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On well-posedness and decay of Active model B

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Abstract. We study the well-posedness and decay estimates of Cauchy problem for Active model B in $\mathbb{R}^3$. First, based on the higher-order norm estimates of solutions and the mollifier technique, we obtain the local existence of unique strong solution. Then, by using pure energy method, one proves the global well-posedness and time decay estimates, provided that the $H^{3/2}$-norm of initial data is sufficiently small.

Keywords: Active model B, well-posedness, decay rate.

1 Introduction

Recently, in order to describe the phase separation among orientationally disordered active particles whose propulsion speed decreases rapidly enough with density, Wittkowski et al. [22] introduced the active model B:

$$\varphi_t = -\nabla \cdot J, \tag{1}$$

$$J = -\nabla \mu + \Gamma, \tag{2}$$

$$\mu = -\varphi + \varphi^3 - \nabla^2 \varphi + \lambda (\nabla \varphi)^2, \tag{3}$$

where $\varphi(x,t)$ is a conserved scalar-order parameter filed at position x and time t. It is worth pointing out that Eq. (1) expresses conservation of φ, Eq. (2) states that its mean current $J - \Gamma$ is proportional to the gradient of a nonequilibrium chemical potential μ obeying Eq. (3) with a constant $\lambda \neq 0$, which is used to violate $\varphi \mapsto -\varphi$ symmetry and time reversal symmetry. Besides, the vector Γ is a Gaussian white noise whose variance we take to be constant. The noise is often neglected altogether for phase-separation studies [3], and we generally ignore it in this paper.

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Combining Eqs. (1)–(3), we obtain the following equation [2, 20]:

$$\varphi_t + \Delta^2\varphi = -\Delta\varphi + \Delta\varphi^3 + \lambda\Delta|\nabla\varphi|^2. \tag{4}$$

Equation (4) is a classical fourth-order nonlinear parabolic equation. If we assume that $\lambda = 0$, it is reduced to the classical Cahn–Hilliard equation [5, 7, 10, 11]

$$\varphi_t + \Delta^2\varphi = \Delta f(\varphi), \tag{5}$$

where $f(\varphi) = \varphi^3 - \varphi$ is the derivative of potential.

It was Caffarelli and Muler [4] who first investigated the Cauchy problem of Eq. (5) in general N dimensions, obtained an $L^\infty(\mathbb{R}^N, \mathbb{R})$-a priori estimate of solution. Latterly, assuming that

$$f(u) \in C^L\big(\bar{B}(\bar{u}, 2r), \mathbb{R}\big), \qquad f(u) = O(1)|u - \bar{u}|^l \quad \text{as } u \to \bar{u},$$

with some positive integer l and $L = \max\{5, N\}$, $\bar{B}(\bar{u}, 2r) = \{u \in \mathbb{R}\colon |u-\bar{u}| \leqslant 2r\}$, by using Hoff and Smoller's idea with a slight modification and Fourier splitting method, Liu et al. [15] considered small data global well-posedness and time decay rate of solutions for the Cauchy problem of Eq. (5) in $\mathbb{R}^N$. Besides, Cholewa and Rodriguez-Bernal [6] exhibited the dissipative mechanism of the Cauchy problem of Eq. (5) in $H^1(\mathbb{R}^N)$. There are also some papers related to the Cauchy problem of Cahn–Hilliard equation; please refer to [8, 9, 14, 23] and the reference therein.

Now, let us turn to Active model B. In [2], by using fixed point argument, Bae and Lee showed the existence of a unique global-in-time solution, Gevrey regularity, and decay properties with small initial data for the Cauchy problem of (4) in critical Besove space and Wiener space. Latterly, for Active model B with the logarithmic Cahn–Hilliard equation, Bae [1] showed the existence of unique global-in-time solutions if the initial data are sufficiently small in Wiener space. In this paper, applying pure energy method [12, 21], we improve their global well-posedness results to critical Sobolev space and establish the optimal decay estimates for strong solutions of system (4).

Due to the first term on the right-hand side of $(4)_1$, which has a destabilizing effect, we cannot deal with its Cauchy problem straightforward [2]. Instead, assuming that $u = \varphi - 1$ (it is motivated by the fact that constants are stationary solutions and so we deal with $(4)_1$ near constant solutions), we obtain the Cauchy problem of $u(x, t)$

$$u_t + \Delta^2 u = 2\Delta u + \Delta u^3 + 3\Delta u^2 + \lambda\Delta|\nabla u|^2, \tag{6}$$

$$u(x, 0) = u_0(x) = \varphi_0(x) - 1. \tag{7}$$

In the following, we consider system (6)–(7) instead of Eq. (4). To choose space to solve system (6)–(7), we first check a scaling-invariant property. On the basis of the left-hand side of (6) and the last term on the right-hand side of (6), a natural scaling is

$$u_\lambda(t, x) \to u\big(\lambda^4 t, \lambda x\big).$$

Hence, we take the Sobolev spaces $H^{3/2}(\mathbb{R}^3)$ related to the scaling-invariant property.

Notation 1. In order to consider system (6)–(7) in critical Sobolev space, we introduce the homogeneous negative index Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$:

$$\dot{H}^{-s}(\mathbb{R}^3) := \{f \in L^2(\mathbb{R}^3)\colon \||\xi|^{-s}\hat{f}(\xi)\|_{L^2} < \infty\}$$

endowed with the norm $\|f\|_{\dot{H}^{-s}} := \||\xi|^{-s}\hat{f}(\xi)\|_{L^2}$. Besides, the symbol Λ^l ($l \in \mathbb{R}$) is defined by

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi,$$

where $\hat{f}$ is the Fourier transform of f. In the following, the letter C will always denote positive constants different in various occurrences.

Next, we establish the local well-posedness of strong solutions to system (6)–(7). More precisely, we prove the following theorem.

