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Positive solutions for a Hadamard-type fractional-order three-point boundary value problem on the half-line*

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Abstract. In this paper, we study a Hadamard-type fractional-order three-point boundary value problem on the half-line. Under some growth conditions concerning the spectral radius of the relevant linear operator, the existence and multiplicity of positive solutions is obtained using a fixed-point method. Our results improve and generalize some results in the literature.

Keywords: Hadamard-type fractional-order differential equations, boundary value problems, positive solutions, fixed-point method.

1 Introduction

In this paper, we study the existence and multiplicity of positive solutions for the Hadamard-type fractional-order three-point boundary value problem on the half-line

$$\begin{aligned} &D_{1+}^{\sigma} z(t) = -\theta(t) f\bigl(t, z(t)\bigr), \quad t \in (1, +\infty), \\ &z(1) = \delta z(1) = 0, \qquad D_{1+}^{\sigma-1} z(+\infty) = b x(\xi), \end{aligned} \tag{1}$$

where D_{1+}^{σ} is the Hadamard fractional derivative of order $\sigma \in (2,3)$, $\delta = t(\mathrm{d}/\mathrm{d}t)$, and b, ξ, θ, f satisfy the following conditions.

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(H0) $b \in [0, +\infty)$, $\xi \in (1, +\infty)$ are constants with $b(\ln \xi)^{\sigma-1} \in [0, \Gamma(\sigma))$;
(H1) $f : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous;
(H2) $\theta(t) : [1, +\infty) \to \mathbb{R}^+$ does not identically vanish on any subinterval of $[1, +\infty)$ and

$$0 < \int_1^{+\infty} \theta(s) \frac{\mathrm{d}s}{s} < +\infty.$$

Fractional-order calculus is a classical research field, which provides a more appropriate description of some natural phenomena. For example, in [1], the authors studied a cancer treatment model given by the Hadamard-type fractional derivative

$$\begin{aligned}
{}^H D^r \rho(t) &= \alpha_1 \rho\left(1 - \frac{\rho}{S_1}\right) - \beta_1 \rho\alpha - \varepsilon D(t)\rho, \\
{}^H D^r \alpha(t) &= \alpha_2 \alpha\left(1 - \frac{\alpha}{S_2}\right) - \beta_2 \rho\alpha - D(t)\alpha, \\
\rho(0) &= \rho_0, \qquad \alpha(0) = \alpha_0,
\end{aligned}$$

where $\rho(t)$ represent the concentration of healthy cells, $\alpha(t)$ is the concentration of cancer cells, and $D(t)$ is the strategy of the radiotherapy.

Recently, fractional-order nonlinear differential equations on an unbounded domain have become an interesting area of research; see, for example, [3–8, 10–13, 16–21, 24–29] and the references therein. In [7] the authors studied the Hadamard-type fractional boundary value problem on the half-line

$$\begin{aligned}
&D_{1+}^{\vartheta} u(t) + p(t) f\bigl(t, u(t), D_{1+}^{\vartheta-1} u(t)\bigr) = 0, \quad n-1 < \vartheta \leqslant n,\ t \in (1, +\infty), \\
&u^{(k)}(1) = 0, \quad 0 \leqslant k \leqslant n-2, \\
&D_{1+}^{\vartheta-1} u(+\infty) = \int_1^{+\infty} g(t) u(t) \frac{\mathrm{d}t}{t} + \sum_{i=1}^{m} \lambda_i I_{1+}^{\beta_i} u(\varsigma),
\end{aligned}$$

and they obtained some multiplicity results for positive solutions via the Bai–Ge fixed point theorem. In [24] the authors used some fixed point theorems of a sum operator in partially ordered Banach spaces to study the local existence and uniqueness of positive solutions for the Hadamard-type fractional differential equation on the half-line

$$\begin{aligned}
&D_{1+}^{\alpha} z(t) + a(t) f\bigl(t, z(t)\bigr) + b(t) g\bigl(t, z(t)\bigr) = 0, \quad t \in (1, +\infty), \\
&z(1) = z'(1) = 0, \qquad D_{1+}^{\alpha-1} z(+\infty) = \sum_{i=1}^{m} \alpha_i I_{1+}^{\beta_i} z(\eta) + c \sum_{j=1}^{n} \sigma_j z(\xi_j).
\end{aligned}$$

The spectral theory of linear operators can be used to study differential equations; see [2, 15, 22, 23, 30]. In [30] the authors studied positive solutions for the fractional integral boundary value problem

$$D_{0+}^{\alpha} z(t) + h(t) f\big(t, z(t)\big) = 0, \quad 0 < t < 1,$$
$$z(0) = z'(0) = z''(0) = 0, \qquad z(1) = \lambda \int\limits_0^{\eta} z(s)\,\mathrm{d}s,$$

where $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies some superlinear and sublinear growth conditions regarding the spectral radius of the linear operator

$$(L_Z z)(t) = \int\limits_0^1 G_Z(t,s) h(s) z(s)\,\mathrm{d}s,$$

and G_Z is the Green's function.

To the authors' knowledge, due to the noncompactness of an infinite interval, there are very few research results on Hadamard-type fractional boundary value problems on the half-line, and even fewer results apply spectral theory methods. Motivated by the aforementioned works, in this paper, we study the existence and multiplicity of positive solutions for the Hadamard-type fractional equations (1) on the half-line. We first study a relevant linear operator and obtain its spectral radius, then we obtain the existence and multiplicity of positive solutions under some growth conditions regarding the spectral radius; see (H3)–(H6) is Section 3.

2 Preliminaries

In this section, we only recall the definition of the Hadamard-type fractional derivative; for more details about the Hadamard-type fractional calculus theory, we refer the reader to [3, 7, 19, 24, 26, 27, 29].

Definition 1. The σ-order Hadamard-type fractional derivative of a function $z : [1, +\infty) \to \mathbb{R}$ is

$$D_{1+}^{\sigma} z(t) = \frac{1}{\Gamma(n-\sigma)} \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int\limits_1^t \left(\ln \frac{t}{s}\right)^{n-\sigma-1} z(s)\,\frac{\mathrm{d}s}{s}, \quad \sigma > 0,\ t > 1,$$

where $n = [\sigma] + 1$, $[\sigma]$ is the integer part of σ.

