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A few generalizations of Kendall's tau. Part II: Intrinsic meaning, properties, and computational aspects

Martynas Manstavičius

Vilnius University, Faculty of Mathematics and Informatics, Institute of Mathematics,
Naugarduko str. 24, LT-03225, Vilnius, Lithuania
martynas.manstavicius@mif.vu.lt

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Abstract. Continuing our investigation on generalizations of Kendall's τ, started in Part I of the paper, here we elaborate on the intrinsic meaning and degree of such polynomial-type concordance measures, as well as present many examples of their computation. In particular, we interpret generalized Kendall's τ_φ as the difference between the average capacities of concordance and discordance, and, for power-type distortion functions φ, we obtain polynomial-type concordance measures of various degree, which could stimulate further research of their characterization as achieved for degree-one polynomial-type concordance measures by Taylor, Edwards, and Mikusiński.

Keywords: Kendall's tau, Scarsini axioms, bivariate copula, transformation, concordance measure, convex capacity, supermodular.

1 Introduction

The idea of probing the object of interest on different scales, using different means, techniques, and measures is ubiquitous in science and its applications in various industries and our daily life, in general. So one should not be surprised to see that many measures of dependence, association, and concordance of random variables, vectors, etc. are suggested and investigated in the literature. In the first part of this paper [16], we have elaborated on various directions that many researchers explored regarding the definitions, properties, and applications of such dependence measures. Therefore, not to repeat ourselves, we refer the reader to that paper and the many references therein. In this part, we will focus on the properties of the suggested generalizations of Kendall's τ, as well as their computations for certain families of bivariate copulas. In particular, we will elaborate on situations when we obtain polynomial-type concordance measures of degree higher than one, which we believe could pave the way for more investigations into how to characterize such measures and answer one of the old questions posed by Taylor and his coauthors [6–8, 18, 24, 25]. In particular, Question 1 in [25, p. 235] reads: "Can we give interesting examples of measures of concordance of degree m for every natural number m?, while Question 2 in the same paper goes a bit further: "Can we characterize measures of concordance of degree one? Of degree m?" These are open problems for n-dimensional ($n \geqslant 2$) copulas, in general, and our contribution is a first step, providing relevant and abundant examples in the bivariate case. There is an analogy between the moments of probability distributions and the values of various concordance measures for copulas, so perhaps in the future one can find a family of concordance measures that could characterize (important) copulas, in particular the independence copula, leading to new statistical tests of independence.

In our work, we will also provide a discussion as to why considered generalizations of concordance measures could rightfully be called as such – concordance in our setting is measured not by using traditional probabilities but rather more general convex capacities, which could have some applications in the economic decision theory. The many examples we present in this paper also highlight the fact that, for many copula families, computation of traditional Spearman's ρ_S, Kendall's τ or their generalizations can be hard, so analytic expressions should not be expected in general. In some cases, though, they are possible.

The rest of this paper is organized as follows: In Section 2 we briefly recall the basic notions and facts about copulas and concordance measures (in the sense of Scarsini) needed to state our main results from Part I of the paper (Theorems 1 and 2) and set the stage for further developments. In Section 3, we discuss the intrinsic meaning of Kendall's τ and the considered generalizations, highlighting the switch from a probability measure, to weigh a random partition of the unit square, to a nonadditive measure (in our case, convex (supermodular) capacity). Section 4 is devoted to computations of generalized Kendall's τ for Farlie–Gumbel–Morgenstern, Plackett, Frank copula families, as well as two subfamilies of Fréchet–Mardia copulas, emphasizing the polynomial-type nature of the proposed concordance measures when φ is a power function. Section 5 concludes.

2 Basic facts from copula theory

We begin by recalling the notion of a bivariate copula. Let $\mathbb{I} := [0, 1]$.

Definition 1. A bivariate copula[1] (a copula for short) C is a function defined on $\mathbb{I}^2$ with values in $\mathbb{I}$ such that

- (boundary conditions) $C(x, 0) = C(0, x) = 0$ and $C(x, 1) = C(1, x) = x$ for any $x \in \mathbb{I}$,
- (2-increasingness) for all $x, x', y, y' \in \mathbb{I}$ with $x \leqslant x'$ and $y \leqslant y'$,

$$V_C\big([x, x'] \times [y, y']\big) = C(x', y') - C(x, y') - C(x', y) + C(x, y) \geqslant 0.$$

The set of bivariate copulas will be denoted by $\mathcal{C}$ (or, more precisely, $\mathcal{C}_2$ if we need explicit dependence on the dimension).

For numerous examples, see [5, 12, 19] and the references therein; among the most important are the comonotonicity copula $M(x, y) = \min\{x, y\} = x \wedge y$, independence

[1] One can also consider n-variate copulas for any $n \geqslant 2$ (see, e.g., [5, 12, 19]), but we will only be concerned with bivariate copulas in this paper.

copula $\Pi(x,y) = xy$, and countermonotonicity copula $W(x,y) = \max\{x+y-1, 0\} = (x+y-1)^+$ for $(x,y) \in \mathbb{I}^2$.

To each (bivariate) copula $C \in \mathcal{C}$, one can associate a Borel measure μ_C such that $\mu_C((0,x] \times (0,y]) = C(x,y)$ for any $x, y \in \mathbb{I}$, and vice versa (see, e.g., [5, Thm. 3.1.2], where the result is stated for a general dimension $d \geqslant 2$). In what follows, integrals with respect to a copula $C \in \mathcal{C}$, e.g., $\int_{\mathbb{I}^2} f \,\mathrm{d}C$, will mean $\int_{\mathbb{I}^2} f \,\mathrm{d}\mu_C$.

On the set of bivariate copulas, one can consider a pointwise partial-order relation defined as follows:

Definition 2. (See [19, Def. 2.8.1].) For any $C_1, C_2 \in \mathcal{C}$, we say that C_1 is *smaller* (resp. *larger*) than C_2 with respect to concordance order, and denote it by $C_1 \prec C_2$ (resp. $C_1 \succ C_2$) if $C_1(x,y) \leqslant C_2(x,y)$ (resp. $C_1(x,y) \geqslant C_2(x,y)$) for any $(x,y) \in \mathbb{I}^2$.

In general, when $d \geqslant 2$, the concordance order is defined as

$$\begin{aligned} C_1 \prec C_2 \quad \Longleftrightarrow \quad & C_1(x_1,\ldots,x_d) \leqslant C_2(x_1,\ldots,x_d) \quad \text{and} \\ & \overline{C}_1(x_1,\ldots,x_d) \leqslant \overline{C}_2(x_1,\ldots,x_d) \quad \forall (x_1,\ldots,x_d) \in \mathbb{I}^d, \end{aligned}$$

where $\overline{C}(x_1,\ldots,x_d) = \mathbf{P}\,(U_1 > x_1, \ldots, U_d > x_d)$, $U_1,\ldots,U_d \sim U(\mathbb{I})$ are uniformly on $\mathbb{I}$ distributed random variables whose copula is C. In other words, $\overline{C}$ is the survival function associated with copula C. For $d = 2$, $\overline{C}(x,y) = 1 - x - y + C(x,y)$, so concordance order for bivariate copulas is equivalent to pointwise order.

Then the famous Fréchet–Hoeffding bounds for copulas (see [19, Eq. (2.2.5)] or [5, Thm. 1.7.3]; for their applications in, e.g., risk management, we can recommend the book by Rüschendorf [20]) can be written succinctly as $W \prec C \prec M$ for any $C \in \mathcal{C}$. Thus (bivariate) copulas M and W are the largest and smallest elements, respectively, with respect to concordance order. In dimension $d > 2$, W is no longer a copula, but the Fréchet–Hoeffding bounds still hold. Also, any concordance measure $\kappa_{X,Y}$ in the sense of Scarsini (Kendall's τ, Spearman's ρ, Gini's γ, etc. are examples; see [19, Def. 5.1.7]), measuring the dependence between continuous random variables X and Y whose copula is C, is nondecreasing with respect to concordance order, hence the name of the measure.

