Nonlinear Analysis: Modelling and Control, Online First, 1–27
https://doi.org/10.15388/namc.2025.30.40067

Existence of solution for a fractional differential system on the chemical graph of glycerol

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Received: October 23, 2024 / **Revised:** February 7, 2025 / **Published online:** April 1, 2025

Abstract. In this paper, we study the chemical graph for an important polyalcoholic compound with the molecular formula $C_3H_8O_3$ by using 0 or 1 to label the elements of its molecular structure graph and formulating the corresponding fractional boundary value problem on each edge of the graph. Under the sense of Caputo's fractional derivatives, the existence of solutions of the fractional boundary value problem on the glycerol graph is investigated by introducing some suitable growth conditions and combing with some fixed point theorems. A specific example is given to verify our results.

Keywords: fractional boundary problem, glycerol graph, fixed point theorem, Caputo fractional derivative.

1 Introduction

Glycerol is an important polyalcoholic compound with the molecular formula $C_3H_8O_3$, which has extensive application in medical, pharmaceutical, and personal care preparations for improving smoothness and providing lubrication or humectant. In addition, it has been shown that adding glycerol to the probiotic Lactobacillus reuteri can increase its production of antimicrobial substances in the human gut [23]. Glycerol has also been incorporated as a component of bioink formulations in the field of bioprinting,

[1,2] The author is supported financially by the Natural Science Foundation of Shandong Province of China (ZR2022AM015) and the ARC Discovery Project grant DP230102079.
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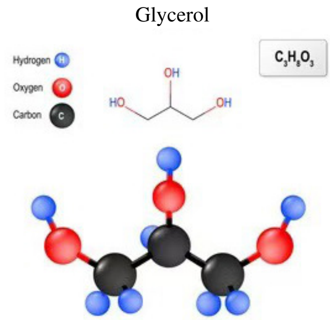
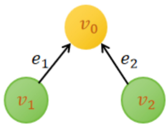


Figure 1. Spatial molecular structure analysis of glycerol.

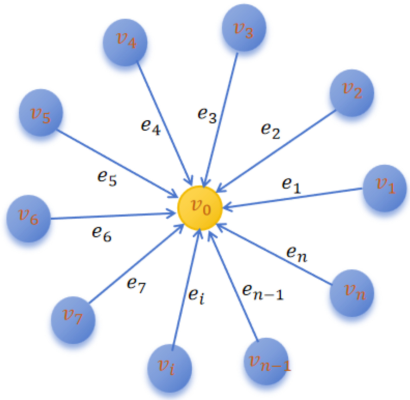
which can add viscosity to the bioink without adding large protein, saccharide, or glycoprotein molecules [3]. From spatial molecular structure analysis of glycerol, it has three hydroxy groups (see Fig. 1) such that glycerol is miscible with water and is hygroscopic in nature [7]. The water solubility makes triglycerides have many uses in life. Chemical graph theory [4, 21] is the best tool to study compound morphology, which can represent any actual or abstract chemical system and becomes an important research area to achieve the consequences of connectivity in chemical networks. The Nobel prize winner Prelog believed that fewer concepts in the natural sciences are more closely related to the notion of graphs than the molecular structural formulae of compounds [19]. Lumer [15] first applied the theory of differential equations to graphs and employed specific local operators to study extended evolution equations on branching spaces. In recent years, some progress has been made in the study of fractional boundary value problems on graphs. For example, in [10], by using some techniques from fixed point theory, Graef et al. established the existence of solutions for a class of fractional boundary value problems on star graphs (see Fig. 2(a))

$$\begin{aligned} &-\mathcal{D}_0^{\alpha} v_i(t) = g_i(t) h_i\big(t, v_i(t)\big), \quad t \in (0, r_i),\ i = 1, 2, \\ &v_1(0) = v_2(0) = 0, \qquad v_1(r_1) = v_2(r_2), \\ &\mathcal{D}_0^{\beta} v_1(r_1) + \mathcal{D}_0^{\beta} v_2(r_2) = 0, \end{aligned} \tag{1}$$

where $\alpha \in (1, 2)$, $\beta \in (0, \alpha)$, $g_i : [0, r_i] \to \mathbb{R}$ is a continuous function on $[0, r_i]$, $g_i(t) \neq 0$, and $h_i : [0, r_i] \times \mathbb{R} \to \mathbb{R}$ is continuous. Define the notion of three-point-star graphs (see Fig. 2(a)), namely, define $V = \{v_0, v_1, v_2\}$ and $E = \{e_1, e_2\}$ as the node set, and define the edge set such that v_0 is a junction node and $\overrightarrow{e_i} = \overrightarrow{v_i v_0}$ is a vertex connecting the nodes v_i to v_0 with an edge of length r_i, $i = 1, 2$. Let $G = V \cup E$ and establish a local coordinate system in $t \in (0, r_i)$ on each edge with vertices e_i and v_0 as the origin, then Graef constructed a nonlinear fractional differential equation (1) and further considered the existence of solutions for Eq. (1) using Banach's contraction principle and Schauder's fixed point theorem. Based on Graef's work, Mehandiratta et al. [16] generalized the three-vertex-star graph to star graphs with n edges (Fig. 2(b)) for a study of the fractional boundary value problem under the Caputo fractional-order derivatives



(a) 2 edges



(b) n edges

Figure 2. A sketch of the star graph G.

sense. A transformation of the translate problems on $(0, r_i)$ to intervals $[0, 1]$ was adopted, and then the study was carried out on the interval $[0, 1]$. Some other works were carried out on networks. For example, Pokornyi [18] studied the second-order scalar ordinary differential equations on a spatial network using geometric graph and the distribution of zeros of solutions of differential inequalities, and an analogue of the Sturm–Liouville oscillation spectral theory was established. The authors in [15, 18] considered differential equations on graphs and used computational and numerical methods to obtain solutions of these equations.

Recently, some researchers investigated the fractional boundary value problems on the molecular graphs of chemical organic matter by utilizing either 0 or 1 to label the elements of a molecule such as ethane graph [1, 9] and cyclohexane graph [2]. This is due to the fact that many new concepts of fractional derivatives and integral operators have been proposed to model natural phenomena, where the existing fractional integral or derivative operators are not sufficient, which leads to that many advanced fractional modelling and analysis techniques have been discussed in the literature, such as in the study of dynamic system model for bioprocess [5, 6, 32], eco-economical processes [20], fractional Kelvin–Voigt model [12], fractional Fourier transforms [8], fractional Brownian motion [14], fractional optimal control problems [26], mathematic properties for fractional problems [11, 25], and so on. Many scholars have continuously promoted the development of nonlinear science by constructing new theoretical frameworks and proposing new research methods such as iterative techniques [28–31], upper-lower solution methods [27, 33, 34], and critical point theory [24] to achieve a series of important results.

Inspired by the above work and a wide range of applications of glycerol in real life [13], in this paper, we are interested in chemical diagrams of glycerol based on chemical graph theory. By observing the spatial molecular structure of glycerol in Fig. 1, we find that the molecular structure of glycerol is a chain structure consisting of three carbon atoms and eight hydrogen atoms, three oxygen atoms and each carbon atom is attached

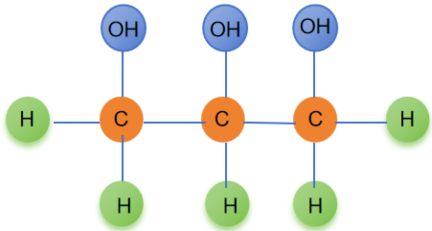


Figure 3. A sketch of the graph representation of glycerol.

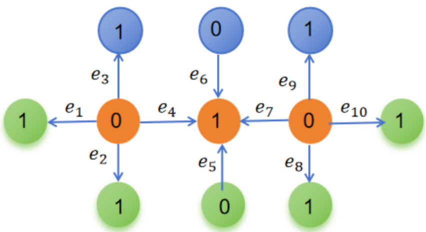


Figure 4. A sketch of the graph representation of glycerol with labeled vertices by 0 or 1.

to a hydroxy group. For the convenience of labelling, we use the carbon atoms, the hydrogen atoms, and the hydroxyl group as the vertices of the graph, while the chemical bonds existing between the atoms are used as the edges of the graph, and we transform spatial molecular structure shown in Fig. 1 into a ichnography as shown in Fig. 3.

