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Convergence analysis of positive solution for Caputo–Hadamard fractional differential equation*

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Abstract. By deriving the expression of Green function and some of its special properties and establishing appropriate substitution and appropriate cone, the existence of unique iterative positive, error estimation, and convergence rate of approximate solution are obtained for singular p-Laplacian Caputo–Hadamard fractional differential equation with infinite-point boundary conditions. Nonlinearities involve derivative terms that make our analysis difficult in the course of this research, and we deal with the difficulty of derivative terms by making appropriate substitutions. An example is given to demonstrate the validity of our main results.

Keywords: Caputo–Hadamard fractional differential model, iterative positive solutions, properties of Green's function, convergence analysis.

1 Introduction

Fractional-order systems have been shown to be more accurate and realistic than integer-order models, and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly, in viscoelasticity, electrochemistry, porous media, and so on. As a result, there has been a significant development in the study of fractional differential equations in recent years; readers can refer to [9, 14–17, 25, 27, 28]. Guo et al. have also made some achievements in this regard, for example, [10–12]. In terms of

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dynamics, we also participated in some work [8, 22, 26, 29] and later prepared to build fractional-order dynamics models and establish the existence theorem of solutions. There are many ways to deal with the solution of fractional differential equations such as spectral analysis [21], Guo–Krasnoselskii's fixed point theorem [20], mixed monotone operator method [24], Bohnenblust–Karlin FP approaches [13], the mountain pass theorem [6], Mawhin's continuation theorem [7], and so on. Up till now, most of the results are obtained in the sense of fractional derivatives such as Caputo and Riemann–Liouville, and there are few models under the Caputo–Hadamard fractional derivatives. Compared with Caputo and Riemann–Liouville fractional derivative, the Caputo–Hadamard fractional-order derivative contains logarithmic function of arbitrary order, which is invariant to dilation on the half-axis. Boutiara [4] studied the following Caputo–Hadamard fractional differential equation:

$$ {}^{CH}D^{r}_{1+}x(t) = f\bigl(t, x(t)\bigr), \quad t \in J = [1, T],\ 0 < r \leqslant 1, $$

with three-point boundary condition

$$ \alpha x(1) + \beta x(T) = \lambda I^{q}_{1+}x(\eta) + \delta, \quad q \in (0, 1], $$

where ${}^{CH}D^{r}_{1+}$ denotes the Caputo–Hadamard fractional derivative, and I^{q}_{1+} denotes the standard Hadamard fractional integral, $0 < r, q \leqslant 1$, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, α, β, λ, and δ are real constants, and $\eta \in (1, T)$. The authors obtained the uniqueness results by means of Boyd and Wong's and Banach's fixed point theorems. Ardjouni [3] discussed the existence and uniqueness of positive solutions of the nonlinear fractional differential equation ${}^{CH}D^{\alpha}_{1+}y(t) = f(t, x(t))$, $t \in J$, with integral boundary conditions $y(1) = b\int_1^{\mathrm{e}} y(s)\,\mathrm{d}s + d$, where $J = [1, \mathrm{e}]$, ${}^{CH}D^{\alpha}_{1+}$ denotes the Caputo–Hadamard fractional derivative, $0 < \alpha \leqslant 1$, $b \geqslant 0$, $d > 0$, and $f : J \times [0, \infty) \to [0, \infty)$ is a given continuous function. The authors discussed the existence and uniqueness of positive solutions by some methods. Derbazi [5] discussed the following Caputo–Hadamard fractional differential problem:

$$ \begin{aligned} &{}^{CH}D^{\alpha}_{1+}u(t) = f\bigl(t, u(t)\bigr), \quad t \in J := [1, T], \\ &a_1 u(1) + b_1 {}^{CH}D^{\gamma}_{1+}u(1) = \lambda_1^{H} I^{\sigma_1}_{1+}u(\eta_1), \quad 1 < \eta_1 < T,\ \sigma_1 > 0, \\ &a_2 u(T) + b_2 {}^{CH}D^{\gamma}_{1+}u(T) = \lambda_2^{H} I^{\sigma_2}_{1+}u(\eta_2), \quad 1 < \eta_1 < T,\ \sigma_2 > 0, \end{aligned} $$

where ${}^{CH}D^{\mu}_{1+}$ is the Caputo–Hadamard fractional derivative of order $\mu \in \{\alpha, \gamma\}$, $1 < \alpha \leqslant 2$, $0 < \gamma \leqslant 1$, ${}^{H}I^{\theta}_{1}$ is the Hadamard fractional integral of order $\theta > 0$, $\theta \in \{\sigma_1, \sigma_2\}$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and a_i, b_i, λ_i, $i = 1, 2$, are suitably chosen real constants. In [23], Makhlouf consider the following Caputo–Hadamard fractional differential equation:

$$ {}^{CH}D^{\alpha}_{1+}\varrho(\vartheta) = f\bigl(\vartheta, \varrho(\vartheta)\bigr) + g\bigl(\vartheta, \varrho(\vartheta)\bigr)\frac{\mathrm{d}W(\vartheta)}{\mathrm{d}\vartheta}, $$

where the initial condition is $\varrho(1) = \omega$, $f : [1, \varLambda] \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable. The authors get the existence and uniqueness of solution of Caputo–Hadamard fractional stochastic differential equations via the Banach fixed point method and the Ulam–Hyers stability, and this equation is analyzed using the generalized and the classical Gronwall inequalities.

Motivated by the excellent results above, this paper will be devoted to considering the infinite-point singular p-Laplacian Caputo–Hadamard fractional differential equation

$$ {}^{H}D_{1^+}^{\beta}\big(\varphi_p\big({}^{CH}D_{1^+}^{\gamma}u\big)\big)(t)+f\big(t,u(t),u'(t)\big)=0,\quad 1<t<\mathrm{e}, \tag{1} $$

under infinite-point boundary condition

$$ \begin{aligned} &u(1)=u'(1)=\cdots=u^{(i-1)}(1)=u^{(i+1)}(1)=\cdots=u^{(n-1)}(1)=0,\\ &u^{(i)}(1)=\sum_{j=1}^{\infty}\eta_j u(\xi_j),\qquad {}^{CH}D_{1^+}^{\gamma}u(1)=0,\\ &\varphi_p\big({}^{CH}D_{1^+}^{\gamma}u(\mathrm{e})\big)=\sum_{j=1}^{\infty}\zeta_j\varphi_p\big({}^{CH}D_{1^+}^{\gamma}u(\xi_j)\big), \end{aligned} \tag{2} $$

where $\beta,\gamma\in\mathbb{R}_+=[0,+\infty)$, $1<\beta\leqslant 2$, $n-1<\gamma\leqslant n$ $(n\geqslant 3)$, $\gamma>i$, p-Laplacian operator φ_p is defined as $\varphi_p(s)=|s|^{p-2}s$, $p,q>1$, $1/p+1/q=1$, and $0<\eta_i,\zeta_i<1$, $1<\xi_i<\mathrm{e}$ $(i=1,2,\dots,\infty)$, $f\in C((1,\mathrm{e})\times\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$ $(\mathbb{R}_+=[0,+\infty))$, $f(t,x_1,x_2)$ has singularity at $t=1,\mathrm{e}$, and ${}^{CH}D_{1^+}^{\beta}u$ is the standard Caputo–Hadamard derivative.

