^{\eta} + \chi\mathrm{e}^{-\eta}}, \quad d \neq 0. \tag{11}$$

Step 3. When we substitute Eq. (10) into Eq. (5), together with Eq. (10), we obtain the necessary constants for Eqs. (4) and (10). Afterwards, by incorporating the acquired parametric limitations into Eqs. (10) and (11), we arrive at straddled solitons, which can be simplified into bright, dark, or singular solitons.

3 Soliton solutions

3.1 The enhanced direct algebraic method

Balancing Q_1'' and Q_1^3 in Eq. (2) implies $N = 1$, and hence,

$$Q_1(\eta) = \alpha_0 + \alpha_1\theta(\eta) + \frac{\beta_1}{\theta(\eta)}. \tag{12}$$

Replace Eq. (12), together with Eq. (7), in Eq. (2). This substitution results in a polynomial denoted as $\theta(\eta)$. When dealing with polynomials, the method includes grouping terms with comparable powers and equating the resultant expression to zero. This process leads to the following set of algebraic equations:

$$\begin{aligned}
&a\alpha_0\left(\varpi_1^2+\varpi_2^2\right) - \beta_1\left(ak^2\tau_3 + 12\alpha_0\alpha_1(b+c+d+d^*)\right)\\
&\quad - a\alpha_1k^2\tau_1 + \alpha_0\omega - 2\alpha_0^3(b+c+d+d^*) = 0,\\
&a\beta_1\left(\varpi_1^2+\varpi_2^2 - 2k^2\tau_2\right) - 6\alpha_0^2\beta_1(b+c+d+d^*)\\
&\quad - 6\alpha_1\beta_1^2(b+c+d+d^*) + \beta_1\omega = 0,\\
&a\alpha_1\left(\varpi_1^2+\varpi_2^2 - 2k^2\tau_2\right) + \alpha_1\omega - 6\alpha_1^2\beta_1(b+c+d+d^*)\\
&\quad - 6\alpha_1\alpha_0^2(b+c+d+d^*) = 0,\\
&-4a\beta_1k^2\tau_0 - 2\beta_1^3(b+c+d+d^*) = 0,\\
&-3a\beta_1k^2\tau_1 - 6\alpha_0\beta_1^2(b+c+d+d^*) = 0,\\
&-3a\alpha_1k^2\tau_3 - 6\alpha_0\alpha_1^2(b+c+d+d^*) = 0,\\
&-2\alpha_1\left(2ak^2\tau_4 + \alpha_1^2(b+c+d+d^*)\right) = 0.
\end{aligned}$$

These equations can be tackled with Mathematica to unveil the unidentified parameters in Eqs. (4) and (12). Consequently, we obtain the exact solutions for Eq. (1).

From now we assume that

$$V(X) = \mu U(X).$$

Case 1. If we set $\tau_0 = \tau_1 = \tau_3 = 0$,

$$\alpha_0 = \beta_1 = 0, \qquad \alpha_1 = \pm\frac{\mathrm{i}\sqrt{2a\tau_4}k}{\sqrt{b+c+d+d^*}}, \qquad \tau_2 = \frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2ak^2},$$

$$\begin{aligned}
U(X) = {}&\pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{k(b+c+d+d^*)}}\,\mathrm{sech}\left[\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a}}(x+y-\nu t)\right]\\
&\times \mathrm{e}^{\mathrm{i}(-\varpi_1x - \varpi_2y + \omega t + \theta_0)},
\end{aligned} \tag{13}$$

$$U(X) = \pm\sqrt{\frac{-a\varpi_1^2 - a\varpi_2^2 - \omega}{k(b+c+d+d^*)}}\,\mathrm{csch}\left[\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a}}(x+y-\nu t)\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{14}$$

Solutions (13) and (14) are a bright and singular solitons with $a\varpi_1^2 + a\varpi_2^2 + \omega > 0$ and $2a > 0$.

Case 2. We set $\tau_0 = \tau_2^2/(4\tau_4)$, $\tau_1 = \tau_3 = 0$.

Result 1.

$$\alpha_0 = \beta_1 = 0, \qquad \alpha_1 = \pm\frac{\mathrm{i}k\sqrt{2a\tau_4}}{\sqrt{b+c+d+d^*}}, \qquad \tau_2 = \frac{a(\varpi_1^2 + \varpi_2^2) + \omega}{2ak^2},$$

$$U(X) = \pm\sqrt{\frac{a(\varpi_1^2 + \varpi_2^2) + \omega}{2(b+c+d+d^*)}}\tanh\left[\sqrt{\frac{-a(\varpi_1^2 + \varpi_2^2) - \omega}{4a}}(x+y-\nu t)\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}, \tag{15}$$

$$U(X) = \pm\sqrt{\frac{a(\varpi_1^2 + \varpi_2^2) + \omega}{2(b+c+d+d^*)}}\coth\left[\sqrt{\frac{-a(\varpi_1^2 + \varpi_2^2) - \omega}{4a}}(x+y-\nu t)\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{16}$$

Solutions (15) and (16) are dark and singular solitons with $a(\varpi_1^2 + \varpi_2^2) + \omega < 0$.

Result 2.

$$\alpha_0 = \alpha_1 = 0, \qquad \beta_1 = \pm\frac{\mathrm{i}(a(\varpi_1^2 + \varpi_2^2) + \omega)}{2\sqrt{2ak^2\tau_4(b+c+d+d^*)}},$$

$$\tau_2 = \frac{a(\varpi_1^2 + \varpi_2^2) + \omega}{2ak^2},$$

$$U(X) = \pm\sqrt{\frac{(a(\varpi_1^2 + \varpi_2^2) + \omega)}{2(b+c+d+d^*)}}\coth\left[\sqrt{\frac{-a(\varpi_1^2 + \varpi_2^2) - \omega}{4a}}(x+y-\nu t)\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}, \tag{17}$$

$$U(X) = \pm\sqrt{\frac{(a(\varpi_1^2 + \varpi_2^2) + \omega)}{2(b+c+d+d^*)}}\tanh\left[\sqrt{\frac{-a(\varpi_1^2 + \varpi_2^2) - \omega}{4a}}(x+y-\nu t)\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{18}$$

Solutions (17) and (18) are singular and dark solitons with $a(\varpi_1^2 + \varpi_2^2) + \omega < 0$.

