Nonlinear Analysis: Modelling and Control, Vol. 60, No. 2, 196–211
https://doi.org/10.15388/namc.2025.60.38471

Vilnius University Press

New formulation of Lyapunov direct method for nonautonomous real-order systems

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Received: April 7, 2023 / **Revised:** January 2, 2025 / **Published online:** January 9, 2025

Abstract. Lyapunov stability analysis of nonautonomous real-order systems is put forward here in the sense of Caputo in a new and different way. We introduce new theorems and inequalities that give stability of constant solutions in the domain of attraction to such systems when attached with random initial time placed on the real axis. We give some examples including an advanced nonlinear Lorenz system to illustrate the results.

Keywords: real-order calculus, nonautonomous real-order system, nonlinear system, random initial time, stability.

1 Introduction

It is widely known that the concept of Newton–Leibniz derivatives of first, second, ..., nth order is quite elementary and plays a crucial role in nonlinear systems and their control (see, e.g., [15, 28]). In accordance, the principle of l'Hôpital–Leibniz derivatives (for example, think of possibility of rational, irrational, real-orders, etc.) has been continually evolving with many applications that include motivated problems in physics, engineering, and other applied areas (we refer to [2, 16, 27]). For instance, many existing real-order (fractional-order) operators have been thoroughly used mainly because of the global nature associated with intrinsic memory features [30] that cannot be found in the elementary Newton–Leibniz derivative concept. In order to understand why these types of real-order operators are important, the readers are referred to an elementary example mentioned in the research findings of Li et al. [22] that gives order-dependent transition and to the nontrivial Leibniz rule discussed by Tarasov in [29] when utilized Caputo derivative operator. On the other hand, many early sophisticated applications dealing with the modelling, analysis, and control are published in [1, 4, 8–10, 14, 16, 17, 24, 26, 27, 32] and can be found in their references.

[1]The author was supported by Indian Institute of Technology Guwahati Post-Doctoral Fellowship provided under grant MATH/IPDF/2020-21/BIKL/01.

It has been continually reported in the literature that stability is a fundamental issue in many problems dealing with predicting dynamics and controlling trajectories. It often plays a basic role in many systems described in terms of real-order systems in the modern areas of dynamics and control of understanding difficult problems. For example, is it possible to control the trajectory of many real-order systems at different initial times?

In short, recent investigations show that the methods and conditions established by many researchers in [5, 11, 21, 22, 31] are not applicable in many instances when it comes to nonautonomous real-order systems. In particular, consider a case where a described system (1) is given and assume that $a < 0$. To the best of the authors' knowledge, this issue has not been investigated yet.

Motivated by the above-mentioned issue, in this paper, we consider the real-order system described by

$$ {}^{C}D^{\omega}_{a+,\,t}x(t) = f\bigl(t, x(t)\bigr), \qquad x(a) = x_a, \tag{1} $$

where ${}^{C}D^{\omega}_{a+,t}x(t) = ({}^{C}D^{\omega}_{a+,t}x_1(t), \dots, {}^{C}D^{\omega}_{a+,t}x_n(t))^{\mathrm{T}}$, $\omega \in (0, 1]$, $a \in \mathbb{R}$, and the function $f : [a, \infty) \times \varOmega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable with respect to its arguments.

The main purpose of this paper is to bring out and establish some new stability results of (1) for an effective analysis. We first introduce new Lyapunov theorems and provide proofs that present the stability of (1). We establish new inequalities that provide some new tools for the application of Lyapunov theorems. For applications, we give some examples that include random initial times to demonstrate the importance of such theorems. In particular, we show that it is possible to stabilize the chaotic trajectory of the real-order Lorenz system using some proposed theorem whenever the initial time is negative.

This paper is organised as follows. We recall some definitions in Section 2. In Section 3, we introduce new Lyapunov theorems. In Section 4, we establish new inequalities. In Section 5, we provide examples, and in Section 6, we draw reasonable conclusions.

2 Notations and definitions

Throughout this paper, we denote the set of real numbers by $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}_+$, the set of natural numbers by $\mathbb{N}$, the Euclidean space by $\mathbb{R}^n$, the Euclidean norm of a vector $x \in \mathbb{R}^n$ by $\|x\|$, the interval $-\infty \leqslant a < b \leqslant \infty$ by I, and the n-times continuously differentiable space by $\mathcal{C}^n$. We recall here the operators mentioned below.

Definition 1. Let $n \in \mathbb{N}$ and $\omega \in \mathbb{R}_+$. The ω-order Riemann–Liouville integral of an integrable function g on I is given by [16, 27]

$$ {}^{RL}I^{-\omega}_{a+,t}g(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t - s)^{\omega-1} g(s)\,\mathrm{d}s, \quad t > a. $$

Definition 2. Let $n \in \mathbb{N}$ and $\omega \in \mathbb{R}_+$. The ω-order Caputo derivative of an $\mathcal{C}^n$-function g on I is given by [2, 16, 27]

$$ {}^C D^{\omega}_{a^+,t} g(t) = \begin{cases} {}^{RL}I^{-(n-\omega)}_{a^+,t} \frac{d^n}{dt^n} g(t) & \text{if } n-1 < \omega < n, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(t) & \text{if } \omega = n. \end{cases} \tag{2} $$

3 Main results

We introduce two new theorems for system (1) that provide key but straightforward statements for the analysis of the system.

