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On a novel type of generalized simulation functions with fixed point results for wide W_s-contractions*

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Abstract. One of the most significant hypotheses in fixed point theory is the nonexpansivity condition of contractive mappings. This property is crucial as operators that do not satisfy this criterion may lack fixed points. In this paper, we propose a novel condition that, within the appropriate framework, can obviate the necessity of imposing the nonexpansivity requirement in the initial hypotheses. By employing this new condition, we illustrate how innovative results can be derived in this area. Finally, we examine the existence and uniqueness of a solution for an elastic beam equation with nonlinear boundary conditions grounded in the introduced fixed point results.

Keywords: elastic beam equations, simulation functions, wide W_s-contractions.

1 Introduction

The field of fixed point theory is a branch of nonlinear analysis that has experienced significant growth in recent years due to the enormous interest on its results, especially when applied to areas such as the determination of solutions to various types of equations (differential, integral, matricial, etc.) or the development of iterative processes that bring the solution of a problem ever closer. Since the appearance of the first results in this line of research (due to Brouwer and Banach), more and more general versions have appeared in more abstract settings, demonstrating the vigor of this scientific field.

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From an abstract point of view, the results in this line of research mainly face the following challenges: first, to prove that an operator has at least one fixed point, and then to study the possible uniqueness of such fixed point. Although it may seem a simple task, the complexity of the current algebraic tools makes each new result in this field a real success.

The fundamental tool to prove the existence of a fixed point is the so-called *contractivity condition*. In general, this is an inequality that, in one way or another, allows us to establish that the distance between the images of two points by the considered operator cannot excessively grow; otherwise, there could be no fixed points. In the last fifty years, more and more general contractivity conditions have been presented, making use of auxiliary functions applied to the contractive condition's main terms. Along these lines, one of the notions that have achieved tremendous success in the last ten years has been the *simulation functions*, originally introduced by Khojasteh, Shukla, and Radenović in 2015 in [10]. Almost immediately after that, many versions of this brilliant concept emerged and have been improved over the last five years; see [2, 3, 5, 7, 9].

One such extension, very recently introduced by Mongkolkeha and Sintunavarat [11] in the setting of b-metric spaces, is the concept of *large $\mathcal{Z}_s$-contraction mapping* (Definition 6), which reduces to a single condition the assumptions imposed on the auxiliary function called *large s-simulation function* (the original simulation functions consisted of three conditions) but, in return, imposes two conditions on the associated contractions, called *large $\mathcal{Z}_s$-contractions* (Definition 7). The curious key about this class of contractions is that there is no single auxiliary function involved in the contractivity condition, but rather that this condition locally depends on a parameter that makes it impossible to control all points in the underlying space. However, returning to what has already been commented, this class of contractions needs to assume the *nonexpansivity condition*, which is natural in the results of fixed point theory but, in a certain way, it is very restrictive.

This fact has made us reflect on the need to look for more general conditions than the nonexpansivity of the contractivity mapping, which, in turn, are compatible with the considered simulation function. Therefore, in this paper, we present a new condition that, in the appropriate framework, can avoid the need to impose the nonexpansivity condition in the initial hypotheses. Using this new condition, we show how new results can be obtained in the field of fixed point theory. Furthermore, we show the exact point at which the nonexpansivity helps to simplify the proofs, thus laying the groundwork for other researchers to pose even more general conditions. With this, we are confident that many other previous theorems can be revised to study if they are still fulfilled by removing such a hypothesis or replacing it with another one that is easier to verify. At the end of this paper, based on the introduced fixed point result, we investigate the existence and uniqueness of a unique solution for an elastic beam equation with nonlinear boundary conditions.

2 Preliminaries

In this section, we introduce the necessary background to understand the main contents of this paper; see also [1]. To start with, we present the definitions and first properties of some classes of auxiliary functions that will be of great help in what follows. Hereafter, let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integer numbers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathbb{R}$ denote the set of all real numbers.

Let X be a nonempty set, and let $T : X \to X$ be a self-operator. Given $m \in \mathbb{N}_0$, let us denote by $T^m : X \to X$ the self-mapping on X, which is defined by

$$T^m = \begin{cases} \text{identity mapping on } X & \text{if } m = 0, \\ T & \text{if } m = 1, \\ T \circ T^{m-1} & \text{if } m \geqslant 2. \end{cases}$$

We refer to T^m as the *m-iteration of T*. Given $x_0 \in X$, the *Picard sequence of T starting from x_0* is the sequence $\{x_n\}_{n \in \mathbb{N}_0} \subseteq X$ defined by $x_n = T^n x_0$ for each $n \in \mathbb{N}_0$.

Definition 1. Given a real number $s \in [1, \infty)$, a *b-metric space with a coefficient s* is a pair (X, d) such that X is a nonempty set, and $d : X \times X \to [0, \infty)$ is a mapping satisfying the following properties for each $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(y, x) = d(x, y)$;
(iii) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

In such a case, we will also say that (X, d, s) is a *b-metric space*. A sequence $\{x_n\} \subseteq X$ on a b-metric space is *convergent to $z \in X$* if $\lim_{n \to \infty} d(x_n, z) = 0$, and it is *Cauchy* if $\lim_{n,m \to \infty} d(x_n, x_m) = 0$. The b-metric space (X, d, s) is *complete* if every Cauchy sequence in X is convergent to a point of X. Moreover, a sequence $\{x_n\}$ in a b-metric space (X, d, s) is *asymptotically regular* if $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$.

Definition 2. (See [8].) A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, where X is a nonempty set, is:

(i) *infinite* if $x_n \neq x_m$ for each $n, m \in \mathbb{N}$ such that $n \neq m$;
(ii) *almost periodic* if there are $n_0, p_0 \in \mathbb{N}$ such that $x_{n_0+r+kp_0} = x_{n_0+r}$ for each $k \in \mathbb{N}$ and all $r \in \{0, 1, \dots, p_0 - 1\}$ (this means that $\{x_n\}_{n \geqslant n_0}$ is a periodic sequence because the terms $\{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots, x_{n_0+p_0-1}\}$ are infinitely repeated in the same order);
(iii) *almost constant* if there is $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for each $n \geqslant n_0$ (this means that $\{x_n\}_{n \geqslant n_0}$ is a constant sequence).