In this paper, the iterative scheme converging to the unique solution will be constructed, and then the estimates on the error and the convergence rate of approximate solution are also obtained. Compared with [30], the equation in this paper is p-Laplacian fractional differential equation, and the method which we used in this paper is iterative sequence. Compared with [18], derivative is involved in the nonlinear terms for BVP (1)–(2), and iterative positive solutions are obtained for BVP (1)–(2).

2 Preliminaries and lemmas

In this section, we introduce definitions and preliminary results, which are used throughout this paper. Now we list a condition below to be used later in the paper.

(H_0) $f:(1,\mathrm{e})\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous and nondecreasing in the second and the third variables, and there exists $r,\sigma\in(0,1)$ such that for all $(t,x,y)\in(1,\mathrm{e})\times[0,+\infty)\times[0,+\infty)$, we have

$$ f(t,rx,ry)\geqslant r^{\sigma}f(t,x,y). \tag{$*$} $$

Remark 1. If (H_0) holds, then for any $c\geqslant 1$, $\sigma\in(0,1)$, and $(t,x,y)\in(1,\mathrm{e})\times[0,+\infty)\times[0,+\infty)$, inequality $(*)$ is equivalent to

$$ f(t,cx,cy)\leqslant c^{\sigma}f(t,x,y). \tag{$**$} $$

Now, we state some lemmas, which are basic and used in this paper.

Definition 1. (See [2, 19].) The Caputo–Hadamard fractional derivative of order $\beta > 0$ of a continuous function $y \in AC([1,T],\mathbb{R})$ is given by

$$ {}^{CH}D_{1^+}^{\beta} y(t) = \frac{1}{\Gamma(n-\beta)} \int\limits_1^t \left(\ln\frac{t}{s}\right)^{\beta-n+1} \left(s\frac{\mathrm{d}}{\mathrm{d}s}\right)^n y(s)\,\frac{\mathrm{d}s}{s}, $$

where $n = [\beta]+1$, $[\beta]$ denotes the integer part of the number β, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2. (See [1, 19].) Let $a > 0$, then the Hadamard-type fractional left integral of order $\beta > 0$ of a function $h : [a,\infty) \to \mathbb{R}$, $h \in L^1[a,\infty)$, is defined by

$$ {}^{H}I_{a^+}^{\beta} h(t) = \frac{1}{\Gamma(\beta)} \int\limits_a^t \left(\ln\frac{t}{\varsigma}\right)^{\beta-1} h(\varsigma)\,\frac{\mathrm{d}\varsigma}{\varsigma}, \quad t \geqslant a. $$

Definition 3. (See [1, 19].) Let $a > 0$, $h : [a,\infty) \to \mathbb{R}$, $t^{n-1}h^{(n-1)}(t) \in AC[a,\infty)$, $n \in \mathbb{N}$, $\beta \in (n-1, n)$. Then the Hadamard fractional left derivative of from $[a,+\infty)$ is defined by

$$ {}^{H}D_{a^+}^{\beta} h(t) = \frac{1}{\Gamma(n-\beta)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int\limits_a^t \left(\ln\frac{t}{\varsigma}\right)^{n-\beta-1} h(\varsigma)\,\frac{\mathrm{d}\varsigma}{\varsigma}, \quad t > a. $$

Lemma 1. (*See* [19].) *Let* $n-1 < \beta \leqslant n$, $n \in \mathbb{N}$, *and* $x \in C^n([1,T])$. *Then*

$$ \left({}^{CH}D_{1^+}^{\beta}\, {}^{H}I_{1^+}^{\beta} x\right)(t) = (x)(t), $$
$$ \left({}^{H}I_{1^+}^{\beta}\, {}^{CH}D_{1^+}^{\beta} x\right)(t) = x(t) + \Sigma_{k=0}^{n-1} c_k (\ln t)^k, $$

where $\beta, c_k \in \mathbb{R}$, $k = 1, 2, \dots, n-1$.

Lemma 2. (*See* [1, 19].) *For* $\beta \in (n-1, n)$, $n \in \mathbb{N}$, $h \in L[a,\infty)$, $a > 0$, *the fractional equation* ${}^{H}D_{a^+}^{\beta} x(t) + \omega(t) = 0$, $t > a$, *has the following expression of solution:*

$$ x(t) = \sum_{i=1}^{n} \chi_i \left(\ln\frac{t}{a}\right)^{\beta-i} - \frac{1}{\Gamma(\beta)} \int\limits_a^t \left(\ln\frac{t}{\varsigma}\right)^{\beta-1} \omega(\varsigma)\,\frac{\mathrm{d}\varsigma}{\varsigma}, \quad t \geqslant a, $$

where $\chi_i \in \mathbb{R}$, $k = 1, 2, \dots, n$.

Let $u(t) = \int_1^t v(t)\,\mathrm{d}t/t$, $v(t) \in C[1,\mathrm{e}]$. Then BVP (1)–(2) reduces to the following modified boundary value problem:

$$ {}^{H}D_{1^+}^{\beta}\left(\varphi_p\left({}^{H}D_{1^+}^{\gamma-1} v(t)\right)\right) + f\left(t, \int\limits_1^t v(t)\,\frac{\mathrm{d}t}{t}, v(t)\right) = 0, \quad 1 < t < \mathrm{e}, \tag{3} $$

with nonlocal boundary conditions

$$\begin{aligned} &v(1) = v'(1) = \cdots = v^{(i-2)}(1) = v^{(i)}(1) = \cdots = v^{(n-2)}(1) = 0, \\ &v^{(i-1)}(\mathrm{e}) = \sum_{j=1}^{\infty} \eta_j v(\xi_j), \qquad {}^{CH}D_{1^+}^{\gamma-1} v(1) = 0, \\ &\varphi_p\big({}^{CH}D_{1^+}^{\gamma-1} v(\mathrm{e})\big) = \sum_{i=1}^{\infty} \zeta_j \varphi_p\big({}^{CH}D_{1^+}^{\gamma-1} v(\xi_j)\big). \end{aligned} \tag{4}$$