First, we introduce the below-mentioned definitions, which will give rigorous understanding of our new theorems that are presented later.

Definition 3 [Point asymptotic stability]. Throughout this paper, the solution or stationary point (SP) $x = 0$ to (1) is said to be point asymptotically stable (PAS) if there exists Ω such that for all $x(a) \in \Omega$, the Euclidean nontrivial measure $\|x(t)\| \to 0$ as $t \to \infty$.

Definition 4 [Point stability]. Throughout this paper, the solution or stationary point (SP) $x = 0$ to (1) is said to be point stable (PS) if there exists a Ω such that for any $x(a) \in \Omega_1$, there is a $\delta > 0$ so that $\|x(a)\| \leqslant \delta \Rightarrow \|x(t)\| \leqslant \epsilon$, where $\epsilon > 0$.

Definitions 3 and 4 can be viewed as different ones as compared to stability definitions found in books [15, 25]. The notions of Definitions 3 and 4 are globally defined on time interval $[a, \infty)$ considering the linear operator (2). These definitions are inevitable in the construction of the proofs of our main theorems.

We introduce here the fundamental theorems of (1). In brief, these theorems can be thought of as "Lyapunov theorems", and the line of approach can be called known "Lyapunov direct method".

Theorem 1 [Fundamental stability theorem I]. *Let $x = 0$ be the SP of (1), and let $x = 0$ in the domain D. Assume that there exists a $\mathcal{C}^1$-function $V(t, x) : [a, \infty) \times D \subseteq \mathbb{R}^n \to [0, \infty)$ satisfying*

(A1) $m_1\|x\|^{r_1} \leqslant V(t, x) \leqslant \mu(t)\|x\|^{r_2}$, *where $m_1, r_1, r_2 > 0$ with $\mu(t)$ continuous on $[a, \infty)$, and $m_1 \leqslant \mu(t)$, $1 \leqslant \mu(t)$, $r_1 \leqslant r_2$, and $r_2 \geqslant 1$;*

(A2) ${}^C D^{\beta}_{a^+,t} V(t, x(t))$ *is uniformly negative definite on the nontrivial solution $x(t)$ of (1), i.e.,*

$$ {}^C D^{\beta}_{a^+,t} V\bigl(t, x(t)\bigr) \leqslant -\alpha < 0 \quad \forall x \in D - \{0\} \quad \forall t > a, \tag{3} $$

where $0 < \beta \leqslant 1$, and some $\alpha > 0$.

Then the SP $x = 0$ to (1) is PAS. When $D = \mathbb{R}^n$ and the result holds, then the SP $x = 0$ is globally PAS.

Proof. Since $V(t, x)$ satisfies inequality (3) of (A2), it suffices to construct an equation such that $x = 0$ should be PAS. We consider

$$ {}^C D^{\beta}_{a^+,t} V\bigl(t, x(t)\bigr) = -\lambda V\bigl(t, x(t)\bigr) - h(t) \quad \forall t > a, \ \forall x \in D - \{0\}, \tag{4} $$

where $\lambda > 0$, and $h : [a, \infty) \to [0, \infty)$ is continuous. Clearly, Eq. (4) satisfies inequality (3). By letting $\bar{W}(t) = V(t, x(t))$, we now define an initial value problem

$$ {}^{C}D^{\beta}_{a^+,t}\bar{W}(t) = -\lambda\bar{W}(t) - h(t), \quad \bar{W}(a) = V_a > 0. \tag{5} $$

Clearly, problem (5) has an analytic solution as follows [16]:

$$ \begin{aligned} \bar{W}(t) &= E_\beta\big(-\lambda(t-a)^\beta\big)\bar{W}(a) \\ &\quad - \int_a^t (t-s)^{\beta-1}E_{\beta,\beta}\big(-\lambda(t-s)^\beta\big)h(s)\,\mathrm{d}s. \end{aligned} \tag{6} $$

Then from (6) one obtains

$$ \lim_{t\to\infty} \bar{W}(t) \leqslant 0. \tag{7} $$

Note that Mittag-Leffler functions $E_\beta(-\lambda(t-a)^\beta)$ and $E_{\beta,\beta}(-\lambda(t-s)^\beta)$ are nonnegative [12], and $E_\beta(-\lambda(t-a)^\beta)$ approaches 0 as $t \to \infty$ [27]. As a result, one gets from (7) and assumption (A1) that

$$ \lim_{t\to\infty} \big\|x(t)\big\| = 0. $$

This completes the proof. □

Theorem 2 [Fundamental stability theorem-II]. *Let $x = 0$ be the SP of* (1)*, and let $x = 0$ in the domain D. Assume that there exists a $\mathcal{C}^1$-function $V(t, x) : [a, \infty) \times D \subseteq \mathbb{R}^n \to [0, \infty)$ satisfying*

- (A1) $m_1\|x\|^{r_1} \leqslant V(t, x) \leqslant \mu(t)\|x\|^{r_2}$, *where $m_1, r_1, r_2 > 0$ with $\mu(t)$ continuous on $[a, \infty)$, and $m_1 \leqslant \mu(t)$, $1 \leqslant \mu(t)$, $r_1 \leqslant r_2$, and $r_2 \geqslant 1$;*
- (A2) ${}^{C}D^{\beta}_{a^+,t}V(t, x(t))$ *is uniformly negative semidefinite on the nontrivial solution $x(t)$ of* (1)*, i.e.,*

$$ {}^{C}D^{\beta}_{a^+,t}V\big(t, x(t)\big) \leqslant -\alpha \leqslant 0 \quad \forall x \in D,\ \forall t > a, \tag{8} $$

where $0 < \beta \leqslant 1$, and some $\alpha \geqslant 0$.