Proposition 1. (*See* [8].) *Every Picard sequence is either infinite or almost periodic.*

Proposition 2. (*See* [6, 8, 15].) *If $\{x_n\}_{n \in \mathbb{N}}$ is a Picard and asymptotically regular sequence in a b-metric space (X, d, s), then $\{x_n\}$ is either infinite or almost constant.*

2.1 Simulation functions

A first approach to the notion of a simulation function was given by Khojasteh et al. [10] in 2015 as follows.

Definition 3. (See [10].) A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a *simulation function* if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Taking into account that the arguments of the function ζ played a symmetric role in (ζ_3), which is not usual in the setting of fixed point theory, some authors slightly modified the previous notion in the following way.

Definition 4. (See [7].) A *simulation function* is a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ_3') If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The family of all simulation functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ in the Roldán López de Hierro et al.'s sense will be denoted by $\mathcal{Z}$. It can be easily deduced from (ζ_2) that

$$\zeta(t,t) < 0 \quad \text{for all } t > 0$$

whatever the simulation function $\zeta \in \mathcal{Z}$.

In 2017, Yamaod and Sintunavarat [13] introduced the new type of simulation functions depending on a constant $s \geqslant 1$ as follows.

Definition 5. (See [13].) Given $s \in [1,\infty)$, a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called an *s-simulation function* if it satisfies the following two conditions:

(S$_2$) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(S$_4$) If $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant s \limsup_{n\to\infty} \beta_n \leqslant s^2 \liminf_{n\to\infty} \alpha_n \tag{1}$$

and

$$0 < \liminf_{n\to\infty} \beta_n \leqslant s \limsup_{n\to\infty} \alpha_n \leqslant s^2 \liminf_{n\to\infty} \beta_n, \tag{2}$$

then $\limsup_{n\to\infty} \zeta(\alpha_n, \beta_n) < 0$.

Recenty, Mongkolkeha and Sintunavarat [11] attempted to extend the class of s-simulation functions by deleting one condition from the above definition as follows.

Definition 6. (See [11].) Given $s \in [1,\infty)$, a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a *large s-simulation function* if it satisfies condition (S$_4$).

The family of all large s-simulation function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ in the above sense will be denoted by C_s. With the assistance of this class, Mongkolkeha and Sintunavarat [11] introduced a new class of self-mappings on a b-metric space, where each mapping in this class has a unique fixed point, provided that the space is complete. Below are the definitions of mappings in this class and the corresponding fixed point results.

Definition 7. (See [11].) Let (X, d) be a b-metric space with the coefficient $s \geqslant 1$ and $\zeta \in C_s$. A mapping $T : X \to X$ is called a *large $\mathcal{Z}_s$-contraction mapping* if the following conditions hold:

(L_1) $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$;

(L_2) For all $\varepsilon > 0$, there is $\zeta \in C_s$ such that

$$\zeta\bigl(d\bigl(Tx, Ty\bigr), d(x, y)\bigr) \geqslant 0 \tag{3}$$

for each $x, y \in X$ satisfying $d(x, y) \geqslant \varepsilon$.

Theorem 1. (*See* [11].) *Let (X, d) be a complete b-metric space with the coefficient $s \geqslant 1$, and let $T : X \to X$ be an large $\mathcal{Z}_s$-simulation contraction mapping. Then T is a Picard mapping, that is, T has a unique fixed point $x_* \in X$, and the Picard sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$, converges to the fixed point x_*.*

3 Some reflections on the successive extensions of the notion of simulation functions

In the field of fixed point theory, two main aims are recurrently studied. Firstly, to prove that a self-mapping has at least one fixed point. Secondly, to deduce that such a fixed point is unique. To do this, it is usual to assume that the self-mapping is *nonexpansive*, that is, the distance between the images of two points of the underlying space from the self-mapping is less than or equal to the distance between such points. The reason is clear: if the mapping is not nonexpansive, then it can be free of fixed points. Let us explain this fact in an algebraic way.

A self-mapping T on a b-metric space (X, d, s) into itself is nonexpansive if $d(Tx, Ty) < d(x, y)$ for each $x, y \in X$ with $x \neq y$ (recall condition (L_1) in Definition 7). If T is not nonexpansive, then it could be free of fixed points. For instance, if $X = \mathbb{N}$ and $Tn = n + 1$ for each $n \in \mathbb{N}$, then T is fixed points free. As a consequence, the nonexpansivity condition is often implicitly assumed in many results in fixed point theory. Nevertheless, two ways are possible: either it is directly assumed under the main hypotheses (as in Theorem 1) or it can also be indirectly deduced from the main hypotheses. The first option is of great help in the proofs of the results in this line of research as we will show later.

With respect to the notion of simulation function, the first and the most important example of such kind of functions is $\zeta_\lambda : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by

$$\zeta_\lambda(t, s) = \lambda s - t \quad \text{for each } t, s \in [0,\infty),$$

where $\lambda \in [0,1)$ is a constant. This is due to the fact that the notion of simulation function originally always had in mind to generalize the Banach contractivity condition, that is, for all $x, y \in X$,

$$d(Tx, Ty) \leqslant \lambda d(x, y) \iff \zeta_\lambda\big(d(Tx, Ty), d(x, y)\big) \geqslant 0.$$

However, as Roldán López de Hierro et al. pointed out in [7], the original notion of simulation function (see Definition 3) entailed a problem: condition (ζ_3) was symmetric in both arguments of ζ, that is, the sequences $\{t_n\}$ and $\{s_n\}$ could swap their roles. As a consequence, for the kind of sequences in axiom (ζ_3), we could deduce both

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0 \quad \text{and} \quad \limsup_{n\to\infty} \zeta(s_n, t_n) < 0.$$

The first of the previous inequalities greatly helps to prove that a Picard sequence is Cauchy. However, the second one has no significance, but it is covered by axiom (ζ_3) in Definition 3, which is a little bit odd. As a consequence, the above-mentioned authors proposed to include the condition "$t_n < s_n$ for all $n \in \mathbb{N}$" on the statement of such property (see Definition 4). This assumption leads to a generalization in which the roles of the arguments of the simulation function are not symmetric. After that, the field of fixed point theory involving simulation functions has experienced tremendous and full of success growth; for instance, see [2, 3, 5, 9, 12, 14]. However, in many cases, the assumptions on extended notions of simulation functions continued being symmetric as in condition (S_4) of Definitions 5.