2.1 Transformations of copulas generated by symmetries of their domain

In relation to the axioms of concordance measures, of particular importance are the transformations of bivariate (or, more generally, multivariate) copulas that are induced by the symmetries of their domain $\mathbb{I}^2$ (or $\mathbb{I}^d$ for $d \in \mathbb{N}$, $d \geqslant 3$). The group of symmetries of the unit square $\mathbb{I}^2$ can be generated by involutions $\pi : \mathbb{I}^2 \to \mathbb{I}^2$ (permutation, i.e., reflection with respect to the main diagonal) and $\sigma_1 : \mathbb{I}^2 \to \mathbb{I}^2$ (partial reflection with respect to the axis $x = 1/2$) given by

$$\pi(x,y) = (y,x) \quad \text{and} \quad \sigma_1(x,y) = (1-x,\, y).$$

Involution means that $\pi^2 = \sigma_1^2 = e$, the identity transformation. Also, one can get the partial reflection $\sigma_2(x,y) = (x,\, 1-y)$ with respect to the axis $y = 1/2$ as $\sigma_2(x,y) =$

$(\pi \circ \sigma_1 \circ \pi)(x, y)$. Combining the two reflections, we get the so-called total reflection

$$\varsigma(x, y) = (\sigma_1 \circ \sigma_2)(x, y) = (\sigma_2 \circ \sigma_1)(x, y) = (1 - x,\, 1 - y).$$

Altogether the group of symmetries of the unit square, also called the dihedral group D_4, has $8 = 2!\,2^2$ elements:

$$D_4 = \{e,\, \pi,\, \sigma_1,\, \sigma_2,\, \varsigma,\, \pi \circ \sigma_1,\, \pi \circ \sigma_2,\, \pi \circ \varsigma\}.$$

It is important to note that D_4 is not commutative since

$$\pi \circ \sigma_1 = \sigma_2 \circ \pi.$$

Given a symmetry $\xi \in D_4$, there is a corresponding transformation $\xi^* : \mathcal{C} \to \mathcal{C}$ given by

$$\xi^*(C)(x, y) := \mu_C\bigl(\xi\bigl([0, x] \times [0, y]\bigr)\bigr), \quad (x, y) \in \mathbb{I}^2. \tag{1}$$

For the partial reflections σ_1, σ_2 and total reflection ς, one easily gets

$$\begin{aligned}
\sigma_1^*(C)(u, v) &= \mu_C\bigl([1 - u,\, 1] \times [0, v]\bigr) \\
&= \mu_C\bigl([0, 1] \times [0, v]\bigr) - \mu_C\bigl([0,\, 1 - u] \times [0, v]\bigr) \\
&= C(1, v) - C(1 - u,\, v) = v - C(1 - u,\, v), \\
\sigma_2^*(C)(u, v) &= C(u, 1) - C(u,\, 1 - v) = u - C(u,\, 1 - v), \\
\varsigma^*(C)(u, v) &= u + v - 1 + C(1 - u,\, 1 - v), \quad u, v \in \mathbb{I},
\end{aligned}$$

while the transpose of C is given by $C^{\mathrm{T}}(u, v) := \pi^*(C)(u, v) = C(v, u)$. Note that $\varsigma^*(C)(u, v) = \widehat{C}(u, v)$, the survival copula corresponding to C.

2.2 Scarsini's axioms of concordance measures

Henceforth we will deal with the family of functionals on the set of copulas $\mathcal{C}$, which measure the "degree of association" of continuous random variables having a given copula and preserve concordance order. This family was axiomatized by Scarsini in 1984 (see [21, 22]; for extensions to the multidimensional case, see [4, 24]).

Definition 3. (See [5, Def. 2.4.7].) A measure of concordance is a mapping $\kappa : \mathcal{C} \to \mathbb{R}$ such that

(κ_1) κ is defined for every copula $C \in \mathcal{C}$,
(κ_2) for every $C \in \mathcal{C}$, $\kappa(C) = \kappa(C^{\mathrm{T}})$,
(κ_3) $\kappa(C_1) \leqslant \kappa(C_2)$ whenever $C_1 \prec C_2$,
(κ_4) $\kappa(C) \in [-1, 1]$,
(κ_5) $\kappa(\mathit{\Pi}) = 0$,
(κ_6) $\kappa(\sigma_1^*(C)) = \kappa(\sigma_2^*(C)) = -\kappa(C)$ for the partial reflections σ_1, σ_2 and any $C \in \mathcal{C}$,
(κ_7) (continuity) if $C_n \to C$ uniformly[2] as $n \to \infty$, then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.

[2]For copulas, pointwise convergence is enough.

One can observe that some authors, e.g., Nelsen [19, Def. 5.1.17, Property 2] and Fuchs [10, Sect. 3], also require

(κ_5') $\kappa(M) = 1$,

which can be achieved by a simple normalization, i.e., defining the new measure $\kappa'(C) := \kappa(C)/\kappa(M)$, if the original concordance measure κ does not satisfy this condition and is nontrivial ($\kappa(C) \neq 0$ for some $C \in \mathcal{C}$).

The most often used concordance measures are Spearman's ρ, Kendall's τ, Gini's γ, Blomqvist's β; see [19, Chap. 5], [5, Sect. 2.4]. On the other hand, Spearman's foot-rule is not a concordance measure; see [19, Exr. 5.21].

These measures are succinctly defined in terms of the so-called *biconvex form* given by

$$[C, D] := \int_{\mathbb{I}^2} C \,\mathrm{d}D, \quad C, D \in \mathcal{C}, \tag{2}$$

which is linear in each place with respect to convex combinations of copulas, hence the terminology. In fact (see [19]), for a copula $C \in \mathcal{C}$,

- Spearman's ρ is given by $\rho_S(C) = 12[C, \Pi] - 3 = 12[C - \Pi, \Pi]$,
- Kendall's τ is defined as $\tau(C) = 4[C, C] - 1 = 4([C, C] - [\Pi, \Pi])$,
- Gini's γ is $\gamma(C) = 4([C, M] + [C, W]) - 2$.

In our earlier work [15], we have constructed several generalizations of Spearman's ρ, Gini's γ, etc., while generalizations of Kendall's τ were presented in [16]. The latter were based on an appropriate distortion function $\varphi : [0, 1] \to \mathbb{R}$ and the form

$$[C, D]_\varphi := \int_{\mathbb{I}^2} \varphi(C) \,\mathrm{d}D = \bigl[\varphi(C), D\bigr], \quad C, D \in \mathcal{C}.$$

When $\varphi(x) = x$, $x \in \mathbb{I}$, $[C, D]_\varphi = [C, D]$, the usual biconvex form in (2) (see [9]), used to define various concordance measures for bivariate copulas. In fact, for Kendall's τ, we have

$$\tau(C) = 4[C, C] - 1 = \sum_{\xi \in R} (-1)^{|\xi|} [\xi^*(C), \xi^*(C)], \quad C \in \mathcal{C}, \tag{3}$$

where R denotes the commutative subgroup of D_4 generated by partial reflections of $\mathbb{I}^2$, that is, $R = \{e, \sigma_1, \sigma_2, \varsigma = \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 \mid \sigma_1^2 = \sigma_2^2 = e\}$, and $\xi^*(C)$ is defined in (1). At the level of random variables (i.e., if U and V are random variables distributed uniformly on the interval $\mathbb{I}$ and joined by copula $C \in \mathcal{C}$, $(U, V) \sim C$),

$$(1 - U, V) \sim \sigma_1^*(C), \quad (U, 1 - V) \sim \sigma_2^*(C), \quad \text{and} \quad (1 - U, 1 - V) \sim \varsigma^*(C).$$

We generalized Kendall's τ in [16], replacing $[C, D]$ in (3) by $[C, D]_\varphi$ and normalizing appropriately:

$$\tau_\varphi(C) := a_\varphi \sum_{\xi \in R} (-1)^{|\xi|} \bigl[\xi^*(C), \xi^*(C)\bigr]_\varphi, \quad C \in \mathcal{C}, \tag{4}$$

where

$$a_\varphi := \left(2\int_0^1 \bigl(\varphi(x)-\varphi(0)\bigr)\,\mathrm{d}x\right)^{-1}, \tag{5}$$

which is positive as φ is assumed nonconstant, nondecreasing, and convex. In the case, $\varphi(x) = x$ on $\mathbb{I}$, $a_\varphi = 1$, and thus we recover in (4) the usual Kendall's τ in (3). For $\varphi(x) = x^p$ with $p \in \mathbb{N}$, $p \geqslant 2$, one easily gets $a_\varphi = (p+1)/2$.