Theorem 1 [Local well-posedness]. *Suppose that $u_0 \in H^2(\mathbb{R}^3)$. Then there exist a small positive time $T_0 > 0$ and a unique strong solution $u(x,t)$ to system* (6)–(7) *in $\mathbb{R}^3 \times (0,T_0]$ such that*

$$u \in C([0,T_0]; H^2(\mathbb{R}^3)) \cap L^2(0,T_0; H^4(\mathbb{R}^3)).$$

The second goal of this paper is to consider the small initial data global well-posedness for system (6)–(7).

Theorem 2 [Small initial data global well-posedness]. *Assume that $u_0 \in H^N(\mathbb{R}^3)$ ($N \geqslant 2$), C_0 is a positive constant to be determined in* (35)*, and C_1, C_2 are two positive constants to be determined in* (43)*. Then there exists a positive constant*

$$\tilde{\delta}_0 := \min\left\{\frac{1}{2C_0}, \frac{1}{2C_1C_2}\right\}$$

such that if $\|u_0\|_{L^2}^2 + \|\Lambda^{3/2}u_0\|_{L^2}^2 < \tilde{\delta}_0$ holds, then there exists a unique global solution $u(x,t)$ that satisfies for all $t \geqslant 0$,

$$\|u(t)\|_{H^N}^2 + \int_0^t \|\nabla u(t)\|_{H^{N+1}}^2\,\mathrm{d}s \leqslant C\|u_0\|_{H^N}^2. \tag{8}$$

In the end, by using the negative Sobolev norm estimates and the pure energy method, we show that the solutions of problem (6)–(7) satisfy some negative algebraic decay estimates.

Theorem 3 [Decay]. *Let $N \geqslant 2$. Assume that $u_0 \in H^N(\mathbb{R}^3) \cap \dot{H}^{-s}(\mathbb{R}^3)$ for some $s \in [0,3/2)$ and $\|u_0\|_{L^2}^2 + \|\Lambda^{3/2}u_0\|_{L^2}^2 < \tilde{\delta}_0$ holds. Then for all $t \geqslant 0$, we have*

$$\|\Lambda^{-s}u(t)\|_{L^2} \leqslant C \tag{9}$$

and

$$\|\Lambda^l u\|_{H^{N-l}} \leqslant C(1+t)^{-(l+s)/2} \quad \textit{for } l = 0,1,\dots,N. \tag{10}$$

Since Hardy–Littlewood–Sobolev theorem implies that for $p \in [1,2)$, $L^p(\mathbb{R}^n) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = n(1/p - 1/2) \in [0, 3/2)$. Then, by Theorem 3, we obtain the optimal decay estimates.

Corollary 1. *Under the assumptions of Theorem 3, if we replace the $\dot{H}^{-s}(\mathbb{R}^n)$ assumption by $u_0 \in L^p(\mathbb{R}^3)$ $(1 \leqslant p \leqslant 2)$, then, for $l = 0, 1, \dots, N$, the following decay estimate holds*:

$$\left\|\Lambda^l u\right\|_{L^2} \leqslant C(1+t)^{-[3(1/p-1/2)/2+l/2]}.$$

The structure of this paper is organized as follows. In Section 2, we introduce some preliminary results, which are useful to prove main results. Section 3 is devoted to prove the local well-posedness of solutions. In Section 4, we prove the small initial data global well-posedness. The proof of decay rate of solutions is shown in Section 5.

2 Preliminaries

The following useful Sobolev embedding theorem will be used in the proofs.

Lemma 1. (*See* [18].) *For $0 \leqslant s < 3/2$, we have*

$$\|u\|_{L^{6/(3-2s)}(\mathbb{R}^3)} \leqslant C\|u\|_{\dot{H}^s(\mathbb{R}^3)} \quad \text{for all } u \in \dot{H}^s\big(\mathbb{R}^3\big).$$

The following Sobolev interpolate inequality was introduced in [17].

Lemma 2. (*See* [17].) *Let $0 \leqslant m, \alpha \leqslant l$, $1 \leqslant q, r \leqslant \infty$. Then we have*

$$\left\|\nabla^\alpha f\right\|_{L^p(\mathbb{R}^3)} \leqslant C\left\|\nabla^m f\right\|^{1-\theta}_{L^q(\mathbb{R}^3)}\left\|\nabla^l f\right\|^{\theta}_{L^r(\mathbb{R}^3)},$$

where $\theta \in [0,1]$, and α satisfies

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta.$$

Here, when $p = \infty$, we require that $0 < \theta < 1$.

The following Kato–Ponce inequality is of great importance in our paper.

Lemma 3. (*See* [13].) *Let $1 < p < \infty$, $s > 0$. There exists a positive constant C such that*

$$\left\|\Lambda^s(fg)\right\|_{L^p(\mathbb{R}^3)} \leqslant C\big[\|f\|_{L^{p_1}(\mathbb{R}^3)}\left\|\Lambda^s g\right\|_{L^{p_2}(\mathbb{R}^3)} + \left\|\Lambda^s f\right\|_{L^{q_1}(\mathbb{R}^3)}\|g\|_{L^{q_2}(\mathbb{R}^3)}\big],$$

where $p_1, q_1, p_2, q_2 \in (1,\infty)$ satisfying $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$.

The Hardy–Littlewood–Sobolev theorem implies the following L^p-type inequality.

Lemma 4. (*See* [19].) *Let $0 \leqslant s < 3/2$, $1 < p \leqslant 2$, and $1/2 + s/3 = 1/p$, then*

$$\|f\|_{\dot{H}^{-s}(\mathbb{R}^3)} \leqslant C\|f\|_{L^p(\mathbb{R}^3)}.$$

The special Sobolev interpolation lemma will be used in the proof of Theorem 2.

