Nonlinear Analysis: Modelling and Control, Vol. 30, No. 2, 333–345
https://doi.org/10.15388/namc.2025.30.38969

Vilnius University Press

Existence of a positive solution with concave and convex components for a system of boundary value problems

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Received: September 30, 2024 / **Revised:** January 29, 2025 / **Published online:** February 26, 2025

Abstract. We prove the existence of at least one positive solution for a system of two nonlinear second-order differential equations with nonlocal boundary conditions. One component of the solution is a concave function, and the other one is a convex function. A recent hybrid Krasnosel'skiĭ–Schauder fixed point theorem is used to prove the existence of a positive solution. To illustrate the applicability of the obtained result, an example is considered.

Keywords: boundary value problem, system of second-order ODEs, positive solutions, hybrid Krasnosel'skiĭ–Schauder fixed point theorem.

1 Introduction

Recently, Infante et al. [10] established a fixed point theorem for operator system that combines well-known Krasnosel'skiĭ–Benjamin fixed point theorem [1] and Schauder's fixed point theorem [11]: consider the operator system

$$x = T_1(x, y), \qquad y = T_2(x, y). \tag{1}$$

If, for fixed y, the operator $T_1(\cdot, y)$ satisfies the conditions of the Krasnosel'skiĭ–Benjamin fixed point theorem and, for fixed x, the operator $T_2(x, \cdot)$ satisfies the conditions of the Schauder's fixed point theorem, then operator system (1) has a fixed point (x, y) such that x is localized in a conical shell, and y is localized in a closed and convex set. The authors in [10] showed the applicability of this result on a system of Hammerstein-type integral operators and, as an example, proved existence of a positive solution for the system of boundary value problems

$$\begin{aligned} &x''(t) + f\bigl(t, x(t), y(t)\bigr) = 0, \quad t \in [0,1],\\ &y''(t) + g\bigl(t, x(t), y(t)\bigr) = 0, \quad t \in [0,1],\\ &x(0) = x(1) = 0 = y'(0) = y(1) + y'(1). \end{aligned}$$

Inspired by the article [10], in this paper, we apply the hybrid Krasnosel'skiĭ–Schauder fixed point theorem [10, Thm. 2.4] to prove the existence of at least one positive solution for the system of differential equations

$$\begin{aligned} x''(t) + f\big(t, x(t), y(t)\big) &= 0, \quad t \in (0,1), \\ y''(t) - g\big(t, x(t), y(t)\big) &= 0, \quad t \in (0,1), \end{aligned} \tag{2}$$

coupled with nonlocal boundary conditions

$$\begin{aligned} \widetilde{a}_0 x(0) - \widetilde{b}_0 x'(0) &= \widetilde{\varphi}_0[x], \qquad a_0 y(0) - b_0 y'(0) = \varphi_0[y] + c_0, \\ \widetilde{a}_1 x(1) + \widetilde{b}_1 x'(1) &= \widetilde{\varphi}_1[x], \qquad a_1 y(1) + b_1 y'(1) = \varphi_1[y] + c_1. \end{aligned} \tag{3}$$

Here a_i, b_i, c_i, $\widetilde{a}_i$, $\widetilde{b}_i$ are nonnegative real constants, $f, g : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$ are continuous functions, $\varphi_i[y] = \int_0^1 y(s)\,\mathrm{d}\Phi_i(s)$ and $\widetilde{\varphi}_i[x] = \int_0^1 x(s)\,\mathrm{d}\widetilde{\Phi}_i(s)$ are linear functionals defined via Riemann–Stieltjes integrals with sign-changing measures, i.e., $\Phi_i, \widetilde{\Phi}_i : [0,1] \to \mathbb{R}$ are functions of bounded variation. So it is possible to consider coefficients of both signs in multipoint conditions (if Φ_i is scale function) and consider functions that may change the sign in integral conditions (if Φ_i' is Riemann-integrable).

We call a pair $(x,y) \in C^2[0,1] \times C^2[0,1]$ a (strictly) positive solution of problem (2), (3) if (x,y) satisfies the differential equations (2), the boundary conditions (3) and $x(t) > 0$, $y(t) > 0$ for all $t \in (0,1)$.

