Nonlinear Analysis: Modelling and Control, Vol. 30, No. 1, 108–118
https://doi.org/10.15388/namc.2025.30.38343

Fixed point theory in RWC–Banach algebras

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Received: January 9, 2023 / **Revised:** December 2, 2024 / **Published online:** January 9, 2025

Abstract. In this paper, we prove some fixed point results for the sum and the product of nonlinear continuous operators acting on an RWC–Banach algebra. Our result is formulated in terms of topological conditions on the operators. An illustrative example on an RWC–Banach algebra, which is not a WC–Banach algebra, is provided.

Keywords: WC–Banach algebra, RWC–Banach algebra, fixed point theorems.

1 Introduction

The concept of a Banach algebra plays an important role in the theory of differential and integral equations; see [2–5, 8]. Several authors have introduced classes of Banach algebras and established fixed point results in the weak topology. In 2010, Ben Amar, Chouayekh, and Jeribi [5] introduced a class of Banach algebra satisfying the following sequential condition:

($\mathcal{P}$) For any sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, $x_n \cdot y_n \rightharpoonup x \cdot y$, where X is a Banach algebra.

They proved some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators. More precisely, the authors consider operator equations having the form $Tx = AxBx + Cx$, where A, B, and C are defined on subsets of a given Banach algebra. In 2014, Banaś and Taoudi [4] extended some of the results established in [5] in the weak topology setting. In particular, the authors introduced the class, probably weaker than property ($\mathcal{P}$), of so-called weakly compact Banach algebras (in short, WC–Banach algebras), i.e., Banach algebras in which the product of two weakly compact subsets is weakly compact. In 2015, Jeribi and Krichen [8] raised the question of whether there exists a WC–Banach algebra in which property ($\mathcal{P}$) fails, and this question remained unanswered until 2019. Banaś and Olszowy [3] gave a positive answer and proved that a Banach algebra X satisfies condition ($\mathcal{P}$) if and only if X is a WC–Banach algebra. In this direction, the authors introduced a new class of Banach algebras, strictly containing that of weakly compact, the so-called relatively weakly compact Banach algebras (in short, RWC–Banach algebras), i.e., Banach algebras in which the product of two relatively weakly compact subsets is relatively weakly compact.

In this paper, we give some examples of an RWC–Banach algebra, which is not a WC–Banach algebra, and we establish a new fixed point theorem for the sum and the product of nonlinear continuous operators acting on an RWC–Banach algebra.

2 Preliminaries and fixed point theory

Throughout the paper, we denote by $\mathbb{R}$ the set of real numbers. The symbol $\mathbb{N}$ stands for the set of natural numbers. Assume that X is a Banach space with the norm $\|\cdot\|$ and the zero element θ. For $r > 0$, the symbol B_r denotes the closed ball centered at θ and with radius r, and $\mathcal{D}(A)$ denotes the domain of an operator A. By the symbol $\mathfrak{M}_X$ we will denote the collection of all nonempty bounded subsets of X, while $\mathfrak{N}^W$ stands for its subfamily consisting of all relatively weakly compact sets. We use the standard notation $\rightharpoonup$ to denote weak convergence and $\to$ to denote strong convergence in X, respectively.

Recall from [8] that a Banach space X is said to be a Banach algebra if it is endowed with the inner operation of multiplication $x \cdot y$ of elements $x, y \in X$, which is associative, bilinear, and such that

$$\|x \cdot y\| \leqslant \|x\| \|y\|.$$

Now recall the concept of the De Blasi measure of weak noncompactness [6] being the function $\mu : \Omega_X \to [0, +\infty)$ defined for all $M \in \mathfrak{M}_X$ in the following way:

$$\mu(M) = \inf\{r > 0\text{: there exits } R \in \mathfrak{N}^W \text{ such that } M \subseteq R + B_r\};$$

for more details, see [6].

Assume that X is a Banach algebra. For two arbitrary subsets U and V of X, we define the product $U \cdot V$ as follows:

$$U \cdot V = \{u \cdot v\text{: } u \in U,\ v \in V\}.$$

Recall from [4] that the product of compacts sets U and V in X is compact. Moreover, it can be shown that the product of a compact set and a weakly compact set is weakly compact. However, the product of two weakly compact sets does not have to be weakly compact as can be seen in the following example.

