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# Distributed constraint optimization for discrete-time multiagent systems with event-triggered communication*

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**Abstract.** This paper investigates the distributed optimization problem (DOP) with equality constraint in discrete-time multiagent systems (MASs) in which the global optimization objective is constituted by the summation of local objective functions. Firstly, by employing the Lagrange multiplier method, we convert the convex optimization problem with equality constraint into a consensus problem of MASs. Secondly, to reduce the communication burden, a type of event-triggered control protocol is proposed to enable all agents achieving consensus. Thirdly, by employing the Lyapunov function method and a set of inequality techniques, we establish some sufficient conditions to ensure that all agents converge to consensus and successfully solve the original DOP. Finally, a numerical simulation example is presented to validate the effectiveness of the theoretical analysis.

**Keywords:** consensus, distributed optimization problem, discrete-time, event-triggered control, multiagent systems.

## 1 Introduction

In recent years, the cooperative control problems of MASs have garnered significant attention from researchers. Distributed consensus [17, 21, 26, 33, 35, 37], as a fundamental problem of cooperative control, aims to design appropriate distributed control protocols to ensure that all agents eventually converge to a common decision value. As an application of distributed consensus, distributed optimization is widely used in

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various practical applications such as resource allocation [30], economic dispatch [13, 18], shortest distance optimization [19], and target tracking [12]. Distributed optimization of MASs is to minimize the global optimization objective, which is composed of local objective functions, through the utilization of solely local computations and information exchanges.

In recent years, many excellent research results have been obtained for distributed optimization problems (DOPs) with different convergence rates. In [40], the distributed convex quadratic optimization problem of MASs with asymptotic convergence rate was studied. In [41], the finite-time DOP of MASs under directed network topology was considered, and a distributed optimization algorithm was proposed. In [36], the authors used the directed network construction method to analyze the fixed-time DOP of MASs on the directed network, and two event-triggering algorithms were designed using the symbolic function and the saturation function, respectively. In [4], the prescribed-time DOP was investigated by using the event-triggered sampling control and the prescribed-time stability theory. In addition, the DOP of MASs with second-order dynamics was considered in [24]. It is worth noting that the above works [4, 24, 36, 40, 41] mainly solved the DOPs by establishing continuous-time models and algorithms.

However, in order to better adapt the needs of engineering practice, the research and development of digital control system is more and more in-depth. The discrete system theory is the basic theory of digital control system. Therefore, the study of cooperative control of discrete-time MASs has more practical application value. To guarantee the asymptotic consensus among agents operating under an asynchronous network structure, a design methodology rooted in Lyapunov function theory was introduced in [23]. Utilizing event-triggered and self-triggered control mechanisms [8], some sufficient conditions were given for ensuring consensus in both centralized and distributed discrete-time event-triggered protocols. As an application of consensus, a distributed subgradient optimization algorithm was proposed in [7] to solve the first-order unconstrained DOP in the undirected network topology. The DOP of MASs with time-varying network topologies was investigated in [20]. One can find that all these research results mentioned above [7, 20] considered unconstrained DOPs. To better simulate some conditional constraints in practical applications and save energy costs, this paper will consider the constrained DOP of discrete-time MASs, which is one of the research motivations of this paper.

With the increasing complexity of practical problems, constrained DOP has gradually become the focus of researchers. The existing constraints mainly include equality constraints [25, 27], inequality constraints [10, 16], and closed convex set constraints [15, 22, 42]. For a class of resource allocation problems, two new distributed discrete-time nonlinear algorithms were proposed in [27]. In addition, the global and local DOPs under time-varying and weighted unbalanced digraph were studied in [10] and [16], respectively. In particular, the work in [10] put forward a new push-sum dual gradient algorithm to study distributed model predictive control for linear discrete-time system networks. In addition, a distributed optimization algorithm based on gradient tracking and projection was proposed in [22]. Based on the idea of consensus, a distributed optimization algorithm based on projection was developed in [42] in which it does not require interaction between gradient information of the local objective functions of the agents. Compared with the above results, since the classical projection strategy is no longer suitable for dealing with optimization problems with closed convex set constraints under the gradient tracking framework, a new indirect projection method was designed in [15], and it is proved that the algorithm can achieve linear convergence rate when introducing a fixed step size.