Notice that the two requirements on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in property (S_4), that is,

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant s \limsup_{n\to\infty} \beta_n \leqslant s^2 \liminf_{n\to\infty} \alpha_n \tag{4}$$

and

$$0 < \liminf_{n\to\infty} \beta_n \leqslant s \limsup_{n\to\infty} \alpha_n \leqslant s^2 \liminf_{n\to\infty} \beta_n \tag{5}$$

are jointly equivalent to assume, in a simpler way, that

$$0 < \limsup_{n\to\infty} \beta_n \leqslant s \liminf_{n\to\infty} \alpha_n \quad \text{and} \quad 0 < \limsup_{n\to\infty} \alpha_n \leqslant s \liminf_{n\to\infty} \beta_n. \tag{6}$$

In the proof of Theorem 1, they were decisive two facts in order to control the key sequences $\{\alpha_n\}$ and $\{\beta_n\}$: on the one hand, the assumption that T is nonexpansive, and, on the other hand, the joint conditions (4) and (5).

In the following section, we will introduce a new fixed point result in which the nonexpansiveness condition is replaced by a more general hypothesis. Furthermore, in order to give a distinct significance to the arguments of the simulation function, we will modify an exponent in one inequality of (6), which will produce a completely different notion.

4 Wide s-simulation functions obtaining new results in fixed point theory

The main aims of this section are to present the new type of generalized simulation functions and to define a novel family of contractions using the idea of the new generalized simulation functions. This family of contractions is to determined by avoiding the condition that mappings in this family are nonexpansive. Furthermore, this family also highlights that the arguments of simulation functions have not to play a symmetric role.

4.1 Wide s-simulation functions

The following one is the kind of auxiliary functions we will use in the main results.

Definition 8. Given $s \in [1, \infty)$, a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be a *wide s-simulation function* if it satisfies the following property:

(S_5) If $\{t_k\}$ and $\{s_k\}$ are bounded sequences in $(0, \infty)$ such that

$$0 < \limsup_{k\to\infty} s_k \leqslant s \liminf_{k\to\infty} t_k \tag{7}$$

and

$$0 < \limsup_{k\to\infty} t_k \leqslant s^3 \liminf_{k\to\infty} s_k, \tag{8}$$

then $\limsup_{k\to\infty} \zeta(t_k, s_k) < 0$.

We denote by W_s the family of all wide s-simulation functions.

We advise the reader that the usage of distinct exponents on the right-hand sides of the previous inequalities is not an erratum. This is, in fact, the reason why the arguments of ζ do not play the same role.

Remark 1. We highlight that the sequences $\{t_k\}$ and $\{s_k\}$ in Definition 8 must be bounded to avoid the case in which all the involved limits (inferior and superior) are ∞.

Remark 2. If ζ is a function satisfying property (S_5), then it also satisfies property (S_4) (because all the pairs of sequences satisfying the hypotheses of (S_4) also satisfy the hypotheses of (S_5)). In this sense, condition (S_4) is more general than condition (S_5), which implies that

$$W_s \subseteq C_s. \tag{9}$$

However, from our point of view, (S_5) better explains that the function ζ has not to satisfy symmetric conclusions such that

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0 \quad \text{and} \quad \limsup_{n\to\infty} \zeta(s_n, t_n) < 0$$

under the hypotheses of axiom (ζ_3). In Example 2, we will prove that the inclusion $W_2 \subseteq C_2$ is strict.

The following one is a four-parametric family of wide 2-simulation functions.

Example 1. Given $\varepsilon > 0$, $M > 0$, and $\delta \in (0,1)$, let $\zeta_{\varepsilon,M,\delta,f} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as follows:

$$\zeta_{\varepsilon,M,\delta,f}(t,s) = \begin{cases} Ms - M(2+\delta)t - \varepsilon \\ \quad \text{if } 0 < t < 4(2+\delta)s \text{ and } 0 < s < (2+\delta)t, \\ f(t,s) \quad \text{otherwise,} \end{cases} \tag{10}$$

where $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is an arbitrary function. We claim that $\zeta_{\varepsilon,M,\delta,f} \in W_2$ for any $\varepsilon > 0$, $M > 0$, and $\delta \in (0,1)$. To prove it, let $\{\alpha_n\}$ and $\{\beta_n\}$ be bounded sequences in $(0,\infty)$ satisfying (7) and (8) for $s = 2$. For simplicity, we denote $\ell_\alpha = \liminf_{k\to\infty} \alpha_k$, $L_\alpha = \limsup_{k\to\infty} \alpha_k$, $\ell_\beta = \liminf_{k\to\infty} \beta_k$, and $L_\beta = \limsup_{k\to\infty} \beta_k$ (all of them are finite and positive). Conditions (7) and (8) mean that

$$0 < \ell_\alpha \leqslant 2L_\beta \leqslant 4\ell_\alpha < \infty$$

and

$$0 < \ell_\beta \leqslant 2L_\alpha \leqslant 8\ell_\beta < \infty.$$

Associated to $\delta \in (0,1)$, there is $\delta' \in (0,\delta)$ satisfying $2(1+\delta')/(1-\delta') < 2+\delta$. Furthermore, having in mind the properties of the limits inferior and superior, associated to $\delta' > 0$, there is $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$(1-\delta')\ell_\alpha \leqslant \alpha_n \leqslant (1+\delta')L_\alpha$$

and

$$(1-\delta')\ell_\beta \leqslant \beta_n \leqslant (1+\delta')L_\beta.$$

As a result, for each $n \geqslant n_0$, we obtain

$$\begin{aligned} \alpha_n &\leqslant (1+\delta')L_\alpha \leqslant (1+\delta')8\ell_\beta = 4\frac{2(1+\delta')}{1-\delta'}(1-\delta')\ell_\beta \\ &= 4\frac{2(1+\delta')}{1-\delta'}\beta_n < 4(2+\delta)\beta_n, \end{aligned}$$

and similarly, we have

$$\begin{aligned} \beta_n &\leqslant (1+\delta')L_\beta \leqslant (1+\delta')2\ell_\alpha = \frac{2(1+\delta')}{1-\delta'}(1-\delta')\ell_\alpha \\ &= \frac{2(1+\delta')}{1-\delta'}\alpha_n < (2+\delta)\alpha_n. \end{aligned}$$