Recall the following theorems:

Theorem 1. (*See* [16, *Thm.* 2].) *If $\varphi : [0,1] \to \mathbb{R}$ is a nonconstant, nondecreasing, and convex function such that $\varphi(1) = \varphi(1-) = 1$, $\varphi(0) = 0$, then $\tau_\varphi : \mathcal{C} \to [-1,1]$ is a measure of concordance, generalizing Kendall's τ.*

Theorem 2. (*See* [16, *Thm.* 3].) *Let $\varphi : [0,1] \to \mathbb{R}$ be a nondecreasing function such that $\varphi(1) = \varphi(1-) = 1$, $\varphi(0) = 0$. Assume, furthermore, that φ is nonconstant, differentiable, and convex on $(0,1)$. Then the measure of concordance τ_φ given in* (4) *can be equivalently expressed as follows*:

$$\tau_\varphi(C) = a_\varphi\biggl\{\int_{\mathbb{I}^2} \varphi'\bigl(\overline{C}\bigr)\,\mathrm{d}\Pi - \int_{\mathbb{I}^2} G_\varphi(C)\partial_1 C\partial_2 C\,\mathrm{d}\Pi\biggr\}, \tag{6}$$

where for any $C \in \mathcal{C}$ and $u, v \in (0,1)$,

$$\begin{aligned}
G_\varphi(C)(u,v) &:= \varphi'\bigl(C(u,v)\bigr) + \varphi'\bigl(\tilde{\sigma}_1^*(C)(u,v)\bigr) + \varphi'\bigl(\tilde{\sigma}_2^*(C)(u,v)\bigr) + \varphi'\bigl(\overline{C}(u,v)\bigr);\\
\tilde{\sigma}_1^*(C)(u,v) &= \sigma_1^*(C)(1-u,\,v) = v - C(u,v);\\
\tilde{\sigma}_2^*(C)(u,v) &= \sigma_2^*(C)(u,\,1-v) = u - C(u,v);\\
\overline{C}(u,v) &= 1-u-v+C(u,v) \quad (\textit{survival function corresponding to } C).
\end{aligned}$$

Remark 1. Whenever φ' is convex, letting

$$\begin{aligned}
x_1 &= C(u,v), & x_2 &= u - C(u,v),\\
x_3 &= v - C(u,v), & x_4 &= 1-u-v+C(u,v),
\end{aligned}$$

we have $x_i \in \mathbb{I}$, $i = 1,\dots,4$, $\sum_{i=1}^4 x_i = 1$ for any $u, v \in \mathbb{I}$ and $C \in \mathcal{C}$. Also, if $\boldsymbol{x} = (x_1,x_2,x_3,x_4)$ and $g(\boldsymbol{x}) = \sum_{i=1}^4 \varphi'(x_i)$, then it follows that $g : \mathbb{I}^4 \to \mathbb{R}$ is Schur-convex (cf. [17, p. 92, C.1 Prop.]), and so

$$4\varphi'\left(\frac{1}{4}\right) \leqslant g(\boldsymbol{x}) \leqslant \varphi'(1) + 3\varphi'(0) \tag{7}$$

since

$$\left(\frac14,\frac14,\frac14,\frac14\right) \prec (x_1,x_2,x_3,x_4) \prec (1,0,0,0),$$

where "$\lessdot$" denotes the *majorization* relation[3] for vectors (as defined by Hardy, Littlewood, and Pólya; see, e.g., [17, p. 80, A.1 Def.]), i.e., for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$\begin{aligned} \boldsymbol{x} \lessdot \boldsymbol{y} \quad \Longleftrightarrow \quad & \sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \quad \forall k = 1, 2, \dots, n-1 \quad \text{and} \\ & \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \end{aligned} \tag{8}$$

with $x_{[i]}$, $y_{[i]}$ denoting the ith largest entries of $\boldsymbol{x}$, $\boldsymbol{y}$, respectively. For more information about majorization and its applications, see the book [17]. Furthermore, the bounds in (7) are sharp since for $\varphi(t) = t^2$, we get equalities. Also, if φ' is concave, then g is Schur-concave, and the inequalities in (7) are reversed.

Example 1. As a sanity check, choose $\varphi(x) = x$, $x \in \mathbb{I}$. Then, clearly, $a_\varphi = 1$, $\int_{\mathbb{I}^2} \varphi'(\overline{C})\,\mathrm{d}\Pi = 1$, $G_\varphi(C) = 4$, and so

$$\tau_\varphi(C) = 1 - 4 \int_{\mathbb{I}^2} \partial_1 C\, \partial_2 C\, \mathrm{d}\Pi = \tau(C)$$

as expected (see [19, Eq. (5.1.12)]).

As a nontrivial example, we have

Example 2. Consider $\varphi(x) = x^2$, $x \in \mathbb{I}$. Then $a_\varphi = 3/2$,

$$\int_{\mathbb{I}^2} \varphi'\bigl(\overline{C}\bigr)\,\mathrm{d}\Pi = 2\int_{\mathbb{I}^2} \overline{C}\,\mathrm{d}\Pi = 2\int_{\mathbb{I}^2} C\,\mathrm{d}\Pi = \frac{\rho_S(C) + 3}{6},$$

where ρ_S denotes Spearman's rho. Also,

$$\begin{aligned} G_\varphi(C)(u,v) &= 2\bigl\{C(u,v) + \bigl[v - C(u,v)\bigr] + \bigl[u - C(u,v)\bigr] \\ &\quad + \bigl[1 - u - v + C(u,v)\bigr]\bigr\} \\ &= 2, \end{aligned}$$

and so

$$\tau_\varphi(C) = \frac{3}{2}\biggl(\frac{\rho_S(C) + 3}{6} - 2\int_{\mathbb{I}^2} \partial_1 C \partial_2 C\,\mathrm{d}\Pi\biggr) = \frac{1}{4}\rho_S(C) + \frac{3}{4}\tau(C),$$

a much simpler expression compared to (4) or (6). We also note a curious similarity with a concordance measure considered by Borroni [1, Eq. (27)], where the author obtained $\gamma_{\varphi_\Delta}(C) = (1/4)\rho_S(C) + (3/4)\gamma(C)$ with $\gamma(C)$ being the Gini's gamma of copula C.

[3]The symbol $\lessdot$ is used here instead of the typical $\prec$ adopted in [17] to avoid abuse of notation; cf. Definition 2.

3 Intrinsic meaning of generalized Kendall's τ

In this section, we revisit the meaning of Kendall's τ and elaborate what its generalization provided in Eq. (4) really measures.

