Nonlinear Analysis: Modelling and Control, Vol. 30, No. 3, 517–532
https://doi.org/10.15388/namc.2025.30.39958

Vilnius University Press

Symmetry analysis, soliton solutions, and conservation laws of the $Q(L, m, n)$ equation*

Gangwei Wang[a,b], **Mengyue He**[a], **Qin Zhou**[c], **Abdul H. Kara**[d]

[a] School of Mathematics and Statistics,
Hebei University of Economics and Business,
Shijiazhuang, 050061, China
gangwei@hueb.edu.cn

[b] Center for Differential Equations and Their Applications,
Hebei University of Economics and Business,
Shijiazhuang, Hebei, 050061, China

[c] Research Center of Nonlinear Science,
School of Mathematical and Physical Sciences,
Wuhan Textile University,
Wuhan 430200, China

[d] School of Mathematics, University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg, South Africa

Received: March 19, 2024 / **Revised:** February 13, 2025 / **Published online:** March 26, 2025

Abstract. In the present paper, symmetry and soliton solutions of the $Q(L, m, n)$ equation are investigated. The infinitesimal operator of this equation is obtained by virtue of Lie group analysis. Taking different values of the parameters for the coefficients, the corresponding vector fields are obtained. Subsequently, soliton solutions of this equation are obtained for different parameters relying on the solitary wave ansatz method. According to different parameters, new soliton solutions are obtained. Also, conservation laws are also derived. Reciprocal Bäcklund transformations of conservation laws presented from the known conservation laws.

Keywords: $Q(L, m, n)$ equation, symmetry analysis, solitary wave ansatz, new soliton solution, conservation laws.

1 Introduction

Nonlinear evolution equations (NLEEs) play a central role in many fields, such as mathematics, physics, engineering, and so on because these equations can better describe natural

*This work is supported by Natural Science Foundation of Hebei Province of China (No. A2023207002), "333 Talent Project" of Hebei Province (No. C20221021), Youth Key Program of Hebei University of Economics and Business (2018QZ07), Key Program of Hebei University of Economics and Business (2020ZD11, 2023ZD10, 2024ZD12), and Youth Team Support Program of Hebei University of Economics and Business.

phenomena. In the last few decades, a great many of nonlinear evolution equations have been investigated extensively by various methods, which include, for example, the Hirota bilinear method [8, 11], the inverse scattering transformation method [1], the Darboux transform method [9], the Bäcklund transformations method [20], the Hamiltonian system method [12–14], exponential function method [10], the Lie symmetry method [5, 6, 17, 29–35, 37], and so on. Using these methods, many nonlinear evolution equations are solved. Solitons are known to play an important role in nonlinear science. Switching, explosion, and chaos of multiwavelength soliton states are presented in [28]. The authors of [27] studied dynamic modeling and coded information storage of vector soliton using deep learning method. Solitons and their biperiodic pulsation in ultrafast fiber lasers are investigated in [15]. The authors of [18] proposed a novel physics-informed GAN with gradient penalty (PIGAN-GP) to predict solutions of the 2-coupled mixed derivative nonlinear Schrödinger equation. Nonlinear Schrödinger–Maxwell–Bloch equation [36] is studied via the phPINN. Data-driven vector degenerate and nondegenerate solitons of coupled nonlocal nonlinear Schrödinger equation are investigated via improved PINN algorithm [19]. The stability of solitary traveling wave solutions of the $(3+1)$-dimensional mKdV–ZK equation is investigated in paper [22]. A lot of soliton solutions of the fractional Wazwaz–Benjamin–Bona–Mahony equations obtained in [23]. Some new exact solutions of two kinds of nonlinear Schrödinger equation are derived through the variational principle method and amplitude ansatz method [24]. In [25], variational principle and optical soliton solutions of some types of nonlinear Schrödinger dynamical systems are investigated. In paper [26], two types of nonlinear Schrödinger equations are studied via variational principle method, some soliton solutions also presented. In [7], some new soliton solutions of a coupled generalized nonlinear Schrödinger system are reported using Darboux transformation.

The $Q(L,m,n)$ equation, which will be investigated in the present paper, is given by [21]

$$u_t + bu_x + a\left(u^{m+1}\right)_x + \omega\left[u\left(u^n\right)_{xx}\right]_x + \delta\left[u\left(u^L\right)_{4x}\right]_x = 0. \tag{1}$$

Here in (1) a, b, ω, and δ are constants. This equation contains fifth-order nonlinear dispersion term. It also includes some important nonlinear evolution equations, such as fifth-order KdV equation, generalized KdV equation, and so on. Because this equation contains many important nonlinear evolution equations, it is necessary for us to study this equation. To the best of our knowledge, a systematic study of this equation from a group perspective and the study of the conservation law of this equation have not been reported in the literature so far. The main contribution of this paper: (i) The corresponding infinitesimal transformations of this equation are obtained for different parameters. (ii) Some new soliton solutions were obtained. (iii) Conservation laws and Bäcklund transformations of conservation laws were obtained.

In the next section, for different parameters, we derive the corresponding infinitesimal operator. In Section 3, solitons are described using the solitary wave ansatz method. Conservation laws are derived in Section 4. Based on the obtained conservation laws, reciprocal Bäcklund transformations of conservation laws are displayed in Section 5. Conclusions are presented in Section 6.

2 Symmetry analysis

For the Lie group of point transformation [5, 6, 17, 29],

$$V=\xi^t(x,t,u)\frac{\partial}{\partial t}+\xi^x(x,t,u)\frac{\partial}{\partial x}+\eta_u(x,t,u)\frac{\partial}{\partial u},$$

where

$$\begin{aligned}t^*&=t+\epsilon\xi^t(x,t,u)+O(\epsilon^2),\\x^*&=x+\epsilon\xi^x(x,t,u)+O(\epsilon^2),\\u^*&=u+\epsilon\eta_u(x,t,u)+O(\epsilon^2).\end{aligned}$$

That is to say, for (1), we need consider fifth-order prolongations formula

$$\begin{aligned}Pr^{(6)}V&=V+\eta_u^t\frac{\partial}{\partial u_t}+\eta_u^x\frac{\partial}{\partial u_x}+\eta_u^{xx}\frac{\partial}{\partial u_{xx}}+\eta_v^{xx}\frac{\partial}{\partial v_{xx}}\\&\quad+\eta_u^{xxx}\frac{\partial}{\partial u_{xxx}}+\eta_u^{xxxx}\frac{\partial}{\partial u_{xxxx}}+\eta_u^{xxxxx}\frac{\partial}{\partial u_{xxxxx}},\end{aligned}$$

where η_u^t, η_u^x, η_u^{xx}, η_u^{xxx}, η_u^{xxxx}, and η_u^{xxxxx} are functions to be fixed. For invariance conditions $Pr^{(5)}V(\Delta)=0$, where $\Delta=0$ in Eq. (1). As this equation contains several arbitrary constants, so we need to consider several cases.