Now we label the vertices of the glycerol graph with either 0 or 1 and consider the length of each edge as a unit length (see Fig. 4). In this case, we construct a local coordinate system on the glycerol graph, and therefore treat each edge of this graph as an interval of unit length. To achieve this goal, we assign two labels 0 or 1 to each vertex of the graph. When we move along any edge, the start vertex is 0, and the end vertex is 1. Each vertex is only used as either the start or the end point, and according to this rule, we likewise do not need to normalize the length of each edge through the use of a specific transformation. The labeled graph is shown in Fig. 4. Labeling of the glycerol graph using the above labeling method is followed by testing the existence of solutions for the following fractional boundary value problem on the graph of glycerol:

$$\begin{aligned} &\mathcal{D}_0^\alpha u_i(t) = h_i\big(t, u_i(t), \mathcal{D}_0^\beta u_i(t), u_i'(t)\big), \\ &\lambda_1 \int_0^1 u_i(s)\,\mathrm{d}s + \lambda_2 \int_0^1 u_i'(s)\,\mathrm{d}s = \mathcal{D}_0^\beta u_i(1), \\ &\lambda_1 u_i(0) + \lambda_2 u_i'(1) = \int_0^1 \mathcal{D}_0^\beta u_i(s)\,\mathrm{d}s \end{aligned} \tag{2}$$

with nonzero real constants λ_j, $j = 1, 2$, where $\mathcal{D}_0^\alpha$ and $\mathcal{D}_0^\beta$ are the derivatives of α- and β-order Caputo fractional derivatives, respectively, $\alpha \in (1,2), \beta \in (0,1)$, $h_i: [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots, 10$, is a continuously differentiable function, i represents the number of edges of the glycerol graph, and each edge has a length of $|e_i| = 1$. In a chemical reaction, the change in the concentration of glycerol is often affected by a variety of factors, which can be described by the fractional differential equations (2).

In order to establish the existence of solutions for fractional boundary value problem (2) in this graph, we need to seek for the suitable conditions that can allow problem (2) to have the solution in this graph. To do this, some growth conditions will be introduced to establish the existence of solutions for the boundary value problem (2) in the chemical graph of glycerol shown in Fig. 4. Our results generalize the fractional boundary value problem to more general chemical graphs. Finally, an example is given to illustrate the significance of our results in this research area.

2 Preliminaries and lemmas

In this section, we first review the definitions and lemmas that will be used later in the paper.

Definition 1. (See [17].) Suppose that $h : [0,1] \to \mathbb{R}$ is a continuous function, then the α-order Riemann–Lioville fractional right integral is defined as

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s, \quad \text{where } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Definition 2. (See [17].) Let $h \in AC^n[0,1]$, the α-order Caputo fractional derivative of function h is defined as

$$\mathcal{D}_0^\alpha h(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} h^{(n)}(s)\,\mathrm{d}s, \quad n-1 < \alpha < n,\ n = [\alpha]+1,$$

where $[\alpha]$ is the integer part of α.

Lemma 1. (*See* [17].) *Let* $\varphi \in L^1(0,1)$ *and* $\alpha, \beta > 0$*, then*

(i) $\mathcal{D}_0^\alpha I^\alpha \varphi(t) = \varphi(t)$;
(ii) $I^\alpha I^\beta \varphi(t) = I^{\alpha+\beta}\varphi(t)$;
(iii) $\mathcal{I}^\alpha \mathcal{D}_0^\alpha \varphi(t) = \varphi(t) + b_0 + b_1 t + b_2 t^2 + \cdots + b_{n-1}t^{n-1}$, *where* $n = [\alpha]+1$.

Remark 1. It follows from Lemma 1 that for $\alpha > 0$, the general solution of the fractional differential equation $\mathcal{D}_0^\alpha \varphi(t) = 0$ is given by

$$\varphi(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_{n-1}t^{n-1},$$

where $b_k \in \mathbb{R}$, $k = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

Lemma 2. *Let* $\alpha \in (1,2)$ *and the real-valued functions* φ_i*,* $i = 1,2,\dots,10$*, be continuous on* $C[0,1]$*, then* u_i^* *is a solution of the boundary value problem*

$$\begin{aligned}
&\mathcal{D}_0^\alpha u_i(t) = \varphi_i(t), \quad t \in [0,1],\\
&\lambda_1 \int_0^1 u_i(s)\,\mathrm{d}s + \lambda_2 \int_0^1 u_i'(s)\,\mathrm{d}s = \mathcal{D}_0^\beta u_i(1),\\
&\lambda_1 u_i(0) + \lambda_2 u_i'(1) = \int_0^1 \mathcal{D}_0^\beta u_i(s)\,\mathrm{d}s, \quad i = 1,2,\dots,10,
\end{aligned} \tag{3}$$

if and only if it is a solution of the following fractional integral equation:

$$
\begin{aligned}
u_i^*(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_i(s)\,\mathrm{d}s + \frac{A_0+A_1-\lambda_1 t}{A_0\lambda_1}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \frac{\lambda_1 t-A_1}{A_0\lambda_1}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(s)\,\mathrm{d}s + \frac{A_1-\lambda_1 t}{A_0}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \frac{A_1\lambda_2-\lambda_1\lambda_2 t}{A_0\lambda_1}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \frac{\lambda_1\lambda_2 t-A_1\lambda_2-A_0\lambda_2}{A_0\lambda_1}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(s)\,\mathrm{d}s,
\end{aligned}
\tag{4}
$$

where

$$
A_0 = \frac{\lambda_1\Gamma(3-\beta)+2\beta-2}{2\Gamma(3-\beta)}, \qquad A_1 = \frac{\lambda_2\Gamma(3-\beta)-1}{\Gamma(3-\beta)}.
$$

Proof. Let $u_i^*(t)$ be a solution of problem (3). It follows from Lemma 1 and $\alpha \in (1,2)$ that there are constants $b_0^{(i)}, b_1^{(i)} \in \mathbb{R}$ such that

$$
u_i^*(t) = I_0^\alpha\bigl(\mathcal{D}_0^\alpha\varphi_i(t)\bigr) + b_0^{(i)} + b_1^{(i)}t,
$$

i.e.,

$$
u_i^*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_i(s)\,\mathrm{d}s + b_0^{(i)} + b_1^{(i)}t. \tag{5}
$$

Consequently,

$$
\begin{aligned}
u_i^{*\prime}(t) &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(s)\,\mathrm{d}s + b_1^{(i)}, \\
\mathcal{D}_0^\beta u_i^*(t) &= \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(s)\,\mathrm{d}s + b_1^{(i)}\frac{t^{1-\beta}}{\Gamma(2-\beta)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 u_i^*(s)\,\mathrm{d}s &= \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s + b_0^{(i)} + \frac{1}{2}b_1^{(i)}, \\
\int_0^1 u_i^{*\prime}(s)\,\mathrm{d}s &= \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s + b_1^{(i)},
\end{aligned}
$$

$$\int_0^1 \mathcal{D}_0^\beta u_i^*(s)\,\mathrm{d}s = \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s + b_1^{(i)}\frac{1}{\Gamma(3-\beta)}.$$

By using the boundary conditions and the equations above, we obtain

$$\begin{aligned}
b_0^{(i)} &= \left(\frac{1}{\lambda_1}+\frac{A_1}{A_0\lambda_1}\right)\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad - \left(\frac{A_1\lambda_2}{A_0\lambda_1}+\frac{\lambda_2}{\lambda_1}\right)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(s)\,\mathrm{d}s - \frac{A_1}{A_0\lambda_1}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(s)\,\mathrm{d}s \\
&\quad + \frac{A_1}{A_0}\left(\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s + \frac{\lambda_2}{\lambda_1}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\right), \\
b_1^{(i)} &= \frac{1}{A_0}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(s)\,\mathrm{d}s - \frac{\lambda_1}{A_0}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad - \frac{\lambda_2}{A_0}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s - \frac{1}{A_0}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\varphi_i(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \frac{\lambda_2}{A_0}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_i(s)\,\mathrm{d}s.
\end{aligned}$$

Now, by substituting the values $b_0^{(i)}$, $b_1^{(i)}$ into Eq. (5), one derive that u_i^* is a solution for integral equation (4). Conversely, if u_i^* is a solution of integral equation (4), by using some direct calculations and the same method, it is easy to prove that u_i^* is also a solution for the fractional problem (3). The proof is completed. □

Our main tools are the following fixed point theorems.

Lemma 3. (*See* [22].) *Let X be a Banach space and $F : X \to X$ be a completely continuous operator. Then the set $\{x \in X\colon x = \mu F x, \mu \in (0,1)\}$ is unbounded, or the operator F has at least one fixed point in X.*

Lemma 4. (*See* [22].) *Let Λ be a bounded nonempty closed convex subset of a Banach space X. Suppose γ_1 is compact and continuous and γ_2 is a contraction mapping such that $\gamma_1 u + \gamma_2 v \in \Lambda$ for $u, v \in \Lambda$. Then there exists $\omega \in \Lambda$ such that $\omega = \gamma_1\omega + \gamma_2\omega$.*

3 Main results

Before the claim of the existence results of the fractional boundary value problem (2) on the glycerol graph Fig. 4, we firstly define our work space and give some constants for the convenience in presenting our results.