Lemma 3. *Given $y \in L^1[1,\mathrm{e}] \cap C(1,\mathrm{e})$, then the solution of the equation*

$$ {}^{CH}D_{1^+}^{\gamma-1} v(t) + y(t) = 0, \quad 1 < t < \mathrm{e}, \tag{5}$$

with boundary condition $v^{(i-1)}(\mathrm{e}) = \sum_{j=1}^{\infty} \eta_j v(\xi_j)$ can be expressed by

$$v(t) = \int_1^{\mathrm{e}} G(t,s) y(s) \,\frac{\mathrm{d}s}{s}, \quad t \in [1,\mathrm{e}], \tag{6}$$

where

$$G(t,s) = \frac{1}{\Delta\Gamma(\gamma-1)} \begin{cases} (\ln t)^{i-1}\Gamma(\gamma-1)P(s)(\ln\frac{\mathrm{e}}{s})^{\gamma-i-1} - \Delta(\ln\frac{t}{s})^{\gamma-2}, \\ \quad 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ (\ln t)^{i-1}\Gamma(\gamma-1)P(s)(\ln\frac{\mathrm{e}}{s})^{\gamma-i-1}, \quad 1 \leqslant t \leqslant s \leqslant \mathrm{e}, \end{cases}$$

in which

$$P(s) = \frac{1}{\Gamma(\gamma-i)} - \frac{1}{\Gamma(\gamma-1)} \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\ln\frac{\xi_j}{s}}{\ln\frac{\mathrm{e}}{s}}\right)^{\gamma-2} \left(\ln\frac{\mathrm{e}}{s}\right)^{i-1}, \tag{7}$$

$$\Delta = (i-1)! - \sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^{i-1}. \tag{8}$$

Proof. By means of Lemma 1, we reduce (5) to an equivalent integral equation

$$\begin{aligned} v(t) = {}&-{}^{H}I_{1^+}^{\gamma-1} y(t) + C_1 + C_2(\ln t) + \cdots + C_{i-1}(\ln t)^{i-2} + C_i(\ln t)^{i-1} \\ &+ C_{i+1}(\ln t)^{i} + \cdots + C_n(\ln t)^{n-2} \end{aligned}$$

for $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. From $v(1) = 0$ we have $C_1 = 0$. Taking the derivative of the above formula, we have

$$\begin{aligned} v'(t) = {}&-{}^{H}I_{1^+}^{\gamma-2} y(t) + C_2\frac{1}{t} + \cdots + C_{i-1}(i-2)(\ln t)^{i-3}\frac{1}{t} + (i-1)C_i\frac{1}{t}(\ln t)^{i-2} \\ &+ \cdots + C_n\frac{1}{t}(n-2)(\ln t)^{n-3}. \end{aligned}$$

From $v'(1) = 0$ we have $C_2 = 0$. Taking the derivative of this formula step by step and combining with $v''(1) = \cdots = v^{(i-2)}(1) = v^{(i)}(1) = \cdots = v^{(n-2)}(1) = 0$, we have $C_j = 0$, $j \geqslant 3$, $j \neq i$. Consequently, we get

$$v(t) = C_i(\ln t)^{i-1} - {}^{H}I_{1+}^{\gamma-1}y(t),$$

hence, we have

$$v^{(i-1)}(\mathrm{e}) = (i-1)!C_i - {}^{H}I_{1+}^{\gamma-i}y(\mathrm{e}). \tag{9}$$

On the other hand, combining $v^{(i-1)}(\mathrm{e}) = \sum_{j=1}^{\infty}\eta_j v(\xi_j)$ with (9), we get

$$\begin{aligned} C_i &= \int_1^{\mathrm{e}} \frac{(\ln\frac{\mathrm{e}}{s})^{\gamma-i-1}}{\Gamma(\gamma-i)\Delta} y(s)\,\frac{\mathrm{d}s}{s} - \sum_{j=1}^{\infty}\eta_j \int_1^{\xi_j} \frac{(\ln\frac{\xi_j}{s})^{\gamma-2}}{\Gamma(\gamma-1)\Delta} y(s)\,\frac{\mathrm{d}s}{s} \\ &= \int_1^{\mathrm{e}} \frac{(\ln\frac{\mathrm{e}}{s})^{\gamma-i-1}P(s)}{\Delta} y(s)\,\frac{\mathrm{d}s}{s}, \end{aligned}$$

where $P(s)$ is as in (7), and Δ is as in (8). Hence,

$$\begin{aligned} v(t) &= C_{i+1}(\ln t)^{i-1} - {}^{H}I_{1+}^{\gamma-1}y(t) \\ &= -\int_1^{t} \frac{\Delta(\ln\frac{t}{s})^{\gamma-2}}{\Gamma(\gamma-1)\Delta} y(s)\,\frac{\mathrm{d}s}{s} + \int_1^{\mathrm{e}} \frac{(\ln\frac{\mathrm{e}}{s})^{\gamma-i-1}(\ln t)^{i-1}P(s)}{\Delta} y(s)\,\frac{\mathrm{d}s}{s}. \end{aligned}$$

Therefore, $G(t,s)$ is as in (6). □

Lemma 4. *Let $f \in C((1,\mathrm{e}] \times (0,+\infty)^2, [0,+\infty))$. Then BVP* (3)–(4) *has a unique solution*

$$v(t) = \int_1^{\mathrm{e}} G(t,s)\varphi_q\left(\int_1^{\mathrm{e}} H(s,\tau) f\big(\tau, {}^{H}I_{1+}v(\tau), v(\tau)\big)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \tag{10}$$

where $G(t,s)$ is as in (6)*, and*

$$H(t,s) = \frac{1}{\overline{\Delta}\Gamma(\beta)}\begin{cases} \Gamma(\beta)Q(s)(\ln t)^{\beta-1}(\ln\frac{\mathrm{e}}{s})^{\beta-1} - \Delta(\ln\frac{t}{s})^{\beta-1}, \\ \quad 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ \Gamma(\beta)Q(s)(\ln t)^{\beta-1}(\ln\frac{\mathrm{e}}{s})^{\beta-1}, \quad 1 \leqslant t \leqslant s \leqslant \mathrm{e}, \end{cases}$$

in which

$$Q(s) = \frac{1}{\Gamma(\beta)} - \frac{1}{\Gamma(\beta)}\sum_{s\leqslant\xi_j}\zeta_j\left(\frac{\ln\frac{\xi_j}{s}}{\ln\frac{\mathrm{e}}{s}}\right)^{\gamma-2}, \qquad \overline{\Delta} = 1 - \sum_{j=1}^{\infty}\zeta_j(\ln\xi_j)^{\beta-1}. \tag{11}$$

Proof. Let $\overline{y} \in C[1, \mathrm{e}], \widehat{v}(t) = \varphi_p({}^H D_{1^+}^{\gamma-1} v)(t)$. Consider the boundary value problem

$$\begin{aligned} &{}^H D_{1^H}^{\beta} \widehat{v}(t) + \overline{y}(t) = 0, \quad 1 < t < \mathrm{e}, \\ &\widehat{v}(1) = 0, \qquad v(\mathrm{e}) = \sum_{j=1}^{\infty} \zeta_j \widehat{v}(\xi_j). \end{aligned} \tag{12}$$

By means of the Lemma 2, we reduce (12) to an equivalent integral equation

$$\widehat{v}(t) = -{}^H I_{1^+}^{\beta} \overline{y}(t) + C_1 (\ln t)^{\beta-1} + C_2 (\ln t)^{\beta-2}.$$

From $\widehat{v}(1) = 0$ we have $C_2 = 0$. Consequently, we get

$$\widehat{v}(t) = C_1 (\ln t)^{\beta-1} - {}^H I_{1^+}^{\beta} \overline{y}(t). \tag{13}$$