Case 1. For the general case,

$$\begin{gathered}\bigl(\delta\neq 0,\ (-1+m)m(1+m)(-2+3m)(5m-3L)L\neq 0,\ \omega\neq 0,\ a\neq 0\bigr),\\\bigl(L(-5+3L)\neq 0,\ \delta\neq 0,\ \omega\neq 0,\ a\neq 0,\ m=1\bigr),\\\left(L(-10+9L)\neq 0,\ \delta\neq 0,\ \omega\neq 0,\ a\neq 0,\ m=\frac{2}{3}\right),\\\left(\delta\neq 0,\ u\neq 0,\ m(1+m)\neq 0,\ \omega\neq 0,\ a\neq 0,\ L=\frac{5m}{3}\right),\end{gathered}$$

and one can get

$$\eta_u=0,\qquad \xi^x=c_2,\qquad \xi^t=c_1.$$

Thus, for the general case, the Lie algebra is spanned by the following vector fields:

$$V_1=\frac{\partial}{\partial x},\qquad V_2=\frac{\partial}{\partial t}.$$

It is easily get the one-parameter Lie symmetry group

$$\begin{aligned}G_1&:(x,t,u,v)\to(x-\varepsilon_1,\ t,u,v),\\G_2&:(x,t,u,v)\to(x,\ t-\varepsilon_2,\ u,v).\end{aligned}$$

The corresponding vector fields can be obtained in these next cases, and they will not be listed one by one.

Case 2. If we let $\delta \neq 0$, $m(1+m)(5m-L)L \neq 0$, $a \neq 0$, $\omega = 0$, one can get

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = -\frac{4bmt\mathbf{c}_2}{L-5m} + \frac{(L-m)x\mathbf{c}_2}{L-5m} + \mathbf{c}_3, \qquad \eta_u = \frac{4u\mathbf{c}_2}{-5m+L}.$$

Case 3. If $\delta \neq 0$, $L + L^2 \neq 0$, $a \neq 0$, $m = L/5$, $\omega = 0$, one has

$$\xi^t = \mathbf{c}_2, \qquad \xi^x = -bt\mathbf{c}_1 + x\mathbf{c}_1 + \mathbf{c}_3, \qquad \eta_u = \frac{u\mathbf{c}_1}{m}.$$

Case 4. When $\delta \neq 0$, $L(1+3L)(3L-5m)(-1+m)m(-2+3m) \neq 0$, $\omega \neq 0$, $a = 0$, one can have

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = \frac{2b(L-2m)t\mathbf{c}_2 + Lx\mathbf{c}_2 - mx\mathbf{c}_2 + 3L\mathbf{c}_3 - 5m\mathbf{c}_3}{3L-5m},$$
$$\eta_u = \frac{2u\mathbf{c}_2}{3L-5m}.$$

Case 5. Considering $\delta \neq 0$, $L(1+3L)(-10+9L) \neq 0$, $\omega \neq 0$, $a = 0$, $m = 2/3$, yields

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = \frac{2b(-4+3L)t\mathbf{c}_2 + (-2+3L)x\mathbf{c}_2 + (-10+9L)\mathbf{c}_3}{-10+9L},$$
$$\eta_u = \frac{6u\mathbf{c}_2}{-10+9L}.$$

Case 6. While $\delta \neq 0$, $L + 3L^2 \neq 0$, $\omega \neq 0$, $a = 0$, $m = 3L/5$, one can obtain

$$\xi^t = \mathbf{c}_2, \qquad \xi^x = -bt\mathbf{c}_1 + x\mathbf{c}_1 + \mathbf{c}_3, \qquad \eta_u = \frac{3u\mathbf{c}_1}{m}.$$

Case 7. Let us consider $\delta \neq 0$, $L(-5+3L)(1+3L) \neq 0$, $\omega \neq 0$, $a = 0$, $m = 1$, we get

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = \frac{2b(-2+L)t\mathbf{c}_2 + (-1+L)x\mathbf{c}_2 + (-5+3L)\mathbf{c}_3}{-5+3L},$$
$$\eta_u = \frac{2u\mathbf{c}_2}{-5+3L}.$$

Case 8. If $\delta \neq 0$, $L(1+3L)(5+3L) \neq 0$, $\omega \neq 0$, $a \neq 0$, $m = -1$, one can obtain

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = \frac{2b(2+L)t\mathbf{c}_2 + (1+L)x\mathbf{c}_2 + (5+3L)\mathbf{c}_3}{5+3L},$$
$$\eta_u = \frac{2u\mathbf{c}_2}{5+3L}.$$

Case 9. When $\delta \neq 0$, $\omega \neq 0$, $a \neq 0$, $L = -5/3$, $m = -1$, one can have

$$\xi^t = \mathbf{c}_2, \qquad \xi^x = -bt\mathbf{c}_1 + x\mathbf{c}_1 + \mathbf{c}_3, \qquad \eta_u = -3u\mathbf{c}_1.$$

Case 10. While $\delta \neq 0$, $\omega \neq 0$, $a = 0$, $L = -1/3$, $m = -1/5$, one can derive

$$\xi^t = \mathbf{c}_2, \qquad \xi^x = -bt\mathbf{c}_1 + x\mathbf{c}_1 + \mathbf{c}_3, \qquad \eta_u = -15u\mathbf{c}_1.$$

Case 11. If $\delta \neq 0$, $\omega \neq 0$, $a \neq 0$, $L = -1/3$, $m = -1$, we have

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = \frac{5}{6}bt\mathbf{c}_2 + \frac{x\mathbf{c}_2}{6} + \mathbf{c}_3, \qquad \eta_u = \frac{u\mathbf{c}_2}{2}.$$

