Nonlinear Analysis: Modelling and Control, Vol. 30, No. 2, 270–290
https://doi.org/10.15388/namc.2025.30.38966

Vilnius University Press

Estimations for the convex modular of the aliasing error of nonlinear sampling Kantorovich operators

Danilo Costarelli [1,2], **Mariarosaria Natale** [2], **Gianluca Vinti** [1,3]

University of Perugia, 1, Via Vanvitelli, 06123 Perugia, Italy
danilo.costarelli@unipg.it; mariarosaria.natale@unipg.it;
gianluca.vinti@unipg.it

Received: May 15, 2024 / **Revised:** January 30, 2025 / **Published online:** February 26, 2025

Abstract. In this paper, we establish quantitative estimates for the nonlinear sampling Kantorovich operators in the general setting of modular spaces L_ρ. To achieve this, we consider a notion of modulus of smoothness based on the convex modular functional ρ, which defines the space. The approach proposed is new in the sense that, in the literature, theorems for the order of approximation in L_ρ are mainly qualitative, i.e., are proved considering functions belonging to Lipschitz classes; here the estimates are achieved for every function belonging to the whole L_ρ. To show the effectiveness of the achieved results, several particular cases of modular spaces are presented in detail.

Keywords: modular spaces, modular inequalities, modulus of smoothness, order of approximation, quantitative estimates.

1 Introduction

In the present paper, we focus on studying the order of approximation for the so-called *nonlinear sampling Kantorovich operators* [18], which are defined by

$$(K_w f)(x) := \sum_{k\in\mathbb{Z}} \chi\left(wx - t_k,\ \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\right), \quad x \in \mathbb{R},$$

[1]Partially supported within the project "National Innovation Ecosystem grant ECS00000041 - VITALITY" funded by the European Union – NextGenerationEU under the Italian MUR.

[2]Supported from the projects: 2024 GNAMPA-INDAM "Tecniche di approssimazione in spazi funzionali con applicazioni a problemi di diffusione" (CUP E53C23001670001); PRIN 2022: "AI- and DIP-Enhanced DAta Augmentation for Remote Sensing of Soil Moisture and Forest Biomass (AIDA)" (20229FX3B9, CUP J53D23000660001) funded by the European Union under the Italian National Recovery and Resilience Plan (NRRP) of NextGenerationEU under the Italian MUR.

[3]Partially supported within the project PRIN 2022: "EXPANSION - EXPlainable AI through high eNergy physicS for medical Imaging in ONcology" funded by the European Union under the Italian National Recovery and Resilience Plan (NRRP) of NextGenerationEU under the MUR (2022Z3BT9F, CUP J53D23002530006).

where $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function ensuring convergence of the series, $\chi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ represents a kernel function satisfying certain properties, and $(t_k)_{k\in\mathbb{Z}}$ is a suitable strictly increasing sequence of real numbers with $\Delta_k := t_{k+1} - t_k > 0$, $k \in \mathbb{Z}$. The choice of $(t_k)_{k\in\mathbb{Z}}$ allows to sample signals by an irregular scheme, and in particular, if $t_k = k$, $k \in \mathbb{Z}$, we obtain uniformly spaced reconstruction algorithm.

Recently, the nonlinear operators received a lot of attention within the scientific community. Indeed, from the point of view of the applications, nonlinear version of the sampling Kantorovich operators may be useful in order to approximate nonlinear signals as, for example, a signal generated by an earthquake, an explosion, an eruption of a volcano, and so on. The pioneer works of the theory of nonlinear integral operators, in connection with approximation problems, can be reconducted to Musielak [15]. Later, it has been extensively developed in the monograph by Bardaro, Musielak, and Vinti [2] and studied by other authors; see, e.g., [1, 4, 13].

Concerning the problem of the order of approximation, quantitative estimates for the aliasing error have been recently established in [4] in the setting of Orlicz spaces; and now, we focus on extending such estimations within the broader setting of modular spaces $L_\rho(\mathbb{R})$ using its typical modulus of smoothness. Its definition is given in the monograph [2] (wherein its basic properties are also investigated) with respect to the modular ρ, which generates the involved spaces.

However, in order to establish such estimations, there are quite a few challenges, mainly due to nonlinear setting and to the general framework of modular spaces itself. While the modular convergence results presented in [7] were obtained via density, achieving quantitative estimates necessitates the use of a direct approach. To address this issue, we propose an innovative approach that combines the technical assumptions in [7] into a single condition (see condition (4)) that establishes a relationship among three modulars taken into consideration and the nonlinear kernel χ of K_w. Condition (4) holds in specific cases such as Musielak–Orlicz spaces and function spaces with modulars lacking an integral representation. Therefore, by imposing this and other suitable assumptions we establish the validity of Theorem 2.

The rest of the paper is organized as follows. In Section 4.1 we investigate the case of Musielak–Orlicz spaces, which contain the weighted Orlicz spaces and the Orlicz spaces as well, and in Section 4.2 the spaces of functions equipped by modulars that are not of integral type. Finally, Section 4.3 provides some direct quantitative estimates for the nonlinear sampling Kantorovich operators acting on the set of uniformly continuous and bounded functions.

2 Basic assumptions and notations

Let $(\mathbb{R}, \Sigma_\mathbb{R}, \mu_\mathbb{R})$ be a measure space. Denoted by $X(\mathbb{R})$ the space of all $\Sigma_\mathbb{R}$-measurable real-valued functions on $\mathbb{R}$, a functional $\rho : X(\mathbb{R}) \to [0, +\infty]$ is said to be a *modular* on $X(\mathbb{R})$ if the following conditions hold:

(ρ_1) $\rho(f) = 0$ if and only if $f \equiv 0$ $\mu_\mathbb{R}$-a.e. in $\mathbb{R}$;
(ρ_2) $\rho(-f) = \rho(f)$ for every $f \in X(\mathbb{R})$;

(ρ_3) $\rho(\alpha f + \beta g) \leqslant \rho(f) + \rho(g)$ for every $f, g \in X(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}_0^+$ with $\alpha + \beta = 1$.

By means of the functional ρ we introduce the *modular space* $L_\rho(\mathbb{R})$ defined as follows:

$$L_\rho(\mathbb{R}) := \Bigl\{ f \in X(\mathbb{R}):\ \lim_{\lambda \to 0} \rho(\lambda f) = 0 \Bigr\}.$$

It is easily verifiable that, if ρ is convex, the definition of $L_\rho(\mathbb{R})$ can be equivalently expressed as follows:

$$L_\rho(\mathbb{R}) = \bigl\{ f \in X(\mathbb{R}):\ \rho(\lambda f) < +\infty \text{ for some } \lambda > 0 \bigr\}.$$

It is well known that $L_\rho(\mathbb{R})$ is a vector subspace of $X(\mathbb{R})$, and the most natural notion of convergence on it is called the *modular convergence*. We will say that a net of functions $(f_w)_{w>0} \subset L_\rho(\mathbb{R})$ is *modularly convergent* (or *ρ-convergent*) to a function $f \in L_\rho(\mathbb{R})$ if

$$\lim_{w \to +\infty} \rho\bigl(\lambda(f_w - f)\bigr) = 0$$

for some $\lambda > 0$. The modular convergence is weaker than the convergence induced by the Luxemburg norm generated by the modular ρ. The latter is equivalent to say that the above limit relation is satisfied for any $\lambda > 0$. A general theory of modular spaces can be found in [14, 16].

The following notions concerning modular functionals will be used along the paper; see [2, 15].

We say that a modular ρ is

(a) *monotone* if $\rho(f) \leqslant \rho(g)$ whenever $|f| \leqslant |g|$ for every $f, g \in X(\mathbb{R})$;
(b) *finite* if the characteristic function $\mathbf{1}_A$ of every measurable set A of finite $\mu_{\mathbb{R}}$-measure belongs to $L_\rho(\mathbb{R})$;
(c) *strongly finite* if each $\mathbf{1}_A$ as above belongs to $E_\rho(\Omega)$;
(d) *absolutely finite* if ρ is finite and if for every $\varepsilon, \lambda_0 > 0$, there exists $\delta > 0$ such that $\rho(\lambda_0 \mathbf{1}_B) < \varepsilon$ for every $B \in \Sigma_{\mathbb{R}}$ with $\mu_{\mathbb{R}}(B) < \delta$;
(e) *absolutely continuous* if there is $\alpha > 0$ such that for every $f \in X(\mathbb{R})$ with $\rho(f) < +\infty$, the following two conditions are satisfied:
  (i) for every $\varepsilon > 0$, there is a measurable subset $A \subset \mathbb{R}$ such that $\mu_{\mathbb{R}}(A) < +\infty$ and $\rho(\alpha f \mathbf{1}_{\Omega \setminus A}) < \varepsilon$;
  (ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(\alpha f \mathbf{1}_B) < \varepsilon$ for any measurable subset $B \subset \mathbb{R}$ with $\mu_{\mathbb{R}}(B) < \delta$;
(f) *bounded* if there exists a constant $C \geqslant 1$ and a measurable, bounded function $h : \mathbb{R} \to \mathbb{R}_0^+$ such that $h(t) \to 0$ as $t \to 0$ and $\rho(f(t + \cdot)) \leqslant \rho(Cf) + h(t)$, $t \in \mathbb{R}$, for all $f \in X(\mathbb{R})$ such that $\rho(f) < +\infty$;
(g) *strongly bounded* if ρ is bounded with $h(t) = 0$ for every $t \in \mathbb{R}$.