Example 1. (See [3].) Consider the classical Hilbert space $(l^2, \langle\cdot,\cdot\rangle_{l^2})$ of square-summable sequences of real numbers with the Euclidean norm

$$\|x\|_{l^2} = \big\|(x_n)\big\|_{l^2} = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$$

for $x = (x_n) \in l^2$. Consider the multiplication of elements in l^2 defined by

$$x \cdot y = (x_n) \cdot (y_n) = \left(\sum_{n=1}^{\infty} x_n y_n, 0, 0, 0, \dots\right).$$

The product is well defined, and we have

$$\|x \cdot y\|_{l^2} \leqslant \|x\|_{l^2} \|y\|_{l^2}.$$

Thus, l^2 with the above defined operation of multiplication forms a Banach algebra. Now, consider the sequence $w_n = (1 - 1/n)e_n$, $n = 1, 2, \dots$, where the sequence (e_n) is the standard unit vector basis of l^2. Note that $w_n \rightharpoonup \theta$ since for an arbitrary $x = (x_n) \in l^2$, we have

$$\langle x, w_n\rangle_{l^2} = \left(1 - \frac{1}{n}\right) x_n \to 0 \quad \text{as } n \to \infty.$$

Let

$$U = (w_n) \cup \{\theta\} \quad \text{and} \quad V = (w_n) \cup \{\theta\}.$$

The sets U and V are weakly compact in the Banach algebra l^2, however, the product set $U \cdot V = \{(1-1/n)^2 e_1\colon n \geqslant 1\}$ is not weakly compact since $(1-1/n)^2 e_1 \rightharpoonup e_1 \notin U \cdot V$.

This example motivates the definitions of so-called WC and RWC–Banach algebras; see [3].

Definition 1. (See [4].) A Banach algebra X is considered to be a WC–Banach algebra if the product $U \cdot V$ of arbitrary weakly compact sets U and V in X is weakly compact.

Definition 2. (See [3].) A Banach algebra X is called an RWC–Banach algebra if the product $U \cdot V$ of arbitrary relatively weakly compact sets U and V in X is relatively weakly compact.

Remark 1. (See [3].) If X is a WC–Banach algebra, then X is an RWC–Banach algebra. However, if X is the RWC–Banach algebra, then X is not necessarily a WC–Banach algebra as can be seen in the following counterexample:

Consider the Banach algebra l^2 (see Example 1) with the norm

$$\|x\|_{l^2} = \big\|(x_n)\big\|_{l^2} = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$$

and with multiplication

$$x \cdot y = (x_n) \cdot (y_n) = \left(\sum_{n=1}^{\infty} x_n y_n, 0, 0, 0, \dots \right).$$

For arbitrary relatively weakly compact subsets U and V of l^2, the product $U \cdot V$ is a bounded subset of the 1-dimensional subspace $\operatorname{span}(e_1)$, where (e_n) denotes the sequence of canonical vectors in l^2. Thus, l^2 is an RWC–Banach algebra.

On the other hand, by taking $u_n = e_n$, $v_n = e_n$ for $n = 1, 2, \dots$, we have that $u_n \rightharpoonup 0$ and $v_n \rightharpoonup 0$ when $n \to \infty$. However, $u_n \cdot v_n = e_1 \not\rightharpoonup 0$. Therefore, l^2 with the multiplication defined above is not a WC–Banach algebra.

In what follows, we will use the following two conditions for the operator A:

(H1) For any sequence $(x_n) \subset \mathcal{D}(A)$, which is weakly convergent in X, the sequence (Ax_n) has a strongly convergent subsequence in X;

(H2) For each weakly convergent sequence $(x_n) \subset \mathcal{D}(A)$, the sequence (Ax_n) contains a weakly convergent subsequence in X.

Conditions (H1) and (H2) were considered in [7, 8], and for some properties on maps satisfying (H1) and (H2), we refer the reader to [8].

Definition 3. An operator $A : \mathcal{D}(A) \subset X \to X$ is called D-Lipschitzian if there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi(0) = 0$, such that for $x, y \in \mathcal{D}(A)$,

$$\|Ax - Ay\| \leqslant \phi\big(\|x - y\|\big).$$

If $\phi(r) < r$ for $r > 0$, then the operator A is referred as a nonlinear contraction with a contraction function ϕ.