Note that $x'' \leqslant 0$ and $y'' \geqslant 0$ in (2), hence x is concave function, and y is convex function. Since we investigate the existence of positive solutions, it is natural to assume that

$$0 < d = a_0 a_1 + a_0 b_1 + a_1 b_0 \quad \text{and} \quad 0 < \widetilde{d} = \widetilde{a}_0 \widetilde{a}_1 + \widetilde{a}_0 \widetilde{b}_1 + \widetilde{a}_1 \widetilde{b}_0.$$

Systems of boundary value problems were studied by many authors; see, for instance, [4–10,19]. Such systems and their positive solutions are important in modeling biological and medical phenomena and studying chemistry, physics, population, and fluid dynamics; see [4,7,10,19] and references therein. In case of one boundary value problem the standard method is to apply Krasnosel'skiĭ–Benjamin [1] or Guo–Krasnosel'skiĭ [3] fixed point theorem to prove existence of a positive solution. It is possible to apply these theorems for systems of boundary value problems, but these theorems cannot guarantee that each component of the solution is nontrivial.

For systems of boundary value problems, one could apply Krasnosel'skiĭ's fixed point theorem on each component of the solution in a pointwise manner. This result, due to Precup [12], is called the vector version of Krasnosel'skiĭ's fixed point theorem or the Krasnosel'skiĭ–Precup fixed point theorem. For details and applications of this result, we refer the reader to [2, 12–15].

In this article, to prove our main result, we apply the hybrid Krasnosel'skiĭ–Schauder fixed point theorem [10, Thm. 2.4]. Let us recall it here. A nonempty closed convex subset $K \subset X$ of normed space $(X, \|\cdot\|)$ is called a cone if $\lambda x \in K$ for every $x \in K$ and for all $\lambda \geqslant 0$, and $K \cap (-K) = 0$.

Theorem 1 [Hybrid Krasnosel'skiĭ–Schauder]. (*See* [10].) *Let U and V be open and bounded subsets of a cone K of the Banach space X such that $0 \in V \subset \overline{V} \subset U$, and let W be a closed convex subset of the Banach space Y.*

Assume that $T = (T_1, T_2) : (\overline{U} \setminus V) \times W \to K \times W$ is a completely continuous map and there exists $\mathrm{h} \in K \setminus \{0\}$ such that either of the following conditions holds in $(\overline{U} \setminus V) \times W$:

(i) $T_1(x, y) + \mu \mathrm{h} \neq x$ *if* $x \in \partial V$ *and* $\mu > 0$, *and* $T_1(x, y) \neq \lambda x$ *if* $x \in \partial U$ *and* $\lambda > 1$; *or*

(ii) $T_1(x, y) \neq \lambda x$ *if* $x \in \partial V$ *and* $\lambda > 1$, *and* $T_1(x, y) + \mu \mathrm{h} \neq x$ *if* $x \in \partial U$ *and* $\mu > 0$.

Then T has at least one fixed point $(x, y) \in (\overline{U} \setminus V) \times W$.

One of the applications of Theorem 1 in [10, Thm. 3.1] requires the existence of four positive numbers ρ_1, ρ_2 (to define $\overline{U} \setminus V$) and α, β (to define W). In this paper, we do not require the existence of α, β and localize the convex component of the solution in $W = \{y\colon\ y(t) \in [0, Q] \text{ for } t \in [0, 1]\}$, where Q could be calculated. Our result complements the previous results in [10].

The outline of the rest of the paper is as follows. In Section 2, we rewrite problem (2), (3) as a system of equivalent integral equations and show estimates of the corresponding Green's functions. In Section 3, we present sufficient conditions for a convex component of the solution to be positive on $(0, 1)$. Section 4 is devoted to the proof of our main result Theorem 2. In Section 5, we illustrate the obtained result with an example.

2 Preliminaries

The standard approach of obtaining solutions of problem (2), (3) is to construct the Green's functions corresponding to the above mentioned problem and rewrite the problem as a system of equivalent integral equations.

It is known (see, for instance, [8, 17, 20]) that the Green's function G_0 corresponding to problem

$$u''(t) + \hat{f}(t) = 0, \quad a_0 u(0) - b_0 u'(0) = 0 = a_1 u(1) + b_1 u'(1)$$

is given by

$$G_0(t, s) = d \begin{cases} u_0(t) u_1(s), & 0 \leqslant s \leqslant t \leqslant 1, \\ u_0(s) u_1(t), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{4}$$

where

$$u_0(t) = \frac{b_1 + a_1(1 - t)}{d}, \qquad u_1(t) = \frac{b_0 + a_0 t}{d}. \tag{5}$$

Recall that $d = a_0 a_1 + a_0 b_1 + a_1 b_0 > 0$. Let $\widetilde{u}_i$ and $\widetilde{G}_0$ be defined similar to (5) and (4), respectively, in terms of $\widetilde{a}_i$ and $\widetilde{b}_i$. Observe that u_0 is a decreasing function and u_1 is an increasing function.