In what follows, we calculate the Green's function associated with (1). For completeness, we provide the proof of the following lemma.

Lemma 1. (*See* [27]). *Let* $y : [1, +\infty) \to \mathbb{R}^+$ *with* $\int_1^{+\infty} y(s)\,(\mathrm{d}s/s) < +\infty$. *Then the boundary value problem*

$$D_{1+}^{\sigma} z(t) = -y(t), \quad t \in (1, +\infty),$$
$$z(1) = \delta z(1) = 0, \qquad D_{1+}^{\sigma-1} z(+\infty) = b z(\xi)$$

has a solution, which can be expressed as

$$z(t) = \int_1^{+\infty} G(t,s)y(s)\,\frac{\mathrm{d}s}{s},$$

where

$$G(t,s) = g(t,s) + \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}}\,g(\xi,s), \quad t,s \in [1,+\infty),$$

$$g(t,s) = \frac{1}{\Gamma(\sigma)}\begin{cases}(\ln t)^{\sigma-1} - (\ln t - \ln s)^{\sigma-1}, & 1 \leqslant s \leqslant t < +\infty,\\ (\ln t)^{\sigma-1}, & 1 \leqslant t \leqslant s < +\infty.\end{cases}$$

Proof. From [27, Lemma 3.1] we have

$$z(t) = c_1(\ln t)^{\sigma-1} + c_2(\ln t)^{\sigma-2} + c_3(\ln t)^{\sigma-3} - \frac{1}{\Gamma(\sigma)}\int_1^t (\ln t - \ln s)^{\sigma-1}y(s)\,\frac{\mathrm{d}s}{s},$$

where $c_i \in \mathbb{R}$, $i = 1,2,3$. From $z(1) = \delta z(1) = 0$ we find $c_2 = c_3 = 0$. Therefore,

$$z(t) = c_1(\ln t)^{\sigma-1} - \frac{1}{\Gamma(\sigma)}\int_1^t (\ln t - \ln s)^{\sigma-1}y(s)\,\frac{\mathrm{d}s}{s},$$

$$D_{1+}^{\sigma-1}z(t) = c_1\Gamma(\sigma) - \int_1^t y(s)\,\frac{\mathrm{d}s}{s}.$$

Using $D_{1+}^{\sigma-1}z(+\infty) = bx(\xi)$, we have

$$c_1\Gamma(\sigma) - \int_1^{+\infty} y(s)\,\frac{\mathrm{d}s}{s} = c_1 b(\ln\xi)^{\sigma-1} - \frac{b}{\Gamma(\sigma)}\int_1^{\xi}(\ln\xi - \ln s)^{\sigma-1}y(s)\,\frac{\mathrm{d}s}{s},$$

and from (H0) we find

$$c_1 = \frac{1}{\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}}\int_1^{+\infty} y(s)\,\frac{\mathrm{d}s}{s} - \frac{b}{\Gamma(\sigma)[\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}]}\int_1^{\xi}(\ln\xi - \ln s)^{\sigma-1}y(s)\,\frac{\mathrm{d}s}{s}.$$

Consequently, we obtain

$$
\begin{aligned}
z(t) &= \frac{(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} \int_1^{+\infty} y(s)\,\frac{\mathrm{d}s}{s} \\
&\quad - \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)[\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}]} \int_1^{\xi} (\ln\xi-\ln s)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(\sigma)} \int_1^{t} (\ln t-\ln s)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} \\
&= \frac{(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} \int_1^{+\infty} y(s)\,\frac{\mathrm{d}s}{s} \\
&\quad - \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)[\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}]} \int_1^{\xi} (\ln\xi-\ln s)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(\sigma)} \int_1^{t} (\ln t-\ln s)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\sigma)} \int_1^{+\infty} (\ln t)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} \\
&\quad - \frac{1}{\Gamma(\sigma)} \int_1^{+\infty} (\ln t)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} \\
&= \int_1^{+\infty} g(t,s) y(s)\,\frac{\mathrm{d}s}{s} + \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)[\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}]} \\
&\quad \times \left[\int_1^{+\infty} (\ln\xi)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} - \int_1^{\xi} (\ln\xi-\ln s)^{\sigma-1} y(s)\,\frac{\mathrm{d}s}{s} \right] \\
&= \int_1^{+\infty} g(t,s) y(s)\,\frac{\mathrm{d}s}{s} + \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} \int_1^{+\infty} g(\xi,s) y(s)\,\frac{\mathrm{d}s}{s} \\
&= \int_1^{+\infty} G(t,s) y(s)\,\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

This completes the proof. □

Lemma 2. (*See* [27]). *The functions* G *and* g *have the following properties*:

(i) $G, g(t,s)$ *is nonnegative and continuous for* $(t,s) \in [1,+\infty) \times [1,+\infty)$;

(ii) $g(t,s)$ *is increasing with respect to* t;

(iii) *For fixed $k > 1$, $g(t, s)$ satisfies*

$$\min_{t\in[e^{1/k}, e^k]} \frac{g(t,s)}{1+(\ln t)^{\sigma-1}} \geqslant \frac{1}{4k^2(1+k^{\sigma-1})} \sup_{t\in[1,+\infty)} \frac{g(t,s)}{1+(\ln t)^{\sigma-1}}.$$

Let $E = C([1, +\infty), \mathbb{R})$, and let

$$Z = \left\{ z \in E\colon \sup_{t\in[1,+\infty)} \frac{|z(t)|}{1+(\ln t)^{\sigma-1}} < +\infty \right\}$$

equipped with the norm

$$\|z\|_Z = \sup_{t\in[1,+\infty)} \frac{|z(t)|}{1+(\ln t)^{\sigma-1}}.$$

Note that $(Z, \|\cdot\|_Z)$ is a Banach space. In what follows, we use $\|\cdot\|$ to replace $\|\cdot\|_Z$. Let

$$P = \bigl\{ z \in Z\colon\ z(t) \geqslant 0,\ t \in [1, +\infty) \bigr\}.$$

Then P is a cone on Z. For convenience, let $B_\rho = \{z \in Z\colon \|z\| < \rho\}$ for $\rho > 0$, and note that B_ρ is an open ball.