Note that if ρ is convex, then any strongly finite modular is also finite.

In order to establish quantitative estimates for the order of approximation of a family of nonlinear operators, we recall the definition of the modulus of smoothness in modular space $L_\rho(\mathbb{R})$ with respect to the modular ρ. For any fixed $f \in L_\rho(\mathbb{R})$, we denote

$$\omega_\rho(f,\delta) := \sup_{|t|<\delta} \rho\bigl(f(t+\cdot) - f(\cdot)\bigr)$$

with $\delta > 0$. For further considerations, we need to recall the following theorem.

Theorem 1. (*See* [2, *Thm.* 2.4].) *Let ρ be a monotone, absolutely finite, absolutely continuous, and bounded modular on $X(\mathbb{R})$. Then for every function $f \in L_\rho(\Omega)$, there exists a constant $\lambda > 0$ such that*

$$\omega_\rho(\lambda f,\delta) \to 0 \quad \textit{as } \delta \to 0^+.$$

Remark 1. Let $f \in L_\rho(\mathbb{R})$, that is, $\rho(\lambda_f f) < +\infty$ for some $\lambda_f > 0$, where ρ is a convex, monotone, and bounded modular on $X(\mathbb{R})$. It is easy to prove that the function $f(t+\cdot)$, for every fixed $t \in \mathbb{R}$, also lies in $L_\rho(\mathbb{R})$. In fact, taking $\lambda \leqslant \lambda_f/C$, we have

$$\rho\bigl(\lambda f(t+\cdot)\bigr) \leqslant \rho(C\lambda f) + h(t) \leqslant \rho(\lambda_f f) + \|h\|_\infty < +\infty$$

with C and h defined as in (f). Note that, since h is bounded, the above estimate is independent (uniform) with respect to the shift parameter t.

In particular, if ρ is strongly bounded, the following inequality holds:

$$\rho\bigl(\lambda f(t+\cdot)\bigr) \leqslant \rho(\lambda_f f) < +\infty. \tag{1}$$

Now, we give the definition of the class of operators we work with.

Let $\varPi = (t_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers with $-\infty < t_k < t_{k+1} < +\infty$, $\lim_{k\to\pm\infty} t_k = \pm\infty$ and such that there exist $\varDelta, \delta > 0$ for which $\delta \leqslant \varDelta_k := t_{k+1} - t_k \leqslant \varDelta$.

A function $\chi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ will be called (nonlinear) *kernel* if the following conditions hold:

(χ_1) $(\chi(wx - t_k, u))_k \in \ell^1(\mathbb{Z})$ for every $x, u \in \mathbb{R}$ and $w > 0$;

(χ_2) $\chi(x, 0) = 0$ for every $x \in \mathbb{R}$;

(χ_3) χ is an (L,ψ)-Lipschitz kernel, i.e., there exist a measurable function $L : \mathbb{R} \to \mathbb{R}_0^+$ and a nondecreasing function $\psi : \mathbb{R} \to \mathbb{R}_0^+$ with $\psi(0) = 0$ such that

$$\bigl|\chi(x,u) - \chi(x,v)\bigr| \leqslant L(x)\psi\bigl(|u-v|\bigr) \quad \text{for every } x, u, v \in \mathbb{R};$$

(χ_4) there exists $\theta_0 > 0$ such that

$$\mathcal{T}_w(x) := \sup_{u\neq 0} \left| \frac{1}{u} \sum_{k\in\mathbb{Z}} \chi(wx - t_k,\, u) - 1 \right| = O\bigl(w^{-\theta_0}\bigr)$$

as $w \to +\infty$ uniformly with respect to $x \in \mathbb{R}$.

Moreover, we assume that the function L of condition (χ_3) satisfies the following additional assumptions:

(L_1) There exists a convex, monotone, strongly finite, and absolutely continuous modular $\xi_{\mathbb{R}}$ on $X(\mathbb{R})$ such that, defining $L_w(x) := wL(wx)$ for $x \in \mathbb{R}$ and $w > 0$,

(i*) there are $\alpha, \alpha_0, M > 0$ such that, if we denote by $A_{\alpha_0,w} \subset \mathbb{R}$, $w > 0$, the set

$$A_{\alpha_0,w} := \{y \in \mathbb{R}\colon |y| \leqslant w^{-\alpha_0}\},$$

it turns out that, for $w > 0$ sufficiently large,

$$\xi_{\mathbb{R}}(\alpha L_w \mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}) \leqslant M w^{-\alpha_0}. \tag{2}$$

Moreover, we assume that $L_w \in L_{\xi_{\mathbb{R}}}(\mathbb{R})$ for every $w > 0$, and for any fixed $c > 0$, there is $N_c > 0$ such that

$$\xi_{\mathbb{R}}(cL_w) \leqslant N_c \quad \text{for every } w > 0. \tag{3}$$

(L_2) The absolute moment of order 0 is finite, i.e.,

$$m_{0,\Pi}(L) := \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} L(x - t_k) < +\infty.$$

Remark 2. Condition (i*) represents a natural quantitative version of property (i) of the absolute continuity. This new condition becomes necessary to investigate the order of approximation for the class of operators $K_w f$.

Remark 3. Note that, if the kernel χ is of the form $\chi(x, u) = L(x)u$ with L satisfying assumptions (L_1) and (L_2), the operators $K_w f$ reduces to the well-known linear ones.

Thus, for $w > 0$, the nonlinear sampling Kantorovich operators for any given kernel χ are defined by

$$(K_w f)(x) := \sum_{k\in\mathbb{Z}} \chi\left(wx - t_k,\ \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\right), \quad x \in \mathbb{R}$$

for any given locally integrable function $f : \mathbb{R} \to \mathbb{R}$ such that the above series is convergent for every $x \in \mathbb{R}$. It is well known that the above operators are well defined for $f \in L^\infty(\mathbb{R})$ by (L_2) and conditions (χ_2) and (χ_3); see [18].

3 Main result

In the case of functions belonging to modular spaces, the convergence of the nonlinear sampling Kantorovich operators $K_w f$ to f has been established in Theorem 4.5 of [7].

In this section, the focus shifts to the quantitative analysis of the nonlinear Kantorovich sampling operators by using the modulus of smoothness in modular spaces recalled in Section 2. However, in order to establish quantitative estimates, there are quite a few challenges, mainly due to nonlinear setting itself and to the general context of modular spaces. Therefore, it becomes necessary to require some technical conditions on the modulars taken into consideration. However, we will also show that these conditions are satisfied in several concrete cases.

First and foremost, for the operators $K_w f$ to be well defined in the general frame of modular spaces, we need to introduce a growth condition that provides a connection between pairs of modulars on $X(\mathbb{R})$ and the function ψ of condition (χ_3).

Let $\rho_{\mathbb{R}}, \eta_{\mathbb{R}}$ be two modulars on $X(\mathbb{R})$. We suppose that, for every $\lambda \in (0,1)$, there exists a constant $C_\lambda \in (0,1)$ satisfying the inequality

$$\rho_{\mathbb{R}}\big(C_\lambda \psi(\mathrm{g})\big) \leqslant \eta_{\mathbb{R}}(\lambda g) \quad \text{for any } g \in X(\mathbb{R}). \tag{H}$$

Remark 4. If $\eta_{\mathbb{R}}$ is monotone, the previous inequality holds also for any $\lambda > 0$. In fact, if $\lambda \geqslant 1$, taking $\alpha = \lambda/(1+\lambda)$ and the corresponding $C_\alpha \in (0,1)$, we obtain

$$\rho_{\mathbb{R}}\big(C_\alpha \psi(\mathrm{g})\big) \leqslant \eta_{\mathbb{R}}(\alpha g) \leqslant \eta_{\mathbb{R}}(\lambda g).$$

Now, we introduce the following *new condition* in which the modulars $\rho_{\mathbb{R}}$, $\eta_{\mathbb{R}}$ are related to the functions L and ψ of condition (χ_3) and the modular $\xi_{\mathbb{R}}$ arising from condition (L_1).

First, let us recall the classical first-order difference operator defined as

$$(\tau_x f)(y) := f(y) - f(y+x)$$

with $x, y \in \mathbb{R}$. Now, we also define the function M^ψ by

$$M^\psi(x) := \sum_{k\in\mathbb{Z}} L(wx - t_k)\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} (\tau_{x-t_k/w} f)(u)\,\mathrm{d}u\right), \qquad x \in \mathbb{R},$$

which is well defined by (L_2) for any bounded and locally integrable function f.

We assume that there exist three positive constants C_1, C_2, C_3 such that for every $\lambda \in (0,1)$, there is $z_\lambda \in (0,1)$ such that

$$\rho_{\mathbb{R}}\big(z_\lambda M^\psi\big) \leqslant C_1 \xi_{\mathbb{R}}\big(C_2 L_w(\cdot)\eta_{\mathbb{R}}\big(C_3\lambda|\tau_{(\cdot)} f|\big)\big) \tag{4}$$

for sufficiently large $w > 0$, where $(\cdot)$ denotes the variable of the involved functions for which we evaluate the modular.

Thus, we can give an estimate of the modular error of approximation $\rho_{\mathbb{R}}(\mu(K_w f - f))$ for sufficiently small $\mu > 0$.