First, if (X_1, Y_1) and (X_2, Y_2) are independent copies of a continuous random vector (X, Y) such that X has distribution function F, Y has distribution function G, and their copula is C, i.e., $\mathbf{P}(X \leqslant x, Y \leqslant y) = C(F(x), G(y))$, $x, y \in \mathbb{R}$, then it is well known (see, e.g., [19, Thm. 5.1.1]) that Kendall's τ measures the difference between the probabilities of concordance and discordance, that is,

$$\tau(C) = \mathbf{P}\bigl((X_1 - X_2)(Y_1 - Y_2) > 0\bigr) - \mathbf{P}\bigl((X_1 - X_2)(Y_1 - Y_2) < 0\bigr).$$

The expression in (3) has 4 terms, which for uniformly on $\mathbb{I}$ distributed random variables U and V such that $(U, V) \sim C$, can be written as follows:

$$\begin{aligned}
\bigl[e^*(C), e^*(C)\bigr] &= \mathbf{E}C(U,V) = \mathbf{P}(X_1 > X_2,\ Y_1 > Y_2) \\
&= \mathbf{P}\bigl((X_1 - X_2)(Y_1 - Y_2) > 0,\ Y_1 > Y_2\bigr), \\
\bigl[\sigma_1^*(C), \sigma_1^*(C)\bigr] &= \mathbf{E}\bigl[\sigma_1^*(C)(1-U,\ V)\bigr] = \mathbf{E}\bigl[V - C(U,V)\bigr] \\
&= \mathbf{P}\bigl((1 - X_1) > (1 - X_2),\ Y_1 > Y_2\bigr) \\
&= \mathbf{P}\bigl((X_1 - X_2)(Y_1 - Y_2) < 0,\ Y_1 > Y_2\bigr), \\
\bigl[\sigma_2^*(C), \sigma_2^*(C)\bigr] &= \mathbf{E}\bigl[\sigma_2^*(C)(U,\ 1-V)\bigr] = \mathbf{E}\bigl[U - C(U,V)\bigr] \\
&= \mathbf{P}\bigl(X_1 > X_2,\ (1 - Y_1) > (1 - Y_2)\bigr) \\
&= \mathbf{P}\bigl((X_1 - X_2)(Y_1 - Y_2) < 0,\ Y_2 > Y_1\bigr), \\
\bigl[\varsigma^*(C), \varsigma^*(C)\bigr] &= \mathbf{E}\bigl[\varsigma^*(C)(1-U,\ 1-V)\bigr] = \mathbf{E}\bigl[1 - U - V + C(U,V)\bigr] \\
&= \mathbf{P}\bigl((1 - X_1) > (1 - X_2),\ (1 - Y_1) > (1 - Y_2)\bigr) \\
&= \mathbf{P}\bigl((X_1 - X_2)(Y_1 - Y_2) > 0,\ Y_2 > Y_1\bigr),
\end{aligned}$$

so that

$$\bigl[e^*(C), e^*(C)\bigr] + \bigl[\varsigma^*(C), \varsigma^*(C)\bigr] = \text{probability of concordance},$$

and

$$\bigl[\sigma_1^*(C), \sigma_1^*(C)\bigr] + \bigl[\sigma_2^*(C), \sigma_2^*(C)\bigr] = \text{probability of discordance}.$$

Also, given $(U, V) \in \mathbb{I}^2$, we can define random sets (see Fig. 1)

$$\begin{aligned}
A^1_{(U,V)} &= [0, U] \times [0, V], & A^2_{(U,V)} &= [U, 1] \times [0, V], \\
A^3_{(U,V)} &= [U, 1] \times [V, 1], & A^4_{(U,V)} &= [0, U] \times [V, 1].
\end{aligned} \tag{9}$$

Then if μ_C denotes the probability measure induced by copula C, that is, $\mu_C([0, u] \times [0, v]) = C(u, v)$ for any $(u, v) \in \mathbb{I}^2$, which is extended for other Borel sets of $\mathbb{I}^2$ using

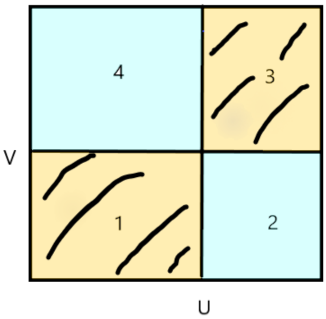

Figure 1. Random partition of $\mathbb{I}^2$ using a point $(U, V) \sim C$. Regions 1 to 4 correspond to sets $A^1_{(U,V)}$ to $A^4_{(U,V)}$ defined in Eq. (9). Regions 1 and 3 (dashed; in yellow) contribute to concordance, while regions 2 and 4 (plain; in cyan) contribute to discordance.

standard measure-theoretic arguments, we get

$$\begin{aligned}\left[e^*(C), e^*(C)\right] + \left[\varsigma^*(C), \varsigma^*(C)\right] &= \mathbf{E}\big(\mu_C(A^1_{(U,V)}) + \mu_C\big(A^3_{(U,V)}\big)\big),\\ \left[\sigma_1^*(C), \sigma_1^*(C)\right] + \left[\sigma_2^*(C), \sigma_2^*(C)\right] &= \mathbf{E}\big(\mu_C\big(A^2_{(U,V)}\big) + \mu_C\big(A^4_{(U,V)}\big)\big),\end{aligned}$$

where the expectations are taken with respect to the joint law of $(U, V) \sim C$.

The generalized Kendall's τ in Eq. (4) can be similarly written as the scaled (by a_φ) difference of two sums:

$$\left[e^*(C), e^*(C)\right]_\varphi + \left[\varsigma^*(C), \varsigma^*(C)\right]_\varphi = \mathbf{E}\big(\varphi\big(\mu_C\big(A^1_{(U,V)}\big)\big) + \varphi\big(\mu_C\big(A^3_{(U,V)}\big)\big)\big)$$

and

$$\left[\sigma_1^*(C), \sigma_1^*(C)\right]_\varphi + \left[\sigma_2^*(C), \sigma_2^*(C)\right]_\varphi = \mathbf{E}\big(\varphi\big(\mu_C\big(A^2_{(U,V)}\big)\big) + \varphi\big(\mu_C\big(A^4_{(U,V)}\big)\big)\big).$$

So in fact, instead of probabilities of concordance and discordance, in the setting of Theorem 1, we are measuring *capacities* (for some applications of convex capacities in economic decision theory, see, e.g., [2, 3, 23]; see also the recent developments of integration theory based on monotone measures [14]) of sets contributing to concordance and discordance as the set function $\nu := \nu_{\varphi,C} = \varphi \circ \mu_C$ defines a (convex) capacity on the class $\mathcal{K}$ of compacts of $\mathbb{I}^2$. Indeed, $\nu(\emptyset) = 0$, $\nu(\mathbb{I}^2) = 1$, and, moreover,

- ν is *monotone*, i.e., if $A, B \in \mathcal{K}$, $A \subset B$, then $\nu(A) \leqslant \nu(B)$. This follows since φ is assumed nondecreasing.
- ν is *supermodular* (also called convex capacity in the literature), i.e., for $A, B \in \mathcal{K}$, we have

  $$\nu(A \cap B) + \nu(A \cup B) \geqslant \nu(A) + \nu(B)$$

  since μ_C is a probability measure and φ is assumed convex. Indeed, if

  $$\begin{gathered} x_1 = \mu_C(A \cap B), \qquad x_2 = \min\big\{\mu_C(A), \mu_C(B)\big\},\\ x_3 = \max\big\{\mu_C(A), \mu_C(B)\big\}, \qquad x_4 = \mu_C(A \cup B),\\ x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4, \qquad x_1 + x_4 = x_2 + x_3,\end{gathered}$$

then $\varphi(x_1) + \varphi(x_4) \geqslant \varphi(x_2) + \varphi(x_3)$ since $(x_2, x_3) \prec (x_1, x_4)$ (cf. (8)) and the function $g((u, v)) = \varphi(u) + \varphi(v)$ is Schur-convex on $\mathbb{I}^2$ (see [17, p. 92, C.1 Prop.]).