From Lemma 1 we define an operator A as

$$(Az)(t) = \int_1^{+\infty} G(t,s)\theta(s) f\bigl(s, z(s)\bigr) \frac{ds}{s}, \quad z \in Z,\ t \in [1, +\infty).$$

Note that G is nonnegative and continuous on $[1, +\infty)\times[1, +\infty)$, and from (H0)–(H2) we obtain that A is a map from P to P, and if there exists $z^{**} \in P\backslash\{0\}$ such that $Az^{**} = z^{**}$, then this z^{**} is a positive solution of (1). Define a linear operator $L : P \to P$ as

$$(Lz)(t) = \int_1^{+\infty} G(t,s)\theta(s) \frac{z(s)}{1+(\ln s)^{\sigma-1}} \frac{ds}{s}, \quad z \in P,\ t \in [1, +\infty).$$

Now, we prove that its spectral radius, denoted by $r(L)$, is positive.

Lemma 3. $r(L) > 0$.

Proof. We first estimate the norm of $L^n = L(L^{n-1})$, $n = 1, 2, \dots, L^0 = I$. Choose $z_0(t) = 1 + (\ln t)^{\sigma-1}$, $t \in [1, +\infty)$ and then $\|z_0\| = 1$. Consequently, we have

$$\begin{aligned}
\|L\| &= \sup_{\|z\|=1} \|Lz\| \geqslant \|Lz_0\| = \sup_{t\in[1,+\infty)} \frac{(Lz_0)(t)}{1+(\ln t)^{\sigma-1}} \\
&\geqslant \frac{b}{\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}} \sup_{t\in[1,+\infty)} \frac{(\ln t)^{\sigma-1}}{1+(\ln t)^{\sigma-1}} \int_1^{+\infty} g(\xi, s)\theta(s) \frac{z_0(s)}{1+(\ln s)^{\sigma-1}} \frac{ds}{s} \\
&= \frac{b}{\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}} \int_1^{+\infty} g(\xi, s)\theta(s) \frac{ds}{s}
\end{aligned}$$

and

$$\begin{aligned}
\|L^2\| &= \sup_{\|z\|=1} \|L^2 z\| \geqslant \|L^2 z_0\| = \sup_{t\in[1,+\infty)} \frac{(L^2 z_0)(t)}{1+(\ln t)^{\sigma-1}} \\
&= \sup_{t\in[1,+\infty)} \frac{1}{1+(\ln t)^{\sigma-1}} \int_1^{+\infty}\int_1^{+\infty} \frac{G(t,s)\theta(s)}{1+(\ln s)^{\sigma-1}} \frac{G(s,\tau)\theta(\tau)}{1+(\ln \tau)^{\sigma-1}} z_0(\tau)\,\frac{\mathrm{d}\tau}{\tau}\,\frac{\mathrm{d}s}{s} \\
&\geqslant \left[\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}\right]^2 \sup_{t\in[1,+\infty)} \frac{(\ln t)^{\sigma-1}}{1+(\ln t)^{\sigma-1}} \\
&\quad\times \int_1^{+\infty}\int_1^{+\infty} \frac{(\ln s)^{\sigma-1} g(\xi,s)\theta(s)}{1+(\ln s)^{\sigma-1}} g(\xi,\tau)\theta(\tau)\,\frac{\mathrm{d}\tau}{\tau}\,\frac{\mathrm{d}s}{s} \\
&= \left[\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}\right]^2 \int_1^{+\infty} \frac{(\ln s)^{\sigma-1} g(\xi,s)\theta(s)}{1+(\ln s)^{\sigma-1}}\,\frac{\mathrm{d}s}{s} \int_1^{+\infty} g(\xi,\tau)\theta(\tau)\,\frac{\mathrm{d}\tau}{\tau}.
\end{aligned}$$

By the method of mathematical induction we obtain

$$\|L^n\| \geqslant \left[\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}\right]^n \left[\int_1^{+\infty} \frac{(\ln s)^{\sigma-1} g(\xi,s)\theta(s)}{1+(\ln s)^{\sigma-1}}\,\frac{\mathrm{d}s}{s}\right]^{n-1} \int_1^{+\infty} g(\xi,\tau)\theta(\tau)\,\frac{\mathrm{d}\tau}{\tau}.$$

Therefore, Gelfand's theorem implies that

$$r(L) = \liminf_{n\to+\infty} \sqrt[n]{\|L^n\|} \geqslant \frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} \int_1^{+\infty} \frac{(\ln s)^{\sigma-1} g(\xi,s)\theta(s)}{1+(\ln s)^{\sigma-1}}\,\frac{\mathrm{d}s}{s} > 0.$$

This completes the proof. □

From Lemma 3 and the Krein–Rutman theorem [14] we have that L has an eigenfunction $\varphi \in P\setminus\{0\}$ corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$, i.e.,

$$\varphi = \lambda_1 L\varphi. \tag{2}$$

Lemma 4. (*See* [9]). *Suppose that $\Omega \subset E$ is a bounded open set and $A : \overline{\Omega}\cap P \to P$ is a continuous and completely continuous (compact) operator. If there exists $z_0 \in P\setminus\{0\}$ such that $z - Az \neq \lambda z_0$ for all $\lambda \geqslant 0$, $z \in \partial\Omega\cap P$, then the fixed point index $i(A,\,\Omega\cap P,\,P) = 0$.*

Lemma 5. (*See* [9]). *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose that $A : \overline{\Omega}\cap P \to P$ is a continuous and completely continuous operator. If $z \neq \lambda Az$ for all $z \in \partial\Omega\cap P$, $0 \leqslant \lambda \leqslant 1$, then the fixed point index $i(A,\,\Omega\cap P,\,P) = 1$.*

3 Main results

Now, we state our main theorems and give their proofs.

Theorem 1. *Suppose that* (H0)–(H2) *and the following conditions hold*:

(H3) $\liminf_{z\to 0^+} f(t,z)/(z/(1+(\ln t)^{\sigma-1})) > \lambda_1$ *uniformly on* $t \in [1,+\infty)$;
(H4) $\limsup_{z\to +\infty} f(t,z)/(z/(1+(\ln t)^{\sigma-1})) < \lambda_1$ *uniformly on* $t \in [1,+\infty)$.