Theorem 2. *Let $\rho_{\mathbb{R}}$ and $\eta_{\mathbb{R}}$ be convex, monotone, absolutely finite, and absolutely continuous modulars satisfying condition* (H)*. In addition, we also assume that $\eta_{\mathbb{R}}$ is strongly bounded and suppose that condition* (4) *is satisfied. For any $f \in L_{\rho_{\mathbb{R}}+\eta_{\mathbb{R}}}(\mathbb{R})$, there exist*

$\mu > 0$, $\lambda > 0$, *and further parameter* $\lambda_1 > 0$ *such that*

$$\rho_{\mathbb{R}}\big(\mu(K_w f - f)\big) \leqslant C_1 N \omega_{\eta_{\mathbb{R}}}\left(C_3\lambda f, \frac{1}{w^{\alpha_0}}\right) + C_1 M w^{-\alpha_0} + \omega_{\eta_{\mathbb{R}}}\left(\frac{\lambda \Delta f}{\delta}, \frac{\Delta}{w}\right) + w^{-\theta_0}\rho_{\mathbb{R}}(\lambda_1 f)$$

for every sufficiently large $w > 0$, *where* C_i, $i = 1, 2, 3$, *are the constants of condition* (4), $N := N_{2C_2}$ *is the constant of condition* (3), M *is the constant of condition* (i*), *and* $\theta_0 > 0$ *is the constant of condition* (χ_4). *In particular, if* $\mu > 0$ *and* $\lambda > 0$ *are sufficiently small, the above inequality implies the modular convergence of* $K_w f$ *to* f.

Proof. Let f be as in the statement, i.e., it belongs both to $L_{\rho_{\mathbb{R}}}(\mathbb{R})$ and $L_{\eta_{\mathbb{R}}}(\mathbb{R})$. Let $\lambda_1 > 0$ such that $\rho_{\mathbb{R}}(\lambda_1 f) < +\infty$. Since $f \in L_{\eta_{\mathbb{R}}}(\mathbb{R})$, we may say that $\lim_{\widetilde{\lambda} \to 0} \eta_{\mathbb{R}}(\widetilde{\lambda} f) = 0$, i.e., for every fixed $\varepsilon > 0$, there exists λ_2 sufficiently small for which $\eta_{\mathbb{R}}(\lambda_2 f) < \varepsilon$. In addition, we recall that $\eta_{\mathbb{R}}$ is strongly bounded, hence definition (g) is satisfied with a suitable constant C. Let now $\lambda_3 \leqslant \lambda_2/C$ be fixed; by (1) it follows that

$$\eta_{\mathbb{R}}(\lambda_3 \widetilde{f}_t) := \eta_{\mathbb{R}}\big(\lambda_3 f(t + \cdot)\big) \leqslant \eta_{\mathbb{R}}(\lambda_2 f) < \varepsilon \tag{5}$$

for every $t \in \mathbb{R}$. Taking $\varepsilon = \alpha/(4C_2)$ in (5), where α is the parameter of (i*), we denote by λ_2 and λ_3 the corresponding constants of (5). Finally, by Theorem 1 there exists $\lambda_4 > 0$ such that $\omega_{\eta_{\mathbb{R}}}(\lambda_4 f, \delta) \to 0$ as $\delta \to 0^+$. Now, considering the constants C_1, C_2, C_3 arising from condition (4), we can fix $\lambda > 0$ such that

$$\lambda < \min\left\{1, \frac{\lambda_1}{2}, \frac{\lambda_2}{2C_3}, \frac{\lambda_3}{2C_3}, \frac{\lambda_4}{C_3}\right\}.$$

Correspondingly to λ, there is $z_\lambda \in (0, 1)$ for which (4) holds and, by condition (H) we know that there exists $C_\lambda \in (0, 1)$ such that $\rho_{\mathbb{R}}(C_\lambda \psi(\mathrm{g})) \leqslant \eta_{\mathbb{R}}(\lambda g)$, $g \in X(\mathbb{R})$. At the same time, by (χ_4) there exist constants $\theta_0, M_1 > 0$ such that

$$\mathcal{T}_w(x) \leqslant M_1 w^{-\theta_0} \tag{6}$$

uniformly with respect to $x \in \mathbb{R}$ for sufficiently large $w > 0$.

Now, defining $d_\lambda := \min\{z_\lambda, C_\lambda\}$, we choose $\mu > 0$ such that

$$\mu \leqslant \min\left\{\frac{d_\lambda}{3}, \frac{d_\lambda}{3m_{0,\Pi}(L)}, \frac{\lambda_1}{3M_1}\right\}.$$

Taking into account that $\rho_{\mathbb{R}}$ is monotone and using (ρ_3), we have

$$\begin{aligned}
&\rho_{\mathbb{R}}\big(\mu(K_w f - f)\big) \\
&\quad \leqslant \rho_{\mathbb{R}}\big(\mu|K_w f - f|\big) \\
&\quad \leqslant \rho_{\mathbb{R}}\left(\mu\left|K_w f - \sum_{k\in\mathbb{Z}} \chi\left(w\cdot - t_k, \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f\left(u + \cdot - \frac{t_k}{w}\right)\mathrm{d}u\right)\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\in\mathbb{Z}}\chi\Biggl(w\cdot-t_k,\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\,\mathrm{d}u\Biggr)\\
&-\sum_{k\in\mathbb{Z}}\chi(w\cdot-t_k,f)+\sum_{k\in\mathbb{Z}}\chi(w\cdot-t_k,f)-f\Biggr|\Biggr)\\
&\leqslant \rho_{\mathbb{R}}\Biggl(3\mu\Biggl|K_w f(\cdot)-\sum_{k\in\mathbb{Z}}\chi\Biggl(w\cdot-t_k,\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\,\mathrm{d}u\Biggr)\Biggr|\Biggr)\\
&+\rho_{\mathbb{R}}\Biggl(3\mu\Biggl|\sum_{k\in\mathbb{Z}}\chi\Biggl(w\cdot-t_k,\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\,\mathrm{d}u\Biggr)\\
&-\sum_{k\in\mathbb{Z}}\chi(w\cdot-t_k,f)\Biggr|\Biggr)+\rho_{\mathbb{R}}\Biggl(3\mu\Biggl|\sum_{k\in\mathbb{Z}}\chi(w\cdot-t_k,f)-f\Biggr|\Biggr)\\
&=: I_1+I_2+I_3,
\end{aligned}
$$

where in the above computations, $(\cdot)$ denotes the variable of the involved functions.

By the (L,ψ)-Lipschitz condition, noting that $3\mu\leqslant z_\lambda$, using assumption (4), we can estimate I_1 as follows:

$$
\begin{aligned}
I_1&=\rho_{\mathbb{R}}\Biggl(3\mu\Biggl|K_w f-\sum_{k\in\mathbb{Z}}\chi\Biggl(w\cdot-t_k,\ \frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\,\mathrm{d}u\Biggr)\Biggr|\Biggr)\\
&=\rho_{\mathbb{R}}\Biggl(3\mu\Biggl|\sum_{k\in\mathbb{Z}}\chi\Biggl(w\cdot-t_k,\ \frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\Biggr)\\
&\quad-\sum_{k\in\mathbb{Z}}\chi\Biggl(w\cdot-t_k,\ \frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\,\mathrm{d}u\Biggr)\Biggr|\Biggr)\\
&\leqslant\rho_{\mathbb{R}}\Biggl(3\mu\sum_{k\in\mathbb{Z}}\Biggl|\chi\Biggl(w\cdot-t_k,\ \frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\Biggr)\\
&\quad-\chi\Biggl(w\cdot-t_k,\ \frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\,\mathrm{d}u\Biggr)\Biggr|\Biggr)\\
&\leqslant\rho_{\mathbb{R}}\Biggl(3\mu\sum_{k\in\mathbb{Z}}L(w\cdot-t_k)\psi\Biggl(\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w}\Bigl|f(u)-f\Bigl(u+\cdot-\frac{t_k}{w}\Bigr)\Bigr|\,\mathrm{d}u\Biggr)\Biggr)\\
&=\rho_{\mathbb{R}}(3\mu M^{\psi})\leqslant C_1\xi_{\mathbb{R}}\bigl(C_2L_w(\cdot)\eta_{\mathbb{R}}\bigl(C_3\lambda|\tau_{(\cdot)}f|\bigr)\bigr).
\end{aligned}
$$

Taking into account condition (ρ_3) and the parameter α_0 arising from (i*), we can rewrite the above term as follows:

$$\begin{aligned}I_1 &\leqslant C_1\xi_{\mathbb{R}}\biggl(\frac{1}{2}2C_2L_w(\cdot)\eta_{\mathbb{R}}\bigl(C_3\lambda|\tau_{(\cdot)}f|\bigr)\mathbf{1}_{A_{\alpha_0,w}}(\cdot)\\&\quad+\frac{1}{2}2C_2L_w\eta_{\mathbb{R}}(\cdot)\bigl(C_3\lambda|\tau_{(\cdot)}f|\bigr)\mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}(\cdot)\biggr)\\&\leqslant C_1\xi_{\mathbb{R}}\bigl(2C_2L_w(\cdot)\eta_{\mathbb{R}}\bigl(C_3\lambda|\tau_{(\cdot)}f|\bigr)\mathbf{1}_{A_{\alpha_0,w}}(\cdot)\bigr)\\&\quad+C_1\xi_{\mathbb{R}}\bigl(2C_2L_w(\cdot)\eta_{\mathbb{R}}\bigl(C_3\lambda|\tau_{(\cdot)}f|\bigr)\mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}(\cdot)\bigr)\\&=: I_{1,1}+I_{1,2}.\end{aligned}$$