- ν is *continuous from the right*, that is, if $A \in \mathcal{K}$ and $\varepsilon > 0$ are arbitrary, then there is an open set $V \supset A$ such that for all $B \in \mathcal{K}$ satisfying $A \subset B \subset V$, $\nu(B) \leqslant \nu(A) + \varepsilon$. This follows from the (uniform) continuity of φ on $\mathbb{I}$ and the regularity of Borel probability measure μ_C. Indeed, given $\varepsilon > 0$, we can find a $\delta = \delta(\varepsilon) > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever $|x - y| < \delta$, $x, y \in \mathbb{I}$. As any Borel measure is outer regular, for a given $A \in \mathcal{K}$, we can find an open set $V \subset \mathbb{I}^2$ such that $\mu_C(A) \leqslant \mu_C(V) \leqslant \mu_C(A) + \delta$. Then for any $B \in \mathcal{K}$ such that $A \subset B \subset V$, using the monotonicity of φ, we obtain

$$\nu(B) = \varphi \circ \mu_C(B) \leqslant \varphi \circ \mu_C(V) \leqslant \varphi \circ \mu_C(A) + \varepsilon = \nu(A) + \varepsilon.$$

4 Computations of τ_φ for various copula families

This section is devoted to various examples when the new concordance measures could be computed analytically or at least be given in a more tractable form than suggested by the general expressions in (4) or (6). In this regard, several copula families stand out, namely those which are ς^*-invariant and which are mapped to copulas in the same family by reflections σ_i^*, $i = 1, 2$. Among such we find the Farlie–Gumbel–Morgenstern (FGM), Plackett, and Frank families. They will be discussed in the following subsections. Finally, we will consider a couple of Fréchet–Mardia subfamilies in relation to concordance measures of polynomial type.

4.1 Farlie–Gumbel–Morgenstern family

The Farlie–Gumbel–Morgenstern (FGM) family of copulas is often chosen to illustrate what could happen if one deviates from independence. This family is very tractable, yet does not provide the full possible range of dependence as measured by Spearman's ρ_S or Kendall's τ. To be more precise, the FGM family is defined (see [13, Sect. 4.29] or [19, Ex. 3.12]) as

$$C_\delta(u, v) := uv\bigl(1 + \delta(1 - u)(1 - v)\bigr), \quad \delta \in [-1, 1],\ u, v \in \mathbb{I}.$$

This family is absolutely continuous with the copula density

$$c_\delta(u, v) = \frac{\partial^2 C_\delta}{\partial u \partial v}(u, v) = 1 + \delta(1 - 2u)(1 - 2v), \quad u, v \in \mathbb{I}.$$

It is also well known (see, e.g., [13, p. 213]) that

$$\rho_S(C_\delta) = \frac{\delta}{3} \in \left[-\frac{1}{3}, \frac{1}{3}\right], \qquad \tau = \frac{2\delta}{9} \in \left[-\frac{2}{9}, \frac{2}{9}\right]$$

and

$$\sigma_i^*(C_\delta) = C_{-\delta}(u, v), \quad i = 1, 2.$$

Table 1. Values of $\tau_\varphi(C_\delta)$ (rounded to 5 places if given as decimal fractions) for $\varphi(t) = t^p$ and various p. Spearman's ρ_S and Kendall's τ are provided for comparison.

p	δ				
	$1/10$	$1/4$	$1/2$	$3/4$	1
2	0.02500	0.06250	0.12500	0.18750	$1/4$
3	0.02400	0.06001	0.12007	0.18023	$884/3675$
4	0.02222	0.05557	0.11122	0.16703	$787/3528$
5	0.02041	0.05104	0.10216	0.15346	$656431/3201660$
10	0.01389	0.03473	0.06952	0.10442	0.13950
ρ_S	$1/30$	$1/12$	$1/6$	$1/4$	$1/3$
τ	$2/90$	$1/18$	$1/9$	$1/6$	$2/9$

Therefore, (4) reduces to

$$\tau_\varphi(C_\delta) = 2a_\varphi\big([C_\delta, C_\delta]_\varphi - [C_{-\delta}, C_{-\delta}]_\varphi\big). \tag{10}$$

In particular, for $\varphi(t) = t^p$, $p \in \mathbb{N}$, one has $a_\varphi = (p+1)/2$ and

$$\tau_\varphi(C_\delta) = (p+1)\int\limits_{\mathbb{I}^2} \big[\big(C_\delta(u,v)\big)^p c_\delta(u,v) - \big(C_{-\delta}(u,v)\big)^p c_{-\delta}(u,v)\big]\,\mathrm{d}u\,\mathrm{d}v.$$

As the integrand is a polynomial in u and v, the evaluation of the latter integral poses no difficulty and could be done using, e.g., Maple software. The exact analytical expression is cumbersome and therefore omitted. A few values of $\tau_\varphi(C_\delta)$ are presented in Table 1.

4.2 Plackett family

Given a parameter $\nu \in [0,\infty)$ and setting $\eta := \nu - 1$, one defines a comprehensive family (i.e., containing W, Π, and M as its members) of Plackett copulas C_ν as follows: for $\nu \neq 1$,

$$C_\nu(u,v) := \frac{1}{2\eta}\big\{1 + \eta(u+v) - \big[\big(1+\eta(u+v)\big)^2 - 4\nu\eta uv\big]^{1/2}\big\}, \quad u, v \in \mathbb{I},$$

while $C_1(u,v) := uv$. This family appears from algebraic considerations and possesses similar symmetries (see [19, Exr. 3.36]):

$$\sigma_i^*(C_\nu)(u,v) = C_{1/\nu}(u,v), \quad \varsigma^*(C_\nu)(u,v) = C_\nu(u,v), \quad u,v \in \mathbb{I}.$$

Thus our suggested concordance measure takes a similar form as in (10):

$$\tau_\varphi(C_\nu) = 2a_\varphi\big([C_\nu, C_\nu]_\varphi - [C_{1/\nu}, C_{1/\nu}]_\varphi\big).$$

Spearman's ρ_S for this family is known and given by

$$\rho_S(C_\nu) = \frac{\nu+1}{\nu-1} - \frac{2\nu}{(\nu-1)^2}\ln\nu$$

(see [19, Exr. 5.8] or [13, p. 164]), yet Kendall's τ does not seem to have a closed-form expression, thus we also do not anticipate a simple analytic form of $\tau_\varphi(C_\nu)$.

4.3 Frank family

Among Archimedean copulas, Frank family is another popular choice in applications due to being comprehensive (see, e.g., [19, p. 118, table notes]). It is defined as

$$C_\theta(u,v) := -\frac{1}{\theta}\ln\left(1+\frac{(\mathrm{e}^{-\theta u}-1)(\mathrm{e}^{-\theta v}-1)}{\mathrm{e}^{-\theta}-1}\right), \quad \theta\in\mathbb{R}\setminus\{0\},\ u,v\in\mathbb{I},$$

and $C_0(u,v) := \Pi(u,v) = uv$.

It is well known [19, Table 4.1 (4.2.5)]) that the (strict) generator $\psi_\theta : [0,1]\to[0,+\infty]$ of this family is given by

$$\psi_\theta(t) = -\ln\frac{\mathrm{e}^{-\theta t}-1}{\mathrm{e}^{-\theta}-1},$$

so that $C_\theta(u,v) = \psi_\theta^{-1}(\psi_\theta(u)+\psi_\theta(v))$. It is also known (see, e.g., [13, p. 166]) that

$$\sigma^*(C_\theta)(u,v) = C_{-\theta}(u,v) \quad \text{and} \quad \varsigma^*(C_\theta)(u,v) = C_\theta(u,v), \quad u,v\in\mathbb{I}.$$

Moreover, among all Archimedean copulas, Frank family is the only which is invariant with respect to the total reflection ς^*.