On the other hand, $\widehat{v}(1) = \sum_{j=1}^{\infty} \zeta_j \widehat{v}(\xi_j)$, and combining this with (13), we get

$$\begin{aligned} C_1 &= \int_1^{\mathrm{e}} \frac{(\ln \frac{\mathrm{e}}{s})^{\beta-1}}{\Gamma(\beta)\overline{\Delta}} \overline{y}(s) \frac{\mathrm{d}s}{s} - \sum_{j=1}^{\infty} \zeta_j \int_1^{\xi_j} \frac{(\ln \frac{\xi_j}{s})^{\beta-1}}{\Gamma(\beta)\overline{\Delta}} \overline{y}(s) \frac{\mathrm{d}s}{s} \\ &= \int_0^1 \frac{(\ln \frac{\mathrm{e}}{s})^{\beta-1} Q(s)}{\overline{\Delta}} \overline{y}(s) \frac{\mathrm{d}s}{s}, \end{aligned}$$

where $Q(s)$ and $\overline{\Delta}$ are as in (11). Hence,

$$\begin{aligned} \widehat{v}(t) &= C_1 (\ln t)^{\beta-1} - {}^H I_{1^+}^{\beta} y(t) \\ &= -\int_1^t \frac{\overline{\Delta}\left(\ln \frac{t}{s}\right)^{\beta-1}}{\Gamma(\beta)\overline{\Delta}} \overline{y}(s) \,\mathrm{d}s + \int_1^{\mathrm{e}} \frac{(\ln \frac{\mathrm{e}}{s})^{\beta-1} Q(s)}{\overline{\Delta}} \overline{y}(s) \,\mathrm{d}s. \end{aligned}$$

Therefore, $H(t, s)$ is as in (10). □

Lemma 5. *The Green function* (6) *has the following property*:

$$\Delta (\ln t)^{i-1} g(s) \leqslant G(t, s) \leqslant a (\ln t)^{i-1}, \quad t, s \in [1, \mathrm{e}].$$

where $g(s) = (\ln(\mathrm{e}/s))^{\gamma-i-1}[1 - (\ln(e/s))^{i-1}]$, $a = 1/\Delta\Gamma(\gamma - i)$.

Proof. First, we prove

$$\begin{aligned} \Delta (\ln t)^{i-1} \left(\ln \frac{\mathrm{e}}{s}\right)^{\gamma-i-1} \left[1 - \left(\ln \frac{\mathrm{e}}{s}\right)^{i-1}\right] &\leqslant \Delta\Gamma(\gamma - 1) G(t, s) \\ &\leqslant (\ln t)^{i-1} \Gamma(\gamma - 1) P(s) \left(\ln \frac{\mathrm{e}}{s}\right)^{\gamma-i-1}. \end{aligned} \tag{14}$$

By direct calculation, we get $P'(s) \geqslant 0$, $s \in [1, \mathrm{e}]$, and so $P(s)$ is nondecreasing with respect to s. For $s \in [1, \mathrm{e}]$, $\gamma - 1 > i$, we get

$$\begin{aligned}
&\Gamma(\gamma-1)P(s)\\
&\quad = (\gamma-2)(\gamma-3)\cdots(\gamma-i) - \sum_{s\leqslant\xi_j}\eta_j\left(\frac{\ln\frac{\xi_j}{s}}{\ln\frac{\mathrm{e}}{s}}\right)^{\gamma-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{i-1}\\
&\quad \geqslant \Gamma(\gamma-1)P(1) = (\gamma-2)(\gamma-3)\cdots(\gamma-i) - \sum_{j=1}^{\infty}\eta_j(\ln\xi_j)^{\gamma-1}\\
&\quad \geqslant (i-1)! - \sum_{j=1}^{\infty}\eta_j(\ln\xi_j)^{i} = \Delta.
\end{aligned} \tag{15}$$

Then we prove (14). The right inequality of (14) is trivial. We have only to prove the left inequality. If $1 \leqslant s \leqslant t \leqslant \mathrm{e}$, we have $\ln t - \ln s \leqslant \ln t - \ln t \ln s = (1 - \ln s)\ln t$, which implies that

$$\left(\ln\frac{t}{s}\right)^{\gamma-2} \leqslant \left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-2}(\ln t)^{\gamma-2}.$$

Then, by $\gamma - 1 > i$ and by (15), one has

$$\begin{aligned}
\Delta\Gamma(\gamma-1)G(t,s) &= (\ln t)^{i-1}P(s)\Gamma(\gamma-1)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1} - \Delta\left(\ln\frac{t}{s}\right)^{\gamma-2}\\
&\geqslant \Delta\left[(\ln t)^{i-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1} - \Delta\left(\ln\frac{t}{s}\right)^{\gamma-2}\right]\\
&\geqslant \Delta\left[(\ln t)^{i-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1} - \left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-2}(\ln t)^{\gamma-2}\right]\\
&\geqslant \Delta(\ln t)^{i-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1}\left[1 - \left(\ln\frac{\mathrm{e}}{s}\right)^{i-1}\right].
\end{aligned}$$

If $1 \leqslant t \leqslant s \leqslant \mathrm{e}$, by (15), then we have

$$\begin{aligned}
\Delta\Gamma(\gamma-1)G(t,s) &= (\ln t)^{i-1}\Gamma(\gamma-1)P(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1}\\
&\geqslant \Delta\left[(\ln t)^{i-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1} - \Delta\left(\ln\frac{t}{s}\right)^{\gamma-2}\right]\\
&\geqslant \Delta\left[(\ln t)^{i-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1} - \left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-2}(\ln t)^{\gamma-2}\right]\\
&\geqslant \Delta(\ln t)^{i-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-i-1}\left[1 - \left(\ln\frac{\mathrm{e}}{s}\right)^{i-1}\right].
\end{aligned}$$

So the left inequality of (14) is proved. Moreover, by $P(s) \leqslant 1/\Gamma(\gamma - i)$, we easily get Lemma 4. □

Lemma 6. *Let* $\overline{\Delta} > 0$*, then the Green functions defined by* (6) *satisfies*:

(i) $H : [1, \mathrm{e}] \times [1, \mathrm{e}] \to \mathbb{R}_+$ *is continuous, and* $H(t, s) > 0$ *for all* $t, s \in (1, \mathrm{e})$;

(ii) *The following inequality holds*:

$$(\ln t)^{\beta-1} H(\mathrm{e}, s) \leqslant H(t, s) \leqslant H(\mathrm{e}, s), \quad t, s \in [1, \mathrm{e}],$$

in which

$$H(\mathrm{e}, s) = \frac{1}{\overline{\Delta}}\left(Q(s) - \frac{\overline{\Delta}}{\Gamma(\beta)}\right)(\ln \mathrm{e} - \ln s)^{\beta-1}.$$

Proof. The proof is similar to that of Lemma 3 in [31], we omit it here. □

Now we define a Banach space E with the norm

$$\|u\| = \max\{u(t), u'(t)\},$$

a cone K on $C[1, \mathrm{e}]$, and an operator as follow. Let

$$K = \{v \in E\colon\ v(t) \geqslant 0,\ t \in [1, \mathrm{e}]\},$$

then K is a normal cone with normality constants 1 in the Banach space E. Next, we define a subset of K by

$$K_{\mathrm{e}} = \{v(t) \in K\colon \text{ there exist two numbers } D_v > 1 > d_v > 0 \\ \text{ such that } d_v \mathrm{e}(t) \leqslant v(t) \leqslant D_v \mathrm{e}(t)\},$$

where $\mathrm{e}(t) = (\ln t)^{i-1}$, and

$$Av(t) = \int_1^{\mathrm{e}} G(t, s)\varphi_q\left(\int_1^{\mathrm{e}} H(s, \tau) f\left(\tau,\ {}^{H}I_{1+}v(\tau),\ v(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}. \tag{16}$$

Problem (3)–(4) has a positive solution if and only if v is a fixed point of A in K_{e}, that is, problem (1)–(2) has a positive solution u.