Then the SP $x = 0$ to (1) *is PS. When $D = \mathbb{R}^n$ and the result holds, then the SP $x = 0$ is globally PS.*

Proof. Employing the procedure of Theorem 1, we consider the equation

$$ {}^{C}D^{\beta}_{a^+,t}V\big(t, x(t)\big) = -\widehat{h}(t) - \lambda V\big(t, x(t)\big) \quad \forall x \in D,\ \forall t > a, \tag{9} $$

where $\lambda \geqslant 0$, and $\widehat{h} : [a, \infty) \to [0, \infty)$ is continuous. Observe that Eq. (9) satisfies inequality (8). By letting $\bar{W}(t) = V(t, x(t))$, we define a fractional differential equation as follows:

$$ {}^{C}D^{\beta}_{a^+,t}\bar{W}(t) = -\lambda\bar{W}(t) - \widehat{h}(t) \tag{10} $$

with initial condition $\bar{W}(a) = V_a > 0$. The explicit solution to (10) is given by [16]

$$\bar{W}(t) = E_\beta\bigl(-\lambda(t-a)^\beta\bigr)\bar{W}(a) - \int_a^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(-\lambda(t-s)^\beta\bigr)\widehat{h}(s)\,\mathrm{d}s. \tag{11}$$

Then from (11) one has

$$\bar{W}(t) \leqslant \bar{W}(a). \tag{12}$$

Note that $E_\beta(-\lambda(t-a)^\beta)$ and $E_{\beta,\beta}(-\lambda(t-s)^\beta)$ are nonnegative [12, 27]:

$$0 < E_\beta\bigl(-\lambda(t-a)^\beta\bigr) \leqslant 1 \quad \forall t \geqslant a.$$

Consequently, from assumption (A1) and (12) one obtains

$$\bigl\|x(t)\bigr\| \leqslant \left[\frac{\mu(a)}{m_1}\bigl\|x(a)\bigr\|^{r_2}\right]^{1/r_1} \quad \forall t \geqslant a.$$

Hence, the SP $x = 0$ is PS. This completes the proof. □

Remark 1. The notion of "uniformly negative definite", with a constant number α, is not standard in Theorems 1 and 2; see [15]. However, one can always construct such a constant α to think of fundamental stability theorems. It should be noted that finding new proofs to Theorems 1 and 2 remains open and quite challenging in qualitative stability theory.

4 Inequalities

We introduce the below-mentioned lemmas that open the use of many Lyapunov functions in the application of Lyapunov direct method.

Lemma 1. *Suppose that V be a real-valued $\mathcal{C}^1$-function on $[a,\infty) \times \Omega \subseteq \mathbb{R}^n$, which is convex w.r.t. its arguments. Let $x : [a,\infty) \to \Omega \subseteq \mathbb{R}^n$ be a continuous function, which is differentiable on (a,∞). Then we have*

$${}^C D^\omega_{a+,t} V\bigl(t, x(t)\bigr) \leqslant \frac{\partial V(t,x(t))}{\partial t}\,{}^C D^\omega_{a+,t} t + \left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}} {}^C D^\omega_{a+,t} x(t) \tag{13}$$

for all $t > a$ and for all $\omega \in (0,1]$.

Proof. To prove inequality (13) by using the Caputo derivative (2), we need to show that the following inequality holds:

$$\int_a^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)} G(s,t)\,\mathrm{d}s \leqslant 0, \tag{14}$$

where

$$G(s,t) = \frac{\partial V(s,x(s))}{\partial s} - \frac{\partial V(t,x(t))}{\partial t} + \left(\frac{\partial V(s,x(s))}{\partial x(s)}\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}s} x(s) - \left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}s} x(s).$$

Define

$$Q(s,t) = V\bigl(s,x(s)\bigr) - V\bigl(t,x(t)\bigr) - \left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}} \bigl(x(s)-x(t)\bigr) \left(\frac{\partial V(t,x(t))}{\partial t}\right)(s-t).$$

Then one obtains

$$\frac{\mathrm{d}}{\mathrm{d}s} Q(s,t) = \frac{\partial V(s,x(s))}{\partial s} + \left(\frac{\partial V(s,x(s))}{\partial x(s)}\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}s} x(s) - \left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}s} x(s) - \frac{\partial V(t,x(t))}{\partial t}.$$

Clearly, $\mathrm{d}/\mathrm{d}\tau Q(s,t) = G(s,t)$. Then inequality (14) becomes

$$\int_a^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)} \frac{\mathrm{d}}{\mathrm{d}s} Q(s,t)\,\mathrm{d}s \leqslant 0.$$