Let $M_i = \{u_i\colon u_i, \mathcal{D}_0^\beta u_i, u_i' \in C[0,1]\}$, which are Banach spaces with the norm

$$\|u_i\|_{M_i} = \sup_{t\in[0,1]} \bigl|u_i(t)\bigr| + \sup_{t\in[0,1]} \bigl|\mathcal{D}_0^\beta u_i(t)\bigr| + \sup_{t\in[0,1]} \bigl|u_i'(t)\bigr|,$$

where $i = 1, 2, \dots, 10$. It is clear that the product space $M = M_1 \times M_2 \times \cdots \times M_{10}$ is also a Banach space with the norm

$$\|u\|_M = \bigl\|(u_1, u_2, \dots, u_{10})\bigr\|_M = \sum_{i=1}^{10} \|u_i\|_{M_i}.$$

Next, from Lemma 2 we introduce an operator $T : M \to M$ defined by

$$\begin{aligned} &T(u_1, u_2, \dots, u_{10})(t)\\ &\quad = \bigl(T_1(u_1, u_2, \dots, u_{10})(t), T_2(u_1, u_2, \dots, u_{10})(t), \dots, T_{10}(u_1, u_2, \dots, u_{10})(t)\bigr), \end{aligned}$$

where

$$\begin{aligned} &T_i(u_1, u_2, \dots, u_{10})(t)\\ &\quad = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_i(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s))\,\mathrm{d}s\\ &\qquad + \frac{A_0 + A_1 - \lambda_1 t}{A_0\lambda_1} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h_i\bigl(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\bigr)\,\mathrm{d}\xi\,\mathrm{d}s\\ &\qquad + \frac{\lambda_1 t - A_1}{A_0\lambda_1} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h_i\bigl(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\bigr)\,\mathrm{d}s\\ &\qquad + \frac{A_1 - \lambda_1 t}{A_0} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} h_i\bigl(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\bigr)\,\mathrm{d}\xi\,\mathrm{d}s\\ &\qquad + \frac{A_1\lambda_2 - \lambda_1\lambda_2 t}{A_0\lambda_1} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} h_i\bigl(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\bigr)\,\mathrm{d}\xi\,\mathrm{d}s\\ &\qquad + \frac{\lambda_1\lambda_2 t - A_1\lambda_2 - A_0\lambda_2}{A_0\lambda_1} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h_i\bigl(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\bigr)\,\mathrm{d}s \end{aligned}$$

for all $t \in [0,1]$, $u_i \in M_i$, $i = 1, 2, \dots, 10$.

For computational convenience, we define the following notations:

$$\begin{aligned} F_0^* = {}& \frac{1}{\Gamma(\alpha+1)} + \frac{|A_0| + |A_1| + |\lambda_1|}{|A_0|\Gamma(\alpha-\beta+2)} + \frac{|A_1| + |\lambda_1|}{|A_0||\lambda_1|\Gamma(\alpha-\beta+1)} + \frac{|A_1| + |\lambda_1|}{|A_0|\Gamma(\alpha+2)}\\ &+ \frac{|A_1||\lambda_2| + |\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha+1)} + \frac{|A_1||\lambda_2| + |A_0||\lambda_2| + |\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha)}, \end{aligned} \tag{6}$$

$$F_1^* = \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{|A_0|\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1|}{\Gamma(\alpha+2)} + \frac{|\lambda_2|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+2)} + \frac{|\lambda_2|}{\Gamma(\alpha)}\right), \tag{7}$$

$$F_2^* = \frac{1}{\Gamma(\alpha)} + \frac{1}{|A_0|}\left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1|}{\Gamma(\alpha+2)} + \frac{|\lambda_2|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+2)} + \frac{|\lambda_2|}{\Gamma(\alpha)}\right). \tag{8}$$

Theorem 1. *Assume that $h_1, h_2, \dots, h_{10} : [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist constants $L_i > 0$, $i = 1,2,\dots,10$, such that $|h_i(t,x,y,z)| \leqslant L_i$, where $(t,x,y,z)\in[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $i=1,2,\dots,10$. Then the fractional boundary value problem* (2) *has a solution on each edge of the graph of glycerol shown in Fig.* 4.

Proof. In order to derive the solution of problem (2) on the graph of glycerol shown in Fig. 4, we shall seek for the fixed point of T in M. To do this, we first show that T is a completely continuous operator. Since $h_1, h_2, \dots, h_{10}$ are continuous in $[0,1]\times\mathbb{R}^3$, the operator T is also continuous. Let $\Omega \in M$ be a bounded set, for any $u = (u_1, u_2, \dots, u_{10}) \in \Omega$, one has

$$\begin{aligned}
&\bigl|(T_i u)(t)\bigr| \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s \\
&\qquad + \frac{|A_0|+|A_1|+|\lambda_1|t}{|A_0||\lambda_1|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\bigl|h_i\bigl(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\bigr)\bigr|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1|t+|A_1|}{|A_0||\lambda_1|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s \\
&\qquad + \frac{|A_1|+|\lambda_1|t}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\bigl|h_i\bigl(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\bigr)\bigr|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|t}{|A_0||\lambda_1|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|h_i\bigl(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\bigr)\bigr|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1||\lambda_2|t+|A_1||\lambda_2|+|A_0||\lambda_2|}{|A_0||\lambda_1|}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad\leqslant L_i\biggl(\frac{1}{\Gamma(\alpha+1)} + \frac{|A_0|+|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha-\beta+2)} + \frac{|A_1|+|\lambda_1|}{|A_0||\lambda_1|\Gamma(\alpha-\beta+1)} + \frac{|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha+2)} \\
&\qquad + \frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha+1)} + \frac{|A_1||\lambda_2|+|A_0||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha)}\biggr) \\
&\quad= L_i F_0^*,
\end{aligned}$$

$$
\begin{aligned}
&\left|\left(\mathcal{D}_0^\beta T_i u\right)(t)\right| \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|h_i\left(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\right)\right|\mathrm{d}s \\
&\qquad+\frac{t^{1-\beta}}{|A_0|\Gamma(2-\beta)}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|h_i\left(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\right)\right|\mathrm{d}s \\
&\qquad+\frac{|\lambda_1|t^{1-\beta}}{|A_0|\Gamma(2-\beta)}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\left|h_i\left(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\right)\right|\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+\frac{|\lambda_2|t^{1-\beta}}{|A_0|\Gamma(2-\beta)}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\left|h_i\left(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\right)\right|\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+\frac{t^{1-\beta}}{|A_0|\Gamma(2-\beta)}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|h_i\left(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\right)\right|\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+\frac{|\lambda_2|t^{1-\beta}}{|A_0|\Gamma(2-\beta)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|h_i\left(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\right)\right|\mathrm{d}s \\
&\quad\leqslant L_i\left(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{|A_0|\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|}{\Gamma(\alpha+2)}\right.\right. \\
&\qquad\left.\left.+\frac{|\lambda_2|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+2)}+\frac{|\lambda_2|}{\Gamma(\alpha)}\right)\right) \\
&\quad= L_i F_1^*,
\end{aligned}
$$

$$
\begin{aligned}
\left|(T_i'u)(t)\right| &\leqslant \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|h_i\left(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\right)\right|\mathrm{d}s \\
&\quad+\frac{1}{|A_0|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|h_i\left(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\right)\right|\mathrm{d}s \\
&\quad+\frac{|\lambda_1|}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\left|h_i\left(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\right)\right|\mathrm{d}\xi\,\mathrm{d}s \\
&\quad+\frac{|\lambda_2|}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\left|h_i\left(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\right)\right|\mathrm{d}\xi\,\mathrm{d}s \\
&\quad+\frac{1}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|h_i\left(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\right)\right|\mathrm{d}\xi\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_2|}{|A_0|} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \big| h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\leqslant L_i \biggl(\frac{1}{\Gamma(\alpha)} + \frac{1}{|A_0|} \biggl(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1|}{\Gamma(\alpha+2)} \\
&\quad + \frac{|\lambda_2|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+2)} + \frac{|\lambda_2|}{\Gamma(\alpha)} \biggr) \biggr) \\
&= L_i F_2^*
\end{aligned}
$$

for all $t \in [0,1]$, $i = 1, 2, \ldots, 10$, where F_0^*, F_1^*, and F_2^* are defined in (6), (7), and (8), respectively. It follows from the above calculation that

$$
\|T_i u\|_{M_i} \leqslant L_i\bigl(F_0^* + F_1^* + F_2^*\bigr),
$$

which implies that

$$
\|Tu\|_M = \sum_{i=1}^{10} \|T_i u\|_{M_i} \leqslant \sum_{i=1}^{10} L_i\bigl(F_0^* + F_1^* + F_2^*\bigr) < \infty,
$$

that is, T is uniformly bounded.