Lemma 1. *Let $A : D(A) \subset X \to X$ be a D-Lipschitzian operator with the D-function ϕ on a Banach space X. Moreover, we assume that A satisfies condition* (H2)*. Then for each set $S \in \mathfrak{M}_X$ such that $S \subset D(A)$, the following inequality is satisfied*:

$$\mu(AS) \leqslant \phi\big(\mu(S)\big).$$

For the proof, we refer the reader to [1].

Lemma 2. *Let M_1 and M_2 be two bounded subsets of an RWC–Banach algebra X. Then we have the following inequality*:

$$\mu(M_1 \cdot M_2) \leqslant \|M_2\| \mu(M_1) + \|M_1\| \mu(M_2) + \mu(M_1)\mu(M_2),$$

where $\|M\|$ denotes the norm of a set M, $M \subseteq X$, i.e., $\|M\| = \sup\{\|x\|, x \in M\}$.

Proof. Assume that M_1 and M_2 are arbitrary bounded subsets of an RWC–Banach algebra X. Let s and t be fixed numbers with $s > \omega(M_1)$ and $t > \omega(M_2)$. Then we can find two relatively weakly compact subsets R_1 and R_2 of X such that

$$M_1 \subseteq R_1 + B_s \quad \text{and} \quad M_2 \subseteq R_2 + B_t. \tag{1}$$

Let us take $z \in M_1 \cdot M_2$. This means that there exist $x \in M_1$ and $y \in M_2$ such that $z = x \cdot y$. In view of (1), there exist $r_1 \in R_1$, $r_2 \in R_2$, $u \in B_s$, and $v \in B_t$ such that $x = r_1 + u$ and $y = r_2 + v$. Then we have

$$\begin{aligned} z &= x \cdot y \\ &= (r_1 + u) \cdot (r_2 + v) = r_1 \cdot r_2 + r_1 \cdot v + u \cdot r_2 + u \cdot v \\ &= r_1 \cdot r_2 + (x - u) \cdot v + u \cdot (y - v) + u \cdot v \\ &= r_1 \cdot r_2 + x \cdot v + u \cdot y - u \cdot v. \end{aligned}$$

This yields the following inclusion:

$$\begin{aligned} M_1 \cdot M_2 &\subseteq R_1 \cdot R_2 + M_1 \cdot B_t + B_s \cdot M_2 + B_s \cdot B_t \\ &\subseteq R_1 \cdot R_2 + \|M_1\|s + \|M_2\|t + st. \end{aligned}$$

The fact that X is an RWC–Banach algebra implies that $R_1 \cdot R_2$ is relatively weakly compact. Now, in view of the definition of the De Blasi measure of weak noncompactness μ, we obtain

$$\mu(M_1 \cdot M_2) \leqslant \|M_1\|s + \|M_2\|t + st.$$

Let $s \to \mu(M_1)$ and $t \to \mu(M_2)$, then we have

$$\mu(M_1 \cdot M_2) \leqslant \|M_2\|\mu(M_1) + \|M_1\|\mu(M_2) + \mu(M_1)\mu(M_2). \qquad \square$$

Lemma 3. (*See* [9].) *Let S be a nonempty, closed, and convex subset of a Banach space X, and let $A : S \to S$ be a continuous operator satisfying condition* (H1). *If AS is relatively weakly compact, then A has at least one fixed point in S.*

Theorem 1. *Let X be a RWC–Banach algebra, and let S be a nonempty, bounded, closed, and convex subset of X. Consider three operators A, B, C such that $A, C : X \to X$ and $B : S \to X$, which satisfy the following conditions*:

(i) *The operators A and C satisfy condition* (H2) *and are D-Lipschitzian with the D-functions ϕ_A and ϕ_C, respectively.*

(ii) *A is a regular operator* (*i.e., for all $x \in X$, Ax is invertible*).

(iii) *The operator B is continuous on S and satisfies condition* (H1); *and the set $B(S)$ is relatively weakly compact.*

(iv) *For each $y \in S$, the following implication holds*:

$$x = Ax \cdot By + Cx \implies x \in S.$$

(v) *For any $r > 0$, the following inequality is satisfied*:

$$L\phi_A(r) + \phi_C(r) < r, \quad L = \big\|B(S)\big\|.$$

Under the above assumptions, the operator equation $x = Ax \cdot Bx + Cx$ has at least one solution in the set S.