Since $0 < \alpha_n < 4(2+\delta)\beta_n$ and $0 < \beta_n < (2+\delta)\alpha_n$, then, for all $n \geqslant n_0$,

$$\begin{aligned} \zeta_{\varepsilon,M,\delta,f}(\alpha_n,\beta_n) &= M\beta_n - M(2+\delta)\alpha_n - \varepsilon \\ &= M\big[\beta_n - (2+\delta)\alpha_n\big] - \varepsilon \leqslant -\varepsilon. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \zeta_{\varepsilon,M,\delta,f}(\alpha_n,\beta_n) \leqslant -\varepsilon < 0,$$

so condition (S_5) holds. Hence, $\zeta_{\varepsilon,M,\delta,f} \in W_2$, and taking into account (9), then $\zeta_{\varepsilon,M,\delta,f} \in C_2$.

Next, we show that the family C_2 is strictly greater than W_2.

Example 2. Given $\varepsilon > 0$, $M > 0$, and $\delta \in (0,1)$, let $\tilde{\zeta}_{\varepsilon,M,\delta,f} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as follows:

$$\tilde{\zeta}_{\varepsilon,M,\delta,f}(t,s) = \begin{cases} Ms - M(2+\delta)t - \varepsilon \\ \quad \text{if } 0 < t < (2+\delta)s \text{ and } 0 < s < (2+\delta)t, \\ f(t,s) \quad \text{otherwise,} \end{cases}$$

where $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is an arbitrary function. Reasoning as in Example 1, it can be checked that $\tilde{\zeta}_{\varepsilon,M,\delta,f} \in C_2$. Next, we suppose that $f(48,6) = 0$, and we prove that $\tilde{\zeta}_{\varepsilon,M,\delta,f} \notin W_2$. Let us consider the sequences $\{t_n\}$ and $\{s_n\}$ defined as

$$t_n = \begin{cases} 48 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd;} \end{cases} \qquad s_n = 6 \quad \text{for all } n \in \mathbb{N}.$$

Then $\{t_n\}, \{s_n\} \subset (0,\infty)$ are bounded sequences such that

$$0 < \limsup_{k\to\infty} s_k = 6 = 2 \cdot 3 = s \liminf_{k\to\infty} t_k$$

and

$$0 < \limsup_{k\to\infty} t_k = 48 = 8 \cdot 6 = s^3 \liminf_{k\to\infty} s_k.$$

Notice that the point $(t_0, s_0) = (48, 6)$ does not satisfy $t < (2+\delta)s$, so $\tilde{\zeta}_{\varepsilon,M,\delta}(48,6) = f(48,6) = 0$. Hence

$$\begin{aligned} \limsup_{k\to\infty} \zeta(t_k, s_k) &= \max\bigl\{\tilde{\zeta}_{\varepsilon,M,\delta}(3,6), \tilde{\zeta}_{\varepsilon,M,\delta}(48,6)\bigr\} \\ &\geqslant \tilde{\zeta}_{\varepsilon,M,\delta}(48,6) = f(48,6) = 0. \end{aligned}$$

This inequality demonstrates that $\tilde{\zeta}_{\varepsilon,M,\delta,f} \notin W_2$.

4.2 Fixed point theory for wide W_s-contractions

We introduce in this subsection the family of contractions in which we are interested.

Definition 9. Let (X, d, s) be a b-metric space with $s \geqslant 1$. A mapping $T : X \to X$ is called a *wide W_s-contraction* if the following conditions hold:

- (L_1') For each $x \in X$, the limit $\lim_{n\to\infty} d(T^n x, T^{n+1} x)$ exists (that is, it is a real finite number);
- (L_2') For all $\varepsilon > 0$, there is $\zeta \in W_s$ such that inequality (3) is satistied for each $x, y \in X$, where $d(x,y) \geqslant \varepsilon$.

In the above definition, condition (L_1') is more general than (L_1). Clearly (L_1) implies (L_1') because if $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$, then for each $x \in X$, we obtain

$$\begin{aligned} 0 &\leqslant d\bigl(T^{n+1}x, T^{n+2}x\bigr) \leqslant d\bigl(T^n x, T^{n+1}x\bigr) \\ &\leqslant \cdots \leqslant d\bigl(Tx, T^2x\bigr) \leqslant d(x,Tx). \end{aligned}$$

Therefore, the sequence $\{d(T^n x, T^{n+1}x)\}_{n\in\mathbb{N}}$ is decreasing and bounded below, so it is convergent. In the following example, we show that the converse is false.

Example 3. Let $X = [2,\infty)$ be endowed with the Euclidean metric d_0, and let $T: X \to X$ be defined by

$$Tx = x + 2 - \frac{1}{x} \quad \text{for each } x \in X.$$

If $x_0 = 2$ and $y_0 = 3$, then

$$\begin{aligned} d_0(Tx_0, Ty_0) &= d_0\left(2 + 2 - \frac{1}{2},\, 3 + 2 - \frac{1}{3}\right) = \left|\frac{7}{2} - \frac{14}{3}\right| \\ &= \frac{7}{6} > 1 = d(x_0, y_0). \end{aligned}$$

Therefore, T is not nonexpansive. However, given $x \in X$, we observe that

$$d_0(x,Tx) = \left| x - \left(x + 2 - \frac{1}{x}\right)\right| = 2 - \frac{1}{x}.$$

Taking into account that T is strictly increasing and $Tx \geqslant x + 1$ for each $x \in X$, we deduce that, given any $x_0 \in X$, we have that $T^n x_0 \to \infty$ as $n \to \infty$, and so

$$\begin{aligned} \lim_{n\to\infty} d_0\bigl(T^n x_0, T^{n+1}x_0\bigr) &= \lim_{n\to\infty} d_0\bigl(T^n x_0, T\bigl(T^n x_0\bigr)\bigr) \\ &= \lim_{n\to\infty}\left(2 - \frac{1}{T^n x_0}\right) = 2. \end{aligned}$$

Hence, T satisfies condition (L_1').