Result 3.

$$\alpha_0 = 0, \qquad \alpha_1 = \mp\frac{\mathrm{i}k\sqrt{2a\tau_4}}{\sqrt{b+c+d+d^*}},$$

$$\beta_1 = \pm\frac{\mathrm{i}\sqrt{a}(k\tau_2)}{\sqrt{2\tau_4(b+c+d+d^*)}}, \qquad \tau_2 = \frac{a(\varpi_1^2 + \varpi_2^2) + \omega}{8ak^2},$$

$$
\begin{aligned}
U(X) = \mp\sqrt{\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{8(b+c+d+d^*)}}\Biggl[\tanh\Biggl[\sqrt{\frac{-a(\varpi_1^2+\varpi_2^2)-\omega}{16a}}(x+y-\nu t)\Biggr] \\
- \coth\Biggl[\sqrt{\frac{-a(\varpi_1^2+\varpi_2^2)-\omega}{16a}}(x+y-\nu t)\Biggr]\Biggr] \\
\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)},
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
U(X) = \mp\sqrt{\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{8(b+c+d+d^*)}}\Biggl[\coth\Biggl[\sqrt{\frac{-a(\varpi_1^2+\varpi_2^2)-\omega}{16a}}(x+y-\nu t)\Biggr] \\
- \tanh\Biggl[\sqrt{\frac{-a(\varpi_1^2+\varpi_2^2)-\omega}{16a}}(x+y-\nu t)\Biggr]\Biggr] \\
\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}.
\end{aligned} \tag{20}
$$

Solutions (19) and (20) are straddled dark-singular solitons with $a(\varpi_1^2+\varpi_2^2)+\omega<0$.

Case 3.1. We set $\tau_1=\tau_3=0$, $\tau_0=m^2(1-m^2)\tau_2^2/((2m^2-1)^2\tau_4)$.

Result 1.

$$
\alpha_0=\beta_1=0, \qquad \alpha_1=\pm\frac{\mathrm{i}k\sqrt{2a\tau_4}}{\sqrt{b+c+d+d^*}}, \qquad \tau_2=\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2ak^2},
$$

$$
\begin{aligned}
U(X) = \sqrt{\frac{m^2(a\varpi_1^2+a\varpi_2^2+\omega)}{(2m^2-1)(b+c+d+d^*)}}\,\mathrm{cn}\biggl(\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2a(2m^2-1)}}(x+y-\nu t)\,\Big|\, m\biggr) \\
\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}.
\end{aligned} \tag{21}
$$

For $m=1$, we get

$$
\begin{aligned}
U(X) = \sqrt{\frac{(a\varpi_1^2+a\varpi_2^2+\omega)}{(b+c+d+d^*)}}\,\mathrm{sech}\biggl(\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2a}}(x+y-\nu t)\biggr) \\
\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}.
\end{aligned} \tag{22}
$$

Solutions (21) and (22) are JEDPT and a bright soliton solutions with $a\varpi_1^2+a\varpi_2^2+\omega>0$ and $a>0$.

Result 2.

$$
\alpha_0=\alpha_1=0, \qquad \beta_1=\pm\frac{km\tau_2\sqrt{2a(m^2-1)}}{(1-2m^2)\sqrt{\tau_4(b+c+d+d^*)}},
$$

$$
\tau_2=\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2ak^2},
$$

$$
\begin{aligned}
U(X) = \sqrt{\frac{(a\varpi_1^2+a\varpi_2^2+\omega)(m^2-1)}{(2m^2-1)(b+c+d+d^*)}}\,\mathrm{nc}\biggl(\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2a(2m^2-1)}}(x+y-\nu t)\,\Big|\, m\biggr) \\
\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}.
\end{aligned} \tag{23}
$$

For $m = 0$, we get

$$U(X) = \sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)}{(b + c + d + d^*)}} \sec\left(\sqrt{-\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a}}(x + y - \nu t)\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{24}$$

Solutions (23) and (24) are JEDPT and a singular periodic with $a\varpi_1^2 + a\varpi_2^2 + \omega < 0$ and $a > 0$.

Case 3.2. We set $\tau_1 = \tau_3 = 0$, $\tau_0 = (1 - m^2)\tau_2^2/((2 - m^2)^2\tau_4)$.

Result 1.

$$\alpha_0 = \beta_1 = 0, \qquad \alpha_1 = \pm\frac{\mathrm{i}k\sqrt{2a\tau_4}}{\sqrt{b + c + d + d^*}}, \qquad \tau_2 = \frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2ak^2},$$

$$U(X) = \sqrt{\frac{m^2(a\varpi_1^2 + a\varpi_2^2 + \omega)}{(2 - m^2)(b + c + d + d^*)}} \,\mathrm{dn}\left(\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a(2 - m^2)}}(x + y - \nu t) \,\Big|\, m\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{25}$$

For $m = 1$, we get

$$U(X) = \sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)}{(b + c + d + d^*)}} \,\mathrm{sech}\left(\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a}}(x + y - \nu t)\right) \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{26}$$

Solutions (25) and (26) are JEDPT and a bright soliton solutions with $a\varpi_1^2 + a\varpi_2^2 + \omega > 0$ and $a > 0$.

Result 2.

$$\alpha_0 = \alpha_1 = 0,$$

$$\beta_1 = \pm\frac{k\tau_2\sqrt{2a(m^2 - 1)}}{(m^2 - 2)\sqrt{\tau_4(b + c + d + d^*)}}, \qquad \tau_2 = \frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2ak^2},$$

$$U(X) = \sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)(m^2 - 1)}{m^2(2 - m^2)(b + c + d + d^*)}} \,\mathrm{nd}\left(\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a(2 - m^2)}}(x + y - \nu t) \,\Big|\, m\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{27}$$

Solution (27) is JEDPT with $a\varpi_1^2 + a\varpi_2^2 + \omega > 0$ and $a > 0$.