Proof. Note that from Lemma 3.4 in [2] the operator $T = ((I - C)/A)^{-1}B : S \to X$ is well defined. Next, we show that $T(S) \subset S$. Let $x \in S$. Denote $y = Tx$. This means that

$$y = \left(\frac{I - C}{A}\right)^{-1} Bx.$$

Hence, we get

$$\frac{I - C}{A} y = Bx$$

or, equivalently,

$$Bx = \frac{y - Cy}{Ay}.$$

Further, we have

$$Ay \cdot Bx = y - Cy$$

and, consequently,

$$y = Ay \cdot Bx + Cy.$$

In view of assumption (iv) from the above equality, we conclude that $y \in S$. Thus, $T(S) \subset S$.

Next, we prove that the operator T is continuous on the set S. Indeed, taking into account assumption (iii), we see that it is sufficient to show that the operator $((I - C)/A)^{-1}$ is continuous on the set $B(S)$. Assume that (y_n) is a sequence contained in the set $B(S)$, which is convergent to a point $y \in B(S)$. Let

$$x_n = \left(\frac{I - C}{A}\right)^{-1} y_n, \qquad x = \left(\frac{I - C}{A}\right)^{-1} y.$$

We show that $x_n \to x$ as $n \to \infty$. To prove this, we first observe that

$$\left(\frac{I - C}{A}\right) x_n = y_n.$$

Hence, we obtain

$$(I - C)x_n = y_n \cdot Ax_n$$

and, consequently,

$$x_n = y_n Ax_n + Cx_n. \tag{2}$$

Similarly, we can show that

$$x = y \cdot Ax + Cx. \tag{3}$$

Combining (2) and (3), we get

$$\begin{aligned}
\|x_n - x\| &\leqslant \|y_n \cdot Ax_n - y \cdot Ax\| + \|Cx_n - Cx\| \\
&\leqslant \|y_n \cdot Ax_n - y_n \cdot Ax\| + \|y_n \cdot Ax - y \cdot Ax\| + \|Cx_n - Cx\| \\
&\leqslant L\phi_A\bigl(\|x_n - x\|\bigr) + \|Ax\|\|y_n - y\| + \phi_C\bigl(\|x_n - x\|\bigr).
\end{aligned}$$

Thus,

$$\|x_n - x\| \leqslant L\phi_A\big(\|x_n - x\|\big) + \|Ax\|\|y_n - y\| + \phi_C\big(\|x_n - x\|\big). \tag{4}$$

Now, keeping in mind that $y \in S$, in view of assumption (vi), we deduce that $x \in S$. Taking into account that the set S is bounded and A is D-Lipschitzian, we infer that $A(S)$ is bounded. Thus, there exists a constant $P > 0$ such that $\|Ax\| \leqslant P$ for each $x \in S$. Next, taking the limit superior in (4), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - x\| \leqslant L\phi_A\Big(\limsup_{n\to\infty} \|x_n - x\|\Big) + P \cdot 0 \\ + \phi_C\Big(\limsup_{n\to\infty} \|x_n - x\|\Big).\end{aligned} \tag{5}$$

Suppose that $r = \limsup_{n\to\infty} \|x_n - x\| > 0$. Then from (5) we have

$$r \leqslant L\phi_A(r) + \phi_C(r).$$

However, this contradicts assumption (v), and this yields that $r = 0$. Hence, the operator $((I - C)/A)^{-1}$ is continuous on the set $B(S)$.

Now, we show that the operator T satisfies condition (H1). In order to prove this fact, assume that $(x_n) \subset S$ is a weakly convergent sequence. Since in view of assumption (iii), the operator B satisfies condition (H1), this implies that the sequence (Bx_n) contains a strongly convergent subsequence (Bx_{k_n}). In view of the continuity of the operator $((I - C)/A)^{-1}$ on the set $B(S)$, we obtain that the sequence

$$\left(\left(\frac{I - C}{A}\right)^{-1} Bx_{k_n}\right)$$

is strongly convergent, i.e., the sequence (Tx_{k_n}) is strongly convergent. Thus, the operator T satisfies condition (H1).