Throughout the paper, we assume that

(A1) $0 \leqslant \varphi_0[u_0] \leqslant 1, \quad 0 \leqslant \varphi_0[u_1], \quad 0 \leqslant \varphi_1[u_0], \quad 0 \leqslant \varphi_1[u_1] \leqslant 1,$
$0 \leqslant \widetilde{\varphi}_0[\widetilde{u}_0] \leqslant 1, \quad 0 \leqslant \widetilde{\varphi}_0[\widetilde{u}_1], \quad 0 \leqslant \widetilde{\varphi}_1[\widetilde{u}_0], \quad 0 \leqslant \widetilde{\varphi}_1[\widetilde{u}_1] \leqslant 1,$

(A2) $0 < D = \begin{vmatrix} 1-\varphi_0[u_0] & -\varphi_1[u_0] \\ -\varphi_0[u_1] & 1-\varphi_1[u_1] \end{vmatrix}$ and $0 < \widetilde{D} = \begin{vmatrix} 1-\widetilde{\varphi}_0[\widetilde{u}_0] & -\widetilde{\varphi}_1[\widetilde{u}_0] \\ -\widetilde{\varphi}_0[\widetilde{u}_1] & 1-\widetilde{\varphi}_1[\widetilde{u}_1] \end{vmatrix}$,

(A3) $0 \leqslant \mathcal{G}_i(s) = \int_0^1 G_0(t,s)\,\mathrm{d}\Phi_i(t), \quad 0 \leqslant \widetilde{\mathcal{G}}_i(s) = \int_0^1 \widetilde{G}_0(t,s)\,\mathrm{d}\widetilde{\Phi}_i(t), \quad i = 0, 1.$

Here the notation $|A|$ denotes the determinant of a square matrix A. The proof of a following Lemma 1 is standard and omitted. The technique of the proof could be found, for instance, in [2, 5, 16, 18, 21].

Lemma 1. *A function u is a solution of problem*

$$\begin{aligned} &u''(t) + \hat{f}(t) = 0, \\ &a_0 u(0) - b_0 u'(0) = \varphi_0[u] + c_0, \\ &a_1 u(1) + b_1 u'(1) = \varphi_1[u] + c_1 \end{aligned}$$

if and only if u is a solution of the perturbed integral equation

$$u(t) = \int_0^1 G(t,s)\hat{f}(s)\,\mathrm{d}s + \Gamma(t),$$

where

$$G(t,s) = \frac{1}{D}\begin{vmatrix} u_0(t) & 1-\varphi_0[u_0] & -\varphi_1[u_0] \\ u_1(t) & -\varphi_0[u_1] & 1-\varphi_1[u_1] \\ G_0(t,s) & -\mathcal{G}_0(s) & -\mathcal{G}_1(s) \end{vmatrix} \tag{6}$$

and

$$\Gamma(t) = \frac{1}{D}\begin{vmatrix} u_0(t) & 1-\varphi_0[u_0] & -\varphi_1[u_0] \\ u_1(t) & -\varphi_0[u_1] & 1-\varphi_1[u_1] \\ 0 & -c_0 & -c_1 \end{vmatrix}. \tag{7}$$

Note that every element in (6) depends on a_i, b_i, or φ_i. Let $\widetilde{G}$ be defined similar to (6) in terms of $\widetilde{a}_i$, $\widetilde{b}_i$, $\widetilde{\varphi}_i$.

Proposition 1. *A pair (x, y) is a solution of problem* (2), (3) *if and only if (x, y) is a solution of the system of integral equations*

$$\begin{aligned} x(t) &= \int_0^1 \widetilde{G}(t,s) f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \\ y(t) &= -\int_0^1 G(t,s) g\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s + \Gamma(t), \quad t \in [0,1]. \end{aligned}$$

Proof. The proof of this proposition directly follows from Lemma 1. □

Let $\mu(t) = \min\{t, 1-t\}$. It is known that the following is valid (see [5, 10, 21]):

(B1) $G(t,s) \geqslant 0$, $\Gamma(t) \geqslant 0$ and $\widetilde{G}(t,s) \geqslant 0$ for all $(t,s) \in [0,1] \times [0,1]$,
(B2) $\mu(t)\widetilde{G}_0(s,s) \leqslant \widetilde{G}_0(t,s) \leqslant \widetilde{G}_0(s,s)$ for all $(t,s) \in [0,1] \times [0,1]$,
(B3) $\widetilde{u}_i(t) \geqslant \mu(t)\widetilde{u}_i(i)$ for all $t \in [0,1]$ and $i = 0,1$.

Note that $\mu(t) \geqslant \min\{l_1, 1-l_2\}$ for $t \in [l_1, l_2] \subset [0,1]$.