Integrating by parts, one obtains

$$\int_a^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)} \frac{\mathrm{d}}{\mathrm{d}s} Q(s,t)\,\mathrm{d}s = \frac{(t-s)^{-\omega} Q(s,t)}{\Gamma(1-\omega)}\bigg|_{s=t} - \frac{(t-a)^{-\omega} Q(a,t)}{\Gamma(1-\omega)} - \frac{\omega}{\Gamma(1-\omega)} \int_a^t \frac{Q(s,t)}{(t-s)^{\omega+1}}\,\mathrm{d}s. \tag{15}$$

By using l'Hôpital rule, it can be obtained that

$$\frac{(t-s)^{-\omega} Q(s,t)}{\Gamma(1-\omega)}\bigg|_{s=t} = \lim_{s\to t} \frac{(t-s)^{-\omega} Q(s,t)}{\Gamma(1-\omega)} = 0.$$

Further, the convexity of $V(t,x)$ yields $Q(s,t) \geqslant 0$; see [3]. It is immediate that (15) is bounded from above by 0. Thus, we conclude inequality (14). □

Remark 2. Lemma 1 can be found in the work by Wu [33], where the author has not considered initial time a to the interval $(-\infty, 0)$. As a result, the version of Lemma 1 of [33] cannot be applied to system (1) if initial time a is placed on $(-\infty, 0)$. In comparison to the inequality of Wu [33], Lemma 1 provides promising implications to random initial-time system (1).

Lemma 2. *Suppose that V is a real-valued $\mathcal{C}^1$-function on $[a,\infty)\times\Omega\subseteq\mathbb{R}^n$, which is convex w.r.t. Ω. Let $x:[a,\infty)\to\Omega\subseteq\mathbb{R}^n$ be a continuous function, which is differentiable on (a,∞). Then we have*

$$ {}^{C}D^{\omega}_{a^+,t}V\big(t,x(t)\big)\leqslant\left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}}{}^{C}D^{\omega}_{a^+,t}x(t) \tag{16}$$

for all $t>a$ and for all $\omega\in(0,1]$.

Proof. To prove inequality (16), it suffices to show that the following inequality holds:

$$\int\limits_a^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)}G(s,t)\,\mathrm{d}s\leqslant 0, \tag{17}$$

where

$$G(s,t)=\frac{\partial V(s,x(s))}{\partial s}+\left(\frac{\partial V(s,x(s))}{\partial x(s)}\right)^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}s}x(s)-\left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}s}x(s).$$

Set

$$Q(s,t)=V\big(s,x(s)\big)-V\big(t,x(t)\big)-\left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}}\big(x(s)-x(t)\big).$$

Then one obtains

$$\frac{\mathrm{d}}{\mathrm{d}s}Q(s,t)=\frac{\partial V(s,x(s))}{\partial s}+\left(\frac{\partial V(s,x(s))}{\partial x(s)}\right)^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}s}x(s)-\left(\frac{\partial V(t,x(t))}{\partial x(t)}\right)^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}s}x(s).$$

Clearly, $(\mathrm{d}/\mathrm{d}s)Q(s,t)=G(s,t)$. Then (17) reduces to

$$\int\limits_a^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)}\frac{\mathrm{d}}{\mathrm{d}s}Q(s,t)\,\mathrm{d}s\leqslant 0.$$

Utilizing integration of parts, one can compute

$$\begin{aligned}\int\limits_a^t \frac{(t-s)^{-\omega}}{\Gamma(1-\omega)}\frac{\mathrm{d}}{\mathrm{d}s}Q(s,t)\,\mathrm{d}s&=\frac{(t-s)^{-\omega}Q(s,t)}{\Gamma(1-\omega)}\bigg|_{s=t}-\frac{(t-a)^{-\omega}Q(a,t)}{\Gamma(1-\omega)}\\&\quad-\frac{\omega}{\Gamma(1-\omega)}\int\limits_a^t\frac{Q(s,t)}{(t-s)^{\omega+1}}\,\mathrm{d}s.\end{aligned} \tag{18}$$

By using l'Hôpital rule and the convexity of $V(t,x)$, it is immediate that (18) is bounded from above by 0. Thus, we conclude inequality (17). □

5 Applications through examples

In this section, we provide some systems that deal with some applications of derived theorems in the analysis of stability.

Example 1. We consider the following linear nonautonomous positive initial time system:

$$\begin{aligned} {}^{C}D^{\omega}_{1+,\,t}x_1(t) &= -x_1 - \frac{1}{\sqrt{1+t}}x_2(t), \\ {}^{C}D^{\omega}_{1+,\,t}x_2(t) &= x_1(t) - x_2(t) \end{aligned} \tag{19}$$

subject to the initial conditions $x_1(1) = x_{11}$ and $x_2(1) = x_{21}$, where $\omega \in (0, 1]$.