Lemma 5. (*See* [19, 21].) *Let $s \geqslant 0$ and $l \geqslant 0$, then*

$$\left\|\nabla^l f\right\|_{L^2(\mathbb{R}^3)} \leqslant C\left\|\nabla^{l+1} f\right\|^{1-\theta}_{L^2(\mathbb{R}^3)}\|f\|^{\theta}_{\dot{H}^{-s}(\mathbb{R}^3)} \quad \text{with } \theta = \frac{1}{l+1+s}.$$

3 Local well-posedness

First, we derive the following key energy estimates on $\varPhi(t)$ defined by

$$\varPhi(t) \triangleq \big\|u(t)\big\|_{H^2} + 1,$$

which are needed for the proof of Theorem 1.

Proposition 1. *Suppose that* $u_0 \in H^2(\mathbb{R}^3)$. *Let* $u(\cdot, t)$ *be a solution to problem* (6)–(7) *in* $\mathbb{R}^3 \times (0, T]$. *Then there exist a small time* $T_0 \in (0, T)$ *such that*

$$\big\|u(t)\big\|_{H^2}^2 + \int_0^{T_0} \big\|\nabla u(\tau)\big\|_{H^3}^2 \,\mathrm{d}\tau + \int_0^{T_0} \big\|u_\tau(\tau)\big\|_{L^2}^2 \,\mathrm{d}\tau \leqslant C. \tag{11}$$

Proof. Multiplying (6) by u and integrating the resulting equation over $\mathbb{R}^3$, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^3} u\Delta u^3 \,\mathrm{d}x + 3\int_{\mathbb{R}^3} u\Delta u \,\mathrm{d}x + \lambda \int_{\mathbb{R}^3} u\Delta|\nabla u|^2 \,\mathrm{d}x. \end{aligned} \tag{12}$$

Simple calculation shows that

$$\int_{\mathbb{R}^3} u\Delta u^3 \,\mathrm{d}x = -\int_{\mathbb{R}^3} \nabla u \cdot \nabla u^3 \,\mathrm{d}x = -3\int_{\mathbb{R}^3} u^2|\nabla u|^2 \,\mathrm{d}x. \tag{13}$$

By using Sobolev's embedding theorem together with Young's inequality, we derive that

$$\begin{aligned} \lambda \int_{\mathbb{R}^3} u\Delta|\nabla u|^2 \,\mathrm{d}x = \lambda \int_{\mathbb{R}^3} |\nabla u|^2 \Delta u \,\mathrm{d}x &\leqslant \frac{}{4}\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 \\ &\leqslant \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{H^1}^4 \leqslant \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\varPhi^4(t). \end{aligned} \tag{14}$$

Moreover, Hölder's inequality gives

$$\begin{aligned} 3\int_{\mathbb{R}^3} u\Delta u^2 \,\mathrm{d}x &= -3\int_{\mathbb{R}^3} \nabla u \cdot \nabla u^2 \,\mathrm{d}x = -6\int_{\mathbb{R}^3} u|\nabla u|^2 \,\mathrm{d}x \\ &\leqslant \frac{3}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \,\mathrm{d}x + C\int_{\mathbb{R}^3} |\nabla u|^2 = \frac{3}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \,\mathrm{d}x - C\int_{\mathbb{R}^3} u\Delta u \,\mathrm{d}x \\ &\leqslant \frac{3}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \,\mathrm{d}x + \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|u\|_{L^2}^2 \\ &\leqslant \frac{3}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \,\mathrm{d}x + \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\varPhi^2(t). \end{aligned} \tag{15}$$

Summing (12)–(15) up, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + 4\|\nabla u\|_{L^2}^2 \leqslant C\Phi^4(t). \tag{16}$$

Taking Δ to (6), multiplying the resulting equation by Δu and integrating over $\mathbb{R}^3$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2}^2 + \|\Delta^2 u\|_{L^2}^2 + 2\|\nabla\Delta u\|_{L^2}^2 \\ &\quad = \int\limits_{\mathbb{R}^3} \Delta^2 u^3 \cdot \Delta u \,\mathrm{d}x + 3\int\limits_{\mathbb{R}^3} \Delta^2 u^2 \cdot \Delta u \,\mathrm{d}x + \lambda \int\limits_{\mathbb{R}^3} \Delta^2 |\nabla u|^2 \cdot \Delta u \,\mathrm{d}x.\end{aligned} \tag{17}$$

There are three terms on the right-hand side of (17). By using Hölder's inequality, Kato–Ponce inequality, and Sobolev interpolate inequality, we deduce that

$$\begin{aligned}\int\limits_{\mathbb{R}^3} \Delta^2 u^3 \cdot \Delta u \,\mathrm{d}x &= \int\limits_{\mathbb{R}^3} \Delta u^3 \cdot \Delta^2 u \,\mathrm{d}x \\ &\leqslant \big\|\Delta^2 u\big\|_{L^2}\big\|\Delta u^3\big\|_{L^2} \leqslant C\big\|\Delta^2 u\big\|_{L^2}\|\Delta u\|_{L^6}\|u\|_{L^6}^2 \\ &\leqslant C\big\|\Delta^2 u\big\|_{L^2}\big(\big\|\Delta^2 u\big\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\big)\|\nabla u\|_{L^2}^2 \\ &\leqslant \frac{1}{6}\big\|\Delta^2 u\big\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2\|\nabla u\|_{L^2}^8 \\ &\leqslant \frac{1}{6}\big\|\Delta^2 u\big\|_{L^2}^2 + C\Phi^{10}(t),\end{aligned} \tag{18}$$

$$\begin{aligned}3\int\limits_{\mathbb{R}^3} \Delta^2 u^2 \cdot \Delta^2 u \,\mathrm{d}x &= 3\int\limits_{\mathbb{R}^3} \Delta u^2 \cdot \Delta^2 u \,\mathrm{d}x \\ &\leqslant C\big\|\Delta^2 u\big\|_{L^2}\big\|\Delta u^2\big\|_{L^2} \leqslant C\big\|\Delta^2 u\big\|_{L^2}\|\Delta u\|_{L^6}\|u\|_{L^3} \\ &\leqslant C\big\|\Delta^2 u\big\|_{L^2}\big(\big\|\Delta^2 u\big\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\big)\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2} \\ &\leqslant \frac{1}{6}\big\|\Delta^2 u\big\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|u\|_{L^2}^2 \\ &\leqslant \frac{1}{6}\big\|\Delta^2 u\big\|_{L^2}^2 + C\Phi^6(t),\end{aligned} \tag{19}$$