Case 3.3. We set $\tau_1 = \tau_3 = 0$, $\tau_0 = m^2\tau_2^2/((m^2 + 1)^2\tau_4)$.

Result 1.

$$\alpha_0 = \beta_1 = 0, \qquad \alpha_1 = \pm\frac{\mathrm{i}k\sqrt{2a\tau_4}}{\sqrt{b + c + d + d^*}}, \qquad \tau_2 = \frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2ak^2},$$

$$U(X) = \sqrt{\frac{m^2(a\varpi_1^2 + a\varpi_2^2 + \omega)}{(m^2+1)(b+c+d+d^*)}}\,\mathrm{sn}\left(\sqrt{-\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a(m^2+1)}}(x+y-\nu t)\;\middle|\; m\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{28}$$

For $m = 1$, we get

$$U(X) = \sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)}{2(b+c+d+d^*)}}\tanh\left(\sqrt{-\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{4a}}(x+y-\nu t)\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{29}$$

Solutions (28) and (29) are JEDPT and a dark soliton solutions with $a\varpi_1^2 + a\varpi_2^2 + \omega < 0$ and $a > 0$

Result 2.

$$\alpha_0 = \alpha_1 = 0,$$
$$\beta_1 = \pm\frac{\mathrm{i}\sqrt{2a}km\tau_2}{(1+m^2)\sqrt{\tau_4(b+c+d+d^*)}}, \qquad \tau_2 = \frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2ak^2},$$

$$U(X) = \sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{(1+m^2)(b+c+d+d^*)}}\,\mathrm{ns}\left(\sqrt{-\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a(m^2+1)}}(x+y-\nu t)\;\middle|\; m\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{30}$$

For $m = 1$, we get

$$U(X) = \sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)}{2(b+c+d+d^*)}}\coth\left(\sqrt{-\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{4a}}(x+y-\nu t)\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{31}$$

For $m = 0$, we get

$$U(X) = \sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{(b+c+d+d^*)}}\csc\left(\sqrt{-\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a}}(x+y-\nu t)\right) \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{32}$$

Solutions (30), (31), and (32) are JEDPT, a singular soliton, and a singular periodic solutions with $a\varpi_1^2 + a\varpi_2^2 + \omega < 0$ and $a > 0$.

Case 4. We set $\tau_1 = \tau_3 = 0$.

Result 1.

$$\alpha_0 = \beta_1 = 0, \qquad \alpha_1 = \pm\frac{\mathrm{i}k\sqrt{2a\tau_4}}{\sqrt{b+c+d+d^*}}, \qquad \tau_2 = \frac{a(\varpi_1^2 + \varpi_2^2) + \omega}{2ak^2},$$

$$U(X) = \pm k\sqrt{\frac{-2a}{b+c+d+d^*}}\left[\frac{3\wp'(k(x+y-\nu t);g_2,g_3)}{[6\wp(k(x+y-\nu t);g_2,g_3)+\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2}]}\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)},$$

$$U(X) = \pm k\sqrt{\frac{-2a\tau_4\tau_0}{b+c+d+d^*}}\left[\frac{[6\wp(k(x+y-\nu t);g_2,g_3)+\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2}]}{3\wp'(k(x+y-\nu t);g_2,g_3)}\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}, \quad \tau_0>0,$$

where

$$g_2 = \frac{(a(\varpi_1^2+\varpi_2^2)+\omega)^2}{48a^2k^4} + \tau_0\tau_4$$

and

$$g_3 = \frac{a(\varpi_1^2+\varpi_2^2)+\omega}{432ak^2}\left(36\tau_0\tau_4 - \left(\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2}\right)^2\right).$$

Result 2.

$$\alpha_0=\alpha_1=0, \qquad \beta_1 = \pm\frac{\mathrm{i}k\sqrt{2a\tau_0}}{\sqrt{b+c+d+d^*}}, \qquad \tau_2 = \frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2},$$

$$U(X) = \pm k\sqrt{\frac{-2a\tau_0\tau_4}{b+c+d+d^*}}\left[\frac{[6\wp(k(x+y-\nu t);g_2,g_3)+\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2}]}{3\wp'(k(x+y-\nu t);g_2,g_3)}\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}, \quad \tau_4>0,$$

$$U(X) = \pm k\sqrt{\frac{-2a}{b+c+d+d^*}}\left[\frac{3\wp'(k(x+y-\nu t);g_2,g_3)}{[6\wp(k(x+y-\nu t);g_2,g_3)+\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2}]}\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)},$$

where

$$g_2 = \frac{(a(\varpi_1^2+\varpi_2^2)+\omega)^2}{48a^2k^4} + \tau_0\tau_4$$

and

$$g_3 = \frac{a(\varpi_1^2+\varpi_2^2)+\omega}{432ak^2}\left(36\tau_0\tau_4 - \left(\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2ak^2}\right)^2\right).$$

Case 5. We set $\tau_0=\tau_1=0$.