Next, we show that the set $T(S)$ is relatively weakly compact. First, notice that after converting of the equality

$$T = \left(\frac{I - C}{A}\right)^{-1} B,$$

we obtain that $T = AT \cdot B + CT$. Further, taking into account the properties of the De Blasi measure of weak noncompactness [6], we get

$$\mu(T(S)) \leqslant \mu\big(A\big(T(S)\big) \cdot B(S)\big) + \mu\big(C\big(T(S)\big)\big).$$

Hence, in view of Lemma 2, we obtain

$$\begin{aligned}\mu\big(T(S)\big) \leqslant \big\|B(S)\big\|\mu\big(A\big(T(S)\big)\big) + \big\|A\big(T(S)\big)\big\|\mu\big(B(S)\big) \\ + \mu\big(C\big(T(S)\big)\big).\end{aligned} \tag{6}$$

In the above estimate, we utilized the facts that the set $B(S)$ is bounded (since $B(S)$ is relatively weakly compact) and the set $T(S)$ is also bounded (since $T(S) \subset S$). Moreover, keeping in mind the assumption that the operator A is D-Lipschitzian, we deduce that the set $A(T(S))$ is bounded. Further, in view of assumption (i), the operator C is D-Lipschizian and satisfies condition (H2). Hence, in view of Lemma 1, we derive the estimate

$$\mu\big(C\big(T(S)\big)\big) \leqslant \phi_C\big(\mu\big(T(S)\big)\big). \tag{7}$$

Next, observe that according to assumption (iii), the set $B(S)$ is relatively weakly compact. This yields

$$\mu\big(B(S)\big) = 0. \tag{8}$$

Now, let us notice that taking into account the boundedness of the set $B(S)$, we can define the finite constant $L = \|B(S)\|$. Consequently, keeping in mind assumption (i) and Lemma 1, we obtain

$$\big\|B(S)\big\|\mu\big(A\big(T(S)\big)\big) \leqslant L\phi_A\big(\mu\big(T(S)\big)\big). \tag{9}$$

Further, combining (7)–(9) and taking into account estimate (6), we get

$$\mu\big(T(S)\big) \leqslant L\phi_A\big(\mu\big(T(S)\big)\big) + \phi_C\big(\mu\big(T(S)\big)\big).$$

According to hypothesis (v), the last equation becomes

$$\mu\big(T(S)\big) \leqslant L\phi_A\big(\mu\big(T(S)\big)\big) + \phi_C\big(\mu\big(T(S)\big)\big) < \mu\big(T(S)\big),$$

which is a contradiction, and therefore, $\mu(T(S)) = 0$.

Now the operator T satisfies the hypotheses of Lemma 3, so there exists a $x \in S$ such that $x = Tx$. Hence, we obtain

$$x = \left(\frac{I - C}{A}\right)^{-1} Bx$$

and, consequently,

$$\left(\frac{I - C}{A}\right)x = Bx.$$

From the above equality we get

$$x - Cx = Bx \cdot Ax.$$

Finally, we have

$$x = Ax \cdot Bx + Cx.$$

This completes the proof of our theorem. □

Remark 2. Consider the case where $A = 1_X$ in which 1_X represents the unit element of the RWC–Banach algebra X. We have the following particular version of Krasnoselskii's-type fixed point theorem.

Corollary 1. *Let X be a RWC–Banach algebra, and let S be a nonempty, bounded, closed, and convex subset of X. Consider two operators B and C such that $B : S \to X$ and $C : X \to X$, which satisfy the following conditions*:

(i) *The operator C is a nonlinear contraction with a contraction function ϕ_C and satisfies condition* (H2).

(ii) *The operator B is continuous on S and satisfies condition* (H1); *and the set $B(S)$ is relatively weakly compact.*

(iii) *For each $y \in S$, the following implication holds*:

$$x = By + Cx \quad \Longrightarrow \quad x \in S.$$

Then $B + C$ has at least one fixed point in S.

3 An example

In this section, we provide an illustrative example that highlights our theoretical results.

Consider the following system:

$$\begin{aligned} x_0 &= \sum_{k=1}^{\infty} \lambda_k x_k x_{k-1} + f(x_0), \\ x_k &= \frac{f(x_k)}{k+1}, \quad k \geqslant 1, \end{aligned} \tag{10}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is a non-null sequence such that $\lambda_n \to 0$, and $f : \mathbb{R} \to \mathbb{R}$ is α-Lipschitzian.

Suppose the following condition are satisfied:

(A1) There exists a $r_0 > 0$ with $r_0^2\|\lambda\|_\infty + (\alpha\pi/\sqrt{6})r_0 + |f(0)|\alpha\pi/\sqrt{6} \leqslant r_0$; here $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ and $\|\lambda\|_\infty = \sup_{n\in\mathbb{N}} |\lambda_n|$.