and

$$\begin{aligned}\lambda\int\limits_{\mathbb{R}^3} \Delta^2 |\nabla u|^2 \cdot \Delta u \,\mathrm{d}x &= \lambda\int\limits_{\mathbb{R}^3} \Delta |\nabla u|^2 \cdot \Delta^2 u \,\mathrm{d}x \\ &\leqslant \lambda\big\|\Delta^2 u\big\|_{L^2}\big\|\Delta|\nabla u|^2\big\|_{L^2} \leqslant C\big\|\Delta^2 u\big\|_{L^2}\|\nabla\Delta u\|_{L^3}\|\nabla u\|_{L^6} \\ &\leqslant C\big\|\Delta^2 u\big\|_{L^2}\big(\big\|\Delta^2 u\big\|_{L^2}^{3/4}\|\Delta u\|_{L^2}^{1/4}\big)\|\Delta u\|_{L^2} \\ &\leqslant \frac{1}{6}\big\|\Delta^2 u\big\|_{L^2}^2 + C\|\Delta u\|_{L^2}^{10} \leqslant \frac{1}{6}\big\|\Delta^2 u\big\|_{L^2}^2 + C\Phi^{10}(t).\end{aligned} \tag{20}$$

Summing (17)–(20), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2}^2 + \|\Delta^2 u\|_{L^2}^2 + 4\|\nabla\Delta u\|_{L^2}^2 \leqslant C\Phi^{10}(t). \tag{21}$$

Note that $\|u\|_{H^2}^2 = C(\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)$. Adding (16) and (21) together gives

$$\begin{aligned}
&\Phi(t) + \int_0^t \bigl(\|\Delta u(s)\|_{L^2}^2 + \|\Delta^2 u(s)\|_{L^2}^2 + 4\|\nabla u(s)\|_{L^2}^2 + 4\|\nabla\Delta u(s)\|_{L^2}^2\bigr)\,\mathrm{d}s \\
&\quad\leqslant \Phi(t_0) + C\int_0^t \Phi^{10}(s)\,\mathrm{d}s,
\end{aligned}$$

Simple calculation shows that there exists a time $T_0 \in (0,T)$ such that

$$\begin{aligned}
&\Phi(t) + \int_0^{T_0} \bigl(\|\Delta u(s)\|_{L^2}^2 + \|\Delta^2 u(s)\|_{L^2}^2 + 4\|\nabla u(s)\|_{L^2}^2 + 4\|\nabla\Delta u(s)\|_{L^2}^2\bigr)\,\mathrm{d}s \\
&\quad\leqslant C(\Phi_0),
\end{aligned} \tag{22}$$

where $C(\Phi_0)$ is a positive constant depending on Φ_0. Moreover, on the basis of (22), Kato–Ponce inequality, and Sobolev's embedding theorem, we easily obtain

$$\begin{aligned}
&\int_0^{T_0} \|u_t\|_{L^2}^2\,\mathrm{d}t \\
&\quad\leqslant C\Biggl(\int_0^{T_0} \|\Delta^2 u\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\Delta u\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\Delta u^3\|_{L^2}^2\,\mathrm{d}t \\
&\qquad + \int_0^{T_0} \|\Delta u^2\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\Delta|\nabla u|^2\|_{L^2}^2\,\mathrm{d}t\Biggr) \\
&\quad\leqslant C\Biggl(\int_0^{T_0} \|\Delta^2 u\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\Delta u\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\Delta u\|_{L^6}^2\|u\|_{L^6}^4\,\mathrm{d}t[0pt] \\
&\qquad + \int_0^{T_0} \|\Delta u\|_{L^6}^2\|u\|_{L^3}^2\,\mathrm{d}t + \int_0^{T_0} \|\nabla\Delta u\|_{L^6}^2\|\nabla u\|_{L^3}^2\,\mathrm{d}t\Biggr) \\
&\quad\leqslant C\Biggl(\int_0^{T_0} \|\Delta^2 u\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\Delta u\|_{L^2}^2\,\mathrm{d}t + \int_0^{T_0} \|\nabla\Delta u\|_{L^2}^2\|\nabla u\|_{L^2}^4\,\mathrm{d}t \\
&\qquad + \int_0^{T_0} \|\nabla\Delta u\|_{L^2}^2\|u\|_{L^2}\|\nabla u\|_{L^2}\,\mathrm{d}t + \int_0^{T_0} \|\Delta^2 u\|_{L^2}^2\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\,\mathrm{d}t\Biggr)
\end{aligned}$$

$$\leqslant C\left(\int_0^{T_0} \|\Delta^2 u\|_{L^2}^2 \,\mathrm{d}t + \int_0^{T_0} \|\Delta u\|_{L^2}^2 \,\mathrm{d}t\right)$$
$$\leqslant C,$$

which, together with (22), yields that (11) holds. This complete the proof. □

Proof of Theorem 1. Since the energy estimates have been derived, the local existence of strong solutions can be established by using the mollifier technique as described in [16] (see, for example, [24]), and we omit the details. Besides, we consider the unique of local strong solutions for problem (6)–(7). Let $u_1, u_2 \in C([0,T_0]; H^2(\mathbb{R}^3)) \cap L^2(0,T_0; H^4(\mathbb{R}^3))$ be two strong solutions of system (6)–(7) with the same initial data $u_1(\cdot,0) = u_2(\cdot,0) = u_0(\cdot,0)$, and set

$$v = u_1 - u_2 \in C\bigl([0,T_0]; H^2\bigl(\mathbb{R}^3\bigr)\bigr) \cap L^2\bigl(0,T_0; H^4\bigl(\mathbb{R}^3\bigr)\bigr).$$