Next, we prove that T is equicontinuous. Let $u = (u_1, u_2, \ldots, u_{10}) \in \Omega$, then for any $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, we have

$$
\begin{aligned}
&\big|(T_i u)(t_2) - (T_i u)(t_1)\big| \\
&\quad \leqslant \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \big| h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \big| h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\qquad + \frac{(t_2-t_1)}{|A_0|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big| h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big| \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{(t_2-t_1)}{|A_0|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big| h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1|(t_2-t_1)}{|A_0|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} \big| h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big| \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|\lambda_2|(t_2-t_1)}{|A_0|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} \big| h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big| \,\mathrm{d}\xi \,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\lambda_2|(t_2-t_1)}{|A_0|}\int\limits_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s\\
&\to 0,\quad t_1\to t_2.
\end{aligned}
$$

By the same method, we also have

$$
\lim_{t_1\to t_2}\bigl|\bigl(\mathcal{D}_0^\beta T_iu\bigr)(t_2)-\bigl(\mathcal{D}_0^\beta T_iu\bigr)(t_1)\bigr|=0\quad\text{and}\quad\lim_{t_1\to t_2}\bigl|(T_i'u)(t_2)-(T_i'u)(t_1)\bigr|=0,
$$

which yield that

$$
\bigl|(Tu)(t_2)-(Tu)(t_1)\bigr|\to 0,\quad t_1\to t_2.
$$

This also proves that T is equicontinuous on $M=M_1\times M_2\times\cdots\times M_{10}$. It follows from the Arzelà–Ascoli theorem that the operator T is completely continuous.

Now we define a subset N of M as follows:

$$
N=:\bigl\{(u_1,u_2,\dots,u_{10})\in M\colon\ (u_1,u_2,\dots,u_{10})=aT(u_1,u_2,\dots,u_{10}),\ a\in(0,1)\bigr\}.
$$

We assert that N is bounded for any $(u_1,u_2,\dots,u_{10})\in M$. In fact, since

$$
(u_1,u_2,\dots,u_{10})=aT(u_1,u_2,\dots,u_{10}),
$$

we have

$$
u_i=aT_i(u_1,u_2,\dots,u_{10})
$$

for all $t\in[0,1]$, $i=1,2,\dots,10$. Thus

$$
\begin{aligned}
&\bigl|u_i(t)\bigr|\\
&\quad=a\bigl|(T_iu)(t)\bigr|\leqslant a\Biggl[\int\limits_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s\\
&\qquad+\frac{|A_0|+|A_1|+|\lambda_1|t}{|A_0||\lambda_1|}\int\limits_0^1\int\limits_0^s\frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\bigl|h_i\bigl(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\bigr)\bigr|\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+\frac{|\lambda_1|t+|A_1|}{|A_0||\lambda_1|}\int\limits_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s\\
&\qquad+\frac{|A_1|+|\lambda_1|t}{|A_0|}\int\limits_0^1\int\limits_0^s\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\bigl|h_i\bigl(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\bigr)\bigr|\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+\frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|t}{|A_0||\lambda_1|}\int\limits_0^1\int\limits_0^s\frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|h_i\bigl(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\bigr)\bigr|\,\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+\frac{|\lambda_1||\lambda_2|t+|A_1||\lambda_2|+|A_0||\lambda_2|}{|A_0||\lambda_1|}\int\limits_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|h_i\bigl(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\bigr)\bigr|\,\mathrm{d}s\Biggr]
\end{aligned}
$$

$$\begin{aligned}
&\leqslant aL_i\biggl(\frac{1}{\Gamma(\alpha+1)}+\frac{|A_0|+|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha-\beta+2)}+\frac{|A_1|+|\lambda_1|}{|A_0||\lambda_1|\Gamma(\alpha-\beta+1)}+\frac{|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha+2)}\\
&\quad+\frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha+1)}+\frac{|A_1||\lambda_2|+|A_0||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha)}\biggr)\\
&= aL_iF_0^*.
\end{aligned}$$

By the same strategy, one gets

$$\bigl|\mathcal{D}_0^\beta u_i(t)\bigr|\leqslant aL_iF_1^*,\qquad \bigl|u_i'(t)\bigr|\leqslant aL_iF_2^*.$$

Therefore,

$$\|u\|_M=\sum_{i=1}^{10}\|u_i\|_{M_i}\leqslant a\sum_{i=1}^{10}L_i\bigl(F_0^*+F_1^*+F_2^*\bigr)<\infty.$$

Thus N is a bounded set, it then follows from Lemma 3 that T has a fixed point in M, which guarantees that the fractional boundary value problem (2) has a solution on the graph of glycerol Fig. 4. □

Now we use the Krasnoselskii fixed point theorem to study the existence result for the fractional boundary value problem (2) on the graph of glycerol Fig. 4. Again, for convenience, the following notations E_0^*, E_1^*, E_2^* are defined:

$$\begin{aligned}
E_0^*&=\frac{|A_0|+|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha-\beta+2)}+\frac{|A_1|+|\lambda_1|}{|A_0||\lambda_1|\Gamma(\alpha-\beta+1)}+\frac{|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha+2)}\\
&\quad+\frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha+1)}+\frac{|A_1||\lambda_2|+|A_0||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha)}, &(9)\\
E_1^*&=\frac{1}{|A_0|\Gamma(2-\beta)}\biggl(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|}{\Gamma(\alpha+2)}+\frac{|\lambda_2|}{\Gamma(\alpha+1)}\\
&\quad+\frac{1}{\Gamma(\alpha-\beta+2)}+\frac{|\lambda_2|}{\Gamma(\alpha)}\biggr), &(10)\\
E_2^*&=\frac{1}{|A_0|}\biggl(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|}{\Gamma(\alpha+2)}+\frac{|\lambda_2|}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+2)}+\frac{|\lambda_2|}{\Gamma(\alpha)}\biggr). &(11)
\end{aligned}$$

Theorem 2. *Suppose that $h_1,h_2,\dots,h_{10}:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist continuous functions $S_1,S_2,\dots,S_{10}:[0,1]\to\mathbb{R}$ such that*

$$\begin{aligned}
&\bigl|h_i(t,u_1,u_2,u_3)-h_i(t,\overline{u}_1,\overline{u}_2,\overline{u}_3)\bigr|\\
&\quad\leqslant S_i(t)\bigl(|u_1-\overline{u}_1|+|u_2-\overline{u}_2|+|u_3-\overline{u}_3|\bigr),\quad i=1,2,3,\dots,10. \qquad (12)
\end{aligned}$$

In addition, assume that there exist continuous maps $W_1,W_2,\dots,W_{10}:[0,1]\to\mathbb{R}$ and continuous nondecreasing functions $V_1,V_2,\dots,V_{10}:[0,+\infty)\to[0,\infty)$ such that

$$\bigl|h_i(t,u_1,u_2,u_3)\bigr|\leqslant W_i(t)V_i\bigl(|u_1|+|u_2|+|u_3|\bigr),\quad i=1,2,3,\dots,10. \qquad (13)$$

Then the fractional boundary value problem (2) *has a solution on each edge of the graph of glycerol shown in Fig.* 4*, provided that*

$$F := \left(E_0^* + E_1^* + E_2^*\right)\sum_{i=1}^{10}\|S_i\| \leqslant 1, \quad \textit{where } \|S_i\| = \sup_{t\in[0,1]}|S_i|. \tag{14}$$

Proof. Denote $\|W_i\| = \sup_{t\in[0,1]}|W_i|$ and take an appropriate real constant such that

$$\sum_{i=1}^{10} V_i\big(\|u_i\|_{M_i}\big)\|W_i\|\{F_0^* + F_1^* + F_2^*\} \leqslant \rho.$$

Define a set

$$N_\rho := \big\{u = (u_1, u_2, \dots, u_{10}) \in M\colon \|u\|_M \leqslant \rho\big\}.$$