Proposition 2. *The Green's function* $\widetilde{G}$ *satisfies*

$$\mu(t)\widetilde{H}(s) \leqslant \widetilde{G}(t,s) \leqslant \widetilde{H}(s), \quad (t,s) \in [0,1] \times [0,1],$$

where

$$\widetilde{H}(s) = \frac{1}{\widetilde{D}} \begin{vmatrix} \widetilde{u}_0(0) & 1 - \widetilde{\varphi}_0[\widetilde{u}_0] & -\widetilde{\varphi}_1[u_0] \\ \widetilde{u}_1(1) & -\widetilde{\varphi}_0[u_1] & 1 - \widetilde{\varphi}_1[\widetilde{u}_1] \\ \widetilde{G}_0(s,s) & -\widetilde{\mathcal{G}}_0(s) & -\widetilde{\mathcal{G}}_1(s) \end{vmatrix}.$$

Proof. First, we expand $\widetilde{H}(s)$ along the first column and see that $\widetilde{H}(s) \geqslant 0$ for $s \in [0,1]$. Next, by (B2) and (B3), we conclude that $\mu(t)\widetilde{H}(s) \leqslant \widetilde{G}(t,s) \leqslant \widetilde{H}(s)$. □

3 Convex component of the solution

In this section, we give sufficient conditions for a convex component of the solution to be nonnegative on $[0,1]$ and positive on $(0,1)$. Let us define the function $h : [0,1] \to \mathbb{R}$ by

$$h(t) = -\frac{t^2}{2} + \frac{a_0}{2d}(2b_1 + a_1)t + \frac{b_0}{2d}(2b_1 + a_1).$$

Lemma 2. *The Green's function G, given by* (6)*, satisfies*

$$\int_0^1 G(t,s)\,\mathrm{d}s = -\frac{t^2}{2} + P_1 t + P_0,$$

where

$$P_1 = \frac{1}{2dD} \begin{vmatrix} -2a_1 & 1 - \varphi_0[u_0] & -\varphi_1[u_0] \\ 2a_0 & -\varphi_0[u_1] & 1 - \varphi_1[u_1] \\ a_0(2b_1 + a_1) & -\varphi_0[h] & -\varphi_1[h] \end{vmatrix} \tag{8}$$

and

$$P_0 = \frac{1}{2dD} \begin{vmatrix} 2b_1 + 2a_1 & 1 - \varphi_0[u_0] & -\varphi_1[u_0] \\ 2b_0 & -\varphi_0[u_1] & 1 - \varphi_1[u_1] \\ b_0(2b_1 + a_1) & -\varphi_0[h] & -\varphi_1[h] \end{vmatrix}. \tag{9}$$

Proof. Observe that

$$\begin{aligned}\int_0^1 G_0(t,s)\,\mathrm{d}s &= d\int_0^t u_0(t)u_1(s)\,\mathrm{d}s + d\int_t^1 u_0(s)u_1(t)\,\mathrm{d}s\\ &= \frac{t(2b_0+a_0t)(b_1+a_1-a_1t)}{2d} + \frac{(1-t)(b_0+a_0t)(2b_1+a_1-a_1t)}{2d}\\ &= -\frac{t^2}{2} + \frac{a_0}{2d}(2b_1+a_1)t + \frac{b_0}{2d}(2b_1+a_1) = h(t)\end{aligned}$$

and

$$\int_0^1 \mathcal{G}_i(s)\,\mathrm{d}s = \int_0^1\int_0^1 G_0(t,s)\,\mathrm{d}\Phi_i(t)\,\mathrm{d}s = \int_0^1\int_0^1 G_0(t,s)\,\mathrm{d}s\ \mathrm{d}\Phi_i(t) = \varphi_i[h], \quad i=0,1.$$

Therefore, by expanding $G(t,s)$ along the first column, we have

$$\begin{aligned}\int_0^1 G(t,s)\,\mathrm{d}s &= \frac{b_1+a_1(1-t)}{d\,D}\begin{vmatrix}-\varphi_0[u_1] & 1-\varphi_1[u_1]\\ -\varphi_0[h] & -\varphi_1[h]\end{vmatrix}\\ &\quad - \frac{b_0+a_0\,t}{d\,D}\begin{vmatrix}1-\varphi_0[u_0] & -\varphi_1[u_0]\\ -\varphi_0[h] & -\varphi_1[h]\end{vmatrix} + h(t) = -\frac{t^2}{2} + P_1t + P_0,\end{aligned}$$

where P_1 and P_0 are given by (8) and (9), respectively. □

Let $Y : C[0,1]\times C[0,1] \to C[0,1]$ be an operator defined via

$$Y(x,y)(t) = -\int_0^1 G(t,s)g\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s + \Gamma(t),$$