$$\alpha_0 = -\frac{\alpha_1\sqrt{\tau_2}}{2\sqrt{\tau_4}}, \qquad \alpha_1 = \pm\frac{k\sqrt{-2a\tau_4}}{\sqrt{b+c+d+d^*}}, \qquad \beta_1 = 0,$$

$$\tau_2 = -\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{ak^2}, \qquad \tau_3 = \frac{2\sqrt{\tau_4(-(a(\varpi_1^2+\varpi_2^2)+\omega))}}{\sqrt{a}k},$$

$$
\begin{aligned}
U(X) = {}& \mp\sqrt{\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2(b+c+d+d^*)}} \\
& \times \left[1 + \frac{2\,\operatorname{sech}^2[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)]}{\pm 2\tanh[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)] + \frac{2\sqrt{\tau_4(-(a(\varpi_1^2+\varpi_2^2)+\omega))}}{\sqrt{a}k}}\right] \\
& \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}, \quad \tau_4>0,
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
U(X) = {}& \mp\sqrt{\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{2(b+c+d+d^*)}} \\
& \times \left[1 - \frac{2\,\operatorname{csch}^2[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)]}{\pm 2\ \coth[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)] + \frac{2\sqrt{\tau_4(-(a(\varpi_1^2+\varpi_2^2)+\omega))}}{\sqrt{a}k}}\right] \\
& \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}, \quad \tau_4>0,
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
U(X) = {}& \mp\frac{\sqrt{(a(\varpi_1^2+\varpi_2^2)+\omega)}}{\sqrt{2(b+c+d+d^*)}} \\
& \times \left[1 - \frac{4\,\operatorname{sech}^2[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)]}{-4 + (1-\tanh[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)])^2}\right] \\
& \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)},
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
U(X) = {}& \mp\frac{\sqrt{(a(\varpi_1^2+\varpi_2^2)+\omega)}}{\sqrt{2(b+c+d+d^*)}} \\
& \times \left[1 - \frac{4\,\operatorname{csch}^2[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)]}{-4 + (1-\coth[\frac{1}{2}\sqrt{-\frac{a(\varpi_1^2+\varpi_2^2)+\omega}{a}}(x+y-\nu t)])^2}\right] \\
& \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x-\varpi_2 y+\omega t+\theta_0)}.
\end{aligned} \tag{36}
$$

Solutions (33), (35) are a straddled bright-dark solitons, and (34), (36) are a straddled singular-singular solitons with $a(\varpi_1^2+\varpi_2^2)+\omega<0$, $a>0$.

3.2 The projective Riccati equations method

Balancing Q_1'' and Q_1^3 in Eq. (2) implies $N=1$. Therefore

$$Q_1(\eta) = \alpha_0 + \alpha_1\theta(\eta) + \beta_1\phi(\eta), \tag{37}$$

where α_0, α_1, and β_1 are constants to be determined such that $\alpha_1 \neq 0$ or $\beta_1 \neq 0$.

Introducing the solution (37), which meets the requirements (8), (9), into Eq. (2), we obtain a series of nonlinear equations as follows:

$$\begin{aligned}
&a\alpha_0\left(\varpi_1^2+\varpi_2^2\right)+\alpha_0\omega-6\alpha_0\beta_1^2(b+c+d+d^*)-2\alpha_0^3(b+c+d+d^*)=0,\\
&6a\alpha_1k^2r-6\alpha_0\alpha_1^2(b+c+d+d^*)+12\alpha_1\beta_1^2r(b+c+d+d^*)\\
&\quad-6\alpha_0\beta_1^2R(r)(b+c+d+d^*)=0,\\
&-4a\alpha_1k^2R(r)-2\alpha_1^3(b+c+d+d^*)-6\alpha_1\beta_1^2R(r)(b+c+d+d^*)=0,\\
&a\beta_1\varpi_1^2+a\beta_1\varpi_2^2-6\alpha_0^2\beta_1(b+c+d+d^*)-2\beta_1^3(b+c+d+d^*)+\beta_1\omega=0,\\
&2a\beta_1k^2r-12\alpha_0\alpha_1\beta_1(b+c+d+d^*)+4\beta_1^3r(b+c+d+d^*)=0,\\
&-4a\beta_1k^2R(r)-6\alpha_1^2\beta_1(b+c+d+d^*)-2\beta_1^3R(r)(b+c+d+d^*)=0,\\
&a\alpha_1\varpi_1^2+a\alpha_1\varpi_2^2-2a\alpha_1k^2+\alpha_1\omega-6\alpha_1\beta_1^2(b+c+d+d^*)\\
&\quad-6\alpha_0^2\alpha_1(b+c+d+d^*)+12\alpha_0\beta_1^2r(b+c+d+d^*)=0.
\end{aligned}$$

Solving this system of algebraic equations yields

Case 1. $R(r)=0$.

$$\alpha_0=\alpha_1=0,$$
$$\beta_1=\pm\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2(b+c+d+d^*)}},\qquad k=\pm\sqrt{\frac{-a\varpi_1^2-a\varpi_2^2-\omega}{a}},$$

$$U(X)=\pm\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2(b+c+d+d^*)}}\tanh\left[\frac{k(x+y-\nu t)}{2}\right]\mathrm{e}^{\mathrm{i}(-\varpi_1x-\varpi_2y+\omega t+\theta_0)},\tag{38}$$
$$U(X)=\pm\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2(b+c+d+d^*)}}\coth\left[\frac{k(x+y-\nu t)}{2}\right]\mathrm{e}^{\mathrm{i}(-\varpi_1x-\varpi_2y+\omega t+\theta_0)}.\tag{39}$$

Solutions (38) and (39) are a dark and a singular solitons.

Case 2. $R(r)=24r^2/25$.

$$\alpha_0=0,\qquad \alpha_1=\frac{2}{5}\sqrt{6}\beta_1r,$$
$$\beta_1=\pm\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2(b+c+d+d^*)}},\qquad k=\pm\frac{\sqrt{-a\varpi_1^2-a\varpi_2^2-\omega}}{\sqrt{a}},$$

$$\begin{aligned}
U(X)=\pm&\sqrt{\frac{a\varpi_1^2+a\varpi_2^2+\omega}{2(b+c+d+d^*)}}\\
&\times\left[\frac{2\sqrt{6}}{5}\frac{5\operatorname{sech}[k(x+y-\nu t)]}{5\operatorname{sech}[k(x+y-\nu t)]\pm1}+\frac{\tanh[k(x+y-\nu t)]}{1\pm5\operatorname{sech}[k(x+y-\nu t)]}\right]\\
&\times\mathrm{e}^{\mathrm{i}(-\varpi_1x-\varpi_2y+\omega t+\theta_0)}.
\end{aligned}\tag{40}$$

Solution (40) is a straddled bright-dark solitons.