It is clear that N_ρ is a nonempty bounded convex subset of M. Now, for all $t \in [0,1]$, define two operators T_1, T_2 on N_ρ:

$$T_1(u_1, u_2, \dots, u_{10})(t) := \big(T_1^{(1)}(u_1, u_2, \dots, u_{10})(t), \dots, T_1^{(10)}(u_1, u_2, \dots, u_{10})(t)\big),$$
$$T_2(u_1, u_2, \dots, u_{10})(t) := \big(T_2^{(1)}(u_1, u_2, \dots, u_{10})(t), \dots, T_2^{(10)}(u_1, u_2, \dots, u_{10})(t)\big),$$

where

$$\big(T_1^{(i)}u\big)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\,\mathrm{d}s,$$

$$\begin{aligned}
\big(T_2^{(i)}u\big)(t) = {}& \frac{A_0 + A_1 - \lambda_1 t}{A_0\lambda_1}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&+ \frac{\lambda_1 t - A_1}{A_0\lambda_1}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\,\mathrm{d}s \\
&+ \frac{A_1 - \lambda_1 t}{A_0}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&+ \frac{A_1\lambda_2 - \lambda_1\lambda_2 t}{A_0\lambda_1}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&+ \frac{\lambda_1\lambda_2 t - A_1\lambda_2 - A_0\lambda_2}{A_0\lambda_1}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Let

$$\overline{V}_i = \sup_{u_i\in M_i} V_i\big(\|u_i\|_{M_i}\big),$$

then for any $\overline{u} = (\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{10}), u = (u_1, u_2, \ldots, u_{10}) \in N_\rho$, we have

$$
\begin{aligned}
&\big|(T_1^{(i)}u)(t) + (T_2^{(i)}u)(t)\big| \\
&\quad \leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\qquad + \frac{|A_0| + |A_1| + |\lambda_1|t}{|A_0||\lambda_1|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big|h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big| \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1|t + |A_1|}{|A_0||\lambda_1|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big|h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\qquad + \frac{|A_1| + |\lambda_1|t}{|A_0|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} \big|h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big| \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|A_1||\lambda_2| + |\lambda_1||\lambda_2|t}{|A_0||\lambda_1|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} \big|h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big| \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1||\lambda_2|t + |A_1||\lambda_2| + |A_0||\lambda_2|}{|A_0||\lambda_1|} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \big|h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big| \,\mathrm{d}s \\
&\quad \leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} W_i V_i\big(\big|\overline{u}_i(s)\big| + \big|\mathcal{D}_0^\beta \overline{u}(s)\big| + \big|\overline{u}_i'(s)\big|\big) \,\mathrm{d}s + \frac{|A_0| + |A_1| + |\lambda_1|t}{|A_0||\lambda_1|} \\
&\qquad \times \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u(\xi)\big| + \big|u_i'(\xi)\big|\big) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1|t + |A_1|}{|A_0||\lambda_1|} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} W_i V_i\big(\big|u_i(s)\big| + \big|\mathcal{D}_0^\beta u(s)\big| + \big|u_i'(s)\big|\big) \,\mathrm{d}s \\
&\qquad + \frac{|A_1| + |\lambda_1|t}{|A_0|} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u(\xi)\big| + \big|u_i'(\xi)\big|\big) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|A_1||\lambda_2| + |\lambda_1||\lambda_2|t}{|A_0||\lambda_1|} \\
&\qquad \times \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u(\xi)\big| + \big|u_i'(\xi)\big|\big) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1||\lambda_2|t + |A_1||\lambda_2| + |A_0||\lambda_2|}{|A_0||\lambda_1|}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} W_i V_i\big(\big|u_i(s)\big| + \big|\mathcal{D}_0^\beta u(s)\big| + \big|u_i'(s)\big|\big)\,\mathrm{d}s \\
&\leqslant \|W_i\|\overline{V}_i\bigg(\frac{1}{\Gamma(\alpha+1)} + \frac{|A_0|+|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha-\beta+2)} + \frac{|A_1|+|\lambda_1|}{|A_0||\lambda_1|\Gamma(\alpha-\beta+1)} \\
&\quad + \frac{|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha+2)} + \frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha+1)} + \frac{|A_1||\lambda_2|+|A_0||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha)}\bigg) \\
&= \|W_i\|\overline{V}_i F_0^*,
\end{aligned}
$$

$$
\begin{aligned}
&\big|\mathcal{D}_0^\beta T_1^{(i)}\overline{u}(t) + \mathcal{D}_0^\beta T_2^{(i)} u(t)\big| \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big|h_i\big(s,\overline{u}_i(s),\mathcal{D}_0^\beta \overline{u}_i(s),\overline{u}_i'(s)\big)\big|\,\mathrm{d}s \\
&\qquad + \frac{t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big|h_i\big(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\big)\big|\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1|t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} \big|h_i\big(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_2|t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} \big|h_i\big(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big|h_i\big(\xi,u_i(\xi),\mathcal{D}_0^\beta u_i(\xi),u_i'(\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_2|t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \big|h_i\big(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\big)\big|\,\mathrm{d}s \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} W_i V_i\big(\big|\overline{u}_i(s)\big| + \big|\mathcal{D}_0^\beta \overline{u}_i(s)\big| + \big|\overline{u}_i'(s)\big|\big)\,\mathrm{d}s \\
&\qquad + \frac{t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} W_i V_i\big(\big|u_i(s)\big| + \big|\mathcal{D}_0^\beta u(s)\big| + \big|u_i'(s)\big|\big)\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_1|t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u(\xi)\big| + \big|u_i^{(\xi)}\big|\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad + \frac{|\lambda_2|t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u(\xi)\big| + \big|u_i'(\xi)\big|\big)\,\mathrm{d}\xi\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1 \int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u(\xi)\big| + \big|u_i'(\xi)\big|\big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&+ \frac{|\lambda_2| t^{1-\beta}}{|A_0|\Gamma(2-\beta)} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} W_i V_i\big(\big|u_i(s)\big| + \big|\mathcal{D}_0^\beta u(s)\big| + \big|u_i'(s)\big|\big)\,\mathrm{d}s \\
\leqslant{}& \|W_i\|\overline{V}_i\bigg(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{|A_0|\Gamma(2-\beta)}\bigg(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1|}{\Gamma(\alpha+2)} \\
&+ \frac{|\lambda_2|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+2)} + \frac{|\lambda_2|}{\Gamma(\alpha)}\bigg)\bigg) \\
={}& \|W_i\|\overline{V}_i F_1^*,
\end{aligned}
$$

$$
\begin{aligned}
&\big|\big(T_1^{(i)}\overline{u}\big)'(t) + \big(T_2^{(i)}u\big)'(t)\big| \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big|h_i\big(s, \overline{u}_i(s), \mathcal{D}_0^\beta \overline{u}_i(s), \overline{u}_i'(s)\big)\big|\,\mathrm{d}s \\
&\qquad+ \frac{1}{|A_0|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\big|h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big|\,\mathrm{d}s \\
&\qquad+ \frac{|\lambda_1|}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}\big|h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+ \frac{|\lambda_2|}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}\big|h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+ \frac{1}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\big|h_i\big(\xi, u_i(\xi), \mathcal{D}_0^\beta u_i(\xi), u_i'(\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+ \frac{|\lambda_2|}{|A_0|}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big|h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\big|\,\mathrm{d}s \\
&\quad\leqslant \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} W_i V_i\big(\big|\overline{u}_i(s)\big| + \big|\mathcal{D}_0^\beta \overline{u}_i(s)\big| + \big|\overline{u}_i'(s)\big|\big)\,\mathrm{d}s \\
&\qquad+ \frac{1}{|A_0|}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} W_i V_i\big(\big|u_i(s)\big| + \big|\mathcal{D}_0^\beta u_i(s)\big| + \big|u_i'(s)\big|\big)\,\mathrm{d}s \\
&\qquad+ \frac{|\lambda_1|}{|A_0|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} W_i V_i\big(\big|u_i(\xi)\big| + \big|\mathcal{D}_0^\beta u_i(\xi)\big| + \big|u_i'(\xi)\big|\big)\,\mathrm{d}\xi\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\lambda_2|}{|A_0|}\int_0^1\int_0^s\frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}W_iV_i\big(|u_i(\xi)|+|\mathcal{D}_0^\beta u_i(\xi)|+|u_i'(\xi)|\big)\,\mathrm{d}\xi\,\mathrm{d}s\\
&+\frac{1}{|A_0|}\int_0^1\int_0^s\frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}W_iV_i\big(\left|u_i(\xi)\right|+\left|\mathcal{D}_0^\beta u_i(\xi)\right|+\left|u_i'(\xi)\right|\big)\,\mathrm{d}\xi\,\mathrm{d}s\\
&+\frac{|\lambda_2|}{|A_0|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}W_iV_i\big(|u_i(s)|+|\mathcal{D}_0^\beta u_i(s)|+|u_i'(s)|\big)\,\mathrm{d}s\\
\leqslant{}&\|W_i\|\overline{V}_i\biggl(\frac{1}{\Gamma(\alpha)}+\frac{1}{|A_0|}\biggl(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|}{\Gamma(\alpha+2)}+\frac{|\lambda_2|}{\Gamma(\alpha+1)}\\
&+\frac{1}{\Gamma(\alpha-\beta+2)}+\frac{|\lambda_2|}{\Gamma(\alpha)}\biggr)\biggr)\\
={}&\|W_i\|\overline{V}_iF_2^*,
\end{aligned}
$$

which implies that

$$
\big\|\left|T_1\overline{u}+T_2u\right|\big\|_M=\sum_{i=1}^{10}\big\|\left|T_1^{(i)}\overline{u}+T_2^{(i)}u\right|\big\|_{M_i}\leqslant\|W_i\|\overline{V}_i(F_0^*+F_1^*+F_2^*)\leqslant\rho.
$$