where $G(t,s)$ is given by (6), and $\Gamma(t)$ is given by (7). Note that

$$\begin{aligned}\Gamma(t) &= \frac{a_1(1-t)}{d\,D}\begin{vmatrix}-\varphi_0[u_1] & 1-\varphi_1[u_1]\\ -c_0 & -c_1\end{vmatrix} - \frac{a_0\,t}{d\,D}\begin{vmatrix}1-\varphi_0[u_0] & -\varphi_1[u_0]\\ -c_0 & -c_1\end{vmatrix}\\ &\quad + \frac{1}{d\,D}\begin{vmatrix}b_1 & 1-\varphi_0[u_0] & -\varphi_1[u_0]\\ b_0 & -\varphi_0[u_1] & 1-\varphi_1[u_1]\\ 0 & -c_0 & -c_1\end{vmatrix}.\end{aligned}$$

We denote

$$Q_2 = \frac{a_1}{d\,D}\begin{vmatrix}-\varphi_0[u_1] & 1-\varphi_1[u_1]\\ -c_0 & -c_1\end{vmatrix}, \qquad Q_1 = \frac{a_0}{d\,D}\begin{vmatrix}1-\varphi_0[u_0] & -\varphi_1[u_0]\\ -c_0 & -c_1\end{vmatrix}, \tag{10}$$

$$Q_0 = \frac{1}{d\,D}\begin{vmatrix}b_1 & 1-\varphi_0[u_0] & -\varphi_1[u_0]\\ b_0 & -\varphi_0[u_1] & 1-\varphi_1[u_1]\\ 0 & -c_0 & -c_1\end{vmatrix}. \tag{11}$$

So that $Y(x,y)(t) = -\int_0^1 G(t,s)g(s,x(s),y(s))\,\mathrm{d}s + Q_2(1-t) + Q_1 t + Q_0$. We set

$$Q_M = \max\{Q_2, Q_1\} + Q_0. \tag{12}$$

Proposition 3. *Let r be a positive real constant, P_1, P_0, Q_2, Q_1, Q_0, Q_M be given by* (8)–(12)*, respectively, and*

$$M = \max\bigl\{g(t,x,y),\ (t,x,y) \in [0,1]\times\Omega\bigr\}, \quad \Omega = [0,r]\times[0,Q_M].$$

Suppose there exists $s \in [0,1]$ such that one of the following is satisfied:

(i) $M(s^2/2 + P_0) + Q_0 \leqslant Q_2 < Q_1 - M(P_1 - 2s)$; *or*
(ii) $M(s^2/2 + P_0) + Q_0 < Q_2 \leqslant Q_1 - M(P_1 - 2s)$.

Then $0 \leqslant Y(x,y)(t) \leqslant Q_M$ for all $(t,x,y) \in [0,1]\times\Omega$. Moreover, $0 < Y(x,y)(t)$ for all $(t,x,y) \in (0,1]\times\Omega$.

Proof. Observe that

$$Y(x,y)(t) \leqslant \Gamma(t) = Q_2(1-t) + Q_1 t + Q_0 \leqslant Q_M, \quad t \in [0,1].$$

To show $0 \leqslant Y(x,y)$, we need to verify that

$$\begin{aligned}\int_0^1 G(t,s)g\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s &\leqslant Q_2(1-t) + Q_1 t + Q_0 \\ &= (Q_1 - Q_2)t + (Q_2 + Q_0), \quad t \in [0,1].\end{aligned}$$

By Lemma 2, we have

$$\int_0^1 G(t,s)g\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s \leqslant M\int_0^1 G(t,s)\,\mathrm{d}s = M\biggl(-\frac{t^2}{2} + P_1 t + P_0\biggr)$$

for all $(t,x,y) \in [0,1]\times\Omega$. Let us denote $\psi(t) = M(-t^2/2 + P_1 t + P_0)$. Now we show $\psi(t) < (Q_1 - Q_0)t + (Q_2 + Q_0)$ for all $t \in (0,1)$, and not strict inequality sign is valid for $t \in [0,1]$. Observe that tangent to the ψ at a point $z \in [0,1]$ is given by

$$M(P_1 - 2z)t + M\biggl(\frac{z^2}{2} + P_0\biggr).$$

Since ψ is concave, we have

$$\psi(t) \leqslant M(P_1 - 2z)t + M\biggl(\frac{z^2}{2} + P_0\biggr), \quad (z,t) \in [0,1]\times[0,1].$$

By assumption, there exists $s \in [0,1]$ such that

$$\psi(t) \leqslant M(P_1 - 2s)t + M\biggl(\frac{s^2}{2} + P_0\biggr) < (Q_1 - Q_2)t + (Q_2 + Q_0), \quad t \in (0,1],$$

$$\psi(0) = M\,P_0 \leqslant M\biggl(P_0 + \frac{s^2}{2}\biggr) \leqslant Q_2 + Q_0,$$

which completes the proof. □

4 Existence of a positive solution

Consider Banach space $C[0,1]$ endowed with the supremum norm

$$\|u\| = \max_{t\in[0,1]} \bigl|u(t)\bigr|.$$

To prove our main result Theorem 2, we use well-known techniques; see, for instance, [10, 21].