While solving the DOP in MASs, it is crucial to not only determine the optimal value but also minimize the communication load to conserve resources. To accomplish this, advancements have been made in controller design, aimed at enhancing both optimization and communication efficiency. In fact, in traditional time-triggered control, each agent regularly communicates with its neighbors to obtain the necessary information. Although this control method is easy to design and analyze, it can also lead to overuse of limited resources in large networks. Surprisingly, the event-triggered control strategy has been greatly improved. As a result, event-triggered control protocols may be more economical than traditional time-triggered protocols. The work in [38] investigated the discrete-time bumpless transfer control problem with a dynamic event-triggered control mechanism in a switching topology. In [11], a discrete-time distributed optimization algorithm with event-triggered communication mechanism was designed to solve the economic scheduling problem. In [6], the DOP of discrete-time MASs was studied through the event-triggered interaction scheme, and the proposed optimization algorithm provided a more relaxed step size selection. To our best knowledge, the DOP of discrete-time MASs with weighted equality constraint has not been studied by using event-triggered control. This is another motivation of this paper.

Inspired by the discussion, this paper studies the DOP of discrete-time MASs with equality constraint in which the equality constraint is represented as the weighted sum of all agent states. The main contributions of this paper are summarized as follows:

1. In the existing research results [3, 10, 14, 31], most of the constrained DOPs in MASs are based on continuous-time models. In many practical applications, although the system itself is a continuous process, the controller can only apply the sampled data obtained in discrete-time since the limited channel bandwidth in the communication system. Therefore, this paper considers the equality constraint DOP under the discrete-time model.
2. Compared with the result of the classical constrained optimization problem in [40], this paper considers the DOP of discrete-time MASs with weighted equality constraint. Especially, if all weight factors in the equality constraint are the same, it can degenerate into the optimization problem in [40]. Therefore, the constraint condition considered in this paper is more extensive. In addition, the Lagrange multiplier approach is used to transform the convex DOP with equality constraints into a consensus problem.
3. In order to reduce the communication load, a control strategy with event-triggered communication mechanism is proposed, which only depends on the states of the agents and neighbors on the event-triggered sequences.

The structure of the paper is as follows. In Section 2, we introduce some preliminaries about graph theory and the problem formulation. A control protocol with event-triggered communication mechanism is proposed and the convergence property for the proposed algorithm is analyzed in Section 3. A numerical simulation is presented in Section 4. Section 5 summarizes the main results of this paper.

**Notations.** In this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ represent the $n$-dimensional real space and the $n \times n$ dimensional set of real matrices, respectively. $I_n$ represents the $n$-dimensional identity matrix. For a vector $x \in \mathbb{R}^n$, $x^{\mathrm{T}}$ is the transpose of $x$. $\mathbb{N}$ denotes the set of natural numbers.

## 2 Preliminaries

### 2.1 Algebraic graph theory

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, \dots, v_n\}$ indicates the set of nodes, and $\mathcal{E}$ represents the edges in which $(v_i, v_j) \in \mathcal{E}$ if there exists an edge between node $v_i$ and node $v_j$. $\mathcal{N}_i = \{v_j \in \mathcal{V}\colon (v_j, v_i) \in \mathcal{E}\}$ denotes the neighbors set of node $v_i$. The adjacency matrix is depicted by $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ in which $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The degree of node $v_i$ is given by $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. The Laplacian matrix $L = [l_{ij}]_{N\times N}$ is given by $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$. It is noted that $\mathcal{A}$ and $L$ are symmetric matrices, and $L$ satisfies $L\mathbf{1}_N = \mathbf{0}_N$. A path from node $v_i$ to $v_j$ is a sequence of edges of form $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \dots,$ $(v_{i_k}, v_j)$ in the graph $\mathcal{G}$ with distinct node $v_{i_k} \in \mathcal{V}$. The graph $\mathcal{G}$ is called connected if there exists a path between any pair of distinct nodes.

### 2.2 Distributed constraint optimization problem

Consider a MAS with $n$ agents, and the main target is solve the following DOP with equality constraint:

$$\min f(x) = \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{n} \psi_i x_i = X_D, \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of agent $i$, and $f(x)$ is the global objective function. $f_i(x_i) = \alpha_i x_i^2 + \beta_i x_i + \varphi_i$ is the local objective function of agent $i$ in which $\alpha_i > 0$, $\beta_i$ and $\varphi_i$ are parameters. $\psi_i > 0$ is the weight parameter, and $X_D$ is a constant.

**Assumption 1.** The communication topology $\mathcal{G}$ among agents is an undirected and connected graph.

**Remark 1.** The constrained DOP, characterized by the state of each agent being confined by specific constraints, is commonly encountered in fields such as resource allocation [5, 32] and economic scheduling problems [34]. In this kind of DOP, the state of each agent is restricted by a certain constraint. Consequently, this paper focuses on the DOP that incorporates weighted state constraints.