3 Main results

To ensure the validity of Lemma 3, we need the following assumption.

(H_1) For $0 < \eta_i, \zeta_i < 1$, $1 < \xi_i < \mathrm{e}$ ($i = 1, 2, \ldots, \infty$), we have

$$\sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^{i-1} < (i-1)!, \qquad \sum_{j=1}^{\infty} \zeta_j (\ln \xi_j)^{\beta-1} < 1.$$

Theorem 1. *Suppose conditions* (H_0)*,* (H_1) *hold. If*

$$0 < \int_1^{\mathrm{e}} f\bigl(\tau, (\ln\tau)^i, (\ln\tau)^{i-1}\bigr) \frac{\mathrm{d}\tau}{\tau} < +\infty, \tag{17}$$

then we have the following conclusions:

(i) *The p-Laplacian fractional-order differential problem* (3)–(4) *has a positive solution $v^*(t) \in K$. Let $u^*(t) = \int_1^t v^*(t)\,\mathrm{d}t/t$, $v(t) \in K$, that is, BVP* (1)–(2) *has a positive solution $u^*(t) \in K_{\overline{\mathrm{e}}}$, where*

$$K_{\overline{\mathrm{e}}} = \biggl\{ u(t) \in K\text{: there exist two numbers } D_u > 1 > d_u > 0 \\ \text{such that } \frac{d_u}{i}\overline{\mathrm{e}}(t) \leqslant u(t) \leqslant \frac{D_u}{i}\overline{\mathrm{e}}(t) \biggr\};$$

(ii) *For any initial value $v_0 \in K_{\mathrm{e}}$, the sequence of functions defined by*

$$v_n = \int_1^{\mathrm{e}} G(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} H(s,\tau) f\bigl(\tau, {}^H I_{1^+} v_{n-1}(\tau), v_{n-1}(\tau)\bigr) \frac{\mathrm{d}\tau}{\tau} \Biggr) \frac{\mathrm{d}s}{s}, \quad n \geqslant 1,$$

converges uniformly to $v^*(t)$ on $[1, \mathrm{e}]$ as $n \to +\infty$, the corresponding sequence $u_n^*(t)$ converges uniformly to $u^*(t)$, and $u^*(t) = \int_1^t v^*(t)\,\mathrm{d}t/t$, $v(t) \in K_{\mathrm{e}}$;

(iii) *There exists an error estimation*

$$\|v_n - v^*\| \leqslant \frac{2}{\sqrt{\epsilon}}\bigl(1 - \epsilon^{(q-1)^n}\bigr)\|v_0\|,$$

that is, $\|u_n - u^*\| \leqslant (2/\sqrt{\epsilon})(1 - \epsilon^{(q-1)^n})\|u_0\|$ and $u_0 = \int_1^t v_0(t)\,\mathrm{d}t/t$, which has the rate of convergence

$$\|v_n - v^*\| = o\bigl(1 - \epsilon^{(q-1)^n}\bigr) \quad \text{as } n \to +\infty,$$

that is,

$$\|u_n - u^*\| = o\bigl(1 - \epsilon^{(q-1)^n}\bigr) \quad \text{as } n \to +\infty,$$

where $0 < \epsilon < 1$ is a positive constant which is determined by the initial value v_0;

(iv) *The positive solution is unique.*

Proof. First, by (16), we know that a fixed point v of the operator A is a solution of the fractional-order integro–differential problem (3)–(4), and let $u(t) = \int_1^t v(t)\,\mathrm{d}t/t$, then $u(t)$ is a solution of the p-Laplacian fractional-order differential problem (1)–(2).

In what follows, we prove that A is well defined and $A : K_{\mathrm{e}} \to K_{\mathrm{e}}$. In fact, according to the definition of K_{e}, for any $v \in K_{\mathrm{e}}$, there are two numbers $0 < d_v < 1 < D_v$ such that

$$d_v\mathrm{e}(t) \leqslant v(t) \leqslant D_v\mathrm{e}(t), \quad t \in [1, \mathrm{e}], \tag{18}$$

hence,

$$
\begin{aligned}
{}^{H}I_{1^+}v(t) &= \int_1^t v(s)\,\frac{\mathrm{d}s}{s} \leqslant D_v(\ln s)^{i-1}\,\frac{\mathrm{d}s}{s} = D_v\int_1^t (\ln s)^{i-1}\,\mathrm{d}(\ln s) \\
&= \frac{D_v}{i}(\ln t)^i, \qquad (19)\\
{}^{H}I_{1^+}v(t) &= \int_1^t v(s)\,\frac{\mathrm{d}s}{s} \geqslant d_v(\ln s)^{i-1}\,\frac{\mathrm{d}s}{s} = d_v\int_1^t (\ln s)^{i-1}\,\mathrm{d}(\ln s) \\
&= \frac{d_v}{i}(\ln t)^i. \qquad (20)
\end{aligned}
$$

Thus, applying Lemmas 5, 6, formulas (17)–(19), and (H_0), we get

$$
\begin{aligned}
&\int_1^{\mathrm{e}} G(t,s)\varphi_q\biggl(\int_1^{\mathrm{e}} H(s,\tau)f\bigl(\tau,\,{}^{H}I_{1^+}v(\tau),\,v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\frac{\mathrm{d}s}{s}\\
&\quad\leqslant \int_1^{\mathrm{e}} a(\ln t)^{i-1}\varphi_q\biggl(\int_1^{\mathrm{e}} H(\mathrm{e},s)f\biggl(\tau,\,\frac{D_v}{i}(\ln\tau)^i,\,D_v(\ln\tau)^{i-1}\biggr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\,\frac{\mathrm{d}s}{s}\\
&\quad\leqslant a\int_1^{\mathrm{e}} \biggl(\frac{(D_v)^\sigma}{\overline{\Delta}\Gamma(\beta)}\biggr)^{q-1}\biggl(\int_1^{\mathrm{e}} f\bigl(\tau,\,(\ln\tau)^i,\,(\ln\tau)^{i-1}\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)^{q-1}\,\frac{\mathrm{d}s}{s} < +\infty. \qquad (21)
\end{aligned}
$$