Consequently,

$$
T_1\overline{u}+T_2u\in N_\rho.
$$

Noticing that T_1 is continuous from the continuity of functions h_i, in what follows, we further show the uniform boundedness of the operator T_1. For any $u\in M$, it follows from (13) that

$$
\begin{aligned}
\left|\big(T_1^{(i)}u\big)(t)\right|&\leqslant\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|h_i\big(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\big)\right|\mathrm{d}s\\
&\leqslant\frac{1}{\Gamma(\alpha+1)}\|W_i\|V_i\big(\left|u_i(t)\right|+\left|\mathcal{D}_0^\beta u(t)\right|+\left|u_i'(t)\right|\big),\\
\left|\mathcal{D}_0^\beta T_1^{(i)}u(t)\right|&\leqslant\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|h_i\big(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\big)\right|\mathrm{d}s\\
&\leqslant\frac{1}{\Gamma(\alpha-\beta+1)}\|W_i\|V_i\big(\left|u_i(t)\right|+\left|\mathcal{D}_0^\beta u(t)\right|+\left|u_i'(t)\right|\big),\\
\left|\big(T_1^{(i)}u\big)'(t)\right|&\leqslant\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|h_i\big(s,u_i(s),\mathcal{D}_0^\beta u_i(s),u_i'(s)\big)\right|\mathrm{d}s\\
&\leqslant\frac{1}{\Gamma(\alpha)}\|W_i\|V_i\big(\left|u_i(t)\right|+\left|\mathcal{D}_0^\beta u(t)\right|+\left|u_i'(t)\right|\big).
\end{aligned}
$$

Consequently, for any $u \in N_\rho$, by the monotonicity of V_i, one derives

$$\begin{aligned}\|T_1 u\|_M &= \sum_{i=1}^{10} \big\| \big| T_1^{(i)} u \big| \big\|_{M_i} \\ &\leqslant \left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \right) \sum_{i=1}^{10} \|W_i\| \overline{V}_i\big(\|u_i\|_{M_i}\big) \\ &\leqslant \left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\alpha+1}{\Gamma(\alpha+1)} \right) \sum_{i=1}^{10} \|W_i\| \overline{V}_i(\rho),\end{aligned}$$

which implies that the operator T_1 is uniformly bounded on N_ρ.

Next, we show that T_1 is compact on N_ρ. Let $u \in N_\rho$ and $t_1, t_2 \in [0,1]$, $t_1 < t_2$, then we have

$$\begin{aligned}&\big| \big(T_1^{(i)} u\big)(t_2) - \big(T_1^{(i)} u\big)(t_1) \big| \\ &\quad\leqslant \left| \int_0^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\, \mathrm{d}s \right. \\ &\qquad \left. - \int_0^{t_1} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\, \mathrm{d}s \right| \\ &\quad\leqslant \left| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\, \mathrm{d}s \right| \\ &\qquad + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big)\, \mathrm{d}s \right| \\ &\quad\leqslant \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \big| h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big) \big|\, \mathrm{d}s \\ &\qquad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \big| h_i\big(s, u_i(s), \mathcal{D}_0^\beta u_i(s), u_i'(s)\big) \big|\, \mathrm{d}s \\ &\quad\leqslant \left(\frac{t_2^\alpha - t_1^\alpha - (t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} \right) \|W_i\| V_i(\rho).\end{aligned}$$

Obviously,

$$\big| \big(T_1^{(i)} u\big)(t_2) - \big(T_1^{(i)} u\big)(t_1) \big| \to 0, \quad t_1 \to t_2.$$

By the similar strategy, we also have

$$\lim_{t_1 \to t_2} \big| \mathcal{D}_0^\beta T_1^{(i)} u(t_2) - \mathcal{D}_0^\beta T_1^{(i)} u(t_1) \big| = 0, \qquad \lim_{t_1 \to t_2} \big| \big(T_1^{(i)} u\big)'(t_2) - \big(T_1^{(i)} u\big)'(t_1) \big| = 0.$$

Thus

$$\left|(T_1u)(t_2)-(T_1u)(t_1)\right|\to 0,\quad t_1\to t_2,$$

which indicates that T_1 is equicontinuous, and then T_1 is a relatively compact operator on N_ρ. By the Arzelà–Ascoli theorem, T_1 is compact on N_ρ.

Finally, we show that T_2 is contractive. To do this, let $\widetilde{u}, u\in N_\rho$, then we have

$$\begin{aligned}
&\left|\left(T_2^{(i)}\widetilde{u}\right)(t)-\left(T_2^{(i)}u\right)(t)\right| \\
&\quad\leqslant \frac{|A_0|+|A_1|+|\lambda_1|t}{|A_0||\lambda_1|}\int_0^1\int_0^s \frac{(s-\xi)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}S_i\left(\left|\widetilde{u}_i(\xi)-u_i(\xi)\right|+\left|\mathcal{D}_0^\beta\widetilde{u}_i(\xi)-\mathcal{D}_0^\beta u_i(\xi)\right|\right.\\
&\qquad\left.+\left|\widetilde{u}_i'(\xi)-u_i'(\xi)\right|\right)\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+\frac{|\lambda_1|t+|A_1|}{|A_0||\lambda_1|}\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}S_i\left(\left|\widetilde{u}_i(s)-u_i(s)\right|+\left|\mathcal{D}_0^\beta\widetilde{u}_i(s)-\mathcal{D}_0^\beta u_i(s)\right|\right.\\
&\qquad\left.+\left|\widetilde{u}_i'(s)-u_i'(s)\right|\right)\mathrm{d}s\\
&\qquad+\frac{|A_1|+|\lambda_1|t}{|A_0|}\int_0^1\int_0^s\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}S_i\left(\left|\widetilde{u}_i(\xi)-u_i(\xi)\right|+\left|\mathcal{D}_0^\beta\widetilde{u}_i(\xi)-\mathcal{D}_0^\beta u_i(\xi)\right|\right.\\
&\qquad\left.+\left|\widetilde{u}_i'(\xi)-u_i'(\xi)\right|\right)\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+\frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|t}{|A_0||\lambda_1|}\int_0^1\int_0^s\frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}S_i\left(\left|\widetilde{u}_i(\xi)-u_i(\xi)\right|+\left|\mathcal{D}_0^\beta\widetilde{u}_i(\xi)-\mathcal{D}_0^\beta u_i(\xi)\right|\right.\\
&\qquad\left.+\left|\widetilde{u}_i'(\xi)-u_i'(\xi)\right|\right)\mathrm{d}\xi\,\mathrm{d}s\\
&\qquad+\frac{|\lambda_1||\lambda_2|t+|A_1||\lambda_2|+|A_0||\lambda_2|}{|A_0||\lambda_1|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}S_i\left(\left|\widetilde{u}_i(s)-u_i(s)\right|\right.\\
&\qquad\left.+\left|\mathcal{D}_0^\beta\widetilde{u}_i(s)-\mathcal{D}_0^\beta u_i(s)\right|+\left|\widetilde{u}_i'(s)-u_i'(s)\right|\right)\mathrm{d}s\\
&\quad\leqslant\|S_i\|\left[\frac{|A_0|+|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha-\beta+2)}+\frac{|A_1|+|\lambda_1|}{|A_0||\lambda_1|\Gamma(\alpha-\beta+1)}+\frac{|A_1|+|\lambda_1|}{|A_0|\Gamma(\alpha+2)}\right.\\
&\qquad\left.+\frac{|A_1||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha+1)}+\frac{|A_1||\lambda_2|+|A_0||\lambda_2|+|\lambda_1||\lambda_2|}{|A_0||\lambda_1|\Gamma(\alpha)}\right]\|\widetilde{u}_i-u_i\|_{M_i}\\
&\quad=\|S_i\|E_0^*\|\widetilde{u}_i-u_i\|_{M_i}.
\end{aligned}$$

By a similar calculation, we have

$$\sup_{t\in[0,1]}\left|\mathcal{D}_0^\beta T_2^{(i)}\widetilde{u}(t)-\mathcal{D}_0^\beta T_2^{(i)}u(t)\right|\leqslant\|S_i\|E_1^*\|\widetilde{u}_i-u_i\|_{M_i},$$

$$\sup_{t\in[0,1]}\left|\left(T_2^{(i)}\widetilde{u}\right)'(t)-\left(T_2^{(i)}u\right)'(t)\right|\leqslant\|S_i\|E_2^*\|\widetilde{u}_i-u_i\|_{M_i},$$

$$\|T_2\widetilde{u}-T_2u\|_M=\sum_{i=1}^{10}\|T_2^{(i)}\widetilde{u}-T_2^{(i)}u\|_{M_i}\leqslant\left(E_0^*+E_1^*+E_2^*\right)\sum_{i=1}^{10}\|S_i\|\|\widetilde{u}_i-u_i\|_{M_i},$$

i.e.,

$$\|T_2\widetilde{u} - T_2 u\|_M \leqslant F\|\widetilde{u} - u\|_M.$$

Thus (14) guarantees that T_2 is contractive on N_ρ. By Lemma 4, T has a fixed point in N_ρ, and, consequently, the fractional boundary value problem (2) has a solution on each edge of the graph of glycerol shown in Fig. 4. □

4 Examples

In this section, we give an example to illustrate our results.