In order to solve the above optimization problem (1), we give the Lagrange function

$$L(x_i, \eta^*) = \sum_{i=1}^{n} f_i(x_i) + \eta^* \left( X_D - \sum_{i=1}^{n} \psi_i x_i \right).$$

Based on the Lagrange multiplier method in [28], the corresponding optimal solution satisfies the following equation:

$$\frac{\partial f_i(x_i)}{\partial x_i} - \psi_i \eta^* = 0. \tag{2}$$

According to (1) and (2),

$$\frac{2\alpha_1 x_1 + \beta_1}{\psi_1} = \frac{2\alpha_2 x_2 + \beta_2}{\psi_2} = \cdots = \frac{2\alpha_n x_n + \beta_n}{\psi_n} = \eta^*. \tag{3}$$

Combining (1) and (3),

$$\eta^* = \frac{X_D + \sum_{i=1}^n \frac{\psi_i \beta_i}{2\alpha_i}}{\sum_{i=1}^n \frac{\psi_i^2}{2\alpha_i}}.$$

The corresponding optimal solution is $x_i^* = (\eta^* \psi_i - \beta_i)/(2\alpha_i)$.

Define $\Im_i(t) = 2\alpha_i x_i(t) + \beta_i/\psi_i$, then formula (3) can be rewritten as $\Im_1(t) = \Im_2(t) = \cdots = \Im_n(t) = \eta^*$. Observing the above formula, it is found that the DOP (1) transforms into a consensus problem. That is, the constrained DOP (1) can be solved when $\Im_1(t), \Im_2(t), \ldots, \Im_n(t)$ converge to $\eta^*$.

## 3 Main results

According to the above statement, we propose a distributed algorithm to achieve consensus of the intermediate variable $\Im_i(t)$ in this section.

Consider the following discrete-time dynamics:

$$\begin{aligned} &\Im_i(t+1) = \Im_i(t) + u_i(t), \\ &x_i(t) = \frac{\psi_i \Im_i(t) - \beta_i}{2\alpha_i}, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{4}$$

where $\Im_i(0) = (2\alpha_i x_i(0) + \beta_i)/\psi_i$, $x_i(0)$ is the initial value of $x_i(t)$ and satisfies $\sum_{i=1}^n \psi_i x_i(0) = X_D$.

**Remark 2.** Most of the existing research results are given based on continuous-time model. However, the study of the DOP of discrete-time MASs is of practical importance. On the one hand, the digital control systems are widely used in engineering practice, and discrete system theory is the cornerstone of these systems. On the other hand, in numerous practical applications, while the system may operate as a continuous process, the controller can only utilize sampled data obtained in discrete-time due to the constrained bandwidth of the communication system's channel.

On the basis of achieving consensus, to maximize the service life of the network, this paper proposes an event-triggered control method to reduce the communication between the agents and the neighbors. In this part, $t_0^i, t_1^i, \ldots, t_k^i, \ldots$ refer to the event-triggered time series of $i$th agent, $t_k^i \in \mathbb{N}$. At the triggering instant $t_k^i$, agent $i$ stores its trigger state $\Im_i(t_k^i)$, and at the same time, passes it to its neighbors. When $t \in [t_k^i, t_{k+1}^i)$, agent $j$

can send out its latest sampling state $\Im_j(t^j_{k'})$ to agent $i$, where $t^j_{k'}$ represents the latest sampling instant of the agent $j$ before $t$. $\hat{\Im}_i(t)$ denotes the last broadcast state of agent $i$ at time step $t$, which can be described in the following form: $\hat{\Im}_i(t) = \Im_i(t^i_k)$, $t \in [t^i_k, t^i_{k+1})$. Specifically, the distributed control input of agent $i$ is given as follows:

$$u_i(t) = -k_i \sum_{j\in N_i} a_{ij}\big(\hat{\Im}_i(t) - \hat{\Im}_j(t)\big), \tag{5}$$

where $k_i > 0$ is the constant control parameter.

For the agent $i$, the next event-triggered instant $t^i_{k+1}$ is described by

$$t^i_{k+1} = \inf\big\{t > t^i_k\colon\ e_i^2(t) > r_i \hat{q}_i(t) w_i^2(t)\big\}, \tag{6}$$

where $r_i$ is a positive parameter, and $e_i(t) = \hat{\Im}_i(t) - \Im_i(t)$ is the measurement error. $w_i(t)$ and $\hat{q}_i(t)$ are represented by

$$w_i(t) = \sum_{j\in N_i} a_{ij}\big(\hat{\Im}_i(t) - \hat{\Im}_j(t)\big)$$

and

$$\hat{q}_i(t) = \min\biggl\{\sum_{j\in N_i} a_{ij}\big\|\hat{\Im}_i(t) - \hat{\Im}_j(t)\big\|^2,\ M\biggr\},$$

where $M$ is a positive constant.