Case 12. Let $L \neq 0$, $\delta \neq 0$, $a \neq 0$, $m = 0$, one gets

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = x\mathbf{c}_2 + \mathbf{c}_3, \qquad \eta_u = \frac{4u\mathbf{c}_2}{L}.$$

Case 13. For $m(1+5m) \neq 0$, $\delta \neq 0$, $\omega \neq 0$, $a = 0$, $L = -1/3$, one can present

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = \frac{2b(1+6m)t\mathbf{c}_2 + (1+3m)x\mathbf{c}_2 + 3(1+5m)\mathbf{c}_3}{3+15m},$$
$$\eta_u = -\frac{2u\mathbf{c}_2}{1+5m}.$$

Case 14. When $(a=0,\ m=0,\ L\delta + 3L^2\delta \neq 0)$, $(a=0,\ \omega=0,\ Lm\delta + 3L^2m\delta \neq 0)$, $(m=-1,\ \omega=0,\ aL\delta + 3aL^2\delta \neq 0)$, we obtain

$$\xi^t = \mathbf{c}_2 + t\mathbf{c}_3, \qquad \xi^x = x\mathbf{c}_1 + bt(-\mathbf{c}_1 + \mathbf{c}_3) + \mathbf{c}_4, \qquad \eta_u = \frac{u(5\mathbf{c}_1 - \mathbf{c}_3)}{L}.$$

Case 15. If $(a=0,\ L=-1/3,\ m=0,\ \delta \neq 0)$, $(a=0,\ L=-1/3,\ \omega=0,\ m\delta \neq 0)$, $(L=-1/3,\ m=-1,\ \omega=0,\ a\delta \neq 0)$, one derives

$$\xi^t = \mathbf{c}_2 + t\mathbf{c}_3, \qquad \xi^x = x\mathbf{c}_1 + bt(-\mathbf{c}_1 + \mathbf{c}_3) + \mathbf{c}_4, \qquad \eta_u = 3u(-5\mathbf{c}_1 + \mathbf{c}_3).$$

Case 16. While $(L=0,\ m=-1/2, a\omega \neq 0)$, $(m=-1/2,\ \delta=0,\ aL\omega \neq 0)$, one can give

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = bt\mathbf{c}_2 + \mathbf{c}_3, \qquad \eta_u = 2u\mathbf{c}_2.$$

Case 17. When $(L=0,\ am\omega + 3am^2\omega + 2am^3\omega \neq 0)$, $(\delta=0,\ aLm\omega + 3aLm^2\omega + 2aLm^3\omega \neq 0)$, one derives

$$\xi^t = \mathbf{c}_1 + t\mathbf{c}_2, \qquad \xi^x = bt\mathbf{c}_2 + \mathbf{c}_3, \qquad \eta_u = -\frac{u\mathbf{c}_2}{m}.$$

Case 18. If $(L=0,\ m=-1,\ a\omega \neq 0)$, $(m=-1,\ \delta=0,\ aL\omega \neq 0)$, one can obtain

$$\xi^t = \mathbf{c}_2 + t\mathbf{c}_3, \qquad \xi^x = x\mathbf{c}_1 + bt(-\mathbf{c}_1 + \mathbf{c}_3) + \mathbf{c}_4, \qquad \eta_u = u(-3\mathbf{c}_1 + \mathbf{c}_3).$$

Case 19. While $(a=0,\ L=0,\ m=-1/2,\ \omega \neq 0)$, $(a=0,\ m=-12,\ \delta=0,\ L\omega \neq 0)$, one can derive

$$\xi^t = \mathbf{c}_2 + t\mathbf{c}_3, \qquad \xi^x = x\mathbf{c}_1 + bt(-\mathbf{c}_1 + \mathbf{c}_3) + \mathbf{c}_4, \qquad \eta_u = 2u(-3\mathbf{c}_1 + \mathbf{c}_3).$$

Case 20. For $(a=0,\ L=0,\ m\omega + 2m^2\omega \neq 0)$, $(a=0,\ \delta=0,\ Lm\omega + 2Lm^2\omega \neq 0)$, one can derive

$$\xi^t = \mathbf{c}_2 + t\mathbf{c}_3, \qquad \xi^x = x\mathbf{c}_1 + bt(-\mathbf{c}_1 + \mathbf{c}_3) + \mathbf{c}_4, \qquad \eta_u = \frac{u(3\mathbf{c}_1 - \mathbf{c}_3)}{m}.$$

3 Soliton solutions

In this section, we will focus on the solitary wave solution of (1). In general, it is difficult to find Lax pairs. Thus, without any loss of generality, we looking for the solitary wave solution to (1) as follows [2–4]:

$$u(x,t) = A\frac{1}{\cosh^p \tau} \quad \text{and} \quad \tau = B(x - vt),$$

where A means the amplitude of the soliton solution, while B represents the inverse width of the soliton solutions. v means the soliton velocity. The unknown exponent p will be determined by L, m, and n. One can get

$$\begin{aligned}
0 = {} & ABvp\frac{1}{\cosh^p \tau}\tanh\tau - ABbp\frac{1}{\cosh^p \tau}\tanh\tau \\
& - aA^{m+1}Bp(m+1)\frac{1}{\cosh^{pm+p}\tau}\tanh\tau \\
& - \omega A^{n+1}B^3p^3n^2(n+1)\frac{1}{\cosh^{pn+p}\tau}\tanh\tau \\
& + \omega A^{n+1}B^3pn(pn+1)(pn+p+2)\frac{1}{\cosh^{pn+p+2}\tau}\tanh\tau \\
& - \delta A^{L+1}B^5p^5L^4(L+1)\frac{1}{\cosh^{pL+p}\tau}\tanh\tau \\
& + 2\delta A^{L+1}B^5pL(pL+1)(pL+p+2)\left(p^2L^2+2pL+2\right)\frac{1}{\cosh^{pL+p+2}\tau}\tanh\tau \\
& - \delta A^{L+1}B^5pL(pL+1)(pL+2)(pL+3)(pL+p+4)\frac{1}{\cosh^{pL+p+4}\tau}\tanh\tau. \qquad (2)
\end{aligned}$$

As this equation contains some constants, we will consider THE following cases:

Case I. $L = m$, $m \neq n$.
From (2), letting the exponents $pn + p$ equal $pL + p + 2$, one can get

$$pn + p = pL + p + 2 \quad \Longrightarrow \quad p = \frac{2}{n - L}.$$

To guarantee the existence of soliton solutions, it also needs

$$pn + p + 2 = pL + p + 4.$$

In this way, the coefficient in front of each independent term should be zero. Therefore, from the terms $(1/\cosh^p\tau)\tanh^p\tau$, $(1/\cosh^{pL+p}\tau)\tanh\tau$, $(1/\cosh^{pL+p+2}\tau)\tanh^p\tau$, and $(1/\cosh^{pL+p+4}\tau)\tanh^p\tau$, setting their coefficients to zero, one can get

$$v = b, \qquad B = \frac{n-L}{2L}\sqrt[4]{-\frac{a}{\delta}}.$$

To ensure the existence of the solution, we also require $L = 2n + 1$, in this way,

$$A = \left(2\frac{\varpi n^2(2n+1)}{\delta(15n^3 + 17n^2 + 7n + 1)}\frac{1}{\sqrt{-\frac{a}{\delta}}}\right)^{1/(n+1)}, \qquad B = -\frac{1}{2}\frac{n+1}{2n+1}\sqrt[4]{-\frac{a}{\delta}},$$
$$p = -\frac{2}{n+1},$$

and $L = 3n$,

$$A = \exp\left\{-\frac{1}{4n}\ln\left(-\frac{400\delta a}{9\varpi^2}\right)\right\}, \qquad B = -\frac{1}{3}\sqrt[4]{-\frac{a}{\delta}}, \qquad p = -\frac{1}{n},$$

for this case, it needs to have $a\delta < 0$.

Therefore, the soliton solution of the $Q(L,m,n)$ equation in the case $L = 2n + 1$ is given by

$$u(x,t) = \left(2\frac{\varpi n^2(2n+1)}{\delta(15n^3 + 17n^2 + 7n + 1)}\frac{1}{\sqrt{-\frac{a}{\delta}}}\right)^{1/(n+1)}$$
$$\times \frac{1}{\cosh^{-2/(n+1)}(-\frac{1}{2}\frac{n+1}{2n+1}\sqrt[4]{-\frac{a}{\delta}}(x - bt))},$$

also, for the case $L = 3n$, we get the soliton solution as follows:

$$u(x,t) = \exp\left\{-\frac{1}{4n}\ln\left(-\frac{400\delta a}{9\varpi^2}\right)\right\}\frac{1}{\cosh^{-1/n}(-\frac{1}{3}\sqrt[4]{-\frac{a}{\delta}}(x - bt))}. \tag{3}$$

Let $n = 1$, $a = 1$, $\delta = -1$, $\varpi = 20$, $b = 1$, some figures are listed as follows.

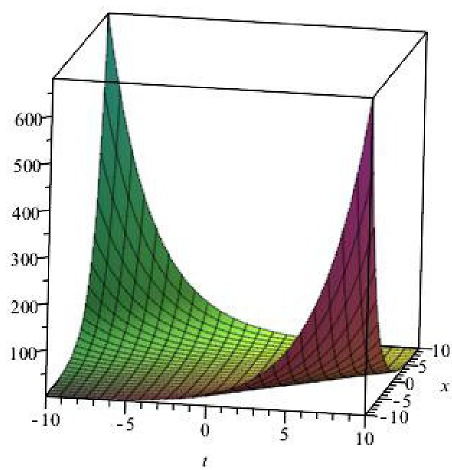

Figure 1. Figure of solution (3).

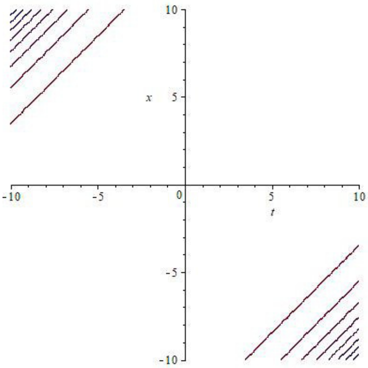

Figure 2. Contour plot of solution (3).

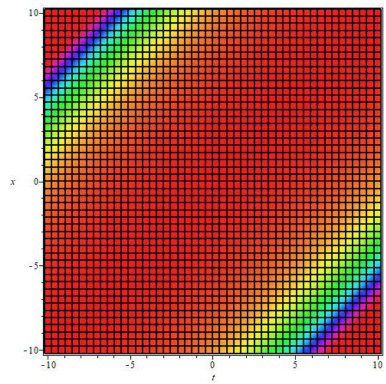

Figure 3. Density plot of solution (3).

Case II. $L = n, m \neq n$.
In this case, considering the exponents $pm + p = pL + p + 4$, one has

$$pm + p = pL + p + 4 \quad \Longrightarrow \quad p = \frac{4}{m - L}.$$

Therefore, for these functions, $(1/\cosh^p \tau) \tanh \tau$, $(1/\cosh^{pn+p} \tau) \tanh \tau$, and $(1/\cosh^{pn+p+2} \tau) \tanh \tau$ are linearly independent, their coefficients should be zero,

$$v = b, \tag{4}$$

$$\begin{aligned} \varpi &= \frac{La(L^2 + 2L + 1)}{B^2(\mathrm{e}^{\ln(A)L})^2 A^2 (L + 3)(L - 1)}, \\ \delta &= -\frac{1}{4} \frac{a(L^2 + 2L + 1)^2}{B^4 L (\mathrm{e}^{\ln(A)L})^2 A^2 (L + 3)(L - 1)}, \qquad m = -(L + 2), \end{aligned} \tag{5}$$

so, we have

$$A = \left(\frac{La(L + 1)^2}{\varpi B^2 (L + 3)(L - 1)} \right)^{1/(2(L+1))} \tag{6}$$

or

$$A = \exp\left\{ -\frac{1}{2(L + 1)} \left(2 \ln 2 - \ln\left(-\frac{a(L + 1)^4}{B^4 L \delta (L + 3)(L - 1)} \right) \right) \right\} \tag{7}$$

and

$$p = -\frac{2}{L + 1}.$$

In this case, the velocity of the soliton is decided by (4), the soliton width is given by (6) or (7), and the amplitude of the soliton is shown by (5).