Let us now focus on $I_{1,1}$. By the definition of the modulus of smoothness on $L_{\eta_{\mathbb{R}}}(\mathbb{R})$, by the convexity of the modular $\xi_{\mathbb{R}}$, and by condition (3) we have

$$I_{1,1}\leqslant C_1\xi_{\mathbb{R}}\bigl(2C_2L_w\,\omega_{\eta_{\mathbb{R}}}\bigl(C_3\lambda f,w^{-\alpha_0}\bigr)\mathbf{1}_{A_{\alpha_0,w}}\bigr).$$

Since $C_3\lambda\leqslant\lambda_4$, $\omega_{\eta_{\mathbb{R}}}(C_3\lambda f,w^{-\alpha_0})\leqslant\omega_{\eta_{\mathbb{R}}}(\lambda_4 f,w^{-\alpha_0})<1$ for w sufficiently large; consequently, by the convexity of the modular we get

$$\begin{aligned}I_{1,1}&\leqslant C_1\omega_{\eta_{\mathbb{R}}}\bigl(C_3\lambda f,w^{-\alpha_0}\bigr)\xi_{\mathbb{R}}(2C_2L_w\mathbf{1}_{A_{\alpha_0,w}})\\&\leqslant C_1\omega_{\eta_{\mathbb{R}}}\bigl(C_3\lambda f,w^{-\alpha_0}\bigr)\xi_{\mathbb{R}}(2C_2L_w\mathbf{1}_{A_{\alpha_0,1}}).\end{aligned}$$

Taking $c=2C_2$, by (3) there exists $N:=N_{2C_2}$ such that

$$\begin{aligned}I_{1,1}&=C_1\omega_{\eta_{\mathbb{R}}}\bigl(C_3\lambda f,w^{-\alpha_0}\bigr)\xi_{\mathbb{R}}(cL_w\mathbf{1}_{A_{\alpha_0,1}})\\&\leqslant C_1N\,\omega_{\eta_{\mathbb{R}}}\bigl(C_3\lambda f,w^{-\alpha_0}\bigr)\end{aligned}$$

for $w>0$ sufficiently large. On the other hand, taking into account condition (ρ_3) for the modular $\eta_{\mathbb{R}}$, for $I_{1,2}$ we can write

$$\begin{aligned}I_{1,2}&=C_1\xi_{\mathbb{R}}\bigl(2C_2L_w(\cdot)\eta_{\mathbb{R}}\bigl(C_3\lambda|\tau_{(\cdot)}f|\bigr)\mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}(\cdot)\bigr)\\&\leqslant C_1\xi_{\mathbb{R}}\bigl(2C_2L_w(\cdot)\eta_{\mathbb{R}}\bigl(C_3\lambda[|f|+|\widetilde{f}_{(\cdot)}|]\bigr)\mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}(\cdot)\bigr)\\&=C_1\xi_{\mathbb{R}}\biggl(2C_2L_w(\cdot)\eta_{\mathbb{R}}\biggl(\frac{1}{2}2C_3\lambda|f|+\frac{1}{2}2C_3\lambda|\widetilde{f}_{(\cdot)}|\biggr)\mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}(\cdot)\biggr)\\&\leqslant C_1\xi_{\mathbb{R}}\bigl(2C_2L_w(\cdot)\bigl[\eta_{\mathbb{R}}\bigl(2C_3\lambda|f|\bigr)+\eta_{\mathbb{R}}\bigl(2C_3\lambda|\widetilde{f}_{(\cdot)}|\bigr)\bigr]\mathbf{1}_{\mathbb{R}\setminus A_{\alpha_0,w}}(\cdot)\bigr),\end{aligned}$$

where we recall that $\widetilde{f}(\cdot)$ is defined in (5). Recalling that both $f\in L_{\eta_{\mathbb{R}}}(\mathbb{R})$ and $\widetilde{f}_x\in L_{\eta_{\mathbb{R}}}(\mathbb{R})$ for every $x\in\mathbb{R}$ (see Remark 1), by (5) we assume that, for every $x\in\mathbb{R}$,

$$\begin{aligned}&\eta_{\mathbb{R}}\bigl(2C_3\lambda|f|\bigr)+\eta_{\mathbb{R}}\bigl(2C_3\lambda|\widetilde{f}_x|\bigr)\\&\quad\leqslant\eta_{\mathbb{R}}\bigl(\lambda_2|f|\bigr)+\eta_{\mathbb{R}}\bigl(\lambda_3|\widetilde{f}_x|\bigr)\leqslant\eta_{\mathbb{R}}\bigl(\lambda_2|f|\bigr)+\eta_{\mathbb{R}}\bigl(\lambda_2|f|\bigr)\\&\quad\leqslant\frac{\alpha}{4C_2}+\frac{\alpha}{4C_2}=\frac{\alpha}{2C_2},\end{aligned}$$

where α is the parameter of condition (i*). Hence, using (2), we get

$$I_{1,2} \leqslant C_1 \xi_{\mathbb{R}}(L_w \alpha \mathbf{1}_{\mathbb{R} \setminus A_{\alpha_0,w}}) \leqslant C_1 M w^{-\alpha_0}.$$

Now, we can proceed estimating I_2. Using assumption (χ_3) and the change of variable $u - t_k/w = y$, we have

$$\begin{aligned}
I_2 &= \rho_{\mathbb{R}}\left(3\mu\left|\sum_{k\in\mathbb{Z}} \chi\left(w\cdot - t_k,\ \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f\left(u + \cdot - \frac{t_k}{w}\right)\mathrm{d}u\right) - \sum_{k\in\mathbb{Z}} \chi\big(w\cdot - t_k,\ f(\cdot)\big)\right|\right) \\
&\leqslant \rho_{\mathbb{R}}\left(3\mu\sum_{k\in\mathbb{Z}}\left|\chi\left(w\cdot - t_k,\ \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f\left(u + \cdot - \frac{t_k}{w}\right)\mathrm{d}u\right) - \chi\big(w\cdot - t_k,\ f(\cdot)\big)\right|\right) \\
&\leqslant \rho_{\mathbb{R}}\left(3\mu\sum_{k\in\mathbb{Z}} L(w\cdot - t_k)\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w}\left|f\left(u + \cdot - \frac{t_k}{w}\right) - f(\cdot)\right|\mathrm{d}u\right)\right) \\
&= \rho_{\mathbb{R}}\left(3\mu\sum_{k\in\mathbb{Z}} L(w\cdot - t_k)\psi\left(\frac{w}{\Delta_k}\int_0^{\Delta_k/w}\big|f(y+\cdot) - f(\cdot)\big|\,\mathrm{d}y\right)\right) \\
&\leqslant \rho_{\mathbb{R}}\left(3\mu\left[\sum_{k\in\mathbb{Z}} L(w\cdot - t_k)\right]\psi\left(\frac{w}{\delta}\int_0^{\Delta/w}\big|f(y+\cdot) - f(\cdot)\big|\,\mathrm{d}y\right)\right) \\
&\leqslant \rho_{\mathbb{R}}\left(3\mu m_{0,\Pi}(L)\psi\left(\frac{w}{\delta}\int_0^{\Delta/w}\big|f(y+\cdot) - f(\cdot)\big|\,\mathrm{d}y\right)\right).
\end{aligned}$$

Then, applying condition (H) and Jensen inequality, since $\eta_{\mathbb{R}}$ is convex, we get

$$\begin{aligned}
I_2 &\leqslant \eta_{\mathbb{R}}\left(\lambda\frac{w}{\delta}\frac{\Delta}{\Delta}\int_0^{\Delta/w}\big|f(y+\cdot) - f(\cdot)\big|\mathrm{d}y\right) \leqslant \frac{w}{\Delta}\int_0^{\Delta/w}\eta_{\mathbb{R}}\left(\lambda\frac{\Delta}{\delta}\big|f(y+\cdot) - f(\cdot)\big|\right)\mathrm{d}y \\
&\leqslant \frac{w}{\Delta}\omega_{\eta_{\mathbb{R}}}\left(\lambda\frac{\Delta}{\delta}f, \frac{\Delta}{w}\right)\int_0^{\Delta/w}\mathrm{d}y = \omega_{\eta_{\mathbb{R}}}\left(\lambda\frac{\Delta}{\delta}f, \frac{\Delta}{w}\right).
\end{aligned}$$

Finally, denoting by $B_0 \subset \mathbb{R}$ the set of all points of $\mathbb{R}$ for which $f \neq 0$ a.e., by using (ρ_1) and condition (χ_4) we can rewrite I_3 as follows:

$$\begin{aligned}
I_3 &= \rho_{\mathbb{R}}\left(3\mu\left|\sum_{k\in\mathbb{Z}}\chi\big(w\cdot - t_k,\ f(\cdot)\mathbf{1}_{B_0}(\cdot)\big) - f(\cdot)\mathbf{1}_{B_0}(\cdot)\right|\right) \\
&= \rho_{\mathbb{R}}\left(3\mu\big|f(\cdot)\big|\left|\frac{1}{|f(\cdot)|}\sum_{k\in\mathbb{Z}}\chi\big(w\cdot - t_k,\ f(\cdot)\big) - 1\right|\mathbf{1}_{B_0}(\cdot)\right)
\end{aligned}$$

$$\leqslant \rho_{\mathbb{R}}\bigl(3\mu\bigl|f(\cdot)\bigr|\mathcal{T}_w(\cdot)\mathbf{1}_{B_0}(\cdot)\bigr) \leqslant \rho_{\mathbb{R}}\bigl(3\mu M_1 f w^{-\theta_0}\bigr)$$
$$\leqslant w^{-\theta_0}\rho_{\mathbb{R}}(3\mu M_1 f) \leqslant w^{-\theta_0}\rho_{\mathbb{R}}(\lambda_1 f)$$

for positive constants M_1 and θ_0 of (6). This completes the proof. □

4 Some applications

In the present section, we study some particular cases of modular spaces to which our approach can be applied.