Proposition 3. *If a wide W_s-contraction has a fixed point, then it is unique.*

Proof. Let (X,d,s) be a b-metric space, and let $T: X \to X$ be a wide W_s-contraction with two distinct fixed points $x_1, x_2 \in X$. Let $\varepsilon = d(x_1,x_2) > 0$. By condition ($\mathrm{L}_2'$), there is $\zeta \in W_s$ such that, for each $x,y \in X$ satisfying $d(x,y) \geqslant \varepsilon = d(x_1,x_2)$, inequality (3) holds. In particular, applying (3) to $x = x_1$ and $y = x_2$ and taking into account that x_1 and x_2 are fixed points of T, we obtain that

$$\zeta\bigl(d(x_1,x_2), d(x_1,x_2)\bigr) = \zeta\bigl(d(Tx_1,Tx_2), d(x_1,x_2)\bigr) \geqslant 0. \tag{11}$$

Let us define $t_n = s_n = d(x_1,x_2) > 0$ for each $n \in \mathbb{N}$. Then $t_n \to d(x_1,x_2)$ and $s_n \to d(x_1,x_2)$ as $n \to \infty$, and so

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = d(x_1,x_2) > 0.$$

Since $s \geqslant 1$, inequalities (7) and (8) are trivial. Having in mind that $\zeta \in W_s$ is a wide s-simulation function, then

$$\zeta\big(d(x_1, x_2), d(x_1, x_2)\big) = \limsup_{k\to\infty} \zeta(t_k, s_k) < 0,$$

which contradicts (11). Therefore, T cannot have two distinct fixed points. □

Next, we prove the main result involving the existence of a unique fixed point for continuous wide W_s-contractions.

Theorem 2. *Every continuous wide W_s-contraction from a complete b-metric space into itself is a Picard operator.*

Proof. Let (X, d, s) be a complete b-metric space, and let $T : X \to X$ be a wide W_s-contraction. Given a point $x_0 \in X$, let $\{x_n = T^n x_0\}_{n\in\mathbb{N}_0}$ be the Picard sequence of T starting from x_0. If there is some $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T, and the uniqueness of such fixed point follows from Proposition 3. On the contrary case, suppose that

$$d(x_n, x_{n+1}) > 0 \quad \text{for each } n \in \mathbb{N}_0. \tag{12}$$

By hypothesis (L_1'), there is $L = \lim_{n\to\infty} d(x_n, x_{n+1}) \geqslant 0$. To prove that $L = 0$, suppose, by contradiction, that $L > 0$. Let $t_n = d(x_{n+1}, x_{n+2}) > 0$ and $s_n = d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}$. Then $t_n \to L$ and $s_n \to L$ as $n \to \infty$, and so

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L > 0.$$

Since $s \geqslant 1$, we have

$$0 < L = \limsup_{n\to\infty} s_n \leqslant sL = s \liminf_{n\to\infty} t_n$$

and

$$0 < L = \limsup_{n\to\infty} t_n \leqslant s^3 L = s^3 \liminf_{n\to\infty} s_n.$$

As a consequence, property (S_5) guarantees that

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{13}$$

On the other hand, taking into account that $d(x_n, x_{n+1}) \to L$ as $n \to \infty$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{L}{2} \leqslant d(x_n, x_{n+1}) \quad \text{for each } n \geqslant n_0. \tag{14}$$

Using (L_2') with $\varepsilon = L/2 > 0$, there is $\zeta \in W_s$ such that, for each $x, y \in X$, where $d(x, y) \geqslant \varepsilon = L/2$, inequality (3) is satisfied. By (14) and (3), it holds that

$$\zeta\big(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})\big) \geqslant 0 \quad \text{for each } n \geqslant n_0,$$

that is,

$$\zeta(t_n, s_n) = \zeta\big(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\big) \geqslant 0 \quad \text{for each } n \geqslant n_0.$$

This condition implies that

$$0 \leqslant \liminf_{n\to\infty} \zeta(t_n, s_n) \leqslant \limsup_{n\to\infty} \zeta(t_n, s_n) \in [0, \infty],$$

which contradicts (13). This contradiction demonstrates that $L = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

In particular, the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is bounded, that is, there is $M > 0$ such that

$$d(x_n, x_{n+1}) \leqslant M \quad \text{for each } n \in \mathbb{N}_0.$$

Using Proposition 2, the Picard sequence $\{x_n\}$ is either infinite or almost constant. By (12), the second case is impossible (it cannot be constant from a term onward). Then it is infinite, that is, $x_n \neq x_m$ for each $n, m \in \mathbb{N}$ such that $n \neq m$.

Next, we prove, by contradiction, that $\{x_n\}$ is a Cauchy sequence in (X, d, s). If this is false, there are $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}_{k\in\mathbb{N}}$ and $\{x_{m(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ such that $k < n(k) < m(k)$ and

$$d(x_{n(k)}, x_{m(k)-1}) < \varepsilon_0 \leqslant d(x_{n(k)}, x_{m(k)}) \quad \text{for each } k \in \mathbb{N}$$

(the first inequality follows by assuming that for each $n(k)$, the number $m(k)$ is the lowest natural number greater than $n(k)$ satisfying the second inequality). Define

$$t_k = d(x_{n(k)}, x_{m(k)}) \quad \text{and} \quad s_k = d(x_{n(k)-1}, x_{m(k)-1}) \quad \text{for each } k \in \mathbb{N}.$$

Clearly, $\{t_k\}_{k\in\mathbb{N}}$ and $\{s_k\}_{k\in\mathbb{N}}$ are sequences in $(0, \infty)$. For the sequence $\{t_k\}_{k\in\mathbb{N}}$, the following holds:

$$\begin{aligned} \varepsilon_0 \leqslant t_k &= d(x_{n(k)}, x_{m(k)}) \\ &\leqslant s\big[d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})\big] \\ &\leqslant s\big[\varepsilon_0 + d(x_{m(k)-1}, x_{m(k)})\big] \leqslant s[\varepsilon_0 + M]. \end{aligned} \tag{15}$$

Since the sequence $\{t_k\}$ is bounded, then it has limit superior and limit superior, and letting $k \to \infty$ in (15), we deduce that

$$0 < \varepsilon_0 \leqslant \liminf_{k\to\infty} t_k \leqslant \limsup_{k\to\infty} t_k \leqslant s\varepsilon_0. \tag{16}$$