Case 3. $R(r) = 5r^2/9$.

$$\alpha_0 = 0, \qquad \alpha_1 = \beta_1 r \frac{\sqrt{5}}{3},$$

$$\beta_1 = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}, \qquad k = \pm\sqrt{\frac{-a\varpi_1^2 - a\varpi_2^2 - \omega}{a}},$$

$$\begin{aligned} U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} &\left[\frac{\sqrt{5}}{3}\frac{3\operatorname{sech}[k(x+y-\nu t)]}{3\operatorname{sech}[k(x+y-\nu t)] \pm 2}\right. \\ &\left. + \frac{2}{2\coth[k(x+y-\nu t)] \pm 3\operatorname{csch}[k(x+y-\nu t)]}\right] \\ &\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \end{aligned} \tag{41}$$

Solution (41) is a straddled bright-singular soliton.

Case 4. $R(r) = r^2 - 1$.

$$\alpha_0 = 0, \qquad \alpha_1 = \beta_1\sqrt{r^2 - 1},$$

$$\beta_1 = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}, \qquad k = \pm\sqrt{\frac{-a\varpi_1^2 - a\varpi_2^2 - \omega}{a}},$$

$$\begin{aligned} U(X) = {}& \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} \\ &\times \left[\frac{4\sqrt{r^2-1}\operatorname{sech}[k(x+y-\nu t)]}{3\tanh[k(x+y-\nu t)] + 4r\operatorname{sech}[k(x+y-\nu t)] + 5}\right. \\ &\left. + \frac{5\tanh[k(x+y-\nu t)] + 3}{3\tanh[k(x+y-\nu t)] + 4r\operatorname{sech}[k(x+y-\nu t)] + 5}\right] \\ &\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)} \end{aligned} \tag{42}$$

or

$$\begin{aligned} U(X) = {}& \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} \\ &\times \left[\frac{\sqrt{r^2-1}\operatorname{sech}[k(x+y-\nu t)]}{r\operatorname{sech}[k(x+y-\nu t)] + 1} + \frac{\tanh[k(x+y-\nu t)]}{r\operatorname{sech}[k(x+y-\nu t)] + 1}\right] \\ &\times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \end{aligned} \tag{43}$$

Solutions (42) and (43) are a straddled bright-dark solitons.

Case 5. $R(r) = r^2 + 1$.

$$\alpha_0 = 0, \qquad \alpha_1 = \beta_1\sqrt{r^2 + 1},$$

$$\beta_1 = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}, \qquad k = \pm\sqrt{\frac{-a\varpi_1^2 - a\varpi_2^2 - \omega}{a}},$$

$$
\begin{aligned}
U(X) = {} & \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} \\
& \times \left[\frac{\sqrt{r^2+1}\,\operatorname{csch}[k(x+y-\nu t)]}{r\operatorname{csch}[k(x+y-\nu t)]+1} + \frac{\coth[k(x+y-\nu t)]}{r\operatorname{csch}[k(x+y-\nu t)]+1}\right] \\
& \times \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}.
\end{aligned} \tag{44}
$$

Solution (44) is a straddled singular-singular soliton.

3.3 The enhanced Kudryashov method

Balancing Q_1'' and Q_1^3 in Eq. (2) implies $N = 1$. Therefore

$$
Q_1(\eta) = \alpha_0 + \alpha_1\theta(\eta) + \beta_1\frac{\theta'(\eta)}{\theta(\eta)}, \tag{45}
$$

where α_0, α_1, and β_1 are constants to be determined such that $\alpha_1 \neq 0$ or $\beta_1 \neq 0$.

Replacing the solution provided by Eq. (45), which fulfills the condition specified in Eq. (10), into Eq. (2), results in the following set of nonlinear equations:

$$
\begin{aligned}
& 6\alpha_0\beta_1^2\chi(b+c+d+d^*) - 6\alpha_0\alpha_1^2(b+c+d+d^*) = 0, \\
& 4a\alpha_1k^2\chi + 6\alpha_1\beta_1^2\chi(b+c+d+d^*) - 2\alpha_1^3(b+c+d+d^*) = 0, \\
& a\beta_1\varpi_1^2 + a\beta_1\varpi_2^2 - 6\alpha_0^2\beta_1(b+c+d+d^*) - 2\beta_1^3(b+c+d+d^*) + \beta_1\omega = 0, \\
& a\alpha_0\left(\varpi_1^2 + \varpi_2^2\right) + \alpha_0\omega - 6\alpha_0\beta_1^2(b+c+d+d^*) - 2\alpha_0^3(b+c+d+d^*) = 0, \\
& -12\alpha_0\alpha_1\beta_1(b+c+d+d^*) = 0, \\
& a\alpha_1\varpi_1^2 + a\alpha_1\varpi_2^2 - 2a\alpha_1k^2 + \alpha_1\omega - 6\alpha_1\beta_1^2(b+c+d+d^*) \\
& \quad - 6\alpha_0^2\alpha_1(b+c+d+d^*) = 0, \\
& 4a\beta_1k^2\chi - 6\alpha_1^2\beta_1(b+c+d+d^*) + 2\beta_1^3\chi(b+c+d+d^*) = 0.
\end{aligned}
$$

Solving this system of algebraic equations concludes the following results.