4.1 Quantitative estimates in Musielak–Orlicz spaces

As a first example, we can consider the well-known Musielak–Orlicz spaces, which have been introduced by Nakano in the 50's and deeply studied by Musielak and Orlicz; see, e.g., [2,12,15,16]. In order to define them, we need to recall the expression of the modular functionals characterizing these spaces.

Let $\varphi : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function satisfying the following conditions:

(ρ_1) $\varphi(\cdot, u)$ is measurable and locally integrable on $\mathbb{R}$ for every $u \in \mathbb{R}_0^+$;

(ρ_2) for every $t \in \mathbb{R}$, $\varphi(t, \cdot)$ is convex on $\mathbb{R}_0^+$ with $\varphi(t, 0) = 0$ and $\varphi(t, u) > 0$ for $u > 0$;

(ρ_3) φ is τ-bounded, i.e., there are a constant $C \geqslant 1$ and a measurable function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ such that for every $t, s \in \mathbb{R}$ and $u \geqslant 0$,

$$\varphi(t - s, u) \leqslant \varphi(t, Cu) + F(t, s).$$

A function φ, as above, is said to be a τ-bounded φ-function; for the sake of brevity, we will simply call it a φ-function. For a sake of simplicity, from now on we only consider φ-functions, which satisfy condition (ρ_3) with $F \equiv 0$.

Then, let φ and ν be two fixed φ-functions, it can be easily shown that

$$\rho_{\mathbb{R}}(f) := I^{\varphi}(f) = \int_{\mathbb{R}} \varphi\bigl(t, \bigl|f(t)\bigr|\bigr)\,\mathrm{d}t, \qquad \eta_{\mathbb{R}}(f) := I^{\nu}(f) = \int_{\mathbb{R}} \nu\bigl(t, \bigl|f(t)\bigr|\bigr)\,\mathrm{d}t$$

are modulars on the space $X(\mathbb{R})$, which satisfy properties (a)–(f) given in Section 2; see, e.g., Examples 2.1(b), 2.2(a), and 2.4 in [2]. Furthermore, $\eta_{\mathbb{R}}$ also satisfies property (g), i.e., it is strongly bounded; in fact, by the change of variable $x + t = y$ and by using (ρ_3) we have

$$\eta_{\mathbb{R}}(f(t + \cdot)) = \int_{\mathbb{R}} \nu\bigl(x, \bigl|f(t + x)\bigr|\bigr)\,\mathrm{d}x = \int_{\mathbb{R}} \nu\bigl(y - t, \bigl|f(y)\bigr|\bigr)\,\mathrm{d}y$$
$$\leqslant \int_{\mathbb{R}} \nu\bigl(y, C\bigl|f(y)\bigr|\bigr)\,\mathrm{d}y = \eta_{\mathbb{R}}(Cf)$$

for every $f \in X(\mathbb{R})$ and $t \in \mathbb{R}$. The modular spaces generated by $\rho_{\mathbb{R}}$ and $\eta_{\mathbb{R}}$ are called *Musielak–Orlicz spaces*, and they are briefly denoted by $\widetilde{L}^{\varphi}(\mathbb{R})$ and $\widetilde{L}^{\nu}(\mathbb{R})$, respectively. In these instances, the *φ-modulus of smoothness* is defined by

$$\omega_{\varphi}(f,\delta) := \sup_{|t|\leqslant\delta} \int_{\mathbb{R}} \varphi\bigl(t, \bigl|f(t+s)-f(t)\bigr|\bigr)\,\mathrm{d}t$$

for functions in $\widetilde{L}^{\varphi}(\mathbb{R})$ (and with the same definition with ν in place of φ, in case of functions $f \in \widetilde{L}^{\nu}(\mathbb{R})$) with $\delta > 0$. By Theorem 1 it is clear that for every $f \in \widetilde{L}^{\varphi}(\mathbb{R})$ (or $\widetilde{L}^{\nu}(\mathbb{R})$), there exists $\lambda > 0$ such that $\omega_{\varphi}(\lambda f,\delta) \to 0$ (or $\omega_{\nu}(\lambda f,\delta)$) as $\delta \to 0^+$.

In the present setting, the growth condition (H) can be rewritten requiring the following inequality involving the φ-functions φ and ν: for every $\lambda \in (0,1)$, there exists $C_\lambda \in (0,1)$ satisfying

$$\varphi\bigl(t, C_\lambda \psi(u)\bigr) \leqslant \nu(t,\lambda u) \tag{H_φ}$$

for every $t \in \mathbb{R}$, $u \in \mathbb{R}_0^+$. For useful examples of φ-functions satisfying condition (H_φ), see, for instance, [2].

Furthermore, we define the nonnegative integral functional

$$\xi_{\mathbb{R}}(f) := \int_{\mathbb{R}} \bigl|f(t)\bigr|\,\mathrm{d}t, \tag{7}$$

where $f \in X(\mathbb{R})$, that is trivially convex, monotone, strongly finite, and absolutely continuous and that generates the L^1-space.

Remark 5. Note that, if $L \in L^1(\mathbb{R})$ and it is bounded in a neighborhood of the origin $0 \in \mathbb{R}$, then assumption (L_1) obviously holds with any $c > 0$ and $N_c = c\|L\|_1$. Moreover, condition (i*) is true when L has compact support, e.g., $\operatorname{supp} L \subset [-R,R]$, $R > 0$ and for $0 < \alpha_0 < 1$. Whereas, if L has not compact support, it may be possible to require the following sufficient condition, which involves its continuous absolute moment:

$$M^{\nu}(L) := \int_{\mathbb{R}} L(u)|u|^{\nu}\,\mathrm{d}u < +\infty$$

for $\nu > 0$; see, e.g., [4]. In this case, (i*) is satisfied with $(1-\alpha_0)\nu$ in place of α_0, and $M = \alpha M^{\nu}(L)$.

Therefore, denoting by γ the characteristic function of the set $[0,1]$ (i.e., $\gamma(u) = 1$ if $u \in [0,1]$, and $\gamma(0) = 0$ otherwise), condition (4) is verified in the present setting. Indeed, in correspondence to a fixed $\lambda > 0$, there exists $C_\lambda \in (0,1)$ by condition (H_φ), and $0 < z_\lambda < \min\{1, C_\lambda/m_{0,\Pi}(L)\}$. Hence, by using Jensen inequality twice, condition (H_φ), the change of variable $wt - t_k = y$, and the τ-boundedness of ν

we have

$$
\begin{aligned}
&\rho_{\mathbb{R}}\bigl(z_\lambda M^\psi\bigr)\\
&\quad= \int_{\mathbb{R}} \varphi\left(t, \left| z_\lambda \sum_{k\in\mathbb{Z}} L(wt-t_k)\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w}\left[f(u)-f\left(u+t-\frac{t_k}{w}\right)\right]\mathrm{d}u\right)\right|\right)\mathrm{d}t\\
&\quad\leqslant \frac{1}{m_{0,\Pi}(L)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}} L(wt-t_k)\\
&\qquad\qquad\times\varphi\left(t,\ z_\lambda m_{0,\Pi}(L)\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w}\left|f(u)-f\left(u+t-\frac{t_k}{w}\right)\right|\mathrm{d}u\right)\right)\mathrm{d}t\\
&\quad\leqslant \frac{1}{m_{0,\Pi}(L)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}} L(wt-t_k)\,\nu\left(t,\ \lambda\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w}\left|f(u)-f\left(u+t-\frac{t_k}{w}\right)\right|\mathrm{d}u\right)\mathrm{d}t\\
&\quad\leqslant \frac{1}{m_{0,\Pi}(L)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}} L(wt-t_k)\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w}\nu\left(t,\ \lambda\left|f(u)-f\left(u+t-\frac{t_k}{w}\right)\right|\right)\mathrm{d}u\,\mathrm{d}t\\
&\quad= \frac{1}{m_{0,\Pi}(L)}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}} L(wy)\frac{w}{\Delta_k}\int_{\mathbb{R}}\nu\left(y+\frac{t_k}{w},\ \lambda\bigl|f(u)-f(u+y)\bigr|\right)\gamma(wu-t_k)\,\mathrm{d}u\,\mathrm{d}y\\
&\quad\leqslant \frac{1}{\delta m_{0,\Pi}(L)}\int_{\mathbb{R}} wL(wy)\int_{\mathbb{R}}\nu\bigl(y,\ C\lambda\bigl|f(u)-f(u+y)\bigr|\bigr)\sum_{k\in\mathbb{Z}}\gamma(wu-t_k)\,\mathrm{d}u\,\mathrm{d}y\\
&\quad\leqslant \frac{m_{0,\Pi}(\gamma)}{\delta m_{0,\Pi}(L)}\int_{\mathbb{R}} wL(wy)\int_{\mathbb{R}}\nu\bigl(y,\ C\lambda\bigl|f(u)-f(u+y)\bigr|\bigr)\,\mathrm{d}u\,\mathrm{d}y\\
&\quad= \frac{m_{0,\Pi}(\gamma)}{\delta m_{0,\Pi}(L)}\xi_{\mathbb{R}}\bigl(L_w(\cdot)\eta_{\mathbb{R}}\bigl(C\lambda|\tau_{(\cdot)}f|\bigr)\bigr)
\end{aligned}
$$

from which (4) follows with $C_1 = m_{0,\Pi}(\gamma)/\delta m_{0,\Pi}(L)$, $C_2 = 1$, and $C_3 = C$.