On the other hand, by Lemmas 5, 6 and formulas (17), (18), and (20), we get

$$
\begin{aligned}
&\int_1^{\mathrm{e}} G(t,s)\varphi_q\biggl(\int_1^{\mathrm{e}} H(s,\tau)f\bigl(\tau,\,{}^{H}I_{1^+}v(\tau),\,v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\,\frac{\mathrm{d}s}{s}\\
&\quad\geqslant \int_1^{\mathrm{e}} \Delta(\ln t)^{i-1}g(s)\varphi_q\biggl(\int_1^{\mathrm{e}} (\ln s)^{\beta-1}H(\mathrm{e},\tau)f\bigl(\tau,\,{}^{H}Iv(\tau),\,v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\frac{\mathrm{d}s}{s}\\
&\quad= \int_1^{\mathrm{e}} \Delta(\ln t)^{i-1}g(s)(\ln s)^{(q-1)(\beta-1)}\\
&\qquad\times\varphi_q\biggl(\int_1^{\mathrm{e}} H(\mathrm{e},\tau)f\bigl(\tau,\,\frac{d_v}{i}(\ln\tau)^i,\,d_v(\ln\tau)^{i-1}\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\,\frac{\mathrm{d}s}{s}\\
&\quad= \int_1^{\mathrm{e}} \Delta(\ln t)^{i-1}g(s)(\ln s)^{(q-1)(\beta-1)}\biggl(\frac{\sigma d_v}{i}\biggr)^{q-1}\\
&\qquad\times\varphi_q\biggl(\int_1^{\mathrm{e}} H(\mathrm{e},\tau)f\bigl(\tau,\,(\ln\tau)^i,\,(\ln\tau)^{i-1}\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$= \int_1^{\mathrm{e}} \Delta g(s)(\ln s)^{(q-1)(\beta-1)}\left(\frac{\sigma d_v}{i}\right)^{q-1} \\ \times \varphi_q\left(\int_1^{\mathrm{e}} H(\mathrm{e},\tau) f\bigl(\tau,\,(\ln\tau)^i,\,(\ln\tau)^{i-1}\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\cdot(\ln t)^{i-1}. \tag{22}$$

Formulas (21), (22) yield that A is well defined, and $A(K_{\mathrm{e}}) \subset K_{\mathrm{e}}$. Now given $v_0 \in K_{\mathrm{e}}$, there exist four positive constants d_{v_0}, D_{v_0}, $\widetilde{d}_{v_0}$, and $\widetilde{D}_{v_0}$ such that

$$d_{v_0}\mathrm{e}(t) \leqslant v_0 \leqslant D_{v_0}\mathrm{e}(t), \qquad \widetilde{d}_{v_0}\mathrm{e}(t) \leqslant Av_0 \leqslant \widetilde{D}_{v_0}\mathrm{e}(t). \tag{23}$$

Then from (23) we have

$$\frac{\widetilde{d}_{v_0}}{D_{v_0}} v_0 \leqslant Av_0 \leqslant \frac{\widetilde{D}_{v_0}}{d_{v_0}} v_0.$$

For $\sigma < 1/(q-1)$, we choose a constant t_0 such that

$$1 < t_0 \leqslant \min\left\{\exp\left\{\left(\frac{\widetilde{d}_{v_0}}{D_{v_0}}\right)^{1/(1-\sigma(q-1))}\right\},\, \exp\left\{\left(\frac{\widetilde{D}_{v_0}}{d_{v_0}}\right)^{1/(1-\sigma(q-1))}\right\}\right\},$$

and for above σ, we have

$$(\ln t_0)^{1-\sigma(q-1)} v_0 \leqslant Av_0 \leqslant \left(\frac{1}{\ln t_0}\right)^{1-\sigma(q-1)} v_0. \tag{24}$$

Take $x_0 = v_0 \ln t_0$, $y_0 = v_0/\ln t_0$, $t_0 \in (1, \mathrm{e})$. Clearly, $x_0 \leqslant y_0$. Now we define the iterative sequence as follows:

$$x_n = Ax_{n-1}, \quad y_n = Ay_{n-1}, \quad n = 1, 2, \dots .$$

By (H_0), we know that A is an increasing operator in v, and

$$\begin{aligned} A(rv) &= \int_1^{\mathrm{e}} G(t,s)\varphi_q\left(\int_1^{\mathrm{e}} H(s,\tau) f\bigl(\tau,\, {}^H I_{1+} rv(\tau),\, rv(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &\geqslant r^{\sigma(q-1)} \int_1^{\mathrm{e}} G(t,s)\varphi_q\left(\int_1^{\mathrm{e}} H(s,\tau) f\bigl(\tau,\, {}^H I_{1+} v(\tau),\, v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &= r^{\sigma(q-1)} Tv, \quad 0 < r < 1, \end{aligned} \tag{25}$$

$$\begin{aligned} A(rv) &= \int_1^{\mathrm{e}} G(t,s)\varphi_q\left(\int_1^{\mathrm{e}} H(s,\tau) f\bigl(\tau,\, {}^H I_{1+} rv(\tau),\, rv(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &\leqslant r^{\sigma(q-1)} \int_1^{\mathrm{e}} G(t,s)\varphi_q\left(\int_1^{\mathrm{e}} H(s,\tau) f\bigl(\tau,\, {}^H I_{1+} rv(\tau),\, rv(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &= r^{\sigma(q-1)} Tv, \quad r \geqslant 1. \end{aligned} \tag{26}$$

It follows from (24)–(26) that

$$x_1 = Ax_0 \geqslant (\ln t_0)^{\sigma(q-1)} Av_0 \geqslant (\ln t_0)v_0 = x_0,$$
$$x_1 = Ax_0 \leqslant \left(\frac{1}{\ln t_0}\right)^{\sigma(q-1)} Av_0 \geqslant \frac{1}{\ln t_0} v_0 = y_0. \tag{27}$$

Hence, by (27), $u_0 \leqslant v_0$, and induction, we have

$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant \cdots y_n \leqslant \cdots \leqslant y_1 \leqslant y_0. \tag{28}$$

Since $x_0 = (\ln t_0)^2 y_0$, we have

$$x_1 = Ax_0 = A((\ln t_0)^2 y_0) \geqslant (\ln t_0)^{2\sigma(q-1)} Ay_0 = (\ln t_0)^{2\sigma(q-1)} y_1.$$