So for this family, we again have

$$\tau_\varphi(C_\theta) = 2a_\varphi\big([C_\theta,C_\theta]_\varphi - [C_{-\theta},C_{-\theta}]_\varphi\big), \tag{11}$$

where by considering Kendall's distribution function of the copula C_θ, namely $K_{C_\theta}(t) := \mathbf{P}(C(U,V)\leqslant t)$, $t\in\mathbb{R}$, $(U,V)\sim C_\theta$, one has

$$[C_\theta,C_\theta]_\varphi = \mathbf{E}\varphi\big(C_\theta(U,V)\big) = \int\limits_{\mathbb{I}^2}\varphi(C_\theta)\,\mathrm{d}C_\theta = \int\limits_0^1\varphi(t)\,\mathrm{d}K_{C_\theta}(t).$$

If, as before, we consider $\varphi(x) = x^p$, $p\in\mathbb{N}$, then the latter integral can be rewritten as

$$\begin{aligned}
[C_\theta,C_\theta]_\varphi &= \varphi(t)K_{C_\theta}(t)\big|_0^1 - \int\limits_0^1 K_{C_\theta}(t)\varphi'(t)\,\mathrm{d}t = 1-\int\limits_0^1\left(t-\frac{\psi_\theta(t)}{\psi'_\theta(t)}\right)\varphi'(t)\,\mathrm{d}t\\
&= \int\limits_0^1\phi(t)\,\mathrm{d}t + \int\limits_0^1\frac{\psi_\theta(t)}{\psi'_\theta(t)}\varphi'(t)\,\mathrm{d}t.
\end{aligned}$$

Note that above we have used the known fact (see [19, Thm. 4.3.4]) that, for an Archimedean copula C, Kendall's distribution function is given by

$$K_C(t) = t-\frac{\psi_C(t)}{\psi'_C(t^+)},$$

and, since concave functions are differentiable almost everywhere, the right-hand side derivative $\psi_C'(t^+)$ can be replaced by $\psi_C'(t)$ inside the integral (see [19, proof of Cor. 5.1.4]). Recalling that $a_\varphi = (p+1)/2$ for the considered $\varphi(x)$, we can write (11) as

$$\tau_\varphi(C_\theta) = p(p+1)\int_0^1 \left(\frac{\psi_\theta(t)}{\psi_\theta'(t)} - \frac{\psi_{-\theta}(t)}{\psi_{-\theta}'(t)}\right)t^{p-1}\,\mathrm{d}t,$$

where, after straightforward simplification,

$$\frac{\psi_\theta(t)}{\psi_\theta'(t)} - \frac{\psi_{-\theta}(t)}{\psi_{-\theta}'(t)} = (1-t)\bigl(1-\mathrm{e}^{-\theta t}\bigr) + \frac{2}{\theta}\bigl(\cosh(\theta t)-1\bigr)\ln\frac{1-\mathrm{e}^{-\theta t}}{1-\mathrm{e}^{-\theta}}.$$

As known expressions for Spearman's ρ_S and Kendall's τ, namely

$$\rho_S(C_\theta) = 1 - \frac{12}{\theta}\bigl[D_1(\theta) - D_2(\theta)\bigr], \quad \tau(C_\theta) = 1 - \frac{4}{\theta}\bigl[1 - D_1(\theta)\bigr]$$

(see, e.g., [19, Exr. 5.9]), involve special Debye functions $D_i(x)$, $i = 1, 2$, given by

$$D_k(x) := \frac{k}{x^k}\int_0^x \frac{t^k}{\mathrm{e}^t - 1}\,\mathrm{d}t, \quad k \in \mathbb{N},\ x \geqslant 0,$$

we do not expect simpler expressions for $\tau_\varphi(C_\theta)$.

4.4 Fréchet–Mardia subfamilies

In this section, we will again focus on a special case of the function φ, namely, for any $p \geqslant 1$, we will let $\varphi(x) = x^p$. By taking a convex combination of $A, B \in \mathcal{C}$, namely $K_t := tA + (1-t)B$, $t \in I$, one always has $\xi^*(K_t) = t\xi^*(A) + (1-t)\xi^*(B)$ for any $\xi \in R$ (see Eq. (1) and below in Section 2.1). Moreover,[4]

$$\begin{aligned}&\bigl[\xi^*(K_t), \xi^*(K_t)\bigr]_\varphi \\ &\quad = t\int \bigl(\xi^*(K_t)\bigr)^p\,\mathrm{d}\xi^*(A) + (1-t)\int \bigl(\xi^*(K_t)\bigr)^p\,\mathrm{d}\xi^*(B),\end{aligned} \tag{12}$$

which is always a polynomial in t of degree at most $n+1$, and hence the same is also true about $\tau_\varphi(K_t)$. Nevertheless, not all choices of A and B will give the *maximal* degree of $p+1$. To illustrate this, we will compute $\tau_\varphi(K_t)$ for $K_t := C_t := tM + (1-t)\Pi$ and $K_t = D_t := tM + (1-t)W$, two subfamilies of Fréchet–Mardia copulas. This will affirmatively answer (in the bivariate case) one of Taylor's questions mentioned in the Introduction, namely about examples of concordance measures of polynomial type of any degree $m \geqslant 1$, and will compliment our findings in [15], where examples of certain even-degree polynomial concordance measures were not given. Along the way, the computed analytic expressions for $\tau_\varphi(C_t)$ and $\tau_\varphi(D_t)$ will highlight some similarities and differences of the new concordance measures.

[4] This was kindly pointed out by one of the reviewers.

4.4.1 The case of C_t

First, we have the following:

Theorem 3. *For any integer $p \geqslant 1$ and $\varphi(x) = x^p$,*

$$\tau_\varphi(C_t) = (p+1)\bigl(I_1(t;p) - I_2(t;p)\bigr),$$

where

$$\begin{aligned} I_1(t;p) &:= \bigl[tM + (1-t)\Pi,\ tM + (1-t)\Pi\bigr]_\varphi \\ &= \sum_{m=0}^{p} \frac{p!(p+m)!}{m!(2p+1)!} t^{p-m+1} + 2\sum_{m=0}^{p} \frac{(-1)^m (p!)^2 (1-t)^{m+1}}{(p-m)!(p+2+m)!}, \end{aligned} \tag{13}$$

$$\begin{aligned} I_2(t;p) &:= \bigl[tW + (1-t)\Pi,\ tW + (1-t)\Pi\bigr]_\varphi \\ &= \frac{(p!)^2 t(1-t)^p}{(2p+1)!} + \frac{(p!)^2(1-t)^{p+1}}{(2p+2)!} + \sum_{k=0}^{p} \frac{p!k!(1-t)^{k+1}}{(p+2+k)!}. \end{aligned} \tag{14}$$

Proof. First, recall that the normalizing constant in (5) in the considered setting is $a_\varphi = (p+1)/2$. Next, note that, for any reflection $\xi \in R$,

$$\xi^*(C_t) = t\bigl(M\mathbf{1}_{\xi\in\{e,\varsigma\}} + W\mathbf{1}_{\xi\in\{\sigma_1,\sigma_2\}}\bigr) + (1-t)\Pi,$$

so $\tau_\varphi(C_t) = (p+1)(I_1(t;p) - I_2(t;p))$, where $I_1(t;p)$ and $I_2(t;p)$ are as defined in (13) and (14), respectively, so we only have to compute both of these expressions. Also, notice that, by [16, Lemma 1], $I_1(t;p) \geqslant I_2(t;p)$ for any $t \in \mathbb{I}$ as $tW + (1-t)\Pi \prec tM + (1-t)\Pi$.

Now we compute the needed integrals. First, taking advantage of (12), we have

$$I_1(t;p) = tI_{11}(t;p) + (1-t)I_{12}(t;p), \tag{15}$$

where, using [16, Ex. 1] and properties of the beta function,

$$\begin{aligned} I_{11}(t;p) &:= \int_{\mathbb{I}^2} \bigl(tM + (1-t)\Pi\bigr)^p \,\mathrm{d}M = \int_0^1 \bigl(tu + (1-t)u^2\bigr)^p \,\mathrm{d}u \\ &= \int_0^1 u^p \sum_{m=0}^{p} \binom{p}{m} u^m \bigl(t(1-u)\bigr)^{p-m} \,\mathrm{d}u \\ &= \sum_{m=0}^{p} \binom{p}{m} t^{p-m} \frac{(p+m)!(p-m)!}{(2p+1)!} = \sum_{m=0}^{p} \frac{p!(p+m)!}{m!(2p+1)!} t^{p-m} \end{aligned} \tag{16}$$

and

$$I_{12}(t;p) := \int_{\mathbb{I}^2} \big(tM + (1-t)\varPi\big)^p \,\mathrm{d}\varPi = 2\int_0^1\int_0^u \big(tv + (1-t)uv\big)^p \,\mathrm{d}u\,\mathrm{d}v$$
$$= \frac{2}{p+1}\int_0^1 u^{p+1}\big(t + (1-t)u\big)^p \,\mathrm{d}u =: \frac{2}{p+1} J_p(t;p). \tag{17}$$

Here, using integration by parts, for any integer $k \in \{1,\dots,p\}$, we obtain

$$J_k(t;p) := \int_0^1 u^{2p+1-k}\big(t + (1-t)u\big)^k \,\mathrm{d}u = \frac{1}{2p+2-k} - \frac{k(1-t)}{2p+2-k} J_{k-1}(t;p),$$

where, of course, $J_0(t;p) = \int_0^1 u^{2p+1}\,\mathrm{d}u = 1/(2p+2)$.