Consider the following system of fractional differential equations:

$$\begin{aligned}
\mathcal{D}_0^{1.53}u_1(t) &= \frac{6\mathrm{e}^t|\arcsin u_1(t)|}{24000} + \frac{\mathrm{e}^t|\mathcal{D}_0^{0.03}u_1(t)|}{4000(1+\mathcal{D}_0^{0.03}u_1(t))} + \frac{2\mathrm{e}^t|\arctan u_1'(t)|}{8000},\\
\mathcal{D}_0^{1.53}u_2(t) &= 0.0003\mathrm{e}^t|\sin u_2(t)| + \frac{3\mathrm{e}^t|\arctan \mathcal{D}_0^{0.03}u_2(t)|}{10000} + \frac{6\mathrm{e}^t|u_2'(t)|}{20000(1+u_2'(t))},\\
\mathcal{D}_0^{1.53}u_3(t) &= \frac{2t|u_3(t)|}{10000} + 0.0002t|\sin \mathcal{D}_0^{0.03}u_3(t)| + \frac{4t|\arcsin u_3'(t)|}{20000},\\
\mathcal{D}_0^{1.53}u_4(t) &= \frac{6t|u_4(t)|}{54000} + \frac{5t|\sin \mathcal{D}_0^{0.03}u_4(t)|}{45000} + \frac{t|u_4'(t)|}{9000},\\
\mathcal{D}_0^{1.53}u_5(t) &= 0.000125t|u_5(t)| + \frac{t|\mathcal{D}_0^{0.03}u_5(t)|}{8000} + \frac{2t|\sin u_5'(t)|}{16000},\\
\mathcal{D}_0^{1.53}u_6(t) &= \frac{\mathrm{e}^t|\arctan u_6(t)|}{7500(1+\arctan u_6(t))} + \frac{2\mathrm{e}^t|\mathcal{D}_0^{0.03}u_6(t)|}{15000} + \frac{4\mathrm{e}^t|u_6'(t)|}{30000},\\
\mathcal{D}_0^{1.53}u_7(t) &= \frac{7t|\sin u_7(t)|}{20000} + \frac{14t|\arctan \mathcal{D}_0^{0.03}u_7(t)|}{40000} + 0.00035t|\arctan u_7'(t)|,\\
\mathcal{D}_0^{1.53}u_8(t) &= \frac{\mathrm{e}^t|u_8(t)|}{1000(1+u_8(t))} + \frac{2\mathrm{e}^t|\mathcal{D}_0^{0.03}u_8(t)|}{2000} + 0.001\mathrm{e}^t|\sin u_8'(t)|,\\
\mathcal{D}_0^{1.53}u_9(t) &= \frac{2t|\arcsin u_9(t)|}{5000} + \frac{t|\arctan \mathcal{D}_0^{0.03}u_9(t)|}{2500} + 0.0004t|u_9'(t)|,\\
\mathcal{D}_0^{1.53}u_{10}(t) &= \frac{2\mathrm{e}^t|u_{10}(t)|}{6000} + \frac{\mathrm{e}^t|\mathcal{D}_0^{0.03}u_{10}(t)|}{3000(1+\mathcal{D}_0^{0.03}u_{10}(t))} + \frac{3\mathrm{e}^t|\arcsin u_{10}'(t)|}{9000}
\end{aligned} \tag{15}$$

subject to the boundary conditions

$$\frac{5}{3}\int_0^1 u_i(s)\,\mathrm{d}s + \frac{7}{4}\int_0^1 u_i'(1) = \mathcal{D}_0^{0.03}u_i(1),$$

$$\frac{5}{3}u_i(0) + \frac{7}{4}u_i'(1) = \int_0^1 \mathcal{D}_0^{0.03}u_i(s)\,\mathrm{d}s.$$

From Theorem 2 the fractional boundary value problem (15) has a solution on each edge of the graph of glycerol shown in Fig. 4.

Proof. Let $\alpha = 1.53$, $\beta = 0.03$, $\lambda_1 = 5/3$, $\lambda_2 = 7/4$. Take

$$\begin{aligned}
h_1(t,x,y,z) &= \frac{6\mathrm{e}^t|\arcsin x|}{24000} + \frac{\mathrm{e}^t|y|}{4000(1+y)} + \frac{2\mathrm{e}^t|\arctan z|}{8000},\\
h_2(t,x,y,z) &= 0.0003\mathrm{e}^t|\sin x| + \frac{3\mathrm{e}^t|\arctan y|}{10000} + \frac{6\mathrm{e}^t|z|}{20000(1+u_2'(t))},\\
h_3(t,x,y,z) &= \frac{2t|x|}{10000} + 0.0002t|\sin y| + \frac{4t|\arcsin z|}{20000},\\
h_4(t,x,y,z) &= \frac{6t|x|}{54000} + \frac{5t|\sin y|}{45000} + \frac{t|z|}{9000},\\
h_5(t,x,y,z) &= 0.000125t|x| + \frac{t|y|}{8000} + \frac{2t|\sin z|}{16000},\\
h_6(t,x,y,z) &= \frac{\mathrm{e}^t|\arctan x|}{7500(1+\arctan x)} + \frac{2\mathrm{e}^t|y|}{15000} + \frac{4\mathrm{e}^t|z|}{30000},\\
h_7(t,x,y,z) &= \frac{7t|\sin x|}{20000} + \frac{14t|\arctan y|}{40000} + 0.00035t|\arctan z|,\\
h_8(t,x,y,z) &= \frac{\mathrm{e}^t|x|}{1000(1+x)} + \frac{2\mathrm{e}^t|y|}{2000} + 0.001\mathrm{e}^t|\sin z|,\\
h_9(t,x,y,z) &= \frac{2t|\arcsin x|}{5000} + \frac{t|\arctan y|}{2500} + 0.0004t|z|,\\
h_{10}(t,x,y,z) &= \frac{2\mathrm{e}^t|x|}{6000} + \frac{\mathrm{e}^t|y|}{3000(1+y)} + \frac{3\mathrm{e}^t|\arcsin z|}{9000}.
\end{aligned}$$

In the following, we verify that conditions (12)–(14) are met, and we have for any $\overline{x}, \overline{y}, \overline{z}, x, y, z \in \mathbb{R}$,

$$\begin{aligned}
\bigl|h_1(t,x,y,z) - h_1(t,\overline{x},\overline{y},\overline{z})\bigr| &\leqslant \frac{\mathrm{e}^t}{4000}\bigl(|\arcsin x - \arcsin\overline{x}| + |y-\overline{y}|\\
&\quad + |\arctan z - \arctan\overline{z}|\bigr)\\
&\leqslant \frac{\mathrm{e}^t}{4000}\bigl(|x-\overline{x}| + |y-\overline{y}| + |z-\overline{z}|\bigr),\\
\bigl|h_2(t,x,y,z) - h_2(t,\overline{x},\overline{y},\overline{z})\bigr| &\leqslant \frac{3\mathrm{e}^t}{10000}\bigl(|\sin x - \sin\overline{x}| + |\arctan y - \arctan\overline{y}| + |z-\overline{z}|\bigr)\\
&\leqslant \frac{3\mathrm{e}^t}{10000}\bigl(|x-\overline{x}| + |y-\overline{y}| + |z-\overline{z}|\bigr),\\
\bigl|h_3(t,x,y,z) - h_3(t,\overline{x},\overline{y},\overline{z})\bigr| &\leqslant \frac{t}{5000}\bigl(|x-\overline{x}| + |\sin y - \sin\overline{y}| + |\arcsin z - \arcsin\overline{z}|\bigr)\\
&\leqslant \frac{t}{5000}\bigl(|x-\overline{x}| + |y-\overline{y}| + |z-\overline{z}|\bigr),\\
\bigl|h_4(t,x,y,z) - h_4(t,\overline{x},\overline{y},\overline{z})\bigr| &\leqslant \frac{t}{9000}\bigl(|x-\overline{x}| + |\sin y - \sin\overline{y}| + |z-\overline{z}|\bigr)\\
&\leqslant \frac{t}{9000}\bigl(|x-\overline{x}| + |y-\overline{y}| + |z-\overline{z}|\bigr),
\end{aligned}$$