Result 1.

$$
\alpha_0 = \beta_1 = 0,
$$

$$
\alpha_1 = \pm\sqrt{\frac{\chi(a\varpi_1^2 + a\varpi_2^2 + \omega)}{b+c+d+d^*}}, \qquad k = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2a}}.
$$

Here we obtain the exact solutions of Eq. (2) as follows:

$$
\begin{aligned}
U(X) \pm {} & \sqrt{\frac{\chi(a\varpi_1^2 + a\varpi_2^2 + \omega)}{b+c+d+d^*}}\left[\frac{4d}{4d^2\mathrm{e}^{k(x+y-\nu t)} + \chi\mathrm{e}^{-k(x+y-\nu t)}}\right] \\
& \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}.
\end{aligned} \tag{46}
$$

Setting $\chi = \pm 4d^2$ in solution (46) concludes bright and singular soliton solutions

$$U(X) = \pm\frac{1}{2d}\sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)}{b+c+d+d^*}}\Big[\operatorname{sech}\big(k(x+y-\nu t)\big)\Big] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}$$

and

$$U(X) = \pm\frac{1}{2d}\sqrt{\frac{(a\varpi_1^2 + a\varpi_2^2 + \omega)}{b+c+d+d^*}}\Big[\operatorname{csch}\big(k(x+y-\nu t)\big)\Big] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{47}$$

Result 2.

$$\alpha_0 = \alpha_1 = 0,$$

$$\beta_1 = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}, \qquad k = \pm\sqrt{\frac{-a\varpi_1^2 - a\varpi_2^2 - \omega}{4a}}.$$

Here the exact solutions of Eq. (2) can be expressed as

$$U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}\left[\frac{4d^2\mathrm{e}^{k(x+y-\nu t)} - \chi\mathrm{e}^{-k(x+y-\nu t)}}{4d^2\mathrm{e}^{k(x+y-\nu t)} + \chi\mathrm{e}^{-k(x+y-\nu t)}}\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{48}$$

Setting $\chi = \pm 4d^2$ in solution (48), we have dark and singular soliton solutions

$$U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}\big[\tanh\big(k(x+y-\nu t)\big)\big]\mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}$$

and

$$U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}\big[\coth\big(k(x+y-\nu t)\big)\big]\mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}.$$

Result 3.

$$\alpha_0 = 0, \qquad \alpha_1 = \beta_1\sqrt{-\chi},$$

$$\beta_1 = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}}, \qquad k = \pm\sqrt{\frac{-a\varpi_1^2 - a\varpi_2^2 - \omega}{a}}.$$

We obtain the exact solutions of Eq. (2) as follows:

$$U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} \times \left[\frac{4d\sqrt{-\chi}}{4d^2\mathrm{e}^{\eta} + \chi\mathrm{e}^{-\eta}} + \frac{4d^2\mathrm{e}^{k(x+y-\nu t)} - \chi\mathrm{e}^{-k(x+y-\nu t)}}{4d^2\mathrm{e}^{k(x+y-\nu t)} + \chi\mathrm{e}^{-k(x+y-\nu t)}}\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}. \tag{49}$$

Setting $\chi = \pm 4d^2$ in solution (49), we have straddled bright-dark and singular-singular solitons

$$U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} \times \left[\frac{\sqrt{-\chi}}{2d}\big[\operatorname{sech}\big(k(x+y-\nu t)\big)\big] + \big[\tanh\big(k(x+y-\nu t)\big)\big]\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)},$$

$$U(X) = \pm\sqrt{\frac{a\varpi_1^2 + a\varpi_2^2 + \omega}{2(b+c+d+d^*)}} \times \left[\frac{\sqrt{-\chi}}{2d}\big[\operatorname{csch}\big(k(x+y-\nu t)\big)\big] + \big[\coth\big(k(x+y-\nu t)\big)\big]\right] \times \mathrm{e}^{\mathrm{i}(-\varpi_1 x - \varpi_2 y + \omega t + \theta_0)}.$$

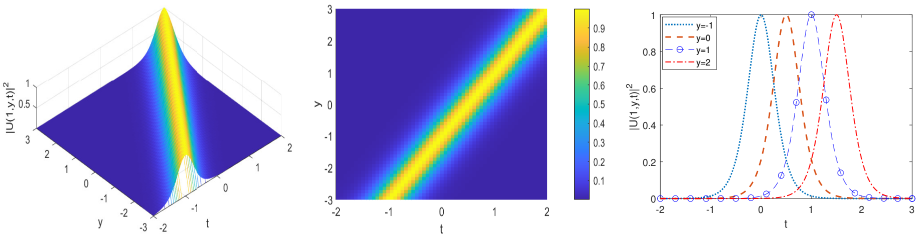
4 Results and discussion

Fig. 1 illustrates the bright soliton described by Eq. (13), where the parameters k, ϖ_1, ϖ_2, a, b, c, d, d^*, and θ_0 are all set to 1, while ω and ν are set to 2, at $x = 1$. Bright solitons are localized wave solutions that arise in nonlinear systems, characterized by their ability to maintain their shape and amplitude while propagating through a medium. These solitons result from a delicate balance between nonlinear effects, which tend to spread the wave, and dispersive effects, which tend to disperse it.

Fig. 2 shows the dark soliton (15) with respect to $\varpi_1 = \varpi_2 = b = c = d = d^* = \theta_0 = 1$, $a = -1$, $\omega = 4$, and $\nu = 2$ in $x = 1$. Dark solitons represent localized wave solutions that arise in nonlinear systems, exhibiting a unique behavior where they create localized regions of reduced intensity or amplitude within a background field. Unlike bright solitons, which form peaks or pulses, dark solitons are characterized by their ability to create troughs or holes in the waveform.

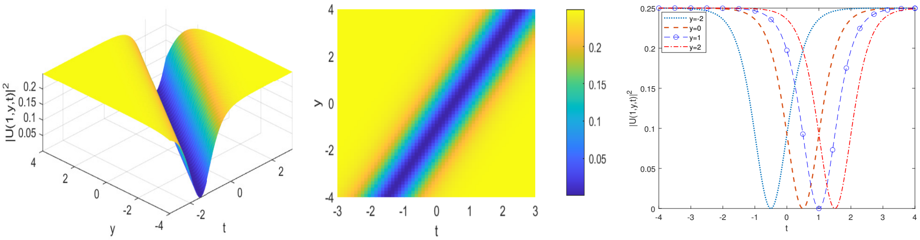
Fig. 3 illustrates the singular soliton described by Eq. (17), where the parameters ϖ_1, ϖ_2, b, c, d, d^*, and θ_0 are all set to 1, a is set to -1, and ω and ν are set to 4 and 2, respectively, at $x = 1$. Singular solitons represent localized wave solutions that arise in nonlinear systems, exhibiting distinctive features where they create localized regions of extreme intensity or amplitude within a background field. These solitons are characterized by their ability to concentrate energy into a single, sharp peak or dip, often leading to mathematical singularities or discontinuities in the solution.