Iterating the above formula, we get for $k \in \{0,\dots,p\}$,

$$J_k(t;p) = \sum_{m=0}^{k} (-1)^m \frac{k!(2p+1-k)!(1-t)^m}{(k-m)!(2p+2+m-k)!}.$$

Substituting into (17) (with $k = p$) yields

$$I_{12}(t;p) = 2\sum_{m=0}^{p} (-1)^m \frac{(p!)^2(1-t)^m}{(p-m)!(p+2+m)!}. \tag{18}$$

Substituting (16) and (18) into (15) yields the expression in (13), as claimed.

Now we deal with $I_2(t;p)$:

$$I_2(t;p) = tI_{21}(t;p) + (1-t)I_{22}(t;p). \tag{19}$$

Using again [16, Ex. 1] and properties of the Beta function gives

$$I_{21}(t;p) := \int_{\mathbb{I}^2} \big(tW + (1-t)\varPi\big)^p \,\mathrm{d}W = \int_0^1 \big((1-t)u(1-u)\big)^p \,\mathrm{d}u$$
$$= \frac{(p!)^2}{(2p+1)!}(1-t)^p. \tag{20}$$

As for $I_{22}(t;p)$, we write

$$I_{22}(t;p) := \int_{\mathbb{I}^2} \big(tW + (1-t)\varPi\big)^p \,\mathrm{d}\varPi = I_{221}(t;p) + I_{222}(t;p), \tag{21}$$

where

$$I_{221}(t;p) := \int_0^1 \int_0^{1-u} \big((1-t)uv\big)^p \,\mathrm{d}u\,\mathrm{d}v = \frac{(p!)^2}{(2p+2)!}(1-t)^p \tag{22}$$

and

$$\begin{aligned} I_{222}(t;p) &:= \int_0^1 \int_{1-u}^1 \big(uv - t(1-u)(1-v)\big)^p \,\mathrm{d}u\,\mathrm{d}v \\ &= \int_0^1 \frac{u^{p+1}[1-((1-t)(1-u))^{p+1}]}{(p+1)(1-(1-t)(1-u))}\,\mathrm{d}u \\ &= \int_0^1 \frac{u^{p+1}}{p+1} \sum_{k=0}^p \big((1-t)(1-u)\big)^k \,\mathrm{d}u = \sum_{k=0}^p \frac{p!k!(1-t)^k}{(p+2+k)!}. \end{aligned} \tag{23}$$

Substituting (22) and (23) into (21) and then combining the result with (20) in (19) leads to the formula in (14), as claimed, finishing the proof. □

Example 3. Using Theorem 3 with $p = 1, 2, 3$, we obtain

- for $p = 1$, we recover Kendall's τ (see [19, Ex. 5.3]), $\tau(C_t) = (t^2 + 2t)/3$;
- for $p = 2$, we obtain $\tau_\varphi(C_t) = (t^2 + 3t)/4$;
- and for $p = 3$, we get $\tau_\varphi(C_t) = (3/70)t^4 + (4/105)t^3 + (41/210)t^2 + (76/105)t$.

Plots of $\tau_\varphi(C_t)$, $t \in \mathbb{I}$, for $\varphi(x) = x^p$, $p = 1, 2, 3$, are illustrated in Fig. 2. For larger values of p, polynomials become more complicated, but the graphs remain surprisingly similar.

From the above example, one can guess that, for even p, $\tau_\varphi(C_t)$ is a polynomial of degree p, while for odd p, we get a polynomial of degree $p + 1$. This is indeed the case, and we have

Corollary 1. *Let $p \geqslant 1$ be an integer, and let $\varphi(x) = x^p$. Then for the Fréchet–Mardia family $\{C_t\}_{t\in\mathbb{I}}$,*

$$\deg \tau_\varphi(C_t) = \begin{cases} p & \text{if } p \in 2\mathbb{Z}; \\ p+1 & \text{if } p \in 2\mathbb{Z}+1. \end{cases}$$

Proof. Looking at (13) and (14), it is clear that $\tau_\varphi(C_t)$ is a polynomial in t of degree at most $p + 1$, as such are both $I_1(t;p)$ and $I_2(t;p)$. So we only have to compute a couple of coefficients, b_{p+1} and b_p, in the expansion $\tau_\varphi(C_t) = b_{p+1}t^{p+1} + b_p t^p + \cdots + b_0$. We have

$$\begin{aligned} b_{p+1} &= (p+1)\bigg(\frac{(p!)^2}{(2p+1)!} - \frac{2(p!)^2}{(2p+2)!} - \frac{(-1)^p(p!)^2}{(2p+1)!} + \frac{2(-1)^p(p!)^2}{(2p+2)!}\bigg) \\ &= \frac{p(1-(-1)^p)(p!)^2}{(2p+1)!} = \begin{cases} 0 & \text{if } p \in 2\mathbb{Z}; \\ \frac{2p(p!)^2}{(2p+1)!} & \text{if } p \in 2\mathbb{Z}+1. \end{cases} \end{aligned}$$

Also, when $p \in 2\mathbb{Z}$, we evaluate b_p to make sure it does not vanish:

$$\begin{aligned}
b_p &= (p+1)\biggl(\frac{(p+1)!p!}{(2p+1)!} + \frac{2(p+1)(p!)^2}{(2p+2)!} - \frac{2(p!)^2}{(2p+1)!} + \frac{(-1)^p p(p!)^2}{(2p+1)!} \\
&\quad + \frac{(-1)^{p+1}(p+1)(p!)^2}{(2p+2)!} + \frac{(-1)^{p+1}p!(p-1)!}{(2p+1)!} + \frac{(-1)^{p+1}(p+1)(p!)^2}{(2p+2)!}\biggr) \\
&= \frac{(p-1)(p-1)!(p+1)!}{(2p)!}.
\end{aligned}$$

□

4.4.2 *The case of D_t*

This time let us consider another subfamily of Fréchet–Mardia copulas, namely $D_t := tM + (1-t)W$, $t \in \mathbb{I}$. We have

Theorem 4. *For copulas D_t, $t \in \mathbb{I}$, and $\varphi(x) = x^p$, $p \geqslant 1$, we have*

$$\tau_\varphi(D_t) = 2^{-p}\Biggl\{t^p - (1-t)^p + \sum_{m=1}^{p} 2^{p-k}\bigl(t^m - (1-t)^m\bigr)\Biggr\}. \tag{24}$$