Case III. $L \neq n \neq m$.
For this case, it can be said that solitons do not exist since $m = 0$, $L = 0$.

4 Conservation laws

The conservation law is, in general, written in the following form:

$$T^t + T^x = 0.$$

We still discuss it in several cases.

Case I. For the general case, we rewrite (1) and get

$$u_t + \left(bu + a\left(u^{m+1}\right) + \omega\left[u\left(u^n\right)_{xx}\right] + \delta\left[u\left(u^L\right)_{4x}\right]\right)_x = 0,$$

therefore, we have the conservation laws

$$T^t = u, \qquad T^x = bu + a\left(u^{m+1}\right) + \omega\left[u\left(u^n\right)_{xx}\right] + \delta\left[u\left(u^L\right)_{4x}\right]. \tag{8}$$

Case II. For the case $n = 1$, $L = 1$, using the multiplier method [5], we can obtain that the multiplier is $c_1 + c_2 \ln u$. Therefore, the conservation laws are

$$T^t = u, \qquad T^x = bu + a(u^{m+1}) + \omega\left[u(u)_{xx}\right] + \delta\left[u(u)_{4x}\right]$$

and

$$\begin{aligned} T^t &= u \ln u - u, \\ T^x &= u^{m+1} a \ln u - \frac{1}{m+1} u^{m+1} a + u u_{xx} \omega \ln u + u u_{xxxx} \delta \ln u \\ &\quad - \frac{1}{2} u_x^2 \omega - u_{xxx} u_x \delta + \frac{1}{2} u_{xx}^{\delta} + \ln u u b - u b. \end{aligned}$$

Case III. For the case $m = 1$, $n = 2$, $L = 2$, considering the multiplier method [5], one can get the fourth-order multiplier

$$\begin{aligned} \Lambda_1 = {} & \frac{2(\delta u_{xxxx} + \omega u_{xx} + \frac{a}{2}) C_1 u^2 + 2 C_3 \delta}{2\delta} \\ & + \frac{(((8 u_x u_{xxx} + 6 u_{xx}^2) C_1 + 2 C_2) \delta + 2 \omega u_x^2 C_1) u}{2\delta}. \end{aligned}$$

Thus, one can get the conservation laws as follows.

For the multiplier 1, one has

$$\begin{aligned} T^t &= u, \\ T^x &= 2u^2 u_{xx} \omega + 2u^2 u_{xxxx} \delta + 2u u_x^2 \omega \\ &\quad + 8 u u_x u_{xxx} \delta + 6 u u_{xx}^2 \delta + u^2 a + ub. \end{aligned}$$

If the multiplier is u, one has

$$\begin{aligned} T^t &= \frac{1}{2} u^2, \\ T^x &= u^2 u_x^2 \omega + 7 u^2 u_{xx}^2 \delta + \frac{1}{2} u^2 b + 6 u_x \delta u^2 u_{xxx} - 4 u u_x^2 u_{xx} \delta \\ &\quad + \frac{2}{3} u^3 a + 2 \delta u^3 u_{xxxx} + 2 \omega u^3 u_{xx} + u_x^4 \delta. \end{aligned}$$

While the multiplier is

$$\Lambda_1 = \frac{u(2\delta u u_{xxxx} + 8\delta u_x u_{xxx} + 6\delta u_{xx}^2 + 2\omega u u_{xx} + 2\omega u_x^2 + au)}{2\delta},$$

one can get

$$\begin{aligned}
T^t &= \frac{u^2(3\delta u u_{xxxx} + 12\delta u_x u_{xxx} + 9\delta u_{xx}^2 + 3\omega u u_{xx} + 3\omega u_x^2 + 2au)}{12\delta},\\
T^x &= \frac{1}{12\delta}\big(12\delta^2 u^4 u_{xxxx}^2 + 96\delta^2 u^3 u_x u_{xxx} u_{xxxx} + 72\delta^2 u^3 u_{xx}^2 u_{xxxx}\\
&\quad + 192\delta^2 u^2 u_x^2 u_{xxx}^2 + 288\delta^2 u^2 u_x u_{xx}^2 u_{xxx} + 108\delta^2 u^2 u_{xx}^4 + 24\delta\omega u^4 u_{xx} u_{xxxx}\\
&\quad + 24\delta\omega u^3 u_x^2 u_{xxxx} + 96\delta\omega u^3 u_x u_{xx} u_{xxx} + 72\delta\omega u^3 u_{xx}^3 + 96\delta\omega u^2 u_x^3 u_{xxx}\\
&\quad + 72\delta\omega u^2 u_x^2 u_{xx}^2 + 12\omega^2 u^4 u_{xx}^2 + 24\omega^2 u^3 u_x^2 u_{xx} + 12\omega^2 u^2 u_x^4 + 12a\delta u^4 u_{xxxx}\\
&\quad + 48a\delta u^3 u_x u_{xxx} + 36a\delta u^3 u_{xx}^2 + 12a\omega u^4 u_{xx} + 12a\omega u^3 u_x^2 + 3a^2 u^4\\
&\quad + 12b\delta u^2 u_x u_{xxx} - 6b\delta u^2 u_{xx}^2 + 24b\delta u u_x^2 u_{xx} - 6b\delta u_x^4 + 6b\omega u^2 u_x^2 + 2abu^3\\
&\quad - 3\delta u^3 u_{txxx} + 3\delta u^2 ut u_{xxx} - 15\delta u^2 u_{tx} u_{xx} - 3\delta u^2 u_{txx} u_x + 18\delta u ut u_x u_{xx}\\
&\quad + 6\delta u u_{tx} u_x^2 - 6\delta u_t u_x^3 - 3\omega u^3 u_{tx} + 3\omega u^2 u_t u_x\big).
\end{aligned}$$

Case IV. If $m = 2$, $n = 2$, $L = 2$, using the same idea, one can obtain the multiplier

$$\begin{aligned}
\Lambda_1 &= C_1 u^2 u_{xxxx} + 4C_1 u u_x u_{xxx} + \frac{C_1 u^2 \omega u_{xx}}{\delta} + 3u_{xx}^2 C_1 u\\
&\quad + \frac{C_1 u \omega u_x^2}{\delta} + \frac{1}{2}\frac{C_1 a u^3}{\delta} + C_2 u + C_3.
\end{aligned}$$