Hence, by induction, we get $x_n \geqslant (\ln t_0)^{2[\sigma(q-1)]^n} y_n$, $n = 0, 1, 2, \dots$. Since K is a normal cone with normality constant 1 and $x_{n+m} - x_n \leqslant y_n - x_n$, for any $m \in \mathbb{N}$, we get

$$\|x_m - x_n\| \leqslant \|y_n - x_n\| \leqslant \left(1 - (\ln t_0)^{2[\sigma(q-1)]^n}\right)\|y_0\| \to 0 \quad \text{as } n \to \infty, \tag{29}$$

which implies that $\{x_n\}$ is a Cauchy sequence, and x_n converges to some $v^* \in K$. By (29) and

$$\|y_n - v^*\| \leqslant \|y_n - x_n\| + \|x_n - v^*\|,$$

we get $y_n \to v^*$. It follows from (28) that $v^* \in K$ is a fixed point of A, and $v^* \in [x_0, y_0]$. Let $u^*(t) = \int_1^t v^*(t)\mathrm{d}t/t$, $v(t) \in K$, then BVP (1)–(2) has a positive solution. Moveover, since $v^* \in K_\mathrm{e}$ and $u(t) = \int_1^t v(t)\mathrm{d}t/t$, $v(t) \in K_\mathrm{e}$, that means $d_v\mathrm{e}(t) \leqslant v(t) \leqslant D_v\mathrm{e}(t)$, then by integral, we have

$$\frac{d_v}{i}(\ln t)^i = \int_1^t d_v\mathrm{e}(t)\,\frac{\mathrm{d}t}{t} \leqslant u(t) = \int_1^t v(t)\,\frac{\mathrm{d}t}{t} \leqslant \int_1^t D_v\mathrm{e}(t)\,\frac{\mathrm{d}t}{t} = \frac{D_v}{i}(\ln t)^i.$$

Then $u^* \in K_{\overline{\mathrm{e}}}$, where

$$K_{\overline{\mathrm{e}}} = \biggl\{u(t) \in K\colon \text{there exist two numbers } D_u > 1 > d_u > 0 \\ \text{such that } \frac{d_u}{i}\overline{\mathrm{e}}(t) \leqslant u(t) \leqslant \frac{D_u}{i}\overline{\mathrm{e}}(t)\biggr\},$$

then statement (i) of Theorem 1 is proved.

Hence, for any initial value $v_0 \in K$, by $x_0 \leqslant v_0 \leqslant y_0$, we have $x_n \leqslant v_n \leqslant y_n$, $n = 1, 2, \dots$. Hence, we get

$$\|v_n - v^*\| \leqslant \|v_n - x_n\| + \|x_n - v^*\| \leqslant 2\|y_n - x_n\| \\ \leqslant 2\left(1 - (\ln t_0)^{2[\sigma(q-1)]^n}\right)\|y_0\|,$$

which implies that the sequence of functions defined by

$$v_n = \int_1^{\mathrm{e}} G(t,s)\varphi_q\left(\int_1^{\mathrm{e}} H(s,\tau) f\left(\tau,\, {}^{H}I_{1+} v_{n-1}(\tau),\, v_{n-1}(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s},$$

$n = 1, 2, \dots$, converges uniformly to the positive solution $v^*(t)$ of BVP (3)–(4) on $[1, \mathrm{e}]$ as $n \to +\infty$. Let $u_n(t) = \int_1^t v_n(t)\,\mathrm{d}t/t$, $v_n \in K$, $n = 1, 2, \dots$. Bring this expression into the above iteration sequence, we get the sequence u_n, and $u_n \to +\infty$ as $n \to +\infty$. Statement (ii) of Theorem 1 is proved.

Moreover, we have the error estimation

$$\|v_n - v^*\| \leqslant \frac{2}{\sqrt{\epsilon}}\left(1 - (\ln t_0)^{2[\sigma(q-1)]^n}\right)\|v_0\|,$$

which has the rate of convergence

$$\|v_n - v^*\| \leqslant o(1 - \left(\ln t_0\right)^{2\left[\sigma(q-1)\right]^n})\|v_0\|,$$

where $0 < \epsilon = (\ln t_0)^2 < 1$ is a constant, which is determined by v_0. Let $u_n(t) = \int_1^t v_n(t)\,\mathrm{d}t/t$, $v_n \in K$, $n = 1, 2, \dots$. There exists an error estimation

$$\|u_n - u^*\| \leqslant \frac{2}{\sqrt{\epsilon}}\left(1 - (\ln t_0)^{2[\sigma(q-1)]^n}\right)\|u_0\|,$$

where $u_0(t) = \int_1^t v_0(t)\,\mathrm{d}t/t$, $v_n \in K$, $n = 1, 2, \dots$, which has the rate of convergence $\|u_n - u^*\| = o(1 - (\ln t_0)^{2[\sigma(q-1)]^n})$, where $0 < \epsilon < 1$ is a positive constant, which is determined by the initial value v_0. Statement (iii) of Theorem 1 is proved.

Next, we prove that the positive solution of problem (3)–(4) is unique. In fact, suppose $\overline{v} \in K$ is any fixed point of A, then we get $A\overline{v} = \overline{v}$. From $\overline{v}, v^* \in K$, the definition of K, and assuming that $t_1 = \sup\{t > 0\colon \overline{v} \geqslant tv^*\}$, we have $1 < t_1 < +\infty$. We assert that $t_1 \geqslant \mathrm{e}$. If not, we get $1 < t < \mathrm{e}$, hence

$$\begin{aligned}\overline{v} &= A\overline{v} \geqslant (\ln t_1)^{\sigma(q-1)} A(t_1 v^*)\\ &= (\ln t_1)^{\sigma(q-1)} A(v^*) = (\ln t_1)^{\sigma(q-1)}(v^*)\end{aligned}$$

since $1 < q < 2$, and we get $(\ln t_1)^{q-1} > \ln t_1$, a contradiction. Thus, we have that $t_1 \geqslant \mathrm{e}$ and $\overline{v} \geqslant v^*$. By the same way, we get $\overline{v} \leqslant v^*$, that is, $\overline{v} = v^*$, and then v^* is a unique fixed point of A in K_{e}, that is, it is a unique positive solution of the fractional-order p-Laplacian differential problem (3)–(4). Statement (iv) of Theorem 1 is proved. Therefore, the proof of Theorem 1 is completed. □

Remark 2. The iterative sequences in Theorem 1 can be chosen arbitrarily, thus we can choose some simple function such 0 or $(\ln t)^{i-1}$, which is useful for computational purpose.

4 An example

Consider the following infinite-point p-Laplacian fractional differential problem:

$$\begin{aligned} &{}^{H}D_{1^+}^{3/2}\bigl(\varphi_3\bigl({}^{CH}D_{1^+}^{7/2}u\bigr)\bigr)(t)+f\bigl(t,u(t),u'(t)\bigr)=0, \quad 1<t<\mathrm{e},\\ &u(1)=u'(1)=u'''(1)=0, \qquad u''(\mathrm{e})=\sum_{j=1}^{\infty}\eta_j u(\xi_j),\\ &{}^{CH}D_{1^+}^{7/2}u(1)=0, \qquad \varphi_3\bigl({}^{CH}D_{1^+}^{7/2}u(\mathrm{e})\bigr)=\sum_{j=1}^{\infty}\zeta_j\varphi_3\bigl({}^{CH}D_{1^+}^{7/2}u(\xi_j)\bigr). \end{aligned} \tag{30}$$