On the other hand, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} 0 < s_k &= d(x_{n(k)-1}, x_{m(k)-1}) \\ &\leqslant s\big[d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)-1})\big] \\ &\leqslant s\big[d(x_{n(k)-1}, x_{n(k)}) + \varepsilon_0\big] \leqslant s[M + \varepsilon_0]. \end{aligned} \tag{17}$$

Hence, the sequence $\{s_k\}$ is also bounded, and letting $k \to \infty$ in (17), we deduce that

$$0 \leqslant \liminf_{k\to\infty} s_k \leqslant \limsup_{k\to\infty} s_k \leqslant s\varepsilon_0. \tag{18}$$

Furthermore, for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\varepsilon_0 \leqslant t_k &= d(x_{n(k)}, x_{m(k)}) \\ &\leqslant s\big[d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})\big] \\ &\leqslant sd(x_{n(k)}, x_{n(k)-1}) + s^2 d(x_{n(k)-1}, x_{m(k)-1}) + s^2 d(x_{m(k)-1}, x_{m(k)}) \\ &= sd(x_{n(k)}, x_{n(k)-1}) + s^2 s_k + s^2 d(x_{m(k)-1}, x_{m(k)}),\end{aligned}$$

and letting $k \to \infty$ in the previous inequality, we deduce that

$$\varepsilon_0 \leqslant s^2 \liminf_{k\to\infty} s_k.$$

In particular,

$$\liminf_{k\to\infty} s_k \geqslant \frac{\varepsilon_0}{s^2} > 0,$$

which, together with (18), implies that

$$0 < \frac{\varepsilon_0}{s^2} \leqslant \liminf_{k\to\infty} s_k \leqslant \limsup_{k\to\infty} s_k \leqslant s\varepsilon_0. \tag{19}$$

Joining (16) and (19), we can prove (7) because

$$0 < \limsup_{k\to\infty} s_k \leqslant s\varepsilon_0 \leqslant s \liminf_{k\to\infty} t_k. \tag{20}$$

In addition to this,

$$0 < \limsup_{k\to\infty} t_k \leqslant s\varepsilon_0 = s^3 \frac{\varepsilon_0}{s^2} \leqslant s^3 \liminf_{k\to\infty} s_k, \tag{21}$$

which proves (8). Taking into account that

$$0 < \frac{\varepsilon_0}{s^2} \leqslant \liminf_{k\to\infty} s_k,$$

there is $k_0 \in \mathbb{N}$ such that

$$0 < \frac{\varepsilon_0}{2s^2} \leqslant s_k \quad \text{for each } k \geqslant k_0.$$

Using $\varepsilon = \varepsilon_0/(2s^2) > 0$ in (L_2'), there is $\zeta \in W_s$ such that, for each $x, y \in X$, where $d(x, y) \geqslant \varepsilon = \varepsilon_0/(2s^2)$, inequality (3) holds. Since

$$d(x_{n(k)-1}, x_{m(k)-1}) = s_k \geqslant \frac{\varepsilon_0}{2s^2} \quad \text{for each } k \geqslant k_0,$$

then

$$\begin{aligned}\zeta(t_k, s_k) &= \zeta\big(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})\big)\\ &= \zeta\big(d(Tx_{n(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, x_{m(k)-1})\big)\\ &\geqslant 0 \quad \text{for each } k \geqslant k_0.\end{aligned}$$

In particular,

$$0 \leqslant \limsup_{k\to\infty} \zeta(t_k, s_k) \in [0, \infty]. \tag{22}$$

On the other hand, using property (S_5) with $\zeta \in W_s$ and taking into account that the sequences $\{t_k\}$ and $\{s_k\}$ satisfy (20) and (21), we deduce that

$$\limsup_{k\to\infty} \zeta(t_k, s_k) < 0,$$

which contradicts (22). This contradiction demonstrates that $\{x_n\}$ is a Cauchy sequence in (X, d, s). As this space is complete, then there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. In this case, as T is continuous, then $x_{n+1} = Tx_n \to Tz$ as $n \to \infty$, and the uniqueness of the limit of a convergent sequence in a b-metric space finally guarantees that $Tz = z$, that is, z is a fixed point of T. The uniqueness of the fixed point follows from Proposition 3. □

Example 4. Let $X = [0, 1] \cup \{10, 10.1\}$ be endowed with $d(x, y) = (x - y)^2$ for all $x, y \in X$. Clearly, d is a b-metric with $s = 2$ (use $x = 0$, $y = 0.5$, and $z = 1$ to check that $s = 2$ cannot be improved with a lesser constant, so d is not a metric). Let us define $T : X \to X$ as follows:

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1],\\ 0 & \text{if } x = 10,\\ 1 & \text{if } x = 10.1.\end{cases}$$

If $x_0 = 10$ and $y_0 = 10.1$, then $d(x_0, y_0) = 0.1^2 = 0.01$ and $d(Tx_0, Ty_0) = d(0, 1) = 1^2 = 1$. Since $d(Tx_0, Ty_0) > d(x_0, y_0)$, then T is not a large $\mathcal{Z}_2$-contraction mapping (condition (L_1) fails), so Theorem 1 is not applicable to T. Nevertheless, let us show that Theorem 2 is applicable. Clearly, T satisfies condition (L_1'). To prove the property (L_2'), given $\varepsilon > 0$, let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined as

$$\zeta(t, s) = \begin{cases} Ms - M(2+\delta)t - \varepsilon \\ \quad \text{if } 0 < t < 4(2+\delta)s \text{ and } 0 < s < (2+\delta)t,\\ 0 \quad \text{otherwise},\end{cases}$$

where $M > 0$ and $\delta \in (0, 1)$ are arbitrary. Notice that ζ is $\zeta_{\varepsilon,M,\delta,f}$ defined as in (10) when $f(t, s) = 0$ for all $t, s \in [0, \infty)$. It was proved in Example 1 that $\zeta \in W_2$. In fact, we claim that $\zeta(d(Tx,Ty),d(x,y))=0$ for all $x,y\in X$ such that $d(x,y)\geqslant\varepsilon$. Suppose that $x,y\in X$ satisfies $d(x,y)\geqslant\varepsilon$. We are going to show that it is impossible that

$$0<d(Tx,Ty)<4(2+\delta)d(x,y) \tag{23}$$

and

$$0<d(x,y)<(2+\delta)d(Tx,Ty). \tag{24}$$

We consider the following cases.