Theorem 2. *Let* Q_M *be given by* (12) *and suppose there exist interval* $[l_1,l_2] \subset [0,1]$ $(m = \min\{l_1, 1-l_2\})$ *and constants* $r_1, r_2 > 0$, $r_1 < r_2$ (*resp.* $r_2/m < r_1$) *such that*

(i) *there exist two continuous functions* $\overline{f}$ *and* $\underline{f}$ *such that*

$$f(t,x,y) \leqslant \overline{f}(t), \quad (t,x,y) \in [0,1]\times[0,r_1]\times[0,Q_M],$$
$$\underline{f}(t) \leqslant f(t,x,y), \quad (t,x,y) \in [l_1,l_2]\times\Bigl[r_2,\frac{r_2}{m}\Bigr]\times[0,Q_M]$$

and

$$r_2 \leqslant \min_{t\in[l_1,l_2]} \int_{l_1}^{l_2} \widetilde{G}(t,s)\underline{f}(s)\,\mathrm{d}s, \qquad \max_{t\in[0,1]} \int_0^1 \widetilde{G}(t,s)\overline{f}(s)\,\mathrm{d}s \leqslant r_1;$$

(ii) *one of the following is valid*:

(a) $M(s^2/2 + P_0) + Q_0 \leqslant Q_2 < Q_1 - M(P_1 - 2s)$; *or*
(b) $M(s^2/2 + P_0) + Q_0 < Q_2 \leqslant Q_1 - M(P_1 - 2s)$,

where P_1, P_0, Q_2, Q_1, Q_0 *are given by* (8)–(11), *respectively,*

$$M = \max\bigl\{g(t,x,y)\colon (t,x,y) \in [0,1]\times\Omega\bigr\},$$
$$\Omega = [0, r_2/m]\times[0,Q_M] \quad \bigl(\textit{resp. } \Omega = [0,r_1]\times[0,Q_M]\bigr).$$

Then boundary value problem (2), (3) *has at least one positive solution* (x,y) *such that*

$$r_1 \leqslant \|x\|, \qquad \min_{t\in[l_1,l_2]} x(t) \leqslant r_2, \qquad \|y\| \leqslant Q_M$$
$$\Bigl(\textit{resp. } r_2 \leqslant \min_{t\in[l_1,l_2]} x(t), \quad \|x\| \leqslant r_1\Bigr).$$

Proof. Let us define cone

$$K = \Bigl\{x \in C[0,1]\colon x(t) \geqslant 0 \text{ for } t\in[0,1],\ \min_{t\in[l_1,l_2]} x(t) \geqslant m\|x\|,\ \widetilde{\varphi}_i[x] \geqslant 0,\ i = 0,1\Bigr\},$$

open sets U_ρ and V_ρ by

$$U_\rho = \bigl\{x\in K\colon \|x\| < \rho\bigr\}, \qquad V_\rho = \Bigl\{x \in K\colon \min_{t\in[l_1,l_2]} x(t) < \rho\Bigr\},$$

closed and convex set W by

$$W = \{y \in C[0,1]\colon\ 0 \leqslant y(t) \leqslant Q_M \text{ for } t \in [0,1]\},$$

and operator $T = (T_1, T_2) : K \times W \to C[0,1] \times C[0,1]$ by

$$T_1(x,y)(t) = \int_0^1 \widetilde{G}(t,s) f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s,$$

$$T_2(x,y)(t) = -\int_0^1 G(t,s) g\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s + \Gamma(t).$$

Boundary value problem (2), (3) has a solution if and only if operator T has a fixed point. By a standard application of Arzelà–Ascoli theorem, it can be proved that T is a completely continuous operator. First, we show that $T_1(x,y)$ satisfies the conditions of the Krasnosel'skiĭ–Benjamin fixed point theorem, and then we show $T : (\overline{V}_{r_2} \setminus U_{r_1}) \times W \to K \times W$.