**Theorem 1.** *For the MAS* (4) *with event-triggered control protocol* (5)–(6)*, if Assumption* 1 *holds and the control parameters satisfy the following condition*:

$$\bar{k}\lambda_n + \bar{r}M\lambda_n^2 < 1, \tag{7}$$

*where* $\bar{k} = \max\{k_i\}$*,* $\bar{r} = \max\{r_i\}$*,* $\lambda_n = \max\{\lambda_i(L)\}$*, then all variables* $\Im_i(t)$ $(i = 1, 2, \dots, n)$ *converge to the same value* $\sum_{i=1}^n(\Im_i(0)/k_i)/\sum_{i=1}^n(1/k_i)$.

*Proof.* Firstly, let $\Im^* = \sum_{i=1}^n(\Im_i(0)/k_i)/\sum_{i=1}^n(1/k_i)$, and let $\theta_i(t) = \Im_i(t) - \Im^*$. Choose the Lyapunov function as $V(t) = \sum_{i=1}^n \theta_i^2(t)/k_i$. Based on (4) and (5), we have

$$\begin{aligned}
&V(t+1)\\
&\quad= \sum_{i=1}^n \frac{1}{k_i}\theta_i^2(t+1) = \sum_{i=1}^n \frac{1}{k_i}\biggl(\Im_i(t) - k_i\sum_{j\in N_i} a_{ij}\big(\hat{\Im}_i(t) - \hat{\Im}_j(t)\big) - \Im^*\biggr)^2\\
&\quad= \sum_{i=1}^n \frac{1}{k_i}\biggl(\Im_i^2(t) - k_i\Im_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\Im}_i(t) - \hat{\Im}_j(t)\big) - \Im^*\Im_i(t)\\
&\qquad - k_i\Im_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\Im}_i(t) - \hat{\Im}_j(t)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ k_i^2\Big(\mathfrak{I}_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big)^2 + \mathfrak{I}^* k_i \sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\\
&- \mathfrak{I}^*\mathfrak{I}_i + \mathfrak{I}^* k_i \sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big) + \mathfrak{I}^{*2}\Big)\\
&= \sum_{i=1}^n \frac{1}{k_i}\Big(\mathfrak{I}_i^2(t) - 2k_i\mathfrak{I}_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big) - 2\mathfrak{I}^*\mathfrak{I}_i(t) + \mathfrak{I}^{*2}\\
&+ k_i^2\Big(\mathfrak{I}_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big)^2 + 2\mathfrak{I}^* k_i \sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big). \qquad (8)
\end{aligned}
$$

Moreover, one has

$$
V(t) = \sum_{i=1}^n \frac{1}{k_i}\theta_i^2(t) = \sum_{i=1}^n \frac{1}{k_i}\big(\mathfrak{I}_i^2(t) - 2\mathfrak{I}^*\mathfrak{I}_i(t) + (\mathfrak{I}^*)^2\big). \qquad (9)
$$

According to (8) and (9), one obtains

$$
\begin{aligned}
&V(t+1) - V(t)\\
&= \sum_{i=1}^n \frac{1}{k_i}\Big(-2k_i\mathfrak{I}_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big) + k_i^2\Big(\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big)^2\\
&+ 2\mathfrak{I}^* k_i \sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big)\\
&= -2\sum_{i=1}^n \mathfrak{I}_i(t)\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big) + \sum_{i=1}^n k_i\Big(\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big)^2\\
&+ 2\mathfrak{I}^* \sum_{i=1}^n\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big).
\end{aligned}
$$

Using Assumption 1, one can obtain that $\sum_{i=1}^n\sum_{j\in N_i} a_{ij}(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)) = 0$.

Let $\bar{k} = \max\{k_i\}$, it yields

$$
\begin{aligned}
&\sum_{i=1}^n k_i\Big(\sum_{j\in N_i} a_{ij}\big(\hat{\mathfrak{I}}_i(t)-\hat{\mathfrak{I}}_j(t)\big)\Big)^2\\
&\qquad \leqslant \bar{k}\big(L\hat{\mathfrak{I}}(t)\big)^{\mathrm{T}}\big(L\hat{\mathfrak{I}}(t)\big) = \bar{k}\hat{\mathfrak{I}}^{\mathrm{T}}(t)L^{\mathrm{T}}L\hat{\mathfrak{I}}(t) \leqslant \bar{k}\lambda_n\hat{\mathfrak{I}}^{\mathrm{T}}(t)L\hat{\mathfrak{I}}(t),
\end{aligned}
$$

where $\lambda_n = \max\{\lambda_i(L)\}$.