Then (1) *has at least one positive solution.*

Proof. From (H3) there is a $r_1 > 0$ such that

$$f(t,z) \geqslant \lambda_1 \frac{z}{1+(\ln t)^{\sigma-1}}, \quad z \in [0,r_1],\ t \in [1,+\infty). \tag{3}$$

For each $z \in \partial B_{r_1} \cap P$, by (3) we have

$$(Az)(t) \geqslant \int_1^{+\infty} G(t,s)\theta(s)\lambda_1 \frac{z(s)}{1+(\ln s)^{\sigma-1}} \frac{\mathrm{d}s}{s} := (L_1 z)(t), \quad t \in [1,+\infty).$$

Note that $r(L_1) = 1$, and from (2) there exists $z^* \in P \setminus \{0\}$ such that

$$L_1 z^* = r(L_1) z^* = z^*. \tag{4}$$

Now, we shall prove that

$$z - Az \neq \mu z^*, \quad z \in \partial B_{r_1} \cap P,\ \mu \geqslant 0. \tag{5}$$

Suppose the contrary. Then there exist $z_0 \in \partial B_{r_1} \cap P$, $\mu_0 \geqslant 0$ such that

$$z_0 - Az_0 = \mu_0 z^*.$$

We only need to consider $\mu_0 > 0$. (Note, $\mu_0 = 0$ implies that $z_0 = Az_0$, i.e., z_0 is a positive fixed point of A, and thus this z_0 is also a positive solution for (1)). Let

$$\mu^* = \sup\{\mu\colon\ z_0 \geqslant \mu z^*\}.$$

Then $\mu^* \geqslant \mu_0 > 0$, $z_0 \geqslant \mu^* z^*$, and from (4) we have

$$z_0 = Az_0 + \mu_0 z^* \geqslant L_1 z_0 + \mu_0 z^* \geqslant L_1 \mu^* z^* + \mu_0 z^* = \mu^* z^* + \mu_0 z^*.$$

This contradicts the definition of μ^*. Therefore, (5) holds, and Lemma 4 implies that

$$i(A,\ B_{r_1} \cap P,\ P) = 0. \tag{6}$$

From (H4) there exist $\varepsilon_1 \in (0,\lambda_1)$ and $c_1 > 0$ such that

$$f(t,z) \leqslant (\lambda_1 - \varepsilon_1) \frac{z}{1+(\ln t)^{\sigma-1}} + c_1, \quad z \geqslant 0,\ t \in [1,+\infty).$$

Then we have

$$(Az)(t) \leqslant \int_1^{+\infty} G(t,s)\theta(s)\left[(\lambda_1-\varepsilon_1)\frac{z(s)}{1+(\ln s)^{\sigma-1}}+c_1\right]\frac{\mathrm{d}s}{s}$$
$$= (\lambda_1-\varepsilon_1)(Lz)(t)+c_1\int_1^{+\infty} G(t,s)\theta(s)\,\frac{\mathrm{d}s}{s}.$$

Let

$$(L_2 z)(t) = (\lambda_1-\varepsilon_1)(Lz)(t) \quad \text{and} \quad \overline{z}(t) = c_1\int_1^{+\infty} G(t,s)\theta(s)\,\frac{\mathrm{d}s}{s}.$$

Then $r(L_2) = 1-\varepsilon_1/\lambda_1 < 1$, which implies that $(I-L_2)^{-1}$ exists and

$$(I-L_2)^{-1} = I + L_2 + L_2^2 + \cdots + L_2^n + \cdots .$$

Keeping in mind the definition of our norm, from (H2) we have

$$\|\overline{z}\| = \sup_{t\in[1,+\infty)} \frac{\overline{z}(t)}{1+(\ln t)^{\sigma-1}}$$
$$\leqslant \sup_{t\in[1,+\infty)} \frac{(\ln t)^{\sigma-1}}{1+(\ln t)^{\sigma-1}} c_1 \int_1^{+\infty}\left[\frac{1}{\Gamma(\sigma)}+\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)\right]\theta(s)\,\frac{\mathrm{d}s}{s}$$
$$= c_1\int_1^{+\infty}\left[\frac{1}{\Gamma(\sigma)}+\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)\right]\theta(s)\,\frac{\mathrm{d}s}{s} < +\infty.$$

Define a set

$$S = \{z\in P\colon\ Az = \mu z,\ \mu\geqslant 1\}.$$

Now, we claim that S is bounded in P. Indeed, if $z\in S$, we have

$$z(t) \leqslant (Az)(t) \leqslant (L_2 z)(t) + \overline{z}(t), \quad t\in[1,+\infty).$$

This implies that $(I-L_2)z \leqslant \overline{z}$. Note that $(I-L_2)^{-1}: P\to P$, and hence we have

$$\|z\| \leqslant \big\|(I-L_2)^{-1}\overline{z}\big\|.$$

As a result, S is bounded as required. Now, we can choose $R_1 > \sup S$ and $R_1 > r_1$ such that

$$Az \neq \mu z, \quad z\in\partial B_{R_1}\cap P,\ \mu\geqslant 1.$$

Therefore, Lemma 5 implies that

$$i(A,\ B_{R_1}\cap P,\ P) = 1. \tag{7}$$

Combining (6) and (7), we have

$$\begin{aligned} i\big(A,\,(B_{R_1}\setminus\overline{B_{r_1}})\cap P,\,P\big) &= i(A,\,B_{R_1}\cap P,\,P)-i(A,\,B_{r_1}\cap P,\,P)\\ &=1-0=1. \end{aligned}$$

Then A has a fixed point in $(B_{R_1}\setminus\overline{B_{r_1}})\cap P$, i.e., (1) has at least one positive solution. This completes the proof. □

Theorem 2. *Suppose that* (H0)–(H2) *and the following conditions hold*:

(H5) $\liminf_{z\to+\infty} f(t,z)(z/(1+(\ln t)^{\sigma-1}))>\lambda_1$ *uniformly on* $t\in[1,+\infty)$;
(H6) $\limsup_{z\to0^+} f(t,z)(z/(1+(\ln t)^{\sigma-1}))<\lambda_1$, *uniformly on* $t\in[1,+\infty)$.