$$
\begin{aligned}
\big|h_5(t,x,y,z)-h_5(t,\overline{x},\overline{y},\overline{z})\big| &\leqslant \frac{t}{8000}\big(|\arcsin x-\arcsin\overline{x}|+|y-\overline{y}|+|\sin z-\sin\overline{z}|\big)\\
&\leqslant \frac{t}{8000}\big(|x-\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big),\\
\big|h_6(t,x,y,z)-h_6(t,\overline{x},\overline{y},\overline{z})|\big| &\leqslant \frac{\mathrm{e}^t}{7500}\big(|\arctan x-\arctan\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big)\\
&\leqslant \frac{\mathrm{e}^t}{7500}\big(|x-\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big),\\
\big|h_7(t,x,y,z)-h_7(t,\overline{x},\overline{y},\overline{z})\big| &\leqslant \frac{7t}{20000}\big(|\sin x-\sin\overline{x}|+|\arcsin y-\arcsin\overline{y}|\\
&\quad+|\arctan z-\arctan\overline{z}|\big)\\
&\leqslant \frac{7t}{20000}\big(|x-\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big),\\
\big|h_8(t,x,y,z)-h_8(t,\overline{x},\overline{y},\overline{z})\big| &\leqslant \frac{\mathrm{e}^t}{1000}\big(|x-\overline{x}|+|y-\overline{y}|+|\sin z-\sin\overline{z}|\big)\\
&\leqslant \frac{\mathrm{e}^t}{1000}\big(|x-\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big),\\
\big|h_9(t,x,y,z)-h_9(t,\overline{x},\overline{y},\overline{z}\big| &\leqslant \frac{t}{2500}\big(|\arcsin x-\arcsin\overline{x}|+|\arctan y-\arctan\overline{y}|\\
&\quad+|z-\overline{z}|\big)\\
&\leqslant \frac{t}{2500}\big(|x-\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big),\\
\big|h_{10}(t,x,y,z)-h_{10}(t,\overline{x},\overline{y},\overline{z})\big| &\leqslant \frac{\mathrm{e}^t}{3000}\big(|x-\overline{x}|+|y-\overline{y}|+|\arcsin z-\arcsin\overline{z}|\big)\\
&\leqslant \frac{\mathrm{e}^t}{3000}\big(|x-\overline{x}|+|y-\overline{y}|+|z-\overline{z}|\big),
\end{aligned}
$$

and then

$$
\begin{aligned}
&S_1=\frac{\mathrm{e}^t}{4000},\quad S_2=\frac{3\mathrm{e}^t}{10000},\quad S_3=\frac{t}{5000},\quad S_4=\frac{t}{9000},\quad S_5=\frac{t}{8000},\\
&S_6=\frac{\mathrm{e}^t}{7500},\quad S_7=\frac{7t}{20000},\quad S_8=\frac{\mathrm{e}^t}{1000},\quad S_9=\frac{t}{2500},\quad S_{10}=\frac{\mathrm{e}^t}{3000}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|S_1\|=\frac{1}{4000},\quad \|S_2\|=\frac{3}{10000},\quad \|S_3\|=\frac{1}{5000},\quad \|S_4\|=\frac{1}{9000},\quad \|S_5\|=\frac{1}{8000},\\
&\|S_6\|=\frac{1}{7500},\quad \|S_7\|=\frac{7}{20000},\quad \|S_8\|=\frac{1}{1000},\quad \|S_9\|=\frac{1}{2500},\quad \|S_{10}\|=\frac{1}{3000}.
\end{aligned}
$$

Let $V_1, V_2, \ldots, V_{10} \equiv 1$ be constant functions. Then we obtain

$$
\begin{aligned}
\big|h_1(t,x,y,z)\big| &\leqslant \frac{\mathrm{e}^t}{4000}\big(|\arcsin x|+|y|+|\arctan z|\big) \leqslant \frac{\mathrm{e}^t}{4000}\big(|x|+|y|+|z|\big),\\
\big|h_2(t,x,y,z)\big| &\leqslant \frac{3\mathrm{e}^t}{10000}\big(|\sin x|+|\arctan y|+|z|\big) \leqslant \frac{3\mathrm{e}^t}{10000}\big(|x|+|y|+|z|\big),
\end{aligned}
$$

$$\begin{aligned}
\left|h_3(t,x,y,z)\right| &\leqslant \frac{t}{5000}\left(|x|+|\sin y|+|\arcsin z|\right)\frac{t}{5000}\left(|x|+|y|+|z|\right),\\
\left|h_4(t,x,y,z)\right| &\leqslant \frac{t}{9000}\left(|x|+|\sin y|+|z|\right) \leqslant \frac{t}{9000}\left(|x|+|y|+|z|\right),\\
\left|h_5(t,x,y,z)\right| &\leqslant \frac{t}{8000}\left(|\arcsin x|+|y|+|\sin z|\right) \leqslant \frac{t}{8000}\left(|x|+|y|+|z|\right),\\
\left|h_6(t,x,y,z)\right| &\leqslant \frac{\mathrm{e}^t}{7500}\left(|\arctan x|+|y|+|z|\right) \leqslant \frac{\mathrm{e}^t}{7500}\left(|x|+|y|+|z|\right),\\
\left|h_7(t,x,y,z)\right| &\leqslant \frac{7t}{20000}\left(|\sin x|+|\sin y|+|\arctan z|\right)\frac{7t}{20000}\left(|x|+|y|+|z|\right)\\
\left|h_8(t,x,y,z)\right| &\leqslant \frac{\mathrm{e}^t}{1000}\left(|x|+|y|+|\sin z|\right) \leqslant \frac{\mathrm{e}^t}{1000}\left(|x|+|y|+|z|\right),\\
\left|h_9(t,x,y,z)\right| &\leqslant \frac{t}{2500}\left(|\arcsin x|+|\arctan y|+|z|\right) \leqslant \frac{\mathrm{e}^t}{2500}\left(|x|+|y|+|z|\right),\\
\left|h_{10}(t,x,y,z)\right| &\leqslant \frac{\mathrm{e}^t}{3000}\left(|x|+|y|+|\arcsin z|\right) \leqslant \frac{\mathrm{e}^t}{3000}\left(|x|+|y|+|z|\right).
\end{aligned}$$

Define the continuous functions $W_1, W_2, \ldots, W_{10} : [0,1] \to \mathbb{R}$ as

$$\begin{aligned}
&W_1 = \frac{\mathrm{e}^t}{4000}, \quad W_2 = \frac{3\mathrm{e}^t}{10000}, \quad W_3 = \frac{t}{5000}, \quad W_4 = \frac{t}{9000}, \quad W_5 = \frac{t}{8000},\\
&W_6 = \frac{\mathrm{e}^t}{7500}, \quad W_7 = \frac{7t}{20000}, \quad W_8 = \frac{\mathrm{e}^t}{1000}, \quad W_9 = \frac{t}{2500}, \quad W_{10} = \frac{\mathrm{e}^t}{3000}.
\end{aligned}$$

It follows from (9)–(11) that

$$E_0^* \approx 27.4893, \quad E_1^* \approx 14.5112, \quad E_2^* \approx 14.3324,$$

which results in

$$F := \left(E_0^* + E_1^* + E_2^*\right)\sum_{i=1}^{10}\|S_i\| \approx 0.1804 \leqslant 1.$$

Therefore, conditions (12)–(14) all hold. According to Theorem 2, the fractional boundary value problem (15) has a solution on each edge of the graph of glycerol shown in Fig. 4. □

5 Conclusion

In the study of star graphs of boundary value problems, a common point for the graphs with edges to other nodes and no edges between other nodes is required [10]. However, the requirements for nodes are more extensive for complex compounds such as glycerol with the molecular formula $C_3H_8O_3$ in chemical graph theory. In this paper, we explore the existence of solutions for fractional boundary value problems on glycerol graphs. By labeling the glycerol graphs and combining various mathematical tools, a rigorous theoretical framework for analyzing the existence of solutions to such problems was successfully constructed. This research provides an important method to define boundary value problems on the edges of the nonstar structural graphs of the chemical compounds of molecules, which can also be applied to a vast range of graph structures such as digraphs of protein networks and some medical technologies. In our further study, we shall focus on more nonlinear problems on graphs with different molecular structures by using nonlinear analysis methods and some numerical techniques.

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