Fig. 4 illustrates the straddled dark-singular solitons (19) with respect to $\varpi_1 = \varpi_2 = b = c = d = d^* = \theta_0 = 1$, $a = -1$, $\omega = 4$, and $\nu = 2$ in $x = 1$. Straddled dark-singular solitons are localized wave solutions that exhibit a unique combination of characteristics from both dark and singular solitons. These solitons manifest as localized regions of reduced intensity or amplitude, akin to dark solitons, while simultaneously featuring extreme concentration of energy into a single, sharp peak or dip, reminiscent of singular solitons.



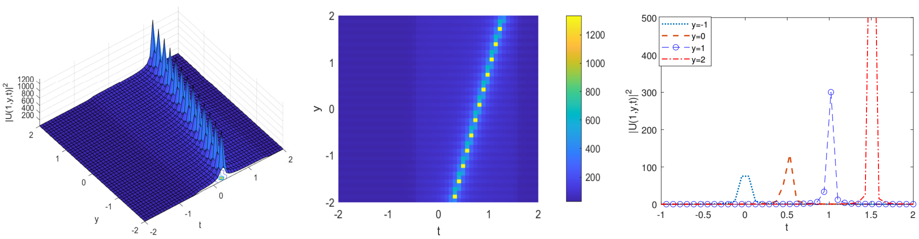
(a) (b) (c)

Figure 1. 3D contour and 2D graphs of bright soliton solution of Eq. (13).



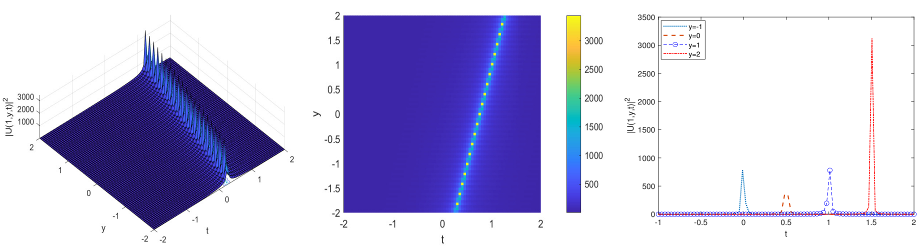
(a) (b) (c)

Figure 2. 3D contour and 2D graphs of dark soliton solution of Eq. (15).



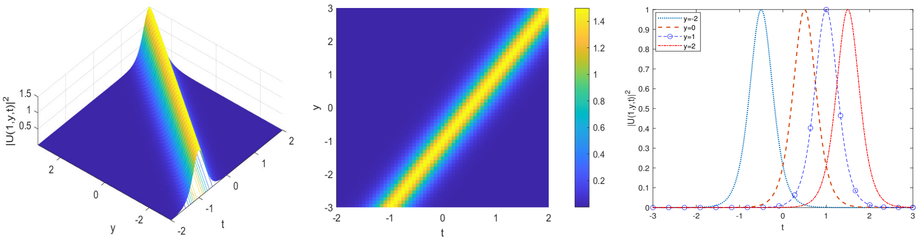
(a) (b) (c)

Figure 3. 3D contour and 2D graphs of singular soliton solution of Eq. (17).



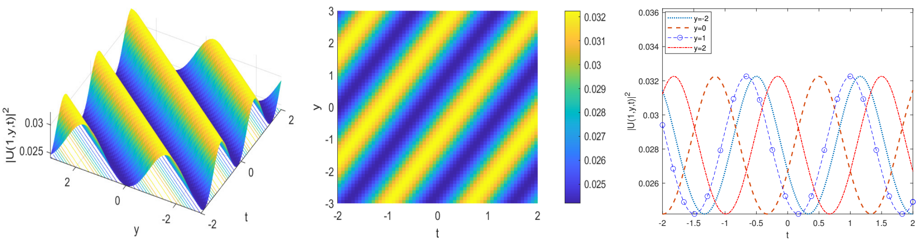
(a) (b) (c)

Figure 4. 3D contour and 2D graphs of straddled dark-singular solitons of Eq. (19).



(a) (b) (c)

Figure 5. 3D contour and 2D graphs of bright soliton solution of Eq. (22).



(a) (b) (c)

Figure 6. 3D contour and 2D graphs of JEDPT solution of Eq. (25).

Fig. 5 depicts the bright soliton described by Eq. (22), where the parameters k, ϖ_1, ϖ_2, a, b, c, d, d^*, and θ_0 are all set to 1, while ω and ν are both set to 2, at $x = 1$.

Moreover, Fig. 6 shows the JEDPT solution (25) with respect to $k = \varpi_1 = \varpi_2 = a = b = c = d = d^* = \theta_0 = 1$, $\omega = \nu = 2$, and $m = 0.25$ in $x = 1$.

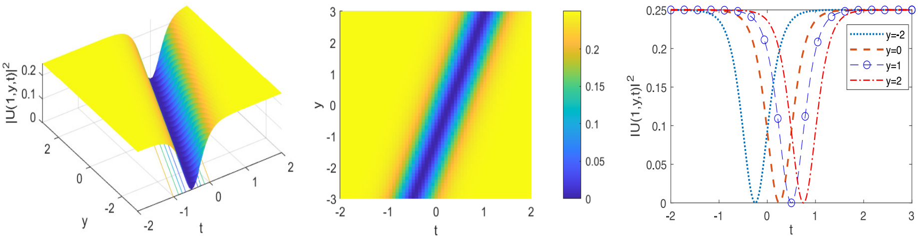
The dark soliton solution (29) with respect to $k = \varpi_1 = \varpi_2 = b = c = d = d^* = \theta_0 = 1$, $a = -1$, $\omega = \nu = 4$, and $x = 1$ are plotted in Fig. 7.