Then (1) *has at least one positive solution.*

Proof. From (H5) there exist $\varepsilon_2,c_2>0$ such that

$$f(t,z)\geqslant(\lambda_1+\varepsilon_2)\frac{z}{1+(\ln t)^{\sigma-1}}-c_2,\quad z\geqslant0,\ t\in[1,+\infty). \tag{8}$$

Next, we shall prove that there is a sufficiently large

$$R_2\geqslant\frac{c_2\int_1^{+\infty}[\frac{1}{\Gamma(\sigma)}+\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)]\theta(s)\frac{\mathrm{d}s}{s}}{\frac{b\varepsilon_2}{4k^2(1+k^{\sigma-1})[\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}]}\int_{\mathrm{e}^{1/k}}^{\mathrm{e}^k}g(\xi,s)\theta(s)\,\frac{\mathrm{d}s}{s}},\quad k>1, \tag{9}$$

such that

$$z-Az\neq\mu\varphi,\quad z\in\partial B_{R_2}\cap P,\ \mu\geqslant0, \tag{10}$$

where φ is defined by (2). Suppose the contrary. Then there exist $z_1\in\partial B_{R_2}\cap P$, $\mu_1\geqslant0$ such that

$$z_1-Az_1=\mu_1\varphi. \tag{11}$$

Note that for fixed $k>1$, we have

$$\begin{aligned} &\min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]}\frac{\frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)}{1+(\ln t)^{\sigma-1}}\\ &\quad\geqslant\frac{1}{k^{\sigma-1}(1+k^{\sigma-1})}\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s),\quad s\in[1,+\infty), \end{aligned}$$

and thus we have

$$\begin{aligned} &\min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]}\frac{G(t,s)}{1+(\ln t)^{\sigma-1}}\\ &\geqslant\frac{1}{4k^2(1+k^{\sigma-1})}\sup_{t\in[1,+\infty)}\frac{g(t,s)}{1+(\ln t)^{\sigma-1}}\\ &\quad+\frac{1}{k^{\sigma-1}(1+k^{\sigma-1})}\sup_{t\in[1,+\infty)}\frac{(\ln t)^{\sigma-1}}{1+(\ln t)^{\sigma-1}}\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)\\ &\geqslant\frac{1}{4k^2(1+k^{\sigma-1})}\sup_{t\in[1,+\infty)}\frac{G(t,s)}{1+(\ln t)^{\sigma-1}},\quad s\in[1,+\infty). \end{aligned} \tag{12}$$

Define a cone

$$P_0 = \left\{ z \in P\colon \min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]} \frac{z(t)}{1+(\ln t)^{\sigma-1}} \geqslant \frac{1}{4k^2(1+k^{\sigma-1})}\|z\| \right\}.$$

Now, we prove that

$$z_1 \in P_0. \tag{13}$$

By (11) and (2) we have

$$\frac{(Az_1)(t)}{1+(\ln t)^{\sigma-1}} = \int_1^{+\infty} \frac{G(t,s)}{1+(\ln t)^{\sigma-1}}\theta(s) f\big(s, z_1(s)\big)\frac{\mathrm{d}s}{s}$$

and

$$\frac{\mu_1\varphi(t)}{1+(\ln t)^{\sigma-1}} = \mu_1\lambda_1 \int_1^{+\infty} \frac{G(t,s)}{1+(\ln t)^{\sigma-1}}\theta(s)\frac{\varphi(s)}{1+(\ln s)^{\sigma-1}}\frac{\mathrm{d}s}{s}.$$

Using (12), we obtain

$$\begin{aligned}
&\min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]} \frac{(Az_1)(t)}{1+(\ln t)^{\sigma-1}} \\
&= \min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]} \int_1^{+\infty} \frac{G(t,s)}{1+(\ln t)^{\sigma-1}}\theta(s) f\big(s, z_1(s)\big)\frac{\mathrm{d}s}{s} \\
&\geqslant \frac{1}{4k^2(1+k^{\sigma-1})} \sup_{t\in[1,+\infty)} \int_1^{+\infty} \frac{G(t,s)}{1+(\ln t)^{\sigma-1}}\theta(s) f\big(s, z_1(s)\big)\frac{\mathrm{d}s}{s} \\
&= \frac{1}{4k^2(1+k^{\sigma-1})}\|Az_1\|.
\end{aligned} \tag{14}$$

Similarly, we obtain

$$\min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]} \frac{\mu_1\varphi(t)}{1+(\ln t)^{\sigma-1}} \geqslant \frac{1}{4k^2(1+k^{\sigma-1})}\|\mu_1\varphi\|.$$

Therefore, we have (13), i.e.,

$$\min_{t\in[\mathrm{e}^{1/k},\mathrm{e}^k]} \frac{z_1(t)}{1+(\ln t)^{\sigma-1}} \geqslant \frac{1}{4k^2(1+k^{\sigma-1})}\|z_1\|.$$

Note that $\|z_1\| = R_2$, and by (9) we have

$$\begin{aligned}
&\varepsilon_2 \int_{\mathrm{e}^{1/k}}^{\mathrm{e}^k} \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} g(\xi,s)\theta(s)\frac{z_1(s)}{1+(\ln s)^{\sigma-1}}\frac{\mathrm{d}s}{s} \\
&\quad - c_2 \int_1^{+\infty} \left[\frac{(\ln t)^{\sigma-1}}{\Gamma(\sigma)} + \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} g(\xi,s)\right]\theta(s)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant (\ln t)^{\sigma-1}\Bigg[\varepsilon_2 \int\limits_{e^{1/k}}^{e^k} \frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} g(\xi,s)\theta(s)\frac{1}{4k^2(1+k^{\sigma-1})}R_2\,\frac{\mathrm{d}s}{s}\\
&\quad - c_2\int_1^{+\infty}\bigg[\frac{1}{\Gamma(\sigma)}+\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)\bigg]\theta(s)\,\frac{\mathrm{d}s}{s}\Bigg]\\
&\geqslant 0.
\end{aligned}
$$