First, we let $V(t, x) = x_1^2 + (1 + 1/\sqrt{1+t})x_2^2$, where $x = (x_1, x_2)^{\mathrm{T}}$. Set $\beta = \omega$. Then we get, by using Lemma 1 for system (1),

$$\begin{aligned} {}^{C}D^{\beta}_{1+,\,t}V\bigl(t, x(t)\bigr) &\leqslant -\frac{1}{2\sqrt{1+t}}x_2^2(t)\,{}^{C}D^{\beta}_{1+,\,t}t + 2x_1(t)\,{}^{C}D^{\beta}_{1+,\,t}x_1(t) \\ &\quad + 2\biggl(1 + \frac{1}{\sqrt{1+t}}\biggr)x_2(t)\,{}^{C}D^{\beta}_{1+,\,t}x_2(t) \\ &= -\frac{1}{2\sqrt{1+t}}x_2^2(t)\frac{(t-1)^{1-\beta}}{\Gamma(2-\beta)} + 2x_1(t)\,{}^{C}D^{\beta}_{1+,\,t}x_1(t) \\ &\quad + 2\biggl(1 + \frac{1}{\sqrt{1+t}}\biggr)x_2(t)\,{}^{C}D^{\beta}_{1+,\,t}x_2(t) \\ &\leqslant -2x_1^2(t) + 2x_1(t)x_2(t) - 2x_2^2(t) \\ &= -\bigl(x_1(t)\;\; x_2(t)\bigr)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &\leqslant -\bigl(x_1^2(t) + x_2^2(t)\bigr) = -r^2(t) \leqslant -r^2 < 0 \end{aligned}$$

for all $(x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2 - \{(0,0)^{\mathrm{T}}\}$ and for all $t > 1$, where $r(t)$ is continuous and satisfies $r^2(t) \geqslant r^2 = x_1^2 + x_2^2$, and r is a positive constant. Set $\alpha = r^2$. Consequently, by Theorem 1, the SP $x = 0$ to (19) should be globally PAS. To see the trajectory of (19), we take a suitable value $\omega = 0.7$ and set the initial conditions as $x_1(1) = 20$ and $x_2(1) = -10$. The obtained trajectory of (19) using the approach in [7] is shown in Fig. 1. It clearly indicates that the state curves approach 0 as time increases toward ∞.

Remark 3. In Example 1, the importance of Theorem 1 is shown for the positive initial time nonautonomous system. Theorem 1 provides a new tool with the point asymptotic analysis. We find that the existing Lyapunov theory, as stated in [5, 11, 21, 22, 31], cannot be applied to (1). In comparison to these previous research, the present theorem provides new platforms for the analysis of the system in large scale and tackling the case when the initial time is positive.

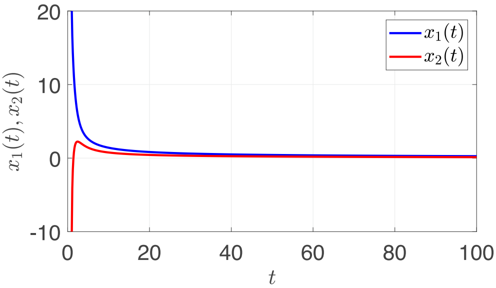

Figure 1. State trajectories of (19) with order $\omega = 0.7$ starting from initial values $x_1(1) = 20$ and $x_2(1) = -10$. It demonstrates PAS.

Example 2. We consider the following nonlinear nonautonomous negative initial time system:

$$ {}^{C}D^{\omega}_{-10^+,t}x(t) = -\frac{1 + \frac{t(t+10)^{1-\omega}}{\Gamma(2-\omega)} + t^2}{(1+t^2)}x(t) - x^3(t) \tag{20}$$

with $x(-10) = x_{-10}$, where $\omega \in (0, 1]$.

We introduce $V(t, x) = x^2 + t^2x^2$ and set $\beta = \omega$. Then, by Lemma 1, one obtains the following on the nontrivial solution $x(t)$ to (20):

$$\begin{aligned}
&{}^{C}D^{\omega}_{-10^+,t}V\bigl(t, x(t)\bigr) \\
&\quad\leqslant 2tx^2(t){}^{C}D^{\omega}_{-10^+,t}t + 2\bigl(1+t^2\bigr)x(t)\left(-\frac{1 + \frac{t(t+10)^{1-\omega}}{\Gamma(2-\omega)} + t^2}{(1+t^2)}x(t)\right) \\
&\qquad - 2\bigl(1+t^2\bigr)x^4(t) \\
&\quad= 2tx^2(t)\frac{(t+10)^{1-\omega}}{\Gamma(2-\omega)} - 2\left(1 + \frac{t(t+10)^{1-\omega}}{\Gamma(2-\omega)} + t^2\right)x^2(t) \\
&\qquad - 2\bigl(1+t^2\bigr)x^4(t) \\
&\quad= -2\bigl(1+t^2\bigr)\bigl(x^2(t) + x^4(t)\bigr) \leqslant -\bigl(1+t^2\bigr)x^2(t) = -r^2(t) \leqslant -r^2 < 0
\end{aligned}$$

for all $x \in \mathbb{R} - \{0\}$ and for all $t > -10$, where $r(t)$ is continuous and satisfies $r^2(t) \geqslant r^2 = x^2$, and r is a positive constant. Set $\alpha = r^2$. Observe that assumptions (A1) and (A2) of Theorem 1 hold. Thus, by Theorem 1, the SP $x = 0$ to (20) should be globally PAS. For the computational demonstration, we consider the values $\omega = 0.8$ and $x(-10) = 20$. The numerical simulation is depicted in Fig. 2. It indicates that the nontrivial solution to (20) approaches 0 as time increases toward ∞.

Remark 4. The importance of Theorem 1 is shown in Example 2 where the initial time is $t = -10$. It provides a new tool for answering the point asymptotic stability. We discover that all the existing Lyapunov theory [5, 11, 21, 22, 31, 33] cannot be applicable to this problem. In comparison to the previous research, the applied theorem provides new platforms for the system analysis in large scale.