Fig. 8 shows the straddled singular-singular solitons solution (33) with respect to $k = \varpi_1 = \varpi_2 = a = b = c = d = d^* = \theta_0 = 1$, $\omega = 3$, $\nu = 2$, and $x = 1$. Straddled singular-singular solitons are localized wave solutions that exhibit a unique combination of characteristics from both singular and singular solitons. These solitons manifest as localized regions of extreme intensity or amplitude, akin to singular solitons, while simultaneously featuring extreme concentration of energy into a single, sharp peak or dip, reminiscent of singular solitons.

Fig. 9 illustrates the straddled bright-dark solitons solution described by equation (43), where the parameters k, ϖ_1, ϖ_2, a, b, c, d, d^*, and θ_0 are all set to 1, while ω and ν are both set to 4, and r is set to 3, at $x = 1$.

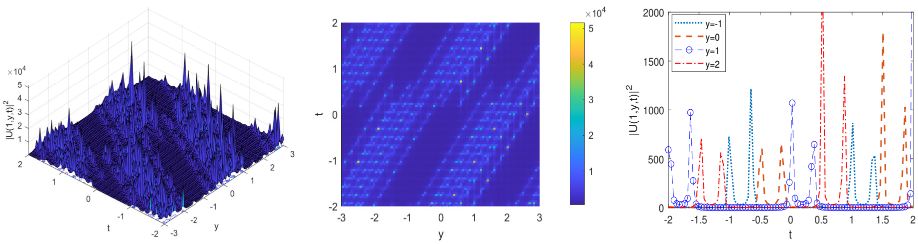
Fig. 10 depicts the singular soliton solution described by Eq. (47), where the parameters k, ϖ_1, ϖ_2, a, b, c, d, d^*, and θ_0 are all set to 1, while ω and ν are both set to 4, at $x = 1$.

The governing model has significant applications in various real-world phenomena, particularly in nonlinear optics, fluid dynamics, and Bose–Einstein condensates. These



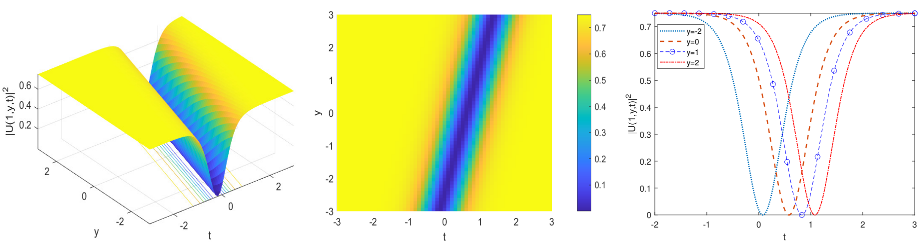
(a) (b) (c)

Figure 7. 3D contour and 2D graphs of dark soliton solution of Eq. (29).



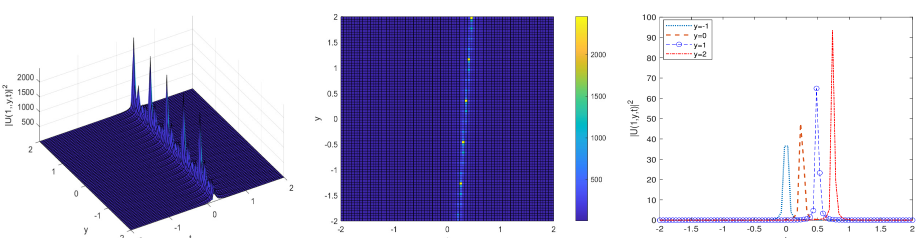
(a) (b) (c)

Figure 8. 3D contour and 2D graphs of straddled singlar-singular solitons solution of Eq. (33).



(a) (b) (c)

Figure 9. 3D contour and 2D graphs of straddled bright-dark solitons solution of Eq. (43).



(a) (b) (c)

Figure 10. 3D contour and 2D graphs of singular soliton solution of Eq. (47).

equations can model light propagation in nonlinear media, allowing for the prediction of soliton formation and interactions, which are essential for optical communication technologies. Additionally, they are used to describe wave dynamics in two-dimensional fluid systems, helping to understand wave stability and behavior in natural resources like oceans and atmospheric phenomena. In quantum physics, these equations provide insights into multicomponent Bose–Einstein condensates, which are crucial for exploring quantum coherence and fundamental quantum behaviors in condensed matter systems.

5 Conclusion

This study has thoroughly analyzed the behavior of optical solitons within $(2+1)$-dimensional generalized coupled nonlinear Schrödinger equations. Utilizing the enhanced direct algebraic method, the enhanced Kudryashov method, and the new projective Riccati equation method, a broad range of soliton solutions have been uncovered, including bright, dark, singular, and straddled variants. Additionally, identifying solutions characterized by Jacobi and Weierstrass elliptic functions has significantly enhanced our comprehension of the intricate dynamics governing optical solitons. The diversity and originality of the soliton solutions discovered in this study highlight the efficacy and resilience of the integration techniques employed and make noteworthy contributions to both the theoretical framework and practical applications in the field of nonlinear optical systems. This paper establishes a firm groundwork for future study and heralds new prospects for developments in optical communications and related technological spheres. The study's results may lead to more research into how optical solitons interact with new materials, such as metamaterials and photonic crystals. This could make it easier to control and keep soliton stability. Moreover, the integration of machine learning techniques with conventional methodologies may generate novel prospects for forecasting soliton behavior in intricate situations. Future studies may concentrate on the experimental validation of the proposed theoretical solutions, perhaps resulting in groundbreaking applications in high-speed data transmission and nonlinear imaging technologies. Interdisciplinary collaborations, particularly between applied physics and engineering, may produce significant advancements in practical applications, hence enhancing our comprehension and application of nonlinear optical systems.

Conflicts of interest. The authors declare no conflicts of interest.

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