Therefore, from (11) and (8) we have

$$
\begin{aligned}
(Az_1)(t) &\geqslant \int\limits_1^{+\infty} G(t,s)\theta(s)\bigg[(\lambda_1+\varepsilon_2)\frac{z_1(s)}{1+(\ln s)^{\sigma-1}} - c_2\bigg]\,\frac{\mathrm{d}s}{s}\\
&\geqslant \lambda_1(Lz_1)(t) + \varepsilon_2 \int\limits_{e^{1/k}}^{e^k} \frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} g(\xi,s)\theta(s)\frac{z_1(s)}{1+(\ln s)^{\sigma-1}}\,\frac{\mathrm{d}s}{s}\\
&\quad - c_2\int\limits_1^{+\infty}\bigg[\frac{(\ln t)^{\sigma-1}}{\Gamma(\sigma)}+\frac{b(\ln t)^{\sigma-1}}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)\bigg]\theta(s)\,\frac{\mathrm{d}s}{s}\\
&\geqslant \lambda_1(Lz_1)(t), \quad t\in[1,+\infty).
\end{aligned}
\tag{15}
$$

From (11) we only need to consider $\mu_1 > 0$. (Note, $\mu_1 = 0$ implies that $z_1 = Az_1$, i.e., z_1 is a positive fixed point of A, and thus this z_1 is also a positive solution for (1).) Let

$$\mu^{**} = \sup\{\mu\colon\ z_1 \geqslant \mu\varphi\}.$$

Then $\mu^{**} \geqslant \mu_1 > 0$, $z_1 \geqslant \mu^{**}\varphi$, and from (2), (11), and (15) we have

$$z_1 = Az_1 + \mu_1\varphi \geqslant \lambda_1 Lz_1 + \mu_1\varphi \geqslant \lambda_1 L\mu^{**}\varphi + \mu_1\varphi = \mu^{**}\varphi + \mu_1\varphi.$$

This contradicts the definition of μ^{**}. Therefore, (10) holds, and Lemma 4 implies that

$$i(A,\ B_{R_2}\cap P,\ P) = 0. \tag{16}$$

From (H6) there is a sufficiently small $r_2 \in (0, R_2)$ such that

$$f(t,z) \leqslant \lambda_1\frac{z}{1+(\ln t)^{\sigma-1}}, \quad z\in[0,r_2],\ t\in[1,+\infty). \tag{17}$$

For each $z \in \partial B_{r_2}\cap P$, by (17) we have

$$(Az)(t) \leqslant \int\limits_1^{+\infty} G(t,s)\theta(s)\lambda_1\frac{z(s)}{1+(\ln s)^{\sigma-1}}\,\frac{\mathrm{d}s}{s} := (L_3 z)(t), \quad t\in[1,+\infty).$$

From the definition of L_3 and (2) we see that

$$r(L_3) = 1. \tag{18}$$

Now, we claim that

$$Az \neq \mu z, \quad z \in \partial B_{r_2} \cap P, \ \mu \geqslant 1. \tag{19}$$

Suppose the contrary. Then there exist $z_2 \in \partial B_{r_2} \cap P$, $\mu_2 \geqslant 1$ such that

$$Az_2 = \mu_2 z_2.$$

Note, $\mu_2 = 1$ implies that $z_2 = Az_2$, i.e., z_2 is a positive fixed point of A, and thus this z_2 is also a positive solution for (1). Therefore, we only consider $\mu_2 > 1$. Consequently,

$$z_2 = \mu_2^{-1} Az_2 \leqslant \mu_2^{-1} L_3 z_2. \tag{20}$$

Note that L_3 is a positive linear operator, and thus we obtain a sequence $\{\mu_2^{-n} L_3^n z_2\}_{n=1}^{\infty}$ such that

$$\mu_2^{-1} L_3 z_2 \leqslant \mu_2^{-1} L_3\big(\mu_2^{-1} L_3 z_2\big) = \mu_2^{-2} L_3^2 z_2 \leqslant \cdots \leqslant \mu_2^{-n} L_3^n z_2 \leqslant \cdots.$$

This, together with (20), implies that

$$\big\|L_3^n\big\| \geqslant \frac{\|L_3^n z_2\|}{\|z_2\|} \geqslant \frac{\mu_2^n \|z_2\|}{\|z_2\|}.$$

Therefore, Gelfand's theorem implies that

$$r(L_3) = \liminf_{n\to+\infty} \sqrt[n]{\|L_3^n\|} \geqslant \liminf_{n\to+\infty} \sqrt[n]{\mu_2^n} = \mu_2 > 1.$$

This contradicts (18), and (19) holds as required. Therefore, Lemma 5 implies that

$$i(A,\, B_{r_2} \cap P,\, P) = 1. \tag{21}$$

Combining (16) and (21), we have

$$\begin{aligned} i\big(A,\, (B_{R_2} \setminus \overline{B_{r_2}}) \cap P,\, P\big) &= i(A,\, B_{R_2} \cap P,\, P) - i(A,\, B_{r_2} \cap P,\, P) \\ &= 0 - 1 = -1. \end{aligned}$$

Then A has a fixed point in $(B_{R_2} \setminus \overline{B_{r_2}}) \cap P$, i.e., (1) has at least one positive solution. This completes the proof. □

Theorem 3. *Suppose that* (H0)–(H2), (H4), (H6), *and the following condition hold*:

(H7) *There exist $\varLambda > 0$ and*

$$\varTheta > \frac{\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}}{b\int_{\mathrm{e}^{1/k}}^{\mathrm{e}^k} g(\xi, s)\theta(s)\frac{\mathrm{d}s}{s}}$$

such that $f(t, z) \geqslant \varTheta\varLambda$ for $z \in [\varLambda/(4k^2(1 + k^{\sigma-1})),\, \varLambda]$, $t \in [\mathrm{e}^{1/k}, \mathrm{e}^k]$, $k > 1$.