Therefore, we have

$$
\begin{aligned}
V(t+1) - V(t) &\leqslant -2\mathfrak{I}^{\mathrm{T}}(t)L\hat{\mathfrak{I}}(t) + \bar{k}\lambda_n\hat{\mathfrak{I}}^{\mathrm{T}}L\hat{\mathfrak{I}}\\
&= -2\big(\hat{\mathfrak{I}}(t) - e(t)\big)^{\mathrm{T}}L\hat{\mathfrak{I}}(T) + \bar{k}\lambda_n\hat{\mathfrak{I}}^{\mathrm{T}}L\hat{\mathfrak{I}}\\
&= -2\hat{\mathfrak{I}}^{\mathrm{T}}(t)L\hat{\mathfrak{I}}(t) + 2e^{\mathrm{T}}(t)L\hat{\mathfrak{I}}(t) + \bar{k}\lambda_n\hat{\mathfrak{I}}^{\mathrm{T}}L\hat{\mathfrak{I}}. \qquad (10)
\end{aligned}
$$

Furthermore, for $a > 0$, one has

$$e^{\mathrm{T}}(t)L\hat{\Im}(t) \leqslant \frac{1}{2a}\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t) + \frac{a}{2}e^{\mathrm{T}}(t)Le(t). \tag{11}$$

Combining to (10) and (11) yields

$$\begin{aligned} V(t+1) - V(t) &\leqslant -2\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t) + \frac{1}{a}\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t) + ae^{\mathrm{T}}(t)Le(t) + \bar{k}\lambda_n\hat{\Im}^{\mathrm{T}}L\hat{\Im} \\ &= \left(-2 + \frac{1}{a} + \bar{k}\lambda_n\right)\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t) + ae^{\mathrm{T}}(t)Le(t) \\ &\leqslant \left(-2 + \frac{1}{a} + \bar{k}\lambda_n\right)\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t) + a\lambda_n e^{\mathrm{T}}(t)e(t). \end{aligned}$$

Based on triggering condition (6), we have

$$V(t+1) - V(t) \leqslant \left(-2 + \frac{1}{a} + \bar{k}\lambda_n\right)\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t) + aM\lambda_n\bar{r}w^{\mathrm{T}}(t)w(t). \tag{12}$$

Since $w_i(t) = \sum_{j\in N_i} a_{ij}(\hat{\Im}_i(t) - \hat{\Im}_j(t)) = \sum_{j=1}^{n} l_{ij}\hat{\Im}_j(t)$, it follows that $w(t) = L\hat{\Im}(t)$. Hence, we have

$$w^{\mathrm{T}}(t)w(t) = \hat{\Im}^{\mathrm{T}}(t)L^{\mathrm{T}}L\hat{\Im}(t). \tag{13}$$

Substituting (12) into (13), one has

$$V(t+1) - V(t) \leqslant \left(-2 + \frac{1}{a} + \bar{k}\lambda_n + aM\bar{r}\lambda_n^2\right)\hat{\Im}^{\mathrm{T}}(t)L\hat{\Im}(t).$$

Choosing $a = 1$ and using condition (7), one can get $V(t+1) - V(t) \leqslant 0$. In combination with $V(t) \geqslant 0$, one can conclude that $V(t) \to 0$ as $t \to \infty$. That means $\lim_{t\to\infty} \Im_i(t) = \Im'$ for $i = 1, 2, \dots, n$ in which $\Im'$ is an unknown constant.

When $\Im_i(t) = \Im'$, it follows that

$$\begin{aligned} \sum_{i=1}^{n}\frac{1}{k_i}\Im_i(t+1) &= \sum_{i=1}^{n}\frac{1}{k_i}\biggl(\Im_i(t) - k_i\sum_{j\in N_i} a_{ij}\bigl(\hat{\Im}_i(t) - \hat{\Im}_j(t)\bigr)\biggr) \\ &= \sum_{i=1}^{n}\frac{1}{k_i}\Im_i(t) - \sum_{i=1}^{n}\sum_{j\in N_i} a_{ij}\bigl(\hat{\Im}_i(t) - \hat{\Im}_j(t)\bigr) \\ &= \sum_{i=1}^{n}\frac{1}{k_i}\Im_i(t). \end{aligned}$$

It indicates that $\sum_{i=1}^{n}\Im_i(t)/k_1$ a is constant, thus

$$\sum_{i=1}^{n}\frac{1}{k_i}\Im_i(t) = \sum_{i=1}^{n}\frac{1}{k_i}\Im_i(0). \tag{14}$$

Based on the above analysis, one has

$$\lim_{t\to\infty}\sum_{i=1}^{n}\frac{1}{k_i}\mathfrak{I}_i(t)=\mathfrak{I}'\left(\sum_{i=1}^{n}\frac{1}{k_i}\right)=\sum_{i=1}^{n}\frac{1}{k_i}\mathfrak{I}_i(0).$$