As a byproduct of Theorem 2, we can deduce the following estimate.

Theorem 3. *Suppose that φ, ν are convex φ-functions satisfying condition H_φ and $f \in \widetilde{L}^{\varphi+\nu}(\mathbb{R})$. Then there exist $\mu > 0$, $\lambda > 0$, and further parameter $\lambda_1 > 0$ such that*

$$
\begin{aligned}
I^\varphi\bigl(\mu(K_w f - f)\bigr) \leqslant{}& \frac{2\|L\|_1 m_{0,\Pi}(\gamma)}{\delta m_{0,\Pi}(L)}\,\omega_\nu\left(C\lambda f, \frac{1}{w^{\alpha_0}}\right) + \frac{m_{0,\Pi}(\gamma)}{\delta m_{0,\Pi}(L)} M w^{-\alpha_0}\\
&+ \omega_\nu\left(\frac{\lambda\Delta f}{\delta}, \frac{\Delta}{w}\right) + I^\varphi[\lambda_1 f] w^{-\theta_0}
\end{aligned}
$$

for every sufficiently large $w > 0$, where $m_{0,\Pi}(L) < +\infty$, $m_{0,\Pi}(\gamma) < +\infty$ since γ is bounded and with compact support, and $\theta_0 > 0$ is the constant of condition (χ_4).

As the weighted Orlicz spaces and the Orlicz spaces are particular instances of the Musielak–Orlicz spaces, with a φ-function of product type, Theorem 3 also offers quantitative estimates in these contexts. Consequently, quantitative order of approximation in L^p-spaces, Zygmund (or interpolation) spaces, and exponential-type spaces can be deduced. For more details concerning these spaces, see e.g., [2, 17].

It is worth noting that quantitative estimates for the case of functions belonging to Orlicz spaces have also been provided in [4] in the multivariate setting.

4.2 Quantitative estimates in modular spaces equipped by modulars that lack integral representation

In the preceding section, we discussed instances of modular spaces characterized by their corresponding modular functionals having integral form. Now, we want to study examples of modulars defined by the supremum operator, i.e., that cannot be expressed in terms of integrals.

Let m be a measure on an interval $[a, b[\subset \mathbb{R}$, where b may be equal to $+\infty$, defined on the σ-algebra of all Lebesgue measurable subsets of $[a, b[$. Let W be a nonempty set of indices, and let $(a_\ell(\cdot))_{\ell\in W}$ be a family of Lebesgue measurable positive real-valued functions on $[a, b[$. Moreover, let $\Phi : [a, b[\times\mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function satisfying the following conditions:

- (Φ_1) $\Phi(x, u)$ is a convex function of $u \geqslant 0$ for every $x \in [a, b[$;
- (Φ_2) $\Phi(x, 0) = 0$, $\Phi(x, u) > 0$ for $u > 0$, and $\Phi(x, u) \to +\infty$ as $u \to +\infty$ for every $x \in [a, b[$;
- (Φ_3) there exists $\lim_{x\to b^-} \Phi(x, u) = \widetilde{\Phi}(u) < +\infty$ for every $u > 0$;
- (Φ_4) $\Phi(x, u)$ is a Lebesgue measurable function on x in $[a, b[$ for every $u \geqslant 0$.

Let Φ and Ψ be two functions as above, we define the functionals $\mathcal{A}^\Phi$ and $\mathcal{A}^\Psi$ by means of the formulas

$$\rho_{\mathbb{R}}(f) := \mathcal{A}^\Phi(f) = \sup_{\ell\in W} \int_a^b a_\ell(x)\biggl[\int_{\mathbb{R}} \Phi\bigl(x, \bigl|f(t)\bigr|\bigr)\,\mathrm{d}t\biggr]\,\mathrm{d}m(x),$$

$$\eta_{\mathbb{R}}(f) := \mathcal{A}^\Psi(f) = \sup_{\ell\in W} \int_a^b a_\ell(x)\biggl[\int_{\mathbb{R}} \Psi\bigl(x, \bigl|f(t)\bigr|\bigr)\,\mathrm{d}t\biggr]\,\mathrm{d}m(x)$$

with $f \in X(\mathbb{R})$. The functionals $\mathcal{A}^\Phi$, $\mathcal{A}^\Psi$ are convex modulars, and under other suitable assumptions (see [2, p. 19, (b) and p. 23, (b)]), they are monotone, strongly finite, absolutely finite, and absolutely continuous. In addition, $\mathcal{A}^\Psi$ is trivially strongly bounded with $C = 1$ since $\mathcal{A}^\Psi(f(t + \cdot)) = \mathcal{A}^\Psi(f)$ for every $t \in \mathbb{R}$.

Herein, condition (H) can be deduced as follows: for every $\lambda \in (0, 1)$, there exists a constant $C_\lambda \in (0, 1)$ such that

$$\Phi\bigl(x, C_\lambda\psi(u)\bigr) \leqslant \Psi(x, \lambda u) \tag{H_Φ}$$

for every $x \in \mathbb{R}$, $u \in \mathbb{R}_0^+$.

Considering the modular $\xi_{\mathbb{R}}$ defined as in (7) and proceeding similarly to the previous section, we have that condition (4) holds with $C_1 = m_{0,\Pi}(\gamma)/\delta m_{0,\Pi}(L)$, $C_2 = 1$, and $C_3 = 1$. Therefore, giving an analogous suitable definition of the modulus of smoothness in this setting, all the assumptions of Theorem 2 are satisfied.

4.3 Quantitative estimates in $C(\mathbb{R})$

Finally, if we consider $C(\mathbb{R})$ the space of all uniformly continuous and bounded functions, then the usual sup-norm $\rho(f) = \|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ is a convex modular in $C(\mathbb{R})$. In such case, we provide a direct quantitative estimate for the rate of convergence of the nonlinear sampling Kantorovich operators for $f \in C(\mathbb{R})$, in terms of the modulus of continuity, defined by

$$\omega(f, \delta) := \sup_{|t|\leqslant\delta} \bigl|f(t+\cdot) - f(\cdot)\bigr|$$

with $\delta > 0$. Furthermore, it is interesting to point out that the following well-known inequality

$$\omega(f, \lambda\delta) \leqslant (1+\lambda)\,\omega(f, \delta) \tag{8}$$

holds with $\delta, \lambda > 0$ (see [3]), which is not satisfied in general for the ρ-modulus of smoothness instead. This property will be of fundamental importance to prove the following theorems.

In addition, in such setting, we need to require a stronger assumption on the function L of (L,ψ)-Lipschitz condition, that is,

(L_2^*) there exists a number $\beta_0 > 0$ such that

$$m_{\beta_0,\Pi}(L) := \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} L(x-t_k)|x-t_k|^{\beta_0} < +\infty,$$

i.e., the absolute moment of order β_0 is finite.

In fact, it is easy to prove that, if $L \in L^1(\mathbb{R})$ is bounded in a neighborhood of the origin $0 \in \mathbb{R}$ as in Remark 5 and satisfies assumption (L_2^*), then $m_{0,\Pi}(L) < +\infty$.

Moreover, from now on, we only consider functions ψ that are concave.

Now, we can prove the following.

Theorem 4. *Let $f \in C(\mathbb{R})$, and let L be a function satisfying condition* (L_2^*) *with $\beta_0 \geqslant 1$. Then we have*

$$\|K_w f - f\|_\infty \leqslant M_2\psi\biggl(\omega\biggl(f, \frac{1}{w}\biggr)\biggr) + M_1\|f\|_\infty w^{-\theta_0}$$

for sufficiently large $w > 0$, where $M_2 := m_{0,\Pi}(L) + \Delta\, m_{0,\Pi}(L) + m_{1,\Pi}(L)$, and $M_1, \theta_0 > 0$ are the constants of condition (χ_4).