Let $x \in \partial U_{r_1}$, i.e., $\|x\| = r_1$. We show that $\|T_1(x,y)\| \leqslant \|x\|$. It is known that this implies $T_1(x,y) \neq \lambda x$ for $\lambda > 1$. Let $(x,y) \in \partial U_{r_1} \times W$ and consider

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 \widetilde{G}(t,s) f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s \leqslant \int_0^1 \widetilde{G}(t,s) \overline{f}(s)\,\mathrm{d}s \\ &\leqslant \max_{t \in [0,1]} \int_0^1 \widetilde{G}(t,s) \overline{f}(s)\,\mathrm{d}s \leqslant r_1. \end{aligned}$$

It follows that $\|T_1(x,y)\| \leqslant r_1 = \|x\|$.

Now, on the contrary, suppose that there exists $x \in \partial V_{r_2}$ such that $T_1(x,y) + \mu \hat{1} = x$ for $\mu > 0$ and $\hat{1} : t \mapsto 1$. Note that since $x \in \partial V_{r_2} \subset K$, we have $m\|x\| \leqslant \min\{x(t),\ t \in [l_1, l_2]\} = r_2$. Hence $\|x\| \leqslant r_2/m$ and

$$r_2 \leqslant x(t) \leqslant \frac{r_2}{m}, \quad t \in [l_1, l_2].$$

Let $(x,y) \in \partial V_{r_2} \times W$ and consider

$$\begin{aligned} x(t) &= \int_0^1 \widetilde{G}(t,s) f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s + \mu \geqslant \int_{l_1}^{l_2} \widetilde{G}(t,s) f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s + \mu \\ &\geqslant \int_{l_1}^{l_2} \widetilde{G}(t,s) \underline{f}(s)\,\mathrm{d}s + \mu \geqslant r_2 + \mu. \end{aligned}$$

Taking the minimum for $t \in [l_1, l_2]$, we get $r_2 \geqslant r_2 + \mu$, which is contradiction.

Now we show $T_1((\overline{V}_{r_2} \setminus U_{r_1}) \times W) \subset K$ and $T_2((\overline{V}_{r_2} \setminus U_{r_1}) \times W) \subset W$. First, note that $T_1(x,y) \geqslant 0$. Let $T_1(x,y)$ achieves maximum value at point t_0, i.e., $\|T_1(x,y)\| = T_1(x,y)(t_0)$. By Proposition 2, we have

$$T_1(x,y)(t) = \int_0^1 \widetilde{G}(t,s) f\big(s, x(s), y(s)\big)\,\mathrm{d}s \geqslant \int_0^1 \mu(t)\widetilde{H}(s) f\big(s, x(s), y(s)\big)\,\mathrm{d}s$$
$$\geqslant \int_0^1 \mu(t)\widetilde{G}(t_0,s) f\big(s, x(s), y(s)\big)\,\mathrm{d}s = \mu(t)\big\|T_1(x,y)\big\|.$$

Hence $T_1(x,y)(t) \geqslant m\|T_1(x,y)\|$ for $t \in [l_1, l_2]$. Next, consider

$$\widetilde{\varphi}_i\big[T_1(x,y)\big] = \int_0^1 \left(\int_0^1 \widetilde{G}(t,s)\,\mathrm{d}\widetilde{\Phi}_i(t) \right) f\big(s, x(s), y(s)\big)\,\mathrm{d}s.$$

By (A1)–(A3), we have $\int_0^1 \widetilde{G}(t,s)\,\mathrm{d}\widetilde{\Phi}_i(t) \geqslant 0$. Therefore $\widetilde{\varphi}_i[T_1(x,y)] \geqslant 0$ and $T_1((\overline{V}_{r_2} \setminus U_{r_1}) \times W) \subset K$.

By Proposition 3, we have $0 \leqslant T_2(x,y)(t) \leqslant Q_M$ for all $(t,x,y) \in [0,1] \times (\overline{V}_{r_2} \setminus U_{r_1}) \times W$ and $0 < T_2(x,y)(t)$ for all $(t,x,y) \in (0,1] \times (\overline{V}_{r_2} \setminus U_{r_1}) \times W$. Thus by hybrid Krasnosel'skiĭ–Schauder fixed point theorem, boundary value problem (2), (3) has at least one positive solution $(x,y) \in (\overline{V}_{r_2} \setminus U_{r_1}) \times W$.