Therefore, we further obtain

$$\lim_{t\to\infty}\mathfrak{I}_i(t)=\mathfrak{I}'=\mathfrak{I}^*=\frac{\sum_{i=1}^{n}\frac{1}{k_i}\mathfrak{I}_i(0)}{\sum_{i=1}^{n}\frac{1}{k_i}}. \tag{15}$$

The proof is completed. □

**Remark 3.** Based on Theorem 1, it is found that when the corresponding parameters satisfy certain conditions, the intermediate variable will achieve consensus and converge to an expression related to the initial variable. We can adjust the initial value so that the state variable eventually converges to a certain expected value. In addition, a type of event-triggered control protocol with dynamic triggering mechanism is proposed in which a minimum function $\hat{q}_i(t)$ is skillfully used so that the frequency of event trigger can be dynamically adjusted.

**Remark 4.** In existing works [29] and [1], the reliable memory and nonfragility were considered in the sampled-data control, and some feasible event-triggered control protocols were proposed to achieve consensus of MASs. In [2], an observer-based control strategy under directed graphs was proposed. Inspired by the above references, we will consider nonfragility and directed network topology in the design of event-triggered optimization control protocols in the future.

**Theorem 2.** *Assuming that all conditions of Theorem* 1 *are satisfied and* $k_i=2\alpha_i/\psi_i^2$ *in the event-triggered control protocol* (5)*, the optimal solution of constrained DOP* (1) *can be obtained under the MAS* (4)*.*

*Proof.* Based on (4) and (15), we have

$$\lim_{t\to\infty}\mathfrak{I}_i(t)=\frac{\sum_{i=1}^{n}\frac{1}{k_i}\left(\frac{2\alpha_i x_i(0)+\beta_i}{\psi_i}\right)}{\sum_{i=1}^{n}\frac{1}{k_i}}.$$

Let $k_i=2\alpha_i/\psi_i^2$, one has

$$\lim_{t\to\infty}\mathfrak{I}_i(t)=\frac{\sum_{i=1}^{n}\psi_i x_i(0)+\sum_{i=1}^{n}\frac{\psi_i\beta_i}{2\alpha_i}}{\sum_{i=1}^{n}\frac{\psi_i^2}{2\alpha_i}}.$$

Due to $\sum_{i=1}^{n}\psi_i x_i(0)=X_D$, one obtains

$$\lim_{t\to\infty}\mathfrak{I}_i(t)=\frac{X_D+\sum_{i=1}^{n}\frac{\psi_i\beta_i}{2\alpha_i}}{\sum_{i=1}^{n}\frac{\psi_i^2}{2\alpha_i}}=\eta^*. \tag{16}$$

Furthermore, it yields

$$X_D = \eta^* \sum_{i=1}^{n} \frac{\psi_i^2}{2\alpha_i} - \sum_{i=1}^{n} \frac{\psi_i \beta_i}{2\alpha_i} = \sum_{i=1}^{n} \frac{\psi_i^2 \eta^* - \psi_i \beta_i}{2\alpha_i} = \lim_{t\to\infty} \sum_{i=1}^{n} \psi_i x_i(t).$$

Based on (14), one has

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{k_i} \Im_i(t) &= \sum_{i=1}^{n} \frac{1}{k_i} \frac{2\alpha_i x_i(t) + \beta_i}{\psi_i} = \sum_{i=1}^{n} \frac{1}{k_i} \frac{2\alpha_i x_i(0) + \beta_i}{\psi_i} * \\ &= \sum_{i=1}^{n} \frac{1}{k_i} \Im_i(0).\end{aligned}$$

Thus $\sum_{i=1}^{n} \psi_i x_i(t) = \sum_{i=1}^{n} \psi_i x_i(0)$ for $t \geqslant 0$. Furthermore, we have $\sum_{i=1}^{n} \psi_i x_i(t) = X_D$ for $t \geqslant 0$. According to (4) and (16), one can get

$$x_i^* = \frac{\eta^* \psi_i - \beta_i}{2\alpha_i} = \frac{1}{2\alpha_i} \left( \frac{\psi_i (X_D + \sum_{i=1}^{n} \frac{\psi_i \beta_i}{2\alpha_i})}{\sum_{i=1}^{n} \frac{\psi_i^2}{2\alpha_i}} - \beta_i \right).$$

The proof is completed. □

In the DOP (1), when $\psi_i = 1$ for $i = 1, 2, \dots, n$, then it can degrade into

$$\min f(x) = \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = X_D. \tag{17}$$

**Corollary 1.** *For the MAS* (4) *with event-triggered control protocol* (5)–(6)*, if Assumption* 1 *holds and the control parameters satisfy the following condition*

$$\bar{k}\lambda_n + \bar{r} M \lambda_n^2 < 1,$$

*where* $\bar{k} = \max\{k_i\}$, $\bar{r} = \max\{r_i\}$, $\lambda_n = \max\{\lambda_i(L)\}$, *then all variables* $\Im_i(t)$ $(i = 1, 2, \dots, n)$ *converge to the same value* $\sum_{i=1}^{n} (\Im_i(0)/k_i) \sum_{i=1}^{n} (1/k_i)$.