Proof. Let $x \in \mathbb{R}$ be fixed. We have

$$\begin{aligned}
&\bigl|(K_w f)(x) - f(x)\bigr| \\
&\quad = \left|\sum_{k\in\mathbb{Z}} \chi\left(wx - t_k,\ \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\right) - f(x)\right| \\
&\quad \leqslant \left|\sum_{k\in\mathbb{Z}} \chi\left(wx - t_k,\ \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,\mathrm{d}u\right) - \sum_{k\in\mathbb{Z}} \chi\bigl(wx - t_k,\ f(x)\bigr)\right| \\
&\qquad + \left|\sum_{k\in\mathbb{Z}} \chi\bigl(wx - t_k,\ f(x)\bigr) - f(x)\right| \\
&\quad = I_1 + I_2. \qquad (9)
\end{aligned}$$

We estimate I_1. Applying condition (χ_3) and taking into account that ψ is nondecreasing, we get

$$\begin{aligned}
I_1 &\leqslant \sum_{k\in\mathbb{Z}} L(wx - t_k)\,\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} \bigl|f(u) - f(x)\bigr|\,\mathrm{d}u\right) \\
&\leqslant \sum_{k\in\mathbb{Z}} L(wx - t_k)\,\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} \omega\bigl(f, |u - x|\bigr)\,\mathrm{d}u\right) \\
&\leqslant \sum_{k\in\mathbb{Z}} L(wx - t_k)\,\psi\left(\frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} \omega\left(f, \frac{1}{w}\right)\bigl[1 + w|u - x|\bigr]\,\mathrm{d}u\right) \\
&= \sum_{k\in\mathbb{Z}} L(wx - t_k)\,\psi\left(\omega\left(f, \frac{1}{w}\right)\left[1 + \frac{w}{\Delta_k}\int_{t_k/w}^{t_{k+1}/w} w|u - x|\,\mathrm{d}u\right]\right)
\end{aligned}$$

for every $w > 0$, where the previous estimate is a consequence of (8) with $\lambda = w|u - x|$ and $\delta = 1/w$. Now, for every $x, u \in \mathbb{R}$, we may write

$$|u - x| \leqslant \left|u - \frac{t_k}{w}\right| + \left|\frac{t_k}{w} - x\right| \leqslant \frac{\Delta}{w} + \frac{|wx - t_k|}{w} \qquad (10)$$

for every $w > 0$; therefore,

$$I_1 \leqslant \sum_{k\in\mathbb{Z}} L(wx - t_k)\,\psi\left(\omega\left(f, \frac{1}{w}\right)\bigl[1 + \Delta + |wx - t_k|\bigr]\right).$$

Since ψ is concave, we have for $u \geqslant 1$,

$$u\psi(v) = u\psi\left(\frac{1}{u} \cdot vu\right) \geqslant u\frac{1}{u}\psi(vu) = \psi(vu) \tag{11}$$

for every $v \geqslant 0$; consequently, we finally get

$$\begin{aligned} I_1 &\leqslant \sum_{k\in\mathbb{Z}} L(wx - t_k)\left[1 + \Delta + |wx - t_k|\right]\psi\left(\omega\left(f, \frac{1}{w}\right)\right) \\ &\leqslant m_{0,\Pi}(L)\,(1 + \Delta)\,\psi\left(\omega\left(f, \frac{1}{w}\right)\right) + m_{1,\Pi}(L)\,\psi\left(\omega\left(f, \frac{1}{w}\right)\right). \end{aligned}$$

Now, setting again $B_0 := \{x \in \mathbb{R}\colon\ f \neq 0\,\text{a.e.}\}$, we can rewrite I_2 as follows:

$$I_2 = \left|\sum_{k\in\mathbb{Z}} \chi\bigl(wx - t_k, f(x)\bigr) - f(x)\right| = \left|\sum_{k\in\mathbb{Z}} \chi\bigl(wx - t_k, f(x)\mathbf{1}_{B_0}(x)\bigr) - f(x)\mathbf{1}_{B_0}(x)\right|.$$

Therefore, by condition (χ_4) there exist constants $M_1, \theta_0 > 0$ such that for sufficiently large $w > 0$,

$$\begin{aligned} I_2 &= \bigl|f(x)\bigr|\left|\frac{1}{|f(x)|}\sum_{k\in\mathbb{Z}} \chi\bigl(wx - t_k, f(x)\mathbf{1}_{B_0}(x)\bigr) - \mathbf{1}_{B_0}(x)\right| \\ &\leqslant \bigl|f(x)\bigr|\mathcal{T}_w(x)\mathbf{1}_{B_0}(x) \leqslant M_1\bigl|f(x)\bigr|w^{-\theta_0} \leqslant M_1\|f\|_\infty w^{-\theta_0} \end{aligned}$$

uniformly with respect to $x \in \mathbb{R}$. This completes the proof. □

Note that if ψ is continuous in 0, from Theorem 4 we can deduce the convergence of the nonlinear sampling Kantorovich operators. Moreover, it is important to underline that the estimate presented in Theorem 4 is valid only when condition (L_2^*) holds with β_0 being greater than or equal to one. However, there exists kernels for which the discrete absolute moments of order $\beta_0 \geqslant 1$ are not finite, but at the same time, condition (L_2^*) is satisfied for some values $0 < \beta_0 < 1$. In such case, Theorem 4 cannot be applied. For this reason, we prove the following.

Theorem 5. *Let $f \in C(\mathbb{R})$, and let L be a function satisfying condition* (L_2^*) *with $0 < \beta_0 < 1$. Then we have*

$$\begin{aligned} \|K_w f - f\|_\infty &\leqslant M_3\,\psi\bigl(\omega\bigl(f, w^{-\beta_0}\bigr)\bigr) + 2^{\beta_0+1}\psi\bigl(\|f\|_\infty\bigr)w^{-\beta_0}m_{\beta_0,\Pi}(L) \\ &\quad + M_1\|f\|_\infty w^{-\theta_0} \end{aligned}$$

for sufficiently large $w > 0$, where $M_3 := m_{0,\Pi}(L) + m_{\beta_0,\Pi}(L) + \Delta^{\beta_0}m_{0,\Pi}(L)$, and $M_1, \theta_0 > 0$ are the constants of condition (χ_4)*.*

Proof. Let $x \in \mathbb{R}$ be fixed. The approximation error $|(K_w f)(x) - f(x)|$ can be decomposed by $I_1 + I_2$ as illustrated in the preliminary steps of the proof of Theorem 4; see (9).

By the same computations it is clear that $I_2 \leqslant M_1 \|f\|_\infty w^{-\theta_0}$, where $M_1, \theta_0 > 0$ are the constants of condition (χ_4). On the other hand, we split the series in I_1 as follows:

$$
\begin{aligned}
I_1 &\leqslant \sum_{k\in\mathbb{Z}} L(wx-t_k)\,\psi\Biggl(\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} \bigl|f(u)-f(x)\bigr|\,\mathrm{d}u\Biggr) \\
&\leqslant \sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\,\psi\Biggl(\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} \bigl|f(u)-f(x)\bigr|\,\mathrm{d}u\Biggr) \\
&\quad + \sum_{|wx-t_k|> w/2} L(wx-t_k)\,\psi\Biggl(\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} \bigl|f(u)-f(x)\bigr|\,\mathrm{d}u\Biggr) \\
&=: I_{1,1} + I_{1,2}.
\end{aligned}
$$

Before estimating $I_{1,1}$, we observe that, for every $u \in [t_k/w, t_{k+1}/w]$ and if $|wx-t_k| \leqslant w/2$, we have

$$
|u-x| \leqslant \left|u-\frac{t_k}{w}\right| + \left|\frac{t_k}{w}-x\right| \leqslant \frac{\Delta}{w} + \frac{1}{2} \leqslant 1
$$

for $w > 0$ sufficiently large, and moreover, since $0 < \beta_0 < 1$, it is also easy to see that

$$
\omega\bigl(f, |u-x|\bigr) \leqslant \omega\bigl(f, |u-x|^{\beta_0}\bigr).
$$

Hence, by using the property for which $\omega(f, \lambda\delta) \leqslant (1+\lambda)\omega(f,\delta)$ with $\lambda = (w|u-x|)^{\beta_0}$ and $\delta = w^{-\beta_0}$ we get

$$
\begin{aligned}
I_{1,1} &\leqslant \sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\,\psi\Biggl(\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} \omega\bigl(f, |u-x|^{\beta_0}\bigr)\,\mathrm{d}u\Biggr) \\
&\leqslant \sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\,\psi\Biggl(\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} \bigl[w^{\beta_0}|u-x|^{\beta_0}+1\bigr]\,\omega\bigl(f, w^{-\beta_0}\bigr)\,\mathrm{d}u\Biggr) \\
&= \sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\,\psi\Biggl(\Biggl[1+\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} w^{\beta_0}|u-x|^{\beta_0}\,\mathrm{d}u\Biggr]\,\omega\bigl(f, w^{-\beta_0}\bigr)\Biggr).
\end{aligned}
$$