Then (1) *has at least two positive solutions.*

Proof. We claim that

$$z - Az \neq \mu \overline{z}^{**}, \quad z \in \partial B_\Lambda \cap P,\ \mu \geqslant 0, \tag{22}$$

where $\overline{z}^{**} \in P_0$ is a fixed element. If (22) is false, then there exist $z_3 \in \partial B_\Lambda \cap P$, $\mu_3 \geqslant 0$ such that

$$z_3(t) = (Az_3)(t) + \mu_3 \overline{z}^{**}(t) \geqslant (Az_3)(t), \quad t \in [1, +\infty). \tag{23}$$

Note that by (14) we also obtain $Az_3 \in P_0$, and thus $z_3 \in P_0$. Note that

$$\frac{z(t)}{1 + (\ln t)^{\sigma-1}} \leqslant \Lambda \quad \text{if } z(t) \leqslant \Lambda,\ t \in [1, +\infty).$$

Therefore, when $z_3 \in \partial B_\Lambda \cap P$, i.e., $\|z_3\| = \Lambda$, we have

$$\frac{\Lambda}{4k^2(1 + k^{\sigma-1})} \leqslant \frac{z_3(t)}{1 + (\ln t)^{\sigma-1}} \leqslant \Lambda, \quad t \in [\mathrm{e}^{1/k}, \mathrm{e}^k],\ k > 1.$$

Consequently, we see that

$$\begin{aligned} \|Az_3\| &= \sup_{t \in [1,+\infty)} \frac{1}{1 + (\ln t)^{\sigma-1}} \int_1^{+\infty} G(t,s)\theta(s) f\bigl(s, z_3(s)\bigr) \frac{\mathrm{d}s}{s} \\ &\geqslant \sup_{t \in [1,+\infty)} \frac{(\ln t)^{\sigma-1}}{1 + (\ln t)^{\sigma-1}} \int_{\mathrm{e}^{1/k}}^{\mathrm{e}^k} \frac{b}{\Gamma(\sigma) - b(\ln \xi)^{\sigma-1}} g(\xi, s)\theta(s)\Theta\Lambda \frac{\mathrm{d}s}{s} \\ &> \Lambda = \|z_3\|. \end{aligned} \tag{24}$$

However, from (23) we have

$$\|z_3\| = \sup_{t \in [1,+\infty)} \frac{z_3(t)}{1 + (\ln t)^{\sigma-1}} \geqslant \sup_{t \in [1,+\infty)} \frac{(Az_3)(t)}{1 + (\ln t)^{\sigma-1}} = \|Az_3\|,$$

and this contradicts (24). Then (22) holds, and Lemma 4 implies that

$$i(A,\ B_\Lambda \cap P,\ P) = 0. \tag{25}$$

Note that we can choose Λ such that $R_1 > \Lambda > r_2$, and then (H4) and (H6) imply that (7), (21) are still satisfied. Consequently, by (25) we have

$$\begin{aligned} i\bigl(A,\ (B_\Lambda \setminus \overline{B_{r_2}}) \cap P,\ P\bigr) &= i(A,\ B_\Lambda \cap P,\ P) - i(A,\ B_{r_2} \cap P,\ P) \\ &= 0 - 1 = -1 \end{aligned}$$

and

$$i\big(A,\,(B_{R_1}\setminus\overline{B_\Lambda})\cap P,\,P\big) = i(A,\,B_{R_1}\cap P,\,P) - i(A,\,B_\Lambda\cap P,\,P\big) \\ = 1-0=1.$$

Therefore, A has a fixed point in $(B_\Lambda\setminus\overline{B_{r_2}})\cap P$ and $(B_{R_1}\setminus\overline{B_\Lambda})\cap P$, respectively. Thus (1) has at least two positive solutions. This completes the proof. □

Theorem 4. *Suppose that* (H0)–(H3), (H5)*, and the following condition hold*:

(H8) *There exist* $\overline{\Lambda}>0$ *and*

$$\widetilde{\Theta}\in\left(0,\ \frac{1}{\int_1^{+\infty}[\frac{1}{\Gamma(\sigma)}+\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)]\theta(s)\,\frac{\mathrm{d}s}{s}}\right)$$

such that $f(t,z)\leqslant\widetilde{\Theta}\overline{\Lambda}$ *for* $z\in[0,\overline{\Lambda}]$, $t\in[1,+\infty)$.

Then (1) *has at least two positive solutions.*

Proof. From (H8) we obtain

$$\frac{z(t)}{1+(\ln t)^{\sigma-1}}\in[0,\overline{\Lambda}]\quad\text{if } z(t)\in[0,\overline{\Lambda}],\ t\in[1,+\infty).$$

Note that when $z\in\partial B_{\overline{\Lambda}}\cap P$, we see that

$$\begin{aligned}\|Az\| &= \sup_{t\in[1,+\infty)}\frac{1}{1+(\ln t)^{\sigma-1}}\int_1^{+\infty}G(t,s)\theta(s)f\big(s,z(s)\big)\,\frac{\mathrm{d}s}{s}\\ &\leqslant \sup_{t\in[1,+\infty)}\frac{(\ln t)^{\sigma-1}}{1+(\ln t)^{\sigma-1}}\int_1^{+\infty}\left[\frac{1}{\Gamma(\sigma)}+\frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}}g(\xi,s)\right]\theta(s)\widetilde{\Theta}\overline{\Lambda}\,\frac{\mathrm{d}s}{s}\\ &< \overline{\Lambda}=\|z\|.\end{aligned}\tag{26}$$

This implies that

$$Az\neq\mu z,\quad z\in\partial B_{\overline{\Lambda}}\cap P,\ \mu\geqslant 1.\tag{27}$$

If (27) is false, then there exist $z_4\in\partial B_{\overline{\Lambda}}\cap P$, $\mu_4\geqslant 1$ such that

$$(Az_4)(t)=\mu_4z_4(t)\geqslant z_4(t),\quad t\in[1,+\infty).$$

Hence, we obtain

$$\|Az_4\|=\sup_{t\in[1,+\infty)}\frac{(Az_4)(t)}{1+(\ln t)^{\sigma-1}}\geqslant\sup_{t\in[1,+\infty)}\frac{z_4(t)}{1+(\ln t)^{\sigma-1}}=\|z_4\|,$$

and this contradicts (26). Then (27) holds, and Lemma 5 implies that

$$i(A,\, B_{\overline{\Lambda}} \cap P,\, P) = 1. \tag{28}$$

Note that we can choose $\overline{\Lambda}$ such that $R_2 > \overline{\Lambda} > r_1$, and then (H3) and (H5) imply that (6), (16) are still satisfied. Consequently, by (28) we have

$$\begin{aligned} i\big(A,\, (B_{\overline{\Lambda}} \setminus \overline{B_{r_1}}) \cap P,\, P\big) &= i(A,\, B_{\overline{\Lambda}} \cap P,\, P) - i(A,\, B_{r_1} \cap P,\, P) \\ &= 1 - 0 = 1 \end{aligned}$$

and

$$\begin{aligned} i\big(A,\, (B_{R_2} \setminus \overline{B_{\overline{\Lambda}}}) \cap P,\, P\big) &= i(A,\, B_{R_2} \cap P,\, P) - i(A,\, B_{\overline{\Lambda}} \cap P,\, P) \\ &= 0 - 1 = -1. \end{aligned}$$