(i) If $x,y\in[0,1]$, then $d(x,y)=(x-y)^2$ and $d(Tx,Ty)=d(x/2,y/2)=(x–y)^2/4$. Therefore,

$$\begin{aligned}0&<(2+\delta)d(Tx,Ty)=(2+\delta)\frac{(x-y)^2}{4}=\frac{2+\delta}{4}(x-y)^2\\&<\frac{3}{4}(x-y)^2<(x-y)^2=d(x,y),\end{aligned}$$

so condition (24) is false.

(ii) If $x\in[0,1]$ and $y\in\{10,10.1\}$, then $d(x,y)=(x-y)^2\geqslant(10-1)^2=81$ and $d(Tx,Ty)=d(0,z)$, where $z\in\{0,1\}$. Hence

$$(2+\delta)d(Tx,Ty)\leqslant(2+\delta)1^2=2+\delta<3<81\leqslant d(x,y),$$

so condition (24) does not hold.

(iii) If $x\in\{10,10.1\}$ and $y\in[0,1]$, the same conclusion holds because d is symmetric.

(iv) If $x,y\in\{10,10.1\}$, then either $x=10$ and $y=10.1$ or vice versa (notice that $x\neq y$ because $d(x,y)\geqslant\varepsilon$). Hence $d(x,y)=d(10,10.1)=0.01$ and $d(Tx,Ty)=d(0,1)=1$, so

$$4(2+\delta)d(x,y)=4(2+\delta)0.01=\frac{2+\delta}{25}<\frac{3}{25}<1=d(Tx,Ty),$$

which means that condition (23) does not hold.

We have checked that if $x,y\in X$ satisfies $d(x,y)\geqslant\varepsilon$, then inequalities (23), (24) cannot hold at the same time. Therefore, $\zeta(d(Tx,Ty),d(x,y))=0$. In particular, T satisfies (L_2'). As it also satisfies (L_1'), then T is a wide W_2-contraction mapping, and as it is continuous, then Theorem 2 is applicable to T to guarantee that is has a unique fixed point.

The continuity of the operator T in Theorem 2 is a strong condition that did not appear in Theorem 1. The reason is that large $\mathcal{Z}_s$-simulation contractions are nonexpansive (recall Definition 7), so it can be proved that, for each $k\in\mathbb{N}$,

$$\begin{aligned}t_k&=d(x_{n(k)},x_{m(k)})=d(Tx_{n(k)-1},Tx_{m(k)-1})\\&<d(x_{n(k)-1},x_{m(k)-1})=s_k.\end{aligned}$$

This inequality greatly simplifies the proof of Theorem 1, and it allowed the authors to use the same exponent in conditions (1) and (2).

In the following result, we replace the continuity condition in Theorem 2 by other general assumptions, involving auxiliary functions. For instance, the following extended version of the nonexpansivity: let Φ be the family of all functions $\varphi : (0,\infty) \to (0,\infty)$ such that

- if $\{t_n\} \subset (0,\infty)$ is a sequence, where $t_n \to 0$ as $n \to \infty$, then $\varphi(t_n) \to 0$ as $n \to \infty$.

Theorem 3. *Let (X,d,s) be a complete b-metric space, and let $T : X \to X$ be a wide W_s-contraction. Suppose that there is a function $\varphi \in \Phi$ such that, for each $x, y \in X$ such that $x \neq y$,*

$$d(Tx,Ty) \leqslant \varphi\big(d(x,y)\big).$$

Then T is a Picard operator.

Proof. Following the arguments of the proof of Theorem 2, we can reduce the proof to the case in which the Picard sequence $\{x_n = T^n x_0\}_{n\in\mathbb{N}_0}$ is infinite and convergent to $z \in X$. As it is infinite, there is $n_0 \in \mathbb{N}$ such that $x_n \neq z$ and $x_n \neq Tz$ for each $n \geqslant n_0$. By hypothesis, since $d(x_n, z) \to 0$ as $n \to \infty$, then $\varphi(d(x_n, z)) \to 0$ as $n \to \infty$, and as

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leqslant \varphi\big(d(x_n, z)\big) \quad \text{for each } n \geqslant n_0,$$

then $d(x_{n+1}, Tz) \to 0$ as $n \to \infty$, that is, $x_{n+1} \to z$ as $n \to \infty$. The uniqueness of the limit of a convergent sequence in a b-metric space finally guarantees that $Tz = z$, that is, z is a fixed point of T. The rest is similar to the proof of Theorem 2. □

A simple way to apply the previous result is the following version.

Corollary 1. *Let (X,d,s) be a complete b-metric space, and let $T : X \to X$ be a wide W_s-contraction. Suppose that there are $a_1, a_2, \ldots, a_m \in (0,\infty)$ such that, for each $x, y \in X$ with $x \neq y$,*

$$d(Tx,Ty) \leqslant a_1 d(x,y) + a_2 d(x,y)^2 + \cdots + a_m d(x,y)^m.$$

Then T is a Picard operator.

Proof. It follows from Theorem 3 by employing the function $\varphi_{a_1,a_2,\ldots,a_m} \in \Phi$ defined by $\varphi_{a_1,a_2,\ldots,a_m}(t) = a_1 t + a_2 t^2 + \cdots + a_m t^m$ for each $t > 0$. □

In the following consequence, we employ a general version of the nonexpansiveness.

Corollary 2. *Let (X,d,s) be a complete b-metric space, and let $T : X \to X$ be a wide W_s-contraction. Suppose that there is $\lambda \in (0,\infty)$ such that, for each $x, y \in X$ with $x \neq y$,*

$$d(Tx,Ty) \leqslant \lambda d(x,y).$$

Then T is a Picard operator.

Proof. It follows from Theorem 3 by employing the function $\varphi_\lambda \in \Phi$ defined by $\varphi_\lambda(t) = \lambda t$ for each $t > 0$. □

Corollary 3. *Let (X, d, s) be a complete b-metric space, and let $T : X \to X$ be a wide W_s-contraction. Suppose that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \leqslant d(x, y).$$

Then T is a Picard operator.