It is easy to see that $v(\cdot,0) = u_1(\cdot,0) - u_2(\cdot,0) = 0$. Then, for system (6)–(7), we have

$$\begin{aligned} v_t + \Delta^2 v + 2\Delta v = {}& \Delta\bigl[v\bigl(u_1^2 + u_1u_2 + u_2^2\bigr)\bigr] + 3\Delta\bigl[v(u_1+u_2)\bigr] \\ & + \lambda\Delta\bigl(\nabla v\cdot\nabla(u_1+u_2)\bigr). \end{aligned} \tag{23}$$

Multiplying both sides of (23) by v, integrating over $\mathbb{R}^3$, we derive that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + 2\|\nabla v\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^3} v\bigl(u_1^2 + u_1u_2 + u_2^2\bigr)\Delta v \,\mathrm{d}x + 3\int_{\mathbb{R}^3} \bigl[v(u_1+u_2)\bigr]\Delta v \,\mathrm{d}x \\ &\qquad + \lambda \int_{\mathbb{R}^3} \bigl(\nabla v\cdot\nabla(u_1+u_2)\bigr)\Delta v \,\mathrm{d}x. \end{aligned} \tag{24}$$

Applying Hölder's inequality, Sobolev interpolate inequality, and Young's inequality, it yields that

$$\begin{aligned} &\int_{\mathbb{R}^3} v\bigl(u_1^2 + u_1u_2 + u_2^2\bigr)\Delta v \,\mathrm{d}x \\ &\quad \leqslant \|\Delta v\|_{L^2}\|v\|_{L^2}\bigl(\|u_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2\bigr) \\ &\quad \leqslant C\|\Delta v\|_{L^2}\|v\|_{L^2}\bigl[\bigl(\|\nabla u_1\|_{L^2}^{1/2}\|\Delta u_1\|_{L^2}^{1/2}\bigr)^2 + \bigl(\|\nabla u_2\|_{L^2}^{1/2}\|\Delta u_2\|_{L^2}^{1/2}\bigr)^2\bigr] \\ &\quad \leqslant C\|\Delta v\|_{L^2}\|v\|_{L^2} \leqslant \frac{1}{6}\|\Delta v\|_{L^2}^2 + C\|v\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} &3\int_{\mathbb{R}^3} \bigl[v(u_1+u_2)\bigr]\Delta v \,\mathrm{d}x \\ &\quad \leqslant C\|\Delta v\|_{L^2}\|v\|_{L^2}\bigl(\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}\bigr) \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|\Delta v\|_{L^2}\|v\|_{L^2}\big[\|\nabla u_1\|_{L^2}^{1/2}\|\Delta u_1\|_{L^2}^{1/2}+\|\nabla u_2\|_{L^2}^{1/2}\|\Delta u_2\|_{L^2}^{1/2}\big]\\
&\leqslant C\|\Delta v\|_{L^2}\|v\|_{L^2}\leqslant \frac{1}{6}\|\Delta v\|_{L^2}^2+C\|v\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda\int\limits_{\mathbb{R}^3}\big(\nabla v\cdot\nabla(u_1+u_2)\big)\Delta v\,\mathrm{d}x\\
&\quad\leqslant \lambda\|\Delta v\|_{L^2}\|\nabla v\|_{L^3}\big(\|\nabla u_1\|_{L^6}+\|\nabla u_2\|_{L^6}\big)\\
&\quad\leqslant C\|\Delta v\|_{L^2}\big(\|\Delta v\|_{L^2}^{3/4}\|v\|_{L^2}^{1/4}\big)\big(\|\Delta u_1\|_{L^2}+\|\Delta u_2\|_{L^2}\big)\\
&\quad\leqslant C\|\Delta v\|_{L^2}^{7/4}\|v\|_{L^2}^{1/4}\leqslant \frac{1}{6}\|\Delta v\|_{L^2}^2+C\|v\|_{L^2}^2.
\end{aligned}
\tag{25}
$$

Combining (26)–(25) together gives

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2+\|\Delta v\|_{L^2}^2+4\|\nabla v\|_{L^2}^2\leqslant C\|v\|_{L^2}^2. \tag{26}
$$

It follows from the Gronwall inequality and $v|_{t=0}=0$ that $v=0$. Hence, the uniqueness of strong solution is complete and the proof is complete. □

4 Small initial data global well-posedness

The aim of this section is to study the global existence of unique strong solution of system (6)–(7), provided that the initial data is sufficiently small. We first prove the following lemma.

Lemma 6. *Assume that C_0 is a positive constant to be determined in* (35). *There exists a sufficiently small constant*

$$
\delta_0:=\frac{1}{2C_0} \tag{27}
$$

such that if $\|u_0\|_{L^2}^2+\|\Lambda^{3/2}u_0\|_{L^2}^2<\delta_0$ holds, then there exists a unique global solution $u(x,t)$ satisfying

$$
\begin{aligned}
&\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2+\int\limits_0^t\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\,\mathrm{d}s\\
&\quad\leqslant C\big(\|u_0\|_{L^2}^2+\big\|\Lambda^{3/2}u_0\big\|_{L^2}^2\big).
\end{aligned}
$$

Proof. Applying Hölder's inequality, Sobolev embedding theorem in $\mathbb{R}^3$, and Young's inequality, we can reestimate (15) as

$$
\begin{aligned}
3\int\limits_{\mathbb{R}^3}u\Delta u^2\,\mathrm{d}x&\leqslant C\|u\|_{L^3}\big\|\Delta u^2\big\|_{L^{3/2}}\leqslant C\|u\|_{L^3}\|u\|_{L^6}\|\Delta u\|_{L^2}\\
&\leqslant C\|u\|_{L^2}^{2/3}\big\|\Lambda^{3/2}u\big\|_{L^2}^{1/3}\big(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\big)\\
&\leqslant C\big(\|u\|_{L^2}+\|\Lambda^{3/2}u\|_{L^2}\big)\big(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\big).
\end{aligned}
\tag{28}
$$