**Corollary 2.** *Assuming that all conditions of Corollary* 1 *are satisfied and* $k_i = 2\alpha_i$ *in the event-triggered control protocol* (5)*, the constrained DOP* (17) *can be solved under the MAS* (4)*. Furthermore, the optimal solution is*

$$x_i^* = \frac{1}{2\alpha_i} \left( \frac{X_D + \sum_{i=1}^{n} \frac{\beta_i}{2\alpha_i}}{\sum_{i=1}^{n} \frac{1}{2\alpha_i} - \beta_i} \right).$$

**Remark 5.** In this paper, we propose an event-triggered control protocol to solve the DOP of MASs. The optimal solution can only be asymptotically reachable. How to design control protocols with fast convergence rates is a challenging problem. In [9, 39], some fixed-time and prescribed-time control protocols were proposed to achieve consensus or synchronization of complex networks. These works provide us with valuable inspiration, and we will attempt to develop fixed-time or prescribed-time optimization protocols to solve constrained DOPs in the future work.

## 4 Simulation example

In this section, a numerical example is given to demonstrate the effectiveness of the proposed algorithm and the feasibility of theoretical analysis.

Consider the DOP (1) with $n = 6$ in which $\psi_1 = 0.5$, $\psi_2 = 0.6$, $\psi_3 = 0.8$, $\psi_4 = 0.4$, $\psi_5 = 0.7$, $\psi_6 = 0.6$, and $X_D = 1.06$. In the local objective function $f_i(x_i) = \alpha_i x_i^2 + \beta_i x_i + \varphi_i$, the corresponding coefficients are $\alpha_1 = 0.05$, $\alpha_2 = 0.036$, $\alpha_1 = 0.096$, $\alpha_4 = 0.04$, $\alpha_5 = 0.1225$, $\alpha_6 = 0.054$, $\beta_1 = 0.05$, $\beta_2 = 0.02$, $\beta_3 = 0.04$, $\beta_4 = 0.06$, $\beta_5 = 0.3$, $\beta_6 = 0.08$, $\varphi_1 = 0.02$, $\varphi_2 = 0.03$, $\varphi_3 = 0.05$, $\varphi_4 = 0.09$, $\varphi_5 = 0.1$, and $\varphi_6 = 0.05$.

In the MAS (4), the communication topology $\mathcal{G}$ is described by Fig. 1 in which all weights are $0.1$. It can be verified that Assumption 1 holds. In the control protocol (5) with event-triggering condition (6), we choose $k_1 = 0.4$, $k_2 = 0.2$, $k_3 = 0.4$, $k_4 = 0.5$, $k_5 = 0.5$,$k_6 = 0.4$, $M = 3$, $r_1 = 0.1$, $r_2 = 0.2$, $r_3 = 0.5$, $r_4 = 0.4$, $r_5 = 0.3$, and $r_6 = 0.6$.

By calculation, one can find that all conditions of Theorems 1 and 2 are satisfied. We choose $\Im_1(0) = 16$, $\Im_2(0) = 10$, $\Im_3(0) = 8$, $\Im_4(0) = 15$, $\Im_5(0) = 18$, and $\Im_6(0) = 12$. According to (4) and (5), the trajectories of $\Im_i(t)$ and control inputs $u_i(t)$ for $i = 1, 2, \dots, 6$ are shown in Figs. 2 and 3, respectively. It can be seen that all intermediate variables $\Im_i(t)$ ultimately achieve consensus, and the distributed control inputs $u_i(t)$ gradually approach zero.

Under the proposed protocol (5), the event-triggering instants for each agent are presented in Fig. 4. Through observation and analysis, it is concluded that our proposed event-triggered control protocol can indeed reduce the communication between agents and the update frequency of the controller. The evolution trajectories of $x_i(t)$ are shown in Fig. 5. The evolution trajectories of weighted equality constraint $\sum_{i=1}^{6} \psi_i x_i(t) = X_D$ and global objective function $f(x)$ are shown in Figs. 6 and 7, respectively. It is found that the weighted equality constraint is always satisfied in the control process, and the global objective function $f(x)$ converges to optimal value is $0.6056$. Therefore, our proposed distributed event-triggered control protocol (5) for the discrete-time MASs (4) can effectively solve the weighted equality constrained DOP (1).