In this way, the conservation laws are given as follows.
For the multiplier 1, one has

$$\begin{aligned}
T^t &= u,\\
T^x &= au^3 + 2\delta u^2 u_{xxxx} + 8\delta u u_x u_{xxx} + 6\delta u u_{xx}^2 + 2\omega u^2 u_{xx} + 2\omega u u_x^2 + bu.
\end{aligned}$$

If the multiplier is u, one has

$$\begin{aligned}
T^t &= \frac{1}{2}u^2,\\
T^x &= u^2 u_x^2 \omega + 7u^2 u_{xx}^2 \delta + \frac{1}{2}u^2 b + 6u_x \delta u^2 u_{xxx} - 4u u_x^2 u_{xx} \delta + \frac{3}{4}u^4 a\\
&\quad + 2\delta u^3 u_{xxxx} + 2\omega u^3 u_{xx} + u_x^4 \delta.
\end{aligned}$$

When the multiplier is

$$\Lambda_1 = \frac{u(2\delta u u_{xxxx} + 8\delta u_x u_{xxx} + 6\delta u_{xx}^2 + 2\omega u u_{xx} + 2\omega u_x^2 + au^2)}{2\delta},$$

one has

$$
\begin{aligned}
T^t &= \frac{u^2(2\delta u u_{xxxx} + 8\delta u_x u_{xxx} + 6\delta u_{xx}^2 + 2\omega u u_{xx} + 2\omega u_x^2 + au^2)}{8\delta}, \\
T^x &= \frac{1}{8\delta}(2a^2u^6 + 8a\delta u^5 u_{xxxx} + 32a\delta u^4 u_x u_{xxx} + 24a\delta u^4 u_{xx}^2 + 8a\omega u^5 u_{xx} \\
&\quad + 8a\omega u^4 u_x^2 + 8\delta^2 u^4 u_{xxxx}^2 + 64\delta^2 u^3 u_x u_{xxx} u_{xxxx} + 48\delta^2 u^3 u_{xx}^2 u_{xxxx} \\
&\quad + 128\delta^2 u^2 u_x^2 u_{xxx}^2 + 192\delta^2 u^2 u_x u_{xx}^2 u_{xxx} + 72\delta^2 u^2 u_{xx}^4 + 16\delta\omega u^4 u_{xx} u_{xxxx} \\
&\quad + 16\delta\omega u^3 u_x^2 u_{xxxx} + 64\delta\omega u^3 u_x u_{xx} u_{xxx} + 48\delta\omega u^3 u_{xx}^3 + 64\delta\omega u^2 u_x^3 u_{xxx} \\
&\quad + 48\delta\omega u^2 u_x^2 u_{xx}^2 + 8\omega^2 u^4 u_{xx}^2 + 16\omega^2 u^3 u_x^2 u_{xx} + 8\omega^2 u^2 u_x^4 + abu^4 \\
&\quad + 8b\delta u^2 u_x u_{xxx} 4b\delta u^2 u_{xx}^2 + 16b\delta u u_x^2 u_{xx} - 4b\delta u_x^4 + 4b\omega u^2 u_x^2 - 2\delta u^3 u_{txxx} \\
&\quad + 2\delta u^2 ut u_{xxx} - 10\delta u^2 u_{tx} u_{xx} - 2\delta u^2 u_{txx} u_x + 12\delta u u_t u_x u_{xx} + 4\delta u u_{tx} u_x^2 \\
&\quad - 4\delta u_t u_x^3 - 2\omega u^3 u_{tx} + 2\omega u^2 ut u_x\big).
\end{aligned}
$$

Case V. When $m = 3$, $n = 3$, $L = 3$, repeating the previous steps yields that

$$\Lambda_1 = C_2 u^2 + C_1.$$

Therefore, for the multiplier 1, one gets

$$
\begin{aligned}
T^t &= u, \\
T^x &= au^4 + 3\delta u^3 u_{xxxx} + 24\delta u^2 u_x u_{xxx} + 18\delta u^2 u_{xx}^2 \\
&\quad + 36\delta u u_x^2 u_{xx} + 3\omega u^3 u_{xx} + 6\omega u^2 u_x^2 + bu.
\end{aligned}
$$

While the multiplier is u^2, we have

$$
\begin{aligned}
T^t &= \frac{1}{3}u^3, \\
T^x &= 18u^4 \delta u_{xxx} u_x + 3\omega u^4 u_x^2 + 21\delta u^4 u_{xx}^2 + 3\omega u^5 u_{xx} \\
&\quad + \frac{1}{3}u^3 b + \frac{2}{3}au^6 + 3u_{xxxx}u^5\delta + 12\delta u^3 u_x^2 u_{xx}.
\end{aligned}
$$

5 Reciprocal Bäcklund transformations of conservation laws

In this section, based on the paper [16], we will investigate the reciprocal Bäcklund transformations of conservation laws for Eq. (1).

The authors of [16] derived the following results:

$$\left(T^t\right)'_{t'} + \left(T^x\right)'_{x'} = 0, \qquad \frac{\partial}{\partial_{t'}} = \frac{F}{T}\frac{\partial}{\partial_x} + \frac{\partial}{\partial_t}, \frac{\partial}{\partial_{x'}} = \frac{1}{T}\frac{\partial}{\partial_x}. \tag{9}$$

From this result, for given conservation laws, one can get reciprocal Bäcklund transformations of conservation laws.

In this case, from (8) one has

$$T^t = u, \qquad T^x = bu + a\big(u^{m+1}\big) + \omega\big[u\big(u^n\big)_{xx}\big] + \delta\big[u\big(u^L\big)_{4x}\big].$$

Therefore, from (9) one can get the reciprocal Bäcklund transformations of conservation laws as follows:

$$\begin{aligned}
\big(T^t\big)' &= \frac{1}{T} = \frac{1}{u},\\
\big(T^x\big)' &= -\frac{F}{T} = -\frac{bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}]}{u}.
\end{aligned} \tag{10}$$

Substituting (10) into (9), one has

$$\begin{aligned}
&\frac{bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}]}{u}\frac{-u_x}{u^2} + \frac{-u_t}{u^2}\\
&\quad + \frac{1}{u}\left(-\frac{bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}]}{u}\right)_x\\
&= \frac{-(bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}])u_x - uu_t}{u^3}\\
&\quad + \frac{-(bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}])_x u}{u^3}\\
&\quad + \frac{u_x(bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}])}{u^3}\\
&= \frac{-uu_t - (bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}])_x u}{u^3}\\
&= \frac{-u(u_t + (bu + a(u^{m+1}) + \omega[u(u^n)_{xx}] + \delta[u(u^L)_{4x}])_x)}{u^3}\\
&= \big(T^t\big)'_{t'} + \big(T^x\big)'_{x'} = 0.
\end{aligned}$$

For other cases, we can get similar results. For the sake of simplicity, we will not list them all.