Proof. As in the proof of Theorem 3, we get, for any reflection $\xi \in R$,

$$\xi^*(D_t) = t\bigl(M\mathbf{1}_{\xi\in\{e,\varsigma\}} + W\mathbf{1}_{\xi\in\{\sigma_1,\sigma_2\}}\bigr) + (1-t)\bigl(W\mathbf{1}_{\xi\in\{e,\varsigma\}} + M\mathbf{1}_{\xi\in\{\sigma_1,\sigma_2\}}\bigr),$$

so

$$\tau_\varphi(D_t) = (p+1)\bigl[J_1(t;p) - J_2(t;p)\bigr],$$

where

$$\begin{aligned}
J_1(t;p) &:= t[D_t, M]_\varphi + (1-t)[D_t, W]_\varphi \\
&= t\int_0^1 \bigl(tu + (1-t)(2u-1)^+\bigr)^p \,\mathrm{d}u + (1-t)\int_0^1 \bigl(t\bigl(u \wedge (1-u)\bigr)\bigr)^p \,\mathrm{d}u \\
&= t\Biggl(t^p\int_0^{1/2} u^p \,\mathrm{d}u + \int_{1/2}^1 \bigl(tu + (1-t)(2u-1)\bigr)^p \,\mathrm{d}u\Biggr) + 2(1-t)t^p\int_0^{1/2} u^p \,\mathrm{d}u \\
&= t\biggl(\frac{t^p}{2^{p+1}(p+1)} + \frac{1-(t/2)^{p+1}}{(p+1)(2-t)}\biggr) + \frac{(1-t)t^p}{2^p(p+1)} \\
&= \frac{t^{p+1}}{2^{p+1}(p+1)} + \frac{t}{2^{p+1}(p+1)}\sum_{k=0}^{p} 2^{p-k}t^k + \frac{(1-t)t^p}{2^p(p+1)}
\end{aligned}$$

and

$$J_2(t;p) := t[D_t, W]_\varphi + (1-t)[D_t, M]_\varphi = J_1(1-t;p).$$

Equation (24) now follows by simplifying

$$2t^{p+1} + 2(1-t)t^p - 2(1-t)^{p+1} - 2t(1-t)^p = 2\bigl(t^p - (1-t)^p\bigr)$$

and after letting $m = k+1$ in the remaining sum over k from 0 to $p-1$. □

Example 4. Using Theorem 4 with $p = 1, 2, 3, 4$, we obtain

- for $p = 1$, we recover Kendall's τ (see [19, Ex. 5.3]), namely $\tau(D_t) = 2t - 1$;
- for $p = 2$, $\tau_\varphi(D_t) = 2t - 1$;
- for $p = 3$, $\tau_\varphi(D_t) = (1/2)t^3 - (3/4)t^2 + (9/4)t - 1$;
- and for $p = 4$, $\tau_\varphi(D_t) = (3/4)t^3 - (9/8)t^2 + (19/8)t - 1$.

Plots of $\tau_\varphi(D_t)$, $t \in \mathbb{I}$, for $\varphi(x) = x^p$, $p = 1, 2, 3, 4$, can be found in Fig. 3. For larger values of p, as in Example 3, polynomials become more complicated, but the graphs again remain surprisingly similar.

More generally, we have

Corollary 2. *Let $p \geqslant 1$ be an integer, and let $\varphi(x) = x^p$. Then for the Fréchet–Mardia family $\{D_t\}_{t\in\mathbb{I}}$,*

$$\deg \tau_\varphi(D_t) = \begin{cases} p - 1 & \textit{if } p \in 2\mathbb{Z}; \\ p & \textit{if } p \in 2\mathbb{Z} + 1. \end{cases}$$

Proof. Inspecting Eq. (24), we see that, for $p \in 2\mathbb{Z} + 1$, the coefficient of t^p is

$$2^{-p}\bigl(2\bigl(1 - (-1)^p\bigr)\bigr) = 2^{-(p-2)} > 0.$$

On the other hand, when $p \in 2\mathbb{Z}$, the coefficient of t^p vanishes, while that of t^{p-1} is

$$2^{-p}\bigl(-2p(-1)^{p-1} + 2\bigl(1 - (-1)^{p-1}\bigr)\bigr) = (p + 2)2^{-(p-1)} > 0. \qquad \square$$

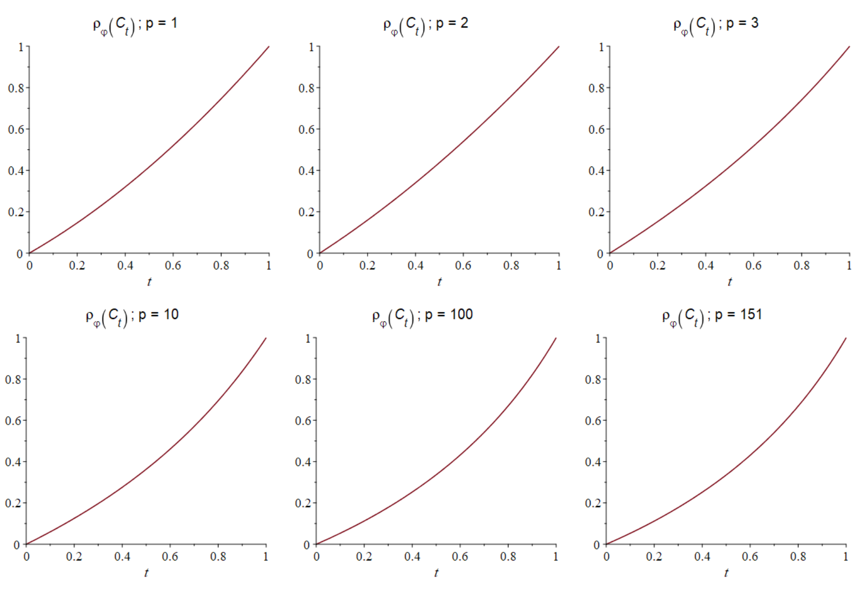

Figure 2. Graphs of $\tau_\varphi(C_t)$, $t \in \mathbb{I}$, for $\varphi(x) = x^p$, $p \in \{1, 2, 3, 10, 100, 151\}$, respectively.

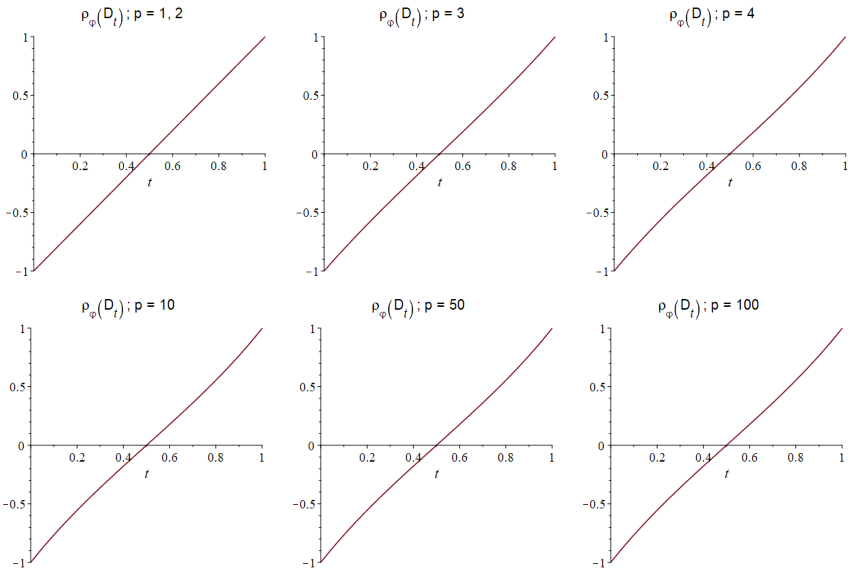

Figure 3. Graphs of $\tau_\varphi(D_t)$, $t \in \mathbb{I}$, for $\varphi(x) = x^p$, $p \in \{1, 2, 3, 4, 10, 50, 100\}$, respectively.

5 Conclusions and future directions

In this part of our investigation of concordance measure constructions, we have provided more examples of computation of generalized Kendall's τ. We have employed mostly power-type distortion functions φ, but other choices are possible, e.g., piecewise linear or polynomial. This is related to copula-based local dependence framework explored in [11]. We have then looked at the intrinsic meaning of our generalizations and established that they are achieved by replacing a probability measure μ_C (induced by a given copula C) with a nonadditive measure (in our case, convex (supermodular) capacity, $\nu = \varphi \circ \mu_C$). Such measures have found their place in the economic decision theory, so we hope that our generalizations could be of use there, too.

Having many examples to work with calls for taking a shot at several open problems about the structure of polynomial-type concordance measures, both bivariate and multivariate; see the works of Edwards, Mikusiński, Taylor [6–8], Fuchs [9, 10] and the references therein. We hope that our work, providing many more polynomial-type concordance measures, will be useful in this direction. A recent work by Borroni [1] also deserves a closer look. We intend to look for more examples and constructions of "generators", as Borroni calls them, used to define concordance measures and provide a different point of view towards what exactly a particular concordance measure measures.

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