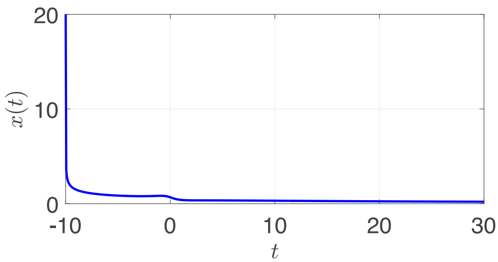

Figure 2. State curve of (20) with order $\omega = 0.8$ starting from initial value $x(-10) = 20$. It demonstrates PAS.

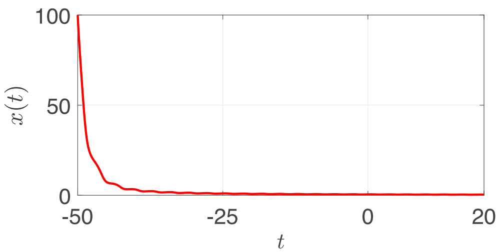

Figure 3. The PAS system (21) where the values $\omega = 0.9$ and initial condition $x(-50) = 100$ are taken.

Example 3. We consider the nonautonomous negative initial time system

$$ {}^{C}D^{\omega}_{-50^+,t}x(t) = -\frac{x(t)}{1+\cos^2(t)} \tag{21} $$

with initial condition $x(-50) = x_{-50}$, where fractional order $\omega \in (0, 1]$.

We propose $V(t, x) = x^2 + \cos^2(t)x^2$ and set order $\beta = \omega$. Then, by using Lemma 2 for system (21), one obtains

$$ \begin{aligned} {}^{C}D^{\omega}_{-50^+,t}V\bigl(t, x(t)\bigr) &\leqslant 2x(t)\bigl(1+\cos^2(t)\bigr)\left(-\frac{x(t)}{1+\cos^2(t)}\right) \\ &= -2x^2(t) \leqslant -x^2(t) = -r^2(t) \leqslant -r^2 < 0 \end{aligned} $$

for all $x \in \mathbb{R} - \{0\}$ and for all $t > -50$, where $r(t)$ is continuous and satisfies $r^2(t) \geqslant r^2 = x^2$, and r is a positive constant. Set $\alpha = r^2$. Thus, according to Theorem 1, the zero SP to (21) should be globally PAS. Here we take a suitable value $\omega = 0.9$ and initial condition $x(-50) = 100$ in system (21).

The evolution of the trajectory of system (21) is presented in Fig. 3. It demonstrates the PAS of zero SP to (21) when $\omega = 0.9$.

Remark 5. Theorem 3 provides a new tool for the analysis of Example 3 when initial time $t = -50$. We observe that the results in [5, 11, 21, 22, 31, 33] are not suitable to be applicable to (3). In contrast to these research, Theorem 3 provides new platforms for the system analysis in large scale.

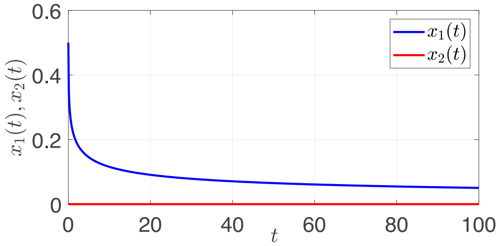

Figure 4. The PS system (22) where the values $\omega = 0.4$ and initial conditions $x_1(0) = 0.5$ and $x_2(0) = 0$ are considered.

Example 4. We consider the following nonautonomous zero initial time system:

$$\begin{aligned}
{}^{C}D^{\omega}_{0+,t}x_1(t) &= x_1(t)\bigl(x_1^2(t) + x_2^2(t) - 1\bigr) - 5\mathrm{e}^{-2t}x_1(t)x_2^3(t),\\
{}^{C}D^{\omega}_{0+,t}x_2(t) &= x_2(t)\bigl(x_1^2(t) + x_2^2(t) - 1\bigr) + 5\mathrm{e}^{-2t}x_1^2(t)x_2^2(t)
\end{aligned} \tag{22}$$

subject to $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, where $\omega \in (0, 1]$.

We demonstrate here by using $V(t, x) = x_1^2 + x_2^2$, where $x = (x_1, x_2)^{\mathrm{T}}$. Subsequently, Lemma 2 yields, along (22),

$$\begin{aligned}
{}^{C}D^{\omega}_{0+,t}V\bigl(t, x(t)\bigr) &\leqslant 2\bigl(x_1^2(t) + x_2^2(t)\bigr)\bigl(x_1^2(t) + x_2^2(t) - 1\bigr)\\
&= 2r^2(t)\bigl(r^2(t) - 1\bigr) \leqslant 2r^2\bigl(r^2 - 1\bigr) = -\alpha \leqslant 0
\end{aligned}$$

for all $x \in \Omega_1$ and for all $t > 0$, where $\Omega_1 = \{(x_1, x_2)^{\mathrm{T}}\colon\ x_1^2 + x_2^2 \leqslant 1\}$, $r(t)$ is continuous and satisfies $r^2(t) \geqslant r^2 = x_1^2 + x_2^2$, and r is a positive constant. Set $\alpha = 2r^2(1 - r^2)$. In order to show the applicability of Theorem 2, we set $\beta = \omega$. Consequently, by Theorem 2, one can immediately ensure that the trivial solution of (22) should be locally PS. For the computational demonstration, we set the value $\omega = 0.4$ and choose the initial conditions $x_1(0) = 0.5$ and $x_2(0) = 0$. The state curves (trajectories) of (22) are presented in Fig. 4, which demonstrates the predictability of (22).