Finally, note that if $r_2/m < r_1$, then the proof is analogous, but now considering the operator T defined in the set $(\overline{U}_{r_1} \setminus V_{r_2}) \times W$. □

5 Example

Consider the system of boundary value problems

$$\begin{aligned}
&x''(t) + \frac{1}{2}x^2(t) + (1-t)y(t) = 0, \quad t \in (0,1),\\
&y''(t) - \frac{1}{6}\big(x(t)y(t) + t\big) = 0, \quad t \in (0,1),\\
&x(0) - 0.5x'(0) = 0, \qquad x'(1) = -\int_0^1 x(t)\sin(2\pi t)\,\mathrm{d}t,\\
&2y(0) - y'(0) = 1, \qquad y(1) = 2y(0.3) - y(0.6) + 1.
\end{aligned}$$

Calculations show that

$$\widetilde{G}(t,s) = \frac{10\pi}{3(2\pi-1)} \begin{vmatrix} 1 & 1 & 0 \\ 0.5+t & 0 & 1-(2\pi)^{-1} \\ \widetilde{G}_0(t,s) & 0 & (\sin(2\pi s) - 2\pi s)/(4\pi^2) \end{vmatrix},$$

$$G(t,s)=\frac{3}{2}\begin{vmatrix}(1-t)/3 & 1 & -1/3\\ (1+2t)/3 & 0 & 1-1/3\\ G_0(t,s) & 0 & -2G_0(0.3,s)+G_0(0.6,s)\end{vmatrix},$$

$$P_1=-\frac{131}{900},\quad P_0=\frac{41}{200},\quad Q_2=\frac{1}{3},\quad Q_1=-\frac{4}{3},\quad Q_0=\frac{2}{3},\quad Q_M=1.$$

Note that assumptions (A1)–(A3) are satisfied. Let $r_1=1.5$, $r_2=6$, $l_1=0.25$, and $l_2=0.75$. In this example,

$$f(t,x,y)=\frac{1}{2}x^2+(1-t)y,$$

$$\overline{f}(t)=\frac{1.5^2}{2}+(1-t)Q_M=1.125+(1-t),$$

$$\underline{f}(t)=\frac{6^2}{2}+(1-t)\cdot 0=18,$$

$$\widetilde{G}_0(t,s)=\begin{cases}0.5+s, & 0\leqslant s\leqslant t\leqslant 1,\\ 0.5+t, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

To calculate integrals that involve $\widetilde{G}(t,s)$, we expand $\widetilde{G}(t,s)$ along the second column (the obtained constants have been rounded to 3 decimal places unless exact).

$$\begin{aligned}&\max_{t\in[0,1]}\int_0^1\widetilde{G}(t,s)\overline{f}(s)\,\mathrm{d}s\\ &\quad=\max_{t\in[0,1]}\int_0^1\biggl(-\frac{(0.5+t)(\sin(2\pi s)-2\pi s)}{4\pi^2}+\biggl(1-\frac{1}{2\pi}\biggr)\widetilde{G}_0(t,s)\biggr)(1.125+1-s)\,\mathrm{d}s\\ &\quad=\max_{t\in[0,1]}\bigl(0.739+1.478t-0.893t^2+0.140t^3\bigr)=1.464\leqslant 1.5=r_1,\end{aligned}$$

$$\begin{aligned}&\min_{t\in[0.25,0.75]}\int_{0.25}^{0.75}\widetilde{G}(t,s)\underline{f}(s)\,\mathrm{d}s\\ &\quad=\min_{t\in[0.25,0.75]}\int_{0.25}^{0.75}18\biggl(-\frac{(0.5+t)(\sin(2\pi s)-2\pi s)}{4\pi^2}+\biggl(1-\frac{1}{2\pi}\biggr)\widetilde{G}_0(t,s)\biggr)\,\mathrm{d}s\\ &\quad=\min_{t\in[0.25,0.75]}\bigl(3.669+12.068t-7.568t^2\bigr)=6.213\geqslant 6=r_2.\end{aligned}$$

Recall that $m=\min\{0.25,1-0.75\}=0.25$ and $r_2/m=24$. Observe that

$$g(t,x,y)=\frac{1}{6}(xy+t)\leqslant\frac{25}{6}=M,\quad (t,x,y)\in[0,1]\times[0,24]\times[0,1].$$

Let $s = 1/5$. Then

$$M\left(\frac{s^2}{2} + P_0\right) - Q_0 \leqslant Q_2 < Q_1 - M(P_1 - 2s),$$
$$\frac{25}{6}\left(\frac{1}{50} + \frac{41}{200}\right) - \frac{2}{3} \leqslant \frac{1}{3} < -\frac{4}{3} - \frac{25}{6}\left(-\frac{131}{900} - \frac{2}{5}\right),$$
$$\frac{13}{48} \leqslant \frac{1}{3} < \frac{203}{216}.$$

All assumptions of Theorem 2 are satisfied, and hence the system of boundary value problems (5) has at least one positive solution (x, y) such that

$$1.5 \leqslant \|x\|, \qquad \min_{t\in[0.25,0.75]} x(t) \leqslant 6, \qquad \|y\| \leqslant 1.$$

Acknowledgment. We would like to thank the referees for their comments, which improved the quality of this paper.

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