6 Conclusions

In this paper, the integral of $Q(L, m, n)$ equation is studied by means of the solitary wave ansatz analysis. Firstly, using the Lie group analysis, the corresponding vector fields are derived for different values of exponents and coefficients. According to the solitary wave assumption method, new soliton solutions are obtained. Also, some conservation laws also presented. The obtained results are of great importance for the study of this equation. In this paper, we study only the case of constant coefficients of this equation, however, for many complex physical phenomena, the case of constant coefficients often does not satisfy our needs. For the case of its corresponding variable coefficients,

$$u_t + f_1(t)u_x + f_2(t)\big(u^{m+1}\big)_x + f_3(t)\big[u\big(u^n\big)_{xx}\big]_x + f_4(t)\big[u\big(u^L\big)_{4x}\big]_x = 0,$$

as well as the case of fractional order,

$$u_t^\alpha + bu_x + a\big(u^{m+1}\big)_x + \omega\big[u\big(u^n\big)_{xx}\big]_x + \delta\big[u\big(u^L\big)_{4x}\big]_x = 0,$$

etc. are worth studying.

In addition to the cases mentioned above, if we assume that $u = v_x$, one can derive the potential equation

$$v_{xt} + bv_{xx} + a\big((v_x)^{m+1}\big)_x + \omega\big[v_x\big((v_x)^n\big)_{xx}\big]_x + \delta\big[v_x\big(t(v_x)^L\big)_{4x}\big]_x = 0$$

and potential systems

$$v_x = u, \qquad -v_t = bu + a\big(u^{m+1}\big) + \omega\big[u\big(u^n\big)_{xx}\big] + \delta\big[u\big(u^L\big)_{4x}\big].$$

All these issues will be reported in future papers.

Author contributions. All authors (G.W., M.H., Q.Z., and A.H.K.) have contributed as follows: writing – original draft preparation, G.W. and M.H.; writing – review and editing, Q.Z. and A.H.K. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. We would like to express our sincere gratitude to the editor and anonymous reviewers for their valuable comments, which have improved this paper.

References

1. M.J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Transformation*, SIAM, Philadelphia, 1981, https://doi.org/10.1137/1.9781611970883.
2. A. Biswas, 1-soliton solution of $(1+2)$-dimensional nonlinear Schröinger's equation in dual-power law media, *Phys. Lett. A*, **372**(38):5941–5943, 2008, https://doi.org/10.1016/j.physleta.2008.07.052.
3. A. Biswas, 1-soliton solution of the $K(m,n)$ equation with generalized evolution, *Phys. Lett. A*, **372**(25):4601–4602, 2008, https://doi.org/10.1016/j.physleta.2008.05.002.
4. A. Biswas, 1-Soliton solution of the $B(m,n)$ equation with generalized evolution, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(8):3226–3229, 2009, https://doi.org/10.1016/j.cnsns.2008.12.025.
5. G.W. Bluman, A Cheviakov, S. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2010, https://doi.org/10.1007/978-0-387-68028-6.
6. S. Dimas, D. Tsoubelis, SYM: A new symmetry-finding package for Mathematica, in N.H. Ibragimov, C. Sophocleous, P.A. Damianou (Eds.), *Proceedings of the 10th International Conference on Modern Group Analysis, Larnaca, Cyprus, 2004*, Univ. Cyprus, Nicosia, 2005, pp. 64–70.

7. C.-C. Ding, L.-W. Zhu, H. Triki, Q. Zhou, Four-wave mixing induced general localized waves for a coupled generalized nonlinear Schrödinger system, *Physica D*, **464**:134191, 2024, `https://doi.org/10.1016/j.physd.2024.134191`.
8. X. Guan, W. Liu, Q. Zhou, A. Biswas, Some lump solutions for a generalized $(3+1)$-dimensional Kadomtsev–Petviashvili equation, *Appl. Math. Comput.*, **366**:124757, 2020, `https://doi.org/10.1016/j.amc.2019.124757`.
9. B. Guo, L. Ling, Q.P. Liu, High-order solutions and generalized Darboux Transformations of derivative nonlinear Schrödinger equations, *Stud. Appl. Math.*, **130**(4):317–344, 2013, `https://doi.org/10.1111/j.1467-9590.2012.00568.x`.
10. J.-H. He, X.-H. Wu, Exp-funtion method for nonlinear wave equations, *Chaos Solitons Fractals*, **30**(3):700–708, 2006, `https://doi.org/10.1016/j.chaos.2006.03.020`.
11. R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett.*, **27**:1192–1194, 1971, `https://doi.org/10.1103/PhysRevLett.27.1192`.
12. W. Hu, Z. Wang, Y. Zhao, Z. Deng, Symmetry breaking of infinite-dimensional dynamic system, *Appl. Math. Lett.*, **103**:106207, 2020, `https://doi.org/10.1016/j.aml.2019.106207`.
13. W. Hu, J. Ye, Z. Deng, Internal resonance of a flexible beam in a spatial tethered system, *J. Sound Vib.*, **475**:115286, 2020, `https://doi.org/10.1016/j.jsv.2020.115286`.
14. W. Hu, C. Zhang, Z. Deng, Vibration and elastic wave propagation in spatial flexible damping panel attached to four special springs, *Commun. Nonlinear Sci. Numer. Simul.*, **84**:105199, 2020, `https://doi.org/10.1016/j.cnsns.2020.105199`.
15. Z.-T. Ju, Z.-Z. Si, X. Yan, C.-Q. Dai, Solitons and their biperiodic pulsation in ultrafast fiber lasers based on CB/GO, *Chin. Phys. Lett.*, **41**(8):084203, 2024, `https://doi.org/10.1088/0256-307X/41/8/084203`.
16. J.G. Kingston, C. Rogers, Reciprocal Bäcklund transformations of conservation laws, *Phys. Lett. A*, **92**(6):261–264, 1982, `https://doi.org/10.1016/0375-9601(82)90081-0`.
17. P.J. Olver, *Application of Lie Group to Differential Equation*, Springer, New York, 1993, `https://doi.org/10.1007/978-1-4612-4350-2`.
18. W.-X. Qiu, K.-L. Geng, B.-W. Zhu, J.-T. Li, C.-Q. Dai, Data-driven forward-inverse problems of the 2-coupled mixed derivative nonlinear Schrödinger equation using deep learning, *Nonlinear Dyn.*, **112**:2024, 10215–10228, `https://doi.org/10.1007/s11071-024-09605-9`.
19. W.-X. Qiu, Z.-Z. Si, D.-S. Mou, C.-Q. Dai, J.-T. Li, W. Liu, Data-driven vector degenerate and nondegenerate solitons of coupled nonlocal nonlinear Schrödinger equation via improved PINN algorithm, *Nonlinear Dyn.*, **113**:4063–4076, 2025, `https://doi.org/10.1007/s11071-024-09648-y`.
20. C. Rogers, W. Shadwiek, *Bäcklund Transformations and Their Application*, Academic Press, New York, 1982.
21. P. Rosenau, D. Levy, Compactons in a class of nonlinearly quintic equations, *Phys. Lett. A*, **252**(6):297–306, 1999, `https://doi.org/10.1016/S0375-9601(99)00012-2`.