Summing up (12)–(14) and (28), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+4\|\nabla u\|_{L^2}^2\\&\quad\leqslant C\big(\|u\|_{L^2}+\|\Lambda^{3/2}u\|_{L^2}\big)\big(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\big).\end{aligned}\tag{29}$$

Taking $\Lambda^{3/2}$ to (6), multiplying the resulting equation by $\Lambda^{3/2}u$, and integrating over $\mathbb{R}^3$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{3/2}u\|_{L^2}^2+\|\Lambda^{7/2}u\|_{L^2}^2+2\|\Lambda^{5/2}u\|_{L^2}^2\\&\quad=\int_{\mathbb{R}^3}\Lambda^{3/2}\Delta u^3\cdot\Lambda^{3/2}u\,\mathrm{d}x+3\int_{\mathbb{R}^3}\Lambda^{3/2}\Delta u^2\cdot\Lambda^{3/2}u\,\mathrm{d}x\\&\qquad+\lambda\int_{\mathbb{R}^3}\Lambda^{3/2}\Delta|\nabla u|^2\cdot\Lambda^{3/2}u\,\mathrm{d}x.\end{aligned}\tag{30}$$

There are three terms on the right-hand side of (17). By using Hölder's inequality, Kato–Ponce inequality, and Sobolev's interpolate inequality, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^3}\Lambda^{3/2}\Delta u^3\cdot\Lambda^{3/2}u\,\mathrm{d}x&=\int_{\mathbb{R}^3}\Lambda^{3/2}u^3\cdot\Lambda^{3/2}\Delta u\,\mathrm{d}x\\&\leqslant\big\|\Lambda^{7/2}u\big\|_{L^2}\big\|\Lambda^{3/2}u^3\big\|_{L^2}\leqslant C\big\|\Lambda^{7/2}u\big\|_{L^2}\big\|\Lambda^{3/2}u\big\|_{L^6}\|u\|_{L^6}^2\\&\leqslant C\big\|\Lambda^{7/2}u\big\|_{L^2}\big\|\Lambda^{5/2}u\big\|_{L^2}\|\nabla u\|_{L^2}^2\\&\leqslant C\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big)\big(\|u\|_{L^2}^{1/3}\big\|\Lambda^{3/2}u\big\|_{L^2}^{2/3}\big)^2\\&\leqslant C\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}\big)^2\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big),\end{aligned}\tag{31}$$

$$\begin{aligned}3\int_{\mathbb{R}^3}\Lambda^{3/2}\Delta u^2\cdot\Lambda^{3/2}\,\mathrm{d}x&=3\int_{\mathbb{R}^3}\Lambda^{3/2}u^2\cdot\Lambda^{3/2}\Delta u\,\mathrm{d}x\\&\leqslant C\big\|\Lambda^{7/2}u\big\|_{L^2}\big\|\Lambda^{3/2}u^2\big\|_{L^2}\leqslant C\big\|\Lambda^{7/2}u\big\|_{L^2}\big\|\Lambda^{3/2}u\big\|_{L^6}\|u\|_{L^3}\\&\leqslant C\big\|\Lambda^{7/2}u\big\|_{L^2}\big\|\Lambda^{5/2}u\big\|_{L^2}\|u\|_{L^2}^{2/3}\big\|\Lambda^{3/2}u\big\|_{L^2}^{1/3}\\&\leqslant C\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}\big)\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big),\end{aligned}\tag{32}$$

and

$$\begin{aligned}&\lambda\int_{\mathbb{R}^3}\Lambda^{3/2}\Delta|\nabla u|^2\cdot\Lambda^{3/2}u\,\mathrm{d}x\\&\quad=\lambda\int_{\mathbb{R}^3}\Lambda^{3/2}|\nabla u|^2\cdot\Lambda^{3/2}\Delta u\,\mathrm{d}x\leqslant\lambda\big\|\Lambda^{7/2}u\big\|_{L^2}^2\big\|\Lambda^{3/2}|\nabla u|^2\big\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\big\|\Lambda^{7/2}u\big\|_{L^2}\|\nabla u\|_{L^3}\big\|\Lambda^{5/2}u\big\|_{L^2} \\
&\leqslant C\big\|\Lambda^{3/2}u\big\|_{L^2}\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big).
\end{aligned} \tag{33}
$$

Combining (30)–(33) together gives

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\Lambda^{3/2}u\big\|_{L^2}^2+\big\|\Lambda^{7/2}u\big\|_{L^2}^2+2\big\|\Lambda^{5/2}u\big\|_{L^2}^2 \\
&\quad\leqslant C\big[\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}\big)^2+\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}\big)+\big\|\Lambda^{3/2}u\big\|_{L^2}\big] \\
&\qquad\times\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big) \\
&\quad\leqslant C\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}\big)\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}+1\big) \\
&\qquad\times\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big).
\end{aligned} \tag{34}
$$

Adding (29) and (34) together, we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2\big)+2\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\|\Delta u\|_{L^2}^2\big)+4\big(\big\|\Lambda^{5/2}u\big\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big) \\
&\quad\leqslant C\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}\big)\big(\|u\|_{L^2}+\big\|\Lambda^{3/2}u\big\|_{L^2}+1\big) \\
&\qquad\times\big(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big).
\end{aligned}
$$

Young's inequality implies that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2\big)+2\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\|\Delta u\|_{L^2}^2\big)+2\big(\big\|\Lambda^{5/2}u\big\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big) \\
&\quad\leqslant\big(1+C_0\big(\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2\big)\big) \\
&\qquad\times\big(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big),
\end{aligned}
$$

where $C_0\in\mathbb{R}^+$ is a constant. Hence,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2\big)+\big(\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big) \\
&\quad\leqslant C_0\big(\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2\big) \\
&\qquad\times\big(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\big\|\Lambda^{7/2}u\big\|_{L^2}^2+\big\|\Lambda^{5/2}u\big\|_{L^2}^2\big).
\end{aligned} \tag{35}
$$