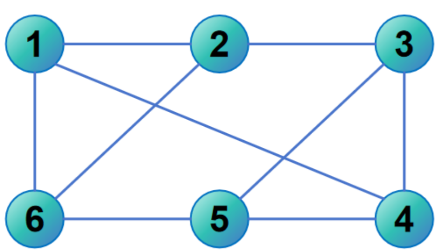


**Figure 1.** Communication topology $\mathcal{G}$.

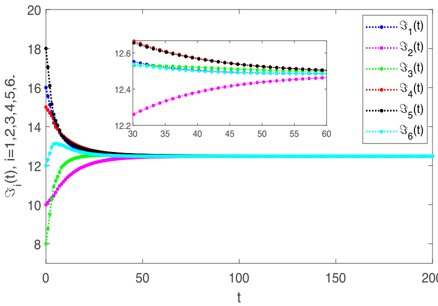


**Figure 2.** The evolution trajectories of $\Im_i(t)$, $i = 1, 2, \ldots, 6$.

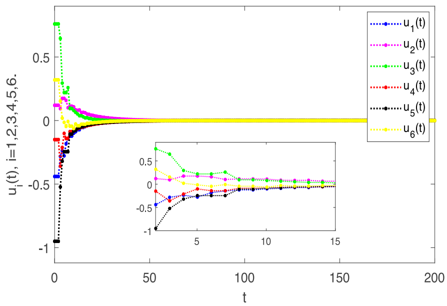


**Figure 3.** The evolution trajectories of control inputs $u_i(t)$.

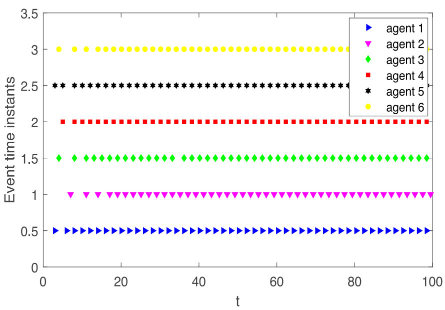


**Figure 4.** Event-triggering instants of all agents.

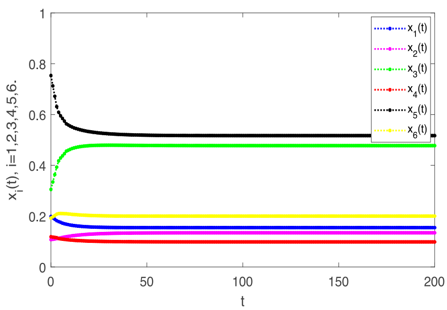


**Figure 5.** The evolution trajectories $x_i(t)$, $i = 1, 2, \ldots, 6$.

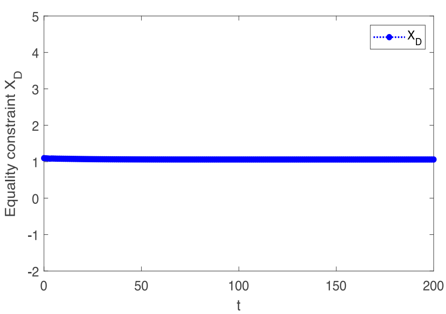


**Figure 6.** Evolutions of equality constraint $\sum_{i=1}^{6} \psi_i x_i(t) = X_D$.

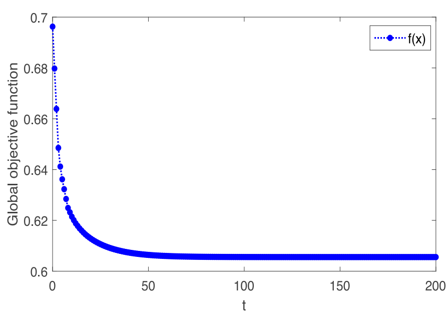


**Figure 7.** The evolution trajectory of $f(x)$.

## 5 Conclusion

This paper considered the weighted equality constrained DOP in discrete-time MASs. Firstly, the constrained DOP was transformed into a consensus problem by applying Lagrange multiplier method. The optimal solution of the constrained DOP can be obtained when all intermediate variables achieved consensus. Secondly, an distributed event-triggered control protocol was proposed, which only depends on states of the agent and its neighbors on the event triggering instants. Thirdly, by using Lyapunov function method and inequality techniques, some sufficient conditions for achieving consensus and solving the constrained DOP were obtained. Finally, a numerical simulation example was given to verify the effectiveness of the proposed protocol. In our future work, we will further improve the proposed control protocol from the perspective of the nonfragility of the sampled-data control and convergence rate, and consider more general constraint DOPs.

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