22. A.R. Seadawy, Stability analysis solutions for nonlinear three-dimensional modified Korteweg–de Vries–Zakharov–Kuznetsov equation in a magnetized electron–positron plasma, *Physica A*, **455**:44–51, 2016, https://doi.org/10.1016/j.physa.2016.02.061.

23. A.R. Seadawy, K.K. Ali, R.I. Nuruddeen, A variety of soliton solutions for the fractional Wazwaz-Benjamin-Bona-Mahony equations, *Results Phys.*, **12**:2234–2241, 2019, https://doi.org/10.1016/j.rinp.2019.02.064.

24. A.R. Seadawy, B.A. Alsaedi, Dynamical stricture of optical soliton solutions and variational principle of nonlinear Schrodinger equation with Kerr law nonlinearity, *Mod. Phys. Lett. B*, **38**(28):2450254, 2024, https://doi.org/10.1142/S0217984924502543.

25. A.R. Seadawy, B.A. Alsaedi, Variational principle and optical soliton solutions for some types of nonlinear Schrödinger dynamical systems, *Int. J. Geom. Methods Mod. Phys.*, **21**(06):2430004, 2024, https://doi.org/10.1142/S0219887824300046.

26. A.R. Seadawy, B.A. Alsaedi, Variational principle for generalized unstable and modify unstable nonlinear Schrödinger dynamical equations and their optical soliton solutions, *Opt. Quant. Electron.*, **56**:844, 2024, https://doi.org/10.1007/s11082-024-06417-4.

27. Z.-Z. Si, D.-L. Wang, B.-W. Zhu, Z.-T. Ju, X.-P. Wang, W. Liu, B.A. Malomed, Y.-Y. Wang, C.-Q. Dai, Deep learning for dynamic modeling and coded information storage of vector-soliton pulsations in mode-locked fiber lasers, *Laser Photonics Rev.*, **18**(12):2400097, 2024, https://doi.org/10.1002/lpor.202400097.

28. Z.-Z. Si, Y.-Y. Wang, C.-Q. Dai, Switching, explosion, and chaos of multi-wavelength soliton states in ultrafast fiber lasers, *Sci. China, Phys. Mech. Astron.*, **67**:274211, 2024, https://doi.org/10.1007/s11433-023-2365-7.

29. K.T. Vu, J. Butcher, J. Carminati, Similarity solutions of partial differential equations using DESOLV, *Comput. Phys. Comm.*, **176**(11–12):682–693, 2007, https://doi.org/10.1016/j.cpc.2007.03.005.

30. G. Wang, A new $(3+1)$-dimensional Schrödinger equation: Derivation, soliton solutions and conservation laws, *Nonlinear Dyn.*, **104**:1595–1602, 2021, https://doi.org/10.1007/s11071-021-06359-6.

31. G. Wang, A novel $(3+1)$-dimensional sine-Gorden and a sinh-Gorden equation: Derivation, symmetries and conservation laws, *Appl. Math. Lett.*, **113**:106768, 2021, https://doi.org/10.1016/j.aml.2020.106768.

32. G. Wang, Symmetry analysis, analytical solutions and conservation laws of a generalized KdV–Burgers–Kuramoto equation and its fractional version, *Fractals*, **29**(04):2150101, 2021, https://doi.org/10.1142/S0218348X21501012.

33. G. Wang, A.-M. Wazwaz, A new (3+1)-dimensional KdV equation and mKdV equation with their corresponding fractional forms, *Fractals*, **30**(4):2250081, 2022, https://doi.org/10.1142/S0218348X22500815.

34. G. Wang, A.-M. Wazwaz, On the modified Gardner type equation and its time fractional form, *Chaos Solitons Fractals*, **123**:127768, 2023, https://doi.org/10.1016/j.chaos.2021.111694.

35. G. Wang, K. Yang, H. Gu, F. Guan, A.H. Kara, A $(2+1)$-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions, *Nucl. Phys. B*, **953**:114956, 2020, `https://doi.org/10.1016/j.nuclphysb.2020.114956`.

36. S.-Y. Xu, Q. Zhou, W. Liu, Prediction of soliton evolution and equation parameters for NLS–MB equation based on the phPINN algorithm, *Nonlinear Dyn.*, **111**:18401–18417, 2023, `https://doi.org/10.1007/s11071-023-08824-w`.

37. Z. Zhao, L. He, Lie symmetry, nonlocal symmetry analysis, and interaction of solutions of a $(2+1)$-dimensional KdV–mKdV equation, *Theor. Math. Phys.*, **206**:142–162, 2021, `https://doi.org/10.1134/S0040577921020033`.