Keeping in mind condition (27), that is,

$$
\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2<\delta_0=\frac{1}{2C_0},
$$

one may deduce that there exists a time $T>0$ such that for all $t\in[0,T)$,

$$
\|u\|_{L^2}^2+\big\|\Lambda^{3/2}u\big\|_{L^2}^2\leqslant\|u_0\|_{L^2}^2+\big\|\Lambda^{3/2}u_0\big\|_{L^2}^2. \tag{36}
$$

Indeed, we can argue (36) by contradiction. Assume that (36) is not true. Hence, there exists a time $T_* < T$ (the first time) such that

$$\left\|u(T_*)\right\|_{L^2} + \left\|\Lambda^{3/2}u(T_*)\right\|_{L^2} = \delta_0 = \frac{1}{2C_0}. \tag{37}$$

Then

$$\left\|u(t)\right\|_{L^2}^2 + \left\|\Lambda^{3/2}u(t)\right\|_{L^2}^2 \leqslant \delta_0 = \frac{1}{2C_0}, \quad 0 \leqslant t \leqslant T_*.$$

On the basis of (35), for $0 \leqslant t \leqslant T_*$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \left\|\Lambda^{3/2}u\right\|_{L^2}^2\right) + \left(\left\|\Lambda^{7/2}u\right\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right) + \left(\left\|\Lambda^{5/2}u\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \\
&\quad \leqslant \frac{1}{2}\left(\left\|\Lambda^{7/2}u\right\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \left\|\Lambda^{5/2}u\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right),
\end{aligned}$$

which implies that

$$\left\|u(T_*)\right\|_{L^2} + \left\|\Lambda^{3/2}u(T_*)\right\|_{L^2} \leqslant \|u_0\|_{L^2}^2 + \left\|\Lambda^{3/2}u_0\right\|_{L^2}^2 < \delta_0 = \frac{1}{2C_0},$$

this is a contradiction with (37). In other words, (36) is true. Hence, by (35), for all $t \in [0, T)$, if

$$\|u_0\|_{L^2}^2 + \left\|\Lambda^{3/2}u_0\right\|_{L^2}^2 < \delta_0,$$

we have

$$\begin{aligned}
&\|u\|_{L^2}^2 + \left\|\Lambda^{3/2}u\right\|_{L^2}^2 + \int_0^t \left(\left\|\Lambda^{7/2}u\right\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \left\|\Lambda^{5/2}u\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \mathrm{d}s \\
&\quad \leqslant C_1\left(\|u_0\|_{L^2}^2 + \left\|\Lambda^{3/2}u_0\right\|_{L^2}^2\right) < C_1\delta_0,
\end{aligned} \tag{38}$$

where $C_1 \in \mathbb{R}^+$ is a positive constant. The proof is complete. □

Next, with the above preparations in hand, we give the proof of Theorem 2.

Proof of Theorem 2. Taking Λ^N to (6), multiplying the resulting equation by $\Lambda^N u$, and integrating over $\mathbb{R}^3$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^N u\right\|_{L^2}^2 + \left\|\Lambda^{N+2}u\right\|_{L^2}^2 + 2\left\|\Lambda^{N+1}u\right\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}^3} \Lambda^N \Delta u^3 \cdot \Lambda^N u \,\mathrm{d}x + 3\int_{\mathbb{R}^3} \Lambda^N \Delta u^2 \cdot \Lambda^N u \,\mathrm{d}x \\
&\qquad + \lambda \int_{\mathbb{R}^3} \Lambda^N \Delta|\nabla u|^2 \cdot \Lambda^N u \,\mathrm{d}x.
\end{aligned} \tag{39}$$

The main tools to deal with the three terms on the right-hand side of (17) are Hölder's inequality, Young's inequality, Kato–Ponce inequality together with Sobolev's interpolate inequality. We have

$$\begin{aligned}\int\limits_{\mathbb{R}^3} \Lambda^N \Delta u^3 \cdot \Lambda^N u \,\mathrm{d}x &\leqslant \big\|\Lambda^{N+1} u\big\|_{L^6} \big\|\Lambda^{N+1} u^3\big\|_{L^{6/5}} \\ &\leqslant C\big\|\Lambda^{N+1} u\big\|_{L^6} \big(\|u\|_{L^3}^2 \big\|\Lambda^{N+1} u\big\|_{L^6}\big) \\ &\leqslant C\big\|\Lambda^{1/2} u\big\|_{L^2}^2 \big\|\Lambda^{N+2} u\big\|_{L^2}^2 \\ &\leqslant C\big(\|u\|_{L^2}^2 + \big\|\Lambda^{3/2} u\big\|_{L^2}^2\big) \big\|\Lambda^{N+2} u\big\|_{L^2}^2, \end{aligned} \tag{40}$$

$$\begin{aligned}3\int\limits_{\mathbb{R}^3} \Lambda^N \Delta u^2 \cdot \Lambda^N u \,\mathrm{d}x &\leqslant 3\big\|\Lambda^{N+1} u\big\|_{L^6} \big\|\Lambda^{N+1} u^2\big\|_{L^{6/5}} \\ &\leqslant C\big\|\Lambda^{N+1} u\big\|_{L^6} \|u\|_{L^3} \big\|\Lambda^{N+1} u\big\|_{L^2} \\ &\leqslant C\big\|\Lambda^{1/2} u\big\|_{L^2} \big(\big\|\Lambda^{N+1} u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2\big) \\ &\leqslant C\big(\|u\|_{L^2} + \big\|\Lambda^{3/2} u\big\|_{L^2}\big) \\ &\quad \times \big(\big\|\Lambda^{N+1} u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2\big), \end{aligned} \tag{41}$$