Remark 6. In Example 4, Theorem 2 is shown to be effective when initial time $t = 0$. On the other hand, we find that [9, Thm. 3], [11, Prop. 6], [31, Thm. 3], and [33, Thm. 3.1] are quite interesting, and they may be applicable to (22).

Example 5. Consider a simple application of engineering control system designed problem as described in (23). Does system (23) give rise to any new behaviour? Is it possible to stabilize any such behaviour by using a linear state feedback control law? Here we set up a well-known Lorenz system as follows:

$$\begin{aligned}
{}^{C}D^{\omega}_{-33+,t}x_1(t) &= 10\bigl(x_2(t) - x_1(t)\bigr) + u_1(t),\\
{}^{C}D^{\omega}_{-33+,t}x_2(t) &= x_1(t)\bigl(28 - x_3(t)\bigr) - x_2(t) + u_2(t),\\
{}^{C}D^{\omega}_{-33+,t}x_3(t) &= x_1(t)x_2(t) - \frac{8}{3}x_3(t) + u_3(t)
\end{aligned} \tag{23}$$

with $x_i(-33) = C_i$ for $i = 1, 2, 3$, where control input $u_i(t)$ needs be designed.

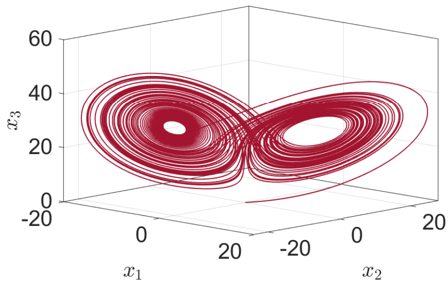

Figure 5. Unpredictability motion of system (23). A typical chaotic trajectory of the chaotic system (23) when $u_i(t) = 0$ for $i = 1, 2, 3$, where the values $\omega = 0.997$ and initial conditions $x_1(-33) = 0.01$, $x_2(-33) = 0.02$, and $x_3(-33) = 0.03$ are taken.

We take $\omega = 0.997$. Whenever the inputs are taken as $u_1(t) = u_2(t) = u_3(t) = 0$, system (23) has three stationary points: $x_{e_1} = (0, 0, 0)^{\mathrm{T}}$, $x_{e_2} = (\sqrt{72}, \sqrt{72}, 27)^{\mathrm{T}}$, $x_{e_3} = (-\sqrt{72}, -\sqrt{72}, 27)^{\mathrm{T}}$, and exhibits a chaotic trajectory (all stationary points are not PAS) as shown in Fig. 5. We design the linear control law $u_i(t) = -k_i x_i(t)$, $k_i > 0$ for $i = 1, 2, 3$, where the values of k_i are required to be chosen. By using the control law, system (23) now get simplified to

$$\begin{aligned}
{}^{C}D^{\omega}_{-33^+,t}x_1(t) &= 10\bigl(x_2(t) - x_1(t)\bigr) - k_1x_1(t),\\
{}^{C}D^{\omega}_{-33^+,t}x_2(t) &= x_1(t)\bigl(28 - x_3(t)\bigr) - x_2(t) - k_2x_2(t),\\
{}^{C}D^{\omega}_{-33^+,t}x_3(t) &= x_1(t)x_2(t) - \frac{8}{3}x_3(t) - k_3x_3(t)
\end{aligned} \tag{24}$$

with $x_i(-33) = C_i$ for $i = 1, 2, 3$. In order to analyse system (24), we invoke Theorem 1. We take autonomous function $V(t, x) = x_1^2 + x_2^2 + x_3^2$, where $x = (x_1, x_2, x_3)^{\mathrm{T}}$. By using Lemma 1 or Lemma 2, we get

$$\begin{aligned}
{}^{C}D^{\omega}_{-33^+,t}V\bigl(t, x(t)\bigr) &\leqslant 2x_1(t)\,{}^{C}D^{\omega}_{-33^+,t}x_1(t) + 2x_2(t)\,{}^{C}D^{\omega}_{-33^+,t}x_2(t)\\
&\quad + 2x_3(t)\,{}^{C}D^{\omega}_{-33^+,t}x_3(t)\\
&= -2k_1x_1^2(t) + 20x_1(t)x_2(t) - 20x_1^2\\
&\quad + 56x_1(t)x_2(t) - 2x_1(t)x_2(t)x_3(t) - 2x_2^2(t)\\
&\quad - 2k_2x_2^2(t) + 2x_3(t)x_1(t)x_2(t) - \frac{16}{3}x_3^2(t) - 2k_3x_3^2(t)\\
&\leqslant -(2k_1 - 18)x_1^2(t) - (2k_2 - 36)x_2^2(t)\\
&\quad - \left(2k_3 + \frac{16}{3}\right)x_3^2(t).
\end{aligned} \tag{25}$$