Proof. Use $\lambda = 1$ in Corollary 2. □

One of the main characteristics of the wide W_s-contractions is that we can never ensure that it exists a unique $\zeta \in W_s$ such that, for each $x, y \in X$, where $x \neq y$, inequality (3) is satisfied. The function $\zeta \in W_s$ in condition (L_2') directly depends on $\varepsilon > 0$, and (3) does not hold in the whole space X, but it must only occur when $d(x, y) \geqslant \varepsilon$. Nevertheless, inspired by condition (ζ_2) that a simulation function must satisfy, we introduce the following kind of wide s-simulation functions.

Definition 10. Given $s \in [1, \infty)$ and $\lambda, \mu \in (0, \infty)$, a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be a *wide* (s, λ, μ)-*simulation function* if it satisfies property (S_5) of Definition 8 and

$$\zeta(t, s) \leqslant \lambda s - \mu t \quad \text{for each } t, s \in (0, \infty). \tag{25}$$

We denote by $W_{s,\lambda,\mu}$ the family of all wide (s, λ, μ)-simulation functions.

Clearly, $W_{s,\lambda,\mu} \subseteq W_s$.

Theorem 4. *Let (X, d, s) be a complete b-metric space, and let $T : X \to X$ be a mapping. Suppose that the following properties are fulfilled*:

- (L_1') *For each $x \in X$, the limit $\lim_{n\to\infty} d(T^n x, T^{n+1} x)$ exists* (*that is, it is a real finite number*);
- (L_2') *Let $\lambda, \mu \in (0, \infty)$, and for all $\varepsilon > 0$, there is $\zeta \in W_{s,\lambda,\mu}$ satisfying inequality* (3) *for each $x, y \in X$, where $d(x, y) \geqslant \varepsilon$.*

Then T is a Picard operator.

Proof. Since $W_{s,\lambda,\mu} \subseteq W_s$, we can follow the arguments of the proof of Theorem 2 to reduce the proof to the case in which for each $x_0 \in X$, the Picard sequence $\{x_n = T^n x_0\}_{n\in\mathbb{N}_0}$ is infinite and convergent to $z \in X$. As it is infinite, there is $n_0 \in \mathbb{N}$ such that $x_n \neq z$ and $x_n \neq Tz$ for each $n \geqslant n_0$. For each $n \in \mathbb{N}_0$ with $n \geqslant n_0$, using (L_2') with $\varepsilon = d(x_n, z) > 0$, there is $\zeta_n \in W_{s,\lambda,\mu}$ such that, for each $x, y \in X$ satisfying $d(x, y) \geqslant \varepsilon = d(x_n, z)$,

$$\zeta_n\big(d(Tx, Ty), d(x, y)\big) \geqslant 0.$$

In particular,

$$\zeta_n\big(d(Tx_n, Tz), d(x_n, z)\big) \geqslant 0 \quad \text{for each } n \in \mathbb{N} \text{ with } n \geqslant n_0.$$

Using (25), for each $n \in \mathbb{N}$ with $n \geqslant n_0$,

$$0 \leqslant \zeta_n\big(d(Tx_n, Tz), d(x_n, z)\big) \leqslant \lambda d(x_n, z) - \mu d(Tx_n, Tz),$$

and as $\lambda, \mu > 0$, then

$$d(Tx_n, Tz) \leqslant \frac{\lambda}{\mu} d(x_n, z) \quad \text{for each } n \in \mathbb{N} \text{ with } n \geqslant n_0.$$

This implies that $d(x_{n+1}, Tz) \to 0$ as $n \to \infty$, and we can reason as in the proof of Theorem 3 to conclude that z is a fixed point of T. □

5 Application to nonlinear elastic beam equations

Our goal in this section is to investigate the existence and uniqueness of a solution for the following fourth-order two-point boundary value problem for elastic beam equations:

$$\begin{aligned} &u''''(t) = f\bigl(t, u(t), u'(t)\bigr) \quad \text{for } 0 < t < 1, \\ &u(0) = u'(0) = 0, \qquad u''(1) = 0, \quad u'''(1) = g\bigl(u(1)\bigr), \end{aligned} \tag{26}$$

where $u \in C([0,1])$ is an unknown function, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given continuous functions. The physical meaning of boundary conditions of (26) are as follows:

(i) The first boundary condition $u(0) = u'(0) = 0$ means that the left end of the beam is fixed;

(ii) The second boundary condition $u''(1) = 0$, $u'''(1) = g(u(1))$ means that the right end of the beam is attached to a bearing device given by the function g.

The proof of the main result in this section is based upon a new fixed point theorem of wide W_s-contractions in the previous section.

Theorem 5. *In addition to problem* (26)*, suppose that* $T : (X, d) \to (X, d)$ *is a wide* W_s*-contraction defined for each* $x \in X$ *by*

$$(Tx)(t) = \int_0^1 G(t,s) f\bigl(s, x(s), x'(s)\bigr)\,\mathrm{d}s - g\bigl(x(1)\bigr)\phi(t)$$

for all $t \in [0,1]$*, where* $X = C([0,1])$ *is the set of all real-valued function defined on* $[0,1]$*,* $d : X \to X$ *is defined by*

$$d(x,y) = \max_{t \in [0,1]} \bigl|x(t) - y(t)\bigr|^p$$

for all $x, y \in X$ *such that* p *is a fixed real-valued constant with* $p \geqslant 1$*,*

$$G(t,s) = \frac{1}{6} \begin{cases} s^2(3t - s) & \text{if } 0 \leqslant s \leqslant t \leqslant 1, \\ t^2(3s - t) & \text{if } 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{27}$$

and $\phi(t) = t^2/2 - t^3/6$ *for all* $t \in [0,1]$*. Then problem* (26) *has a unique solution.*

Proof. It is well known that (X, d) is a complete b-metric space. From [4] the Green function $G(t, s)$ of the linear problem $u''''(t) = 0$ with the boundary conditions in (26) is $G(t, s)$ defined by (27). Then problem (26) is equivalent to the following integral equation:

$$u(t) = \int_0^1 G(t, s) f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s - g\bigl(u(1)\bigr)\phi(t). \tag{28}$$

This implies that the integral problem (28) is equivalent to the fixed point problem with T. Since T is a continuous wide W_s-contraction, Theorem 2 implies the existence of the unique fixed point of T. Therefore, problem (26) has a unique solution. □

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