(A2) $\|\lambda\|_\infty + \alpha\pi/\sqrt{6} < 1$.

Now system (10) will be investigated in the RWC–Banach algebra l^2 endowed with the law defined in Remark 1.

Theorem 2. *Under assumptions* (A1) *and* (A2), *system* (10) *has at least one solution $x = (x_n)_{n\in\mathbb{N}}$ in the space l^2.*

Proof. Consider the subset S of l^2 defined by

$$S = \bigl\{x = (x_n)_{n\in\mathbb{N}}\colon \|x\|_{l^2} \leqslant r_0\bigr\}.$$

Now S is a nonempty, closed, convex, and bounded subset of l^2. Consider the three operators A, $\tilde{B}$, and C defined on l^2:

$$Ax = (0, x_0, x_1, \dots), \qquad \tilde{B}x = (\lambda_0 x_0, \lambda_1 x_1, \dots),$$
$$Cx = \left(f(x_0), \frac{f(x_1)}{2}, \frac{f(x_2)}{3}, \dots\right).$$

We also define the operator B, defined on S by $\tilde{B}_S$, the restriction of $\tilde{B}$ on the ball S.

It can be easily seen that system (10) is equivalent to the equation $x = Ax \cdot Bx + Cx$. To prove our statement, it is sufficient to show that the operators A, B, and C satisfy the hypothesis of Theorem 1.

First, it is clear that the operators A, B, and C are well defined from l^2 to l^2. Now, the linear operator $\tilde{B}$ is compact since $\tilde{B}$ is the limit of the finite rank operator $\tilde{B}_N$, where $\tilde{B}_N(x) = (\lambda_0 x_0, \lambda_1 x_1, \dots, \lambda_N x_N, 0, \dots)$. Then condition (iii) is satisfied. Next, we will consider the operator A. Clearly, A is the right-shift bounded linear operator with $\|A\| = 1$. Also, A is 1-Lipschitzian and satisfies (H2). Indeed, since A is bounded, so by the closed graph theorem A is weakly sequentially continuous, and so it satisfies (H2).

Now for $x, y \in l^2$, we have

$$\begin{aligned}\|Cx - Cy\|_{l^2}^2 &= \sum_{k=0}^{\infty} \frac{|f(x_k) - f(y_k)|^2}{(k+1)^2} \leqslant \sum_{k=0}^{\infty} \frac{\alpha^2 |x_k - y_k|^2}{(k+1)^2} \\ &\leqslant \sum_{k=0}^{\infty} \frac{\alpha^2 \|x - y\|_2^2}{(k+1)^2} \leqslant \frac{\alpha^2 \pi^2}{6} \|x - y\|_{l^2}^2.\end{aligned}$$

Hence, C is $\alpha\pi/\sqrt{6}$-Lipschitzian. It follows that C takes bounded sets of l^2 into bounded sets of l^2, and since l^2 is reflexive, we deduce that C satisfies (H2).

Take x, y, and z in S, then using hypothesis (A1), we have

$$\|Ax \cdot By + Cz\|_{l^2} \leqslant r_0^2 \|\lambda\|_\infty + \frac{\alpha\pi}{\sqrt{6}} r_0 + \left|f(0)\right| \frac{\alpha\pi}{\sqrt{6}} \leqslant r_0.$$

Finally, $\|B(S)\| = \|\lambda\|_\infty$, and using hypothesis (A2), condition (v) of Theorem 1 is satisfied.

Now, applying Theorem 1, we deduce that system (10) has at least one solution $x = (x_n)_{n \in \mathbb{N}}$ in the space l^2. □

4 Conclusion

In this work, we have established a new fixed point theorem for the sum and the product of nonlinear continuous operators acting on RWC–Banach algebras, which are not WC-algebras compact. Now l^2, endowed with the norm and with the multiplication defined in Remark 1, is an example of such an RWC–Banach algebra, and we have applied our theoretical result there.

It is easy to prove that if X is a WC–Banach algebra, then the linear space $C(K, X)$ of all continuous functions acting from a compact Hausdorff space K into X is also a WC–Banach algebra. Hence, we ask: If X is an RWC–Banach algebra, does the space $C(K, X)$ remains an RWC–Banach algebra?

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