We assume the following conditions:

$$k_1 > 9, \qquad k_2 > 18, \qquad k_3 > 0. \tag{26}$$

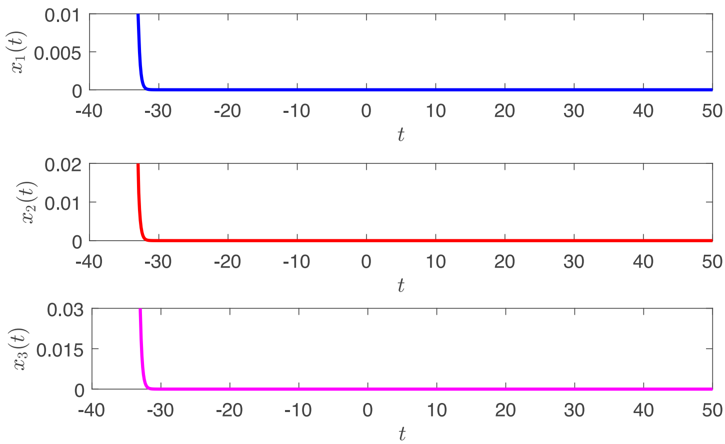

Figure 6. The PAS of chaotic system (24) with a control action, where $\omega = 0.997$, $k_1 = 10$, $k_2 = 20$, and $k_3 = 1$.

Denote by $\lambda = \{(2k_1 - 18), (2k_2 - 36), (2k_3 + 16/3)\}$. Clearly, if conditions (26) hold, one has $\lambda > 0$. Then it follows from (25) that

$$^{C}D^{\omega}_{-33+,t}V\big(t, x(t)\big) \leqslant -\lambda\big(x_1^2(t) + x_2^2(t) + x_3^2(t)\big) = -\lambda r^2(t) \leqslant -\lambda r^2 < 0$$

for all $t > a$ and for all $x \in \mathbb{R}^3 - \{0\}$, where $r(t)$ is continuous and satisfies $r^2(t) \geqslant r^2 = x_1^2 + x_2^2 + x_3^2$, and r is a positive constant. Set $\alpha = \lambda r^2$. Thus, by Theorem 1, the SP x_{e_1} of system (24) is globally PAS if the control parameters satisfy conditions (26). As a result, the control system (23) is globally PAS under the mentioned control law. We now take values of control parameters as $k_1 = 10$, $k_2 = 20$, and $k_3 = 1$. For these choices of control parameters, the numerical responses of system (24) is presented in Fig. 6, where the values $\omega = 0.997$ and initial conditions $x_1(-33) = 0.01$, $x_2(-33) = 0.02$, and $x_3(-33) = 0.03$ are taken. It indicates that the evolution of the chaotic trajectory of the state of system (24) may be controlled to a desired SP x_{e_1} under the influence of linear state feedback control law provided the control parameters satisfy the criteria given by (26).

Remark 7. In Example 5, the famous Lorenz system (see, e.g., [13, 18, 23]) exhibits a trajectory that seems to be unpredictable (irregular) at any future instant when the initial time is -33. Using the linear state feedback control methodology (see, e.g., [18–20]), we demonstrate that it is possible to stabilize the unpredictability of the motion of such a system. Theorem 1 provides a new tool that guarantees the possibility of stabilizing the irregular motion to a regular one.

Remark 8. In [6], Diethelm indicated that, in the formulation of real-world models, e.g., dynamics of a dengue fever outbreak, the biological parameters can make real-order models technically somewhat different because the dimensions of the left-hand side and right-hand side of equations in Examples 1–5 may not be the same. As a result, it is not possible to balance the system, and the exact formulation could be a new and modified one. It is obvious that all the models in Examples 1–5 are theoretical in nature and they seem dimensionally inconsistent. Since the examples are regarded without correction to exact balance between the left-hand and right-hand sides, one can use the suggested theoretical approach in practical real-world applications dealing with quantifying physical quantities while maintaining dimensional consistency.

6 Conclusions

In this paper, we have focused on the stability of nonautonomous real-order systems, where the initial time could be negative, zero, or positive. We have developed Lyapunov theorems for analysing the point stability defined in the sense of the Caputo derivative. These theorems show some new distinctive features such as new inequalities and their wider applicability as demonstrated in Section 5 in contrast to the earlier Lyapunov idea [5, 11, 21, 22, 31, 33]. We believe that these Lyapunov theorems seem like fundamental theorems for real-order systems. Indeed, we have made this claim in the sense that Theorems 1, 2 are fundamental theorems due to their crux of assumptions and proofs. We also believe that the finding of new proofs to Theorems 1 and 2 will certainly enrich the development of advanced Lyapunov theory. As a result, we term the function V associated with Theorems 1 and 2 for system (1) a fundamental function in our investigation. In the end, we have shown that this study has certain advantages in comparison to existing different versions of Lyapunov theorems.

Author contributions. Both authors have equally contributed to the study conception, design, and manuscript preparation. Both authors have read, reviewed, and approved the final manuscript.

Conflicts of interest. Both authors declare that they have no relevant financial or nonfinancial interests to disclose.

Acknowledgment. Authors are immensely grateful to the esteemed reviewers for their insightful comments, which has definitely led to the improvement of the manuscript. The Editor Prof. Linas Petkevičius is profusely thanked for allowing a revision.

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