Therefore, A has a fixed point in $(B_{\overline{\Lambda}} \setminus \overline{B_{r_1}}) \cap P$ and $(B_{R_2} \setminus \overline{B_{\overline{\Lambda}}}) \cap P$, respectively. Thus (1) has at least two positive solutions. This completes the proof. □

Now, we provide some examples to verify our main results. Let $\sigma = 2.5$, $b = 1$, $\xi = \mathrm{e}$, $\theta(t) = 1/(1 + \ln^2 t)$, $t \in [1, +\infty)$. Then $b(\ln \xi)^{\sigma-1} = 1 < \Gamma(\sigma) = 1.33$, $\int_1^{+\infty} \theta(t)\,(\mathrm{d}t/t) = \pi/2$, and thus (H0), (H2) hold.

Example 1. Let $f(t, z) = \sqrt{z}/(1 + (\ln t)^{\sigma-1})$, $z \geqslant 0$, $t \in [1, +\infty)$. Then

$$\liminf_{z \to 0^+} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \liminf_{z \to 0^+} \frac{\frac{\sqrt{z}}{1+(\ln t)^{\sigma-1}}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = +\infty > \lambda_1$$

and

$$\limsup_{z \to +\infty} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \limsup_{z \to +\infty} \frac{\frac{\sqrt{z}}{1+(\ln t)^{\sigma-1}}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = 0 < \lambda_1$$

uniformly on $t \in [1, +\infty)$. Therefore, (H1), (H3)–(H4) hold, and by Theorem 1, (1) has at least one positive solution.

Example 2. Let $f(t, z) = z^2/(1 + (\ln t)^{\sigma-1})$, $z \geqslant 0$, $t \in [1, +\infty)$. Then

$$\liminf_{z \to +\infty} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \liminf_{z \to +\infty} \frac{\frac{z^2}{1+(\ln t)^{\sigma-1}}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = +\infty > \lambda_1$$

and

$$\limsup_{z \to 0^+} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \limsup_{z \to 0^+} \frac{\frac{z^2}{1+(\ln t)^{\sigma-1}}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = 0 < \lambda_1$$

uniformly on $t \in [1, +\infty)$. Therefore, (H1), (H5)–(H6) hold, and by Theorem 2, (1) has at least one positive solution.

Example 3. Let $k = 2$, $\Lambda = 1 + 2\sqrt{2}$ and note $\int_{e^{1/k}}^{e^k} g(\xi, s)\theta(s)\,(ds/s) = 27.69$. Now, let

$$f(t,z) = \begin{cases} 16^2\Lambda z^2, & z \in [0, 1/16],\ t \in [1, +\infty), \\ \Lambda, & z \in [1/16, 1 + 2\sqrt{2}],\ t \in [1, +\infty), \\ \sqrt{\Lambda z}, & z \in [1 + 2\sqrt{2}, +\infty),\ t \in [1, +\infty). \end{cases}$$

Then

$$\limsup_{z\to+\infty} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \limsup_{z\to+\infty} \frac{\sqrt{\Lambda z}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = 0 < \lambda_1$$

and

$$\limsup_{z\to 0^+} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \limsup_{z\to 0^+} \frac{16^2\Lambda z^2}{\frac{z}{1+(\ln t)^{\sigma-1}}} = 0 < \lambda_1$$

uniformly on $t \in [1, +\infty)$. Moreover, if $z \in [\Lambda/(4k^2(1 + k^{\sigma-1})),\ \Lambda]$, $t \in [e^{1/k}, e^k]$, we have

$$f(t,z) = \Lambda \geqslant \Theta\Lambda \quad \text{if } 1 \geqslant \Theta > \frac{\Gamma(\sigma) - b(\ln\xi)^{\sigma-1}}{b\int_{e^{1/k}}^{e^k} g(\xi,s)\theta(s)\,\frac{ds}{s}} = 0.012.$$

Therefore, (H1), (H4), and (H6)–(H7) hold, and Theorem 3 implies that (1) has at least two positive solutions.

Example 4. Let $\overline{\Lambda} = 100$ and

$$f(t,z) = \begin{cases} 2.5\sqrt{z}, & z \in [0, 100],\ t \in [1, +\infty), \\ \frac{z^2}{400}, & z \in [100, +\infty),\ t \in [1, +\infty). \end{cases}$$

Then

$$\liminf_{z\to 0^+} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \liminf_{z\to 0^+} \frac{2.5\sqrt{z}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = +\infty > \lambda_1$$

and

$$\liminf_{z\to+\infty} \frac{f(t,z)}{\frac{z}{1+(\ln t)^{\sigma-1}}} = \liminf_{z\to+\infty} \frac{\frac{z^2}{400}}{\frac{z}{1+(\ln t)^{\sigma-1}}} = +\infty > \lambda_1$$

uniformly on $t \in [1, +\infty)$. Moreover, if $z \in [0, \overline{\Lambda}]$, $t \in [1, +\infty)$, we have

$$f(t,z) \leqslant 25 \leqslant \widetilde{\Theta}\overline{\Lambda} \quad \text{if } \widetilde{\Theta} \in [0.25, 0.254) \subset (0, 0.254),$$
$$\frac{1}{\int_1^{+\infty}[\frac{1}{\Gamma(\sigma)} + \frac{b}{\Gamma(\sigma)-b(\ln\xi)^{\sigma-1}} g(\xi,s)]\theta(s)\,\frac{ds}{s}} = 0.254.$$

Therefore, (H1), (H3), (H5), and (H8) hold, and Theorem 4 implies that (1) has at least two positive solutions.

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