Since ψ is concave and then subadditive, by (11) we have

$$
I_{1,1} \leqslant \sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\Biggl[1+\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} w^{\beta_0}|u-x|^{\beta_0}\,\mathrm{d}u\Biggr]\,\psi\bigl(\omega\bigl(f, w^{-\beta_0}\bigr)\bigr)
$$

$$\leqslant m_{0,\Pi}(L)\,\psi\bigl(\omega\bigl(f,w^{-\beta_0}\bigr)\bigr)$$
$$+\,\psi\bigl(\omega\bigl(f,w^{-\beta_0}\bigr)\bigr)\sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\frac{w}{\Delta_k}\int\limits_{t_k/w}^{t_{k+1}/w} w^{\beta_0}|u-x|^{\beta_0}\,\mathrm{d}u$$

for $w>0$ sufficiently large. By using again (10) and by exploiting the subadditivity of the function $|\cdot|^{\beta_0}$ with $0<\beta_0<1$ we can write

$$I_{1,1}\leqslant \psi\bigl(\omega\bigl(f,w^{-\beta_0}\bigr)\bigr)\Biggl[m_{0,\Pi}(L)+\sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\biggl(w^{\beta_0}\biggl|\frac{t_k}{w}-x\biggr|^{\beta_0}+\Delta^{\beta_0}\biggr)\Biggr]$$
$$\leqslant \psi\bigl(\omega\bigl(f,w^{-\beta_0}\bigr)\bigr)\Biggl[m_{0,\Pi}(L)+\sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)|wx-t_k|^{\beta_0}$$
$$+\,\Delta^{\beta_0}\sum_{|wx-t_k|\leqslant w/2} L(wx-t_k)\Biggr]$$
$$\leqslant \psi\bigl(\omega\bigl(f,w^{-\beta_0}\bigr)\bigr)\bigl[m_{0,\Pi}(L)+m_{\beta_0,\Pi}(L)+\,\Delta^{\beta_0}m_{0,\Pi}(L)\bigr].$$

Finally, for what concerns $I_{1,2}$, we have

$$I_{1,2}\leqslant \psi\bigl(2\|f\|_\infty\bigr)\sum_{|wx-t_k|>w/2} L(wx-t_k)$$
$$\leqslant \psi\bigl(2\|f\|_\infty\bigr)\sum_{|wx-t_k|>w/2} \frac{|wx-t_k|^{\beta_0}}{|wx-t_k|^{\beta_0}}L(wx-t_k)$$
$$\leqslant \biggl(\frac{2}{w}\biggr)^{\beta_0}\psi\bigl(2\|f\|_\infty\bigr)\sum_{|wx-t_k|>w/2} |wx-t_k|^{\beta_0}L(wx-t_k)$$
$$\leqslant 2^{\beta_0+1}\psi\bigl(\|f\|_\infty\bigr)w^{-\beta_0}\;m_{\beta_0,\Pi}(L).$$

Thus, the theorem is proved. □

If we consider the Fejér kernel (see, e.g., [6, 10]), condition (L_2^*) is satisfied only for every $\beta_0<1$ (then $m_{\beta_0,\Pi}(L)=+\infty$ for $\beta_0\geqslant 1$). Therefore, Theorem 4 cannot be applied, while Theorem 5 holds.

Remark 6. In general, it is possible to give a condition on the kernels, which ensures that (L_2^*) holds for $0\leqslant\beta_0<\beta$, for some $\beta<1$, and $m_{\beta_0,\Pi}(L)=+\infty$ for $\beta<\beta_0\leqslant 1$. In this regard, we refer the readers to [6].

Remark 7. In general, in order to construct suitable examples for the nonlinear sampling Kantorovich operators, we consider kernel functions of the following form:

$$\chi(w\underline{x}-t_{\underline{k}},\,u)=L(w\underline{x}-t_{\underline{k}})g_w(u),$$

where $(g_w)_{w>0}$, $g_w:\mathbb{R}\to\mathbb{R}$ is a family of functions satisfying $g_w(u)\to u$ uniformly as $w\to+\infty$ and such that there exists a function ψ with $|g_w(u)-g_w(v)|\leqslant\psi(|u-v|)$ for every $u, v \in \mathbb{R}$ and $w > 0$. For concrete examples of function sequences $(g_w)_{w>0}$ and specific functions L, we refer the readers to [2, 4, 5, 11, 18].

5 Conclusions

In conclusion, we have established a quantitative estimate for nonlinear sampling Kantorovich operators in the general framework of modular spaces. This represents a natural advancement in the field and enables us to give a unifying approach that includes also the linear case and applies to several settings of approximation problems, thanks to the generality of the employed modular spaces. The theory of modular spaces contains, in fact, the Musielak–Orlicz and the Orlicz spaces, which are, for instance, generalizations of the weighted L^p-spaces and the classical L^p-spaces, respectively. Also, the case of variable exponent L^p-spaces can be here included as they are specific instances of Musielak–Orlicz spaces.

Finally, we stress again that the nonlinear sampling Kantorovich operators in the very general setting of modular spaces have been considered for the first time from the *quantitative* point of view in this paper. In the wide already existing literature concerning the *nonlinear* sampling-type operators (also produced by the authors of the present paper), only the modular convergence of such operators have been considered in [7], using a density-based approach that is completely different with respect to the constructive-direct one here proposed. For the sake of completeness, we also highlight the papers [8] and [4, 9] in which the convergence and the degree of approximation have been respectively considered for the multivariate version of such operators in the space of continuous functions and in Orlicz spaces. Again, as observed above, Orlicz spaces are only one of the possible cases of modular spaces, which, in general, represent a more general framework. Also, several approximation results in both the univariate and multivariate cases can be found in the literature for the special case of the *linear* sampling Kantorovich operators (see, e.g., [6]) again arising as special cases of the (more general) nonlinear ones here considered.

Author contributions. All authors (D.C., M.N., and G.V.) have equally contributed to this work for writing, reviewing, and editing. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors thank the anonymous reviewers for their thoughtful comments, which have contributed to improving the quality and clarity of the paper.

References

1. C. Bardaro, I. Mantellini, Modular approximation by sequences of nonlinear integral operators in Musielak-Orlicz spaces, *Atti Semin. Mat. Fis. Univ. Modena*, **46**(Suppl.):403–425, 1998, `http://hdl.handle.net/11391/156015`.

2. C. Bardaro, J. Musielak, G. Vinti, *Nonlinear Integral Operators and Applications*, De Gruyter Ser. Nonlinear Anal. Appl., Vol. 9, De Gruyter, Berlin, 2003, https://doi.org/10.1515/9783110199277.
3. P.L. Butzer, R.J. Nessel, *Fourier Analysis and Approximation, Vol. I*, Pure Appl. Math., Vol. 40, Academic Press, New York, 1971, https://doi.org/10.1007/978-3-0348-7448-9.
4. N. Çetin, D. Costarelli, M. Natale, G. Vinti, Nonlinear multivariate sampling Kantorovich operators: Quantitative estimates in functional spaces, *Dolomites Res. Notes Approx.*, **15**(3): 2035–6803, 2022, https://doi.org/10.14658/pupj-drna-2022-3-3.
5. L. Coroianu, S.G. Gal, L^p-approximation by truncated max-product sampling operators of Kantorovich-type based on Fejér kernel, *J. Integral Equations Appl.*, **29**(2):349–364, 2017, https://doi.org/10.1216/jie-2017-29-2-349.
6. D. Costarelli, A.M. Minotti, G. Vinti, Approximation of discontinuous signals by sampling Kantorovich series, *J. Math. Anal. Appl.*, **450**(2):1083–1103, 2017, https://doi.org/10.1016/j.jmaa.2017.01.066.
7. D. Costarelli, M. Natale, G. Vinti, Convergence results for nonlinear sampling Kantorovich operators in modular spaces, *Numer. Funct. Anal. Optim.*, **44**(12):1276–1299, 2023, https://doi.org/10.1080/01630563.2023.2241143.
8. D. Costarelli, G. Vinti, Approximation by nonlinear multivariate sampling Kantorovich type operators and applications to image processing, *Numer. Funct. Anal. Optim.*, **34**(8):819–844, 2013, https://doi.org/10.1080/01630563.2013.767833.
9. D. Costarelli, G. Vinti, Degree of approximation for nonlinear multivariate sampling Kantorovich operators on some functions spaces, *Numer. Funct. Anal. Optim.*, **36**(8):964–990, 2015, https://doi.org/10.1080/01630563.2015.1040888.
10. D. Cruz-Uribe, P. Hästö, Extrapolation and interpolation in generalized Orlicz spaces, *Trans. Am. Math. Soc.*, **370**(6):4323–4349, 2018, https://doi.org/10.1090/tran/7155.
11. K. Demirci, S. Orhan, Statistical relative approximation on modular spaces, *Results Math.*, **71**:1167–1184, 2017, https://doi.org/10.1007/s00025-016-0548-5.
12. L. Diening, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, *Bull. Sci. Math.*, **129**(8):657–700, 2005, https://doi.org/10.1016/j.bulsci.2003.10.003.
13. H. Karsli, Some convergence results for nonlinear singular integral operators, *Demonstr. Math.*, **46**(4):729–740, 2013, https://doi.org/10.1515/dema-2013-0487.
14. W.M. Kozlowski, *Modular Function Spaces*, Marcel Dekker, New York, 1988.
15. J. Musielak, Nonlinear approximation in some modular function spaces, *Math. Japon.*, **38**(1): 83–90, 1993, https://doi.org/10.11501/10996897.
16. J. Musielak, W. Orlicz, On modular spaces, *Studia Math.*, **18**(1):49–65, 1959, https://doi.org/10.4064/sm-18-1-49-65.
17. M.M. Rao, Z.D. Ren, *Applications of Orlicz Spaces*, CRC Press, Boca Raton, FL, 2002, https://doi.org/10.1201/9780203910863.
18. G. Vinti, L. Zampogni, Approximation by means of nonlinear Kantorovich sampling type operators in Orlicz spaces, *J. Approx. Theory*, **161**(2):511–528, 2009, https://doi.org/10.1016/j.jat.2008.11.011.