Let $v(t)=u'(t)$, then (30) is changed into the following (31):

$$\begin{aligned} &{}^{H}D_{1^+}^{3/2}\bigl(\varphi_3\bigl({}^{CH}D_{1^+}^{5/2}v\bigr)\bigr)(t)+f\left(t,\int_1^t v(t)\,\frac{\mathrm{d}t}{t},\,v(t)\right)=0, \quad 1<t<\mathrm{e},\\ &v(1)=v''(1)=0, \qquad v'(\mathrm{e})=\sum_{j=1}^{\infty}\eta_j v(\xi_j),\\ &{}^{CH}D_{1^+}^{5/2}v(1)=0, \qquad \varphi_3\bigl({}^{CH}D_{1^+}^{5/2}v(\mathrm{e})\bigr)=\sum_{j=1}^{\infty}\zeta_j\varphi_3\bigl({}^{CH}D_{1^+}^{5/2}v(\xi_j)\bigr), \end{aligned} \tag{31}$$

where

$$f\bigl(t,u(t),u'(t)\bigr)=\frac{u^{1/3}(t)+(u'(t))^{1/4}}{(\ln t)^{1/3}},$$

$\gamma=7/2$, $\beta=3/2$, $\varsigma_j=1/(3j^3)$, $\eta_j=1/(2j^2)$, $\xi_j=\mathrm{e}^{1/j^2}$, $i=2$, $p=3$, $q=3/2$.

Let

$$f(t,x,y)=\frac{x^{1/3}+y^{1/4}}{(\ln t)^{1/3}},$$

then $f\in C((1,\mathrm{e})\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, and for any fixed $t\in(1,\mathrm{e})$, $f(t,x,y)$ is nondecreasing in x and y. Now we take $\sigma=2/3$, $r\in[1,\mathrm{e}]$. For any $(t,x,y)\in(1,\mathrm{e})\times[0,+\infty)\times[0,+\infty)$, we get

$$f(t,rx,ry)=\frac{x^{1/3}+y^{1/4}}{(\ln t)^{1/3}}\geqslant r^{2/3}\frac{x^{1/3}+y^{1/4}}{(\ln t)^{1/3}}=r^{2/3}f(t,x,y).$$

Thus condition (H_0) is satisfied.

By simple calculation, we have

$$\varDelta=(i-1)!-\sum_{j=1}^{\infty}\eta_j(\ln\xi_j)^{i-1}=1-\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^4}=0.4586>0,$$

$$\overline{\varDelta}=1-\sum_{j=1}^{\infty}\zeta_j(\ln\xi_j)^{\beta-1}=1-\frac{1}{3}\sum_{j=1}^{\infty}\frac{1}{j^4}\approx 0.6392>0,$$

so $\sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^{i-1} < (i-1)!$, $\sum_{j=1}^{\infty} \zeta_j (\ln \xi_j)^{\beta-1} < 1$. Therefore, condition ($\mathrm{H}_1$) is also satisfied. Now we check condition (17). In fact,

$$\begin{aligned} 0 &< \int_1^{\mathrm{e}} f\bigl(\tau,\, (\ln\tau)^i,\, (\ln\tau)^{i-1}\bigr) \frac{\mathrm{d}\tau}{\tau} = \int_1^{\mathrm{e}} f\bigl(\tau,\, (\ln\tau)^2,\, \ln\tau\bigr) \frac{\mathrm{d}\tau}{\tau} < +\infty \\ &= \int_1^{\mathrm{e}} \frac{(\ln\tau)^{2/3} + (\ln\tau)^{1/4}}{(\ln\tau)^{1/3}} \frac{\mathrm{d}\tau}{\tau} = \frac{81}{44} < +\infty. \end{aligned}$$

Then the p-Laplacian fractional-order differential problem (1)–(2) has a unique positive solution $u^*(t) =^H Iv^*(t)$, $v^* \in K$. Moreover, for any initial value $v_0 \in K$, the sequence of functions defined by

$$v_n = \int_1^{\mathrm{e}} G(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} H(s,\tau)\frac{({}^H I_{1^+} v_{n-1})^{1/3}(t) + (v_{n-1}(t))^{1/4}}{(\ln t)^{1/3}} \frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s},$$

$n = 1, 2, \dots$, converges uniformly to $v^*(t)$ on $[1, \mathrm{e}]$ as $n \to \infty$. Let $u(t) = \int_1^t v(t)\mathrm{d}t/t$, $v(t) \in K$, we have that BVP (1)–(2) has a unique positive solution, and (i) of Theorem 1 is obtained.

Moreover, there exist an error estimation

$$\begin{aligned} \|u_n - u^*\| &= \max_{t\in[1,\mathrm{e}]} \bigl|{}^H I\bigl(v_n(t) - v^*(t)\bigr)\bigr| \leqslant \frac{2}{\sqrt{\epsilon}}\bigl(1 - \epsilon^{[\sigma(q-1)]^n}\bigr)\|v_0\| \\ &= \frac{2}{\sqrt{\epsilon}}\bigl(1 - \epsilon^{(1/3)^n}\bigr)\|v_0\|, \end{aligned}$$

where $\sigma = 2/3$, which has the rate of convergence

$$\|u_n - u^*\| = o\bigl(1 - \epsilon^{(1/3)^n}\bigr),$$

where $0 < \epsilon < 1$ is a positive constant, which is determined by the initial value v_0. Especially, if $v_0 = \ln t$ by computation, we have $0.3065(\ln t) \leqslant Av_0 \leqslant 0.8574(\ln t)$, and (ii) Theorem 1 is obtained.

Take

$$\begin{aligned} t_0 &\leqslant \min\biggl\{\exp\biggl\{\biggl(\frac{\widetilde{d}_{v_0}}{D_{v_0}}\biggr)^{1/(1-\sigma(q-1))}\biggr\},\, \exp\biggl\{\biggl(\frac{\widetilde{D}_{v_0}}{d_{v_0}}\biggr)^{1/(1-\sigma(q-1))}\biggr\}\biggr\} \\ &= \min\{1.1849,\, 2.2120\} = 1.1849, \end{aligned}$$

then we get the error estimation

$$\|u_n - u^*\| \leqslant \frac{2}{\sqrt{\epsilon}}\bigl(1 - \epsilon^{(1/3)^n}\bigr)\|v_0\| = 11.7855 \cdot \bigl(1 - 0.0288^{(1/2)^n}\bigr) \cdot 0.1697$$

and the estimate of convergence rate

$$\|u_n - u^*\| = o\bigl(1 - 0.0288^{(1/2)^n}\bigr).$$

Then (iii) of Theorem 1 is obtained.

5 Conclusions

By deriving the expression of Green function and some special properties, we establish appropriate cone. Since the nonlinearity contains derivative terms, we overcome the difficulty caused by derivative terms by proper substitution. Then the existence of unique iterative positive, error estimation, and convergence rate of approximate solution are obtained for singular p-Laplacian Caputo–Hadamard fractional differential equation with infinite-point boundary conditions. We choose some simple function such 0 or $(\ln t)^{i-1}$, which is useful for computational purpose. The understanding of the properties of the solution and its future application may provide convenient and theoretical guidance.

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