and

$$\begin{aligned}\lambda\int\limits_{\mathbb{R}^3} \Lambda^N \Delta |\nabla u|^2 \cdot \Lambda^N u \,\mathrm{d}x &\leqslant \lambda\big\|\Lambda^{N+1} u\big\|_{L^6} \big\|\Lambda^{N+1} |\nabla u|^2\big\|_{L^{6/5}} \\ &\leqslant C\big\|\Lambda^{N+1} u\big\|_{L^6} \|\nabla u\|_{L^3} \big\|\Lambda^{N+2} u\big\|_{L^2} \\ &\leqslant C\big\|\Lambda^{3/2} u\big\|_{L^2} \big\|\Lambda^{N+2} u\big\|_{L^2}^2 \\ &\leqslant C\big(\|u\|_{L^2} + \big\|\Lambda^{3/2} u\big\|_{L^2}\big) \big\|\Lambda^{N+2} u\big\|_{L^2}^2. \end{aligned} \tag{42}$$

Plugging (40)–(42) into (39), one deduces that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\Lambda^N u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2 + 2\big\|\Lambda^{N+1} u\big\|_{L^2}^2 \\ &\quad \leqslant C\big(\|u\|_{L^2} + \big\|\Lambda^{3/2} u\big\|_{L^2}\big)\big(\|u\|_{L^2} + \big\|\Lambda^{3/2} u\big\|_{L^2} + 1\big) \\ &\qquad \times \big(\big\|\Lambda^{N+1} u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2\big). \end{aligned}$$

By using Young's inequality and (38), we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big\|\Lambda^N u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2 + \big\|\Lambda^{N+1} u\big\|_{L^2}^2 \\ &\quad \leqslant C_2\big(\|u\|_{L^2}^2 + \big\|\Lambda^{3/2} u\big\|_{L^2}^2\big)\big(\big\|\Lambda^{N+1} u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2\big) \\ &\quad \leqslant C_1 C_2\big(\|u_0\|_{L^2}^2 + \big\|\Lambda^{3/2} u_0\big\|_{L^2}^2\big)\big(\big\|\Lambda^{N+1} u\big\|_{L^2}^2 + \big\|\Lambda^{N+2} u\big\|_{L^2}^2\big), \end{aligned} \tag{43}$$

where $C_2 \in \mathbb{R}^+$ is a positive constant. Then, if

$$\|u_0\|_{L^2}^2 + \big\|\Lambda^{3/2} u\big\|_{L^2}^2 < \tilde{\delta}_0 := \min\left\{\frac{1}{2C_0}, \frac{1}{2C_1 C_2}\right\},$$

we have
$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varLambda^N u\right\|_{L^2}^2 + C_3\left(\left\|\varLambda^{N+2}u\right\|_{L^2}^2 + \left\|\varLambda^{N+1}u\right\|_{L^2}^2\right) \leqslant 0, \tag{44}$$
where $C_3 = 1 - C_1C_2\tilde{\delta}_0$. Integrating (44) over $(0,t)$, we obtain (8) and complete the proof. ☐

5 Decay estimates

In order to study the decay estimates of system (6)–(7), we shall derive the evolution of the negative Sobolev norms of the solution.

Lemma 7. *If* $\|u_0\|_{L^2}^2 + \|\varLambda^{3/2}u_0\|_{L^2}^2 < \tilde{\delta}_0$*, then for* $s \in [0,1/2]$*, we have*
$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varLambda^{-s}u\right\|_{L^2}^2 + C\left(\left\|\varLambda^{-s}\nabla u\right\|_{L^2}^2 + \left\|\varLambda^{-s}\Delta u\right\|_{L^2}^2\right) \leqslant C\left\|\nabla\varLambda^{-s}u\right\|_{L^2}\|\nabla u\|_{H^2}^2. \tag{45}$$
Moreover, for $s \in (1/2,3/2)$*, we have*
$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varLambda^{-s}u\right\|_{L^2}^2 + C\left(\left\|\varLambda^{-s}\nabla u\right\|_{L^2}^2 + \left\|\varLambda^{-s}\Delta u\right\|_{L^2}^2\right)\\ &\quad\leqslant C\left\|\nabla\varLambda^{-s}u\right\|_{L^2}\left(\|\nabla u\|_{H^2}^2 + \|\Delta u\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\right).\end{aligned} \tag{46}$$

Proof. Applying $\varLambda^{-s}$ to (6), multiplying the resulting identity by $\varLambda^{-s}u$, summing up and then integrating by parts, we deduce that
$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varLambda^{-s}u\right\|_{L^2}^2 + \left\|\varLambda^{-s}\Delta u\right\|_{L^2}^2 + 2\left\|\varLambda^{-s}\nabla u\right\|_{L^2}^2\\ &\quad= \int_{\mathbb{R}^3}\varLambda^{-s}\Delta u^3\cdot\varLambda^{-s}u\,\mathrm{d}x + 3\int_{\mathbb{R}^3}\varLambda^{-s}\Delta u^2\cdot\varLambda^{-s}u\,\mathrm{d}x\\ &\qquad+ \lambda\int_{\mathbb{R}^3}\varLambda^{-s}\Delta|\nabla u|^2\cdot\varLambda^{-s}u\,\mathrm{d}x.\end{aligned} \tag{47}$$

The main tool to estimate the nonlinear terms in the right-hand side of (47) is the Hardy–Littlewood–Sobolev theorem. This forces us to require that $s \in (0,3/2)$. If $s \in (0,1/2]$, we easily obtain $1/2+s/3 < 1$ and $3/s \geqslant 6$. Then, applying Hardy–Littlewood–Sobolev theorem, Kato–Ponce inequality together with Hölder's and Young's inequalities, it yields that
$$\begin{aligned}&\int_{\mathbb{R}^n}\varLambda^{-s}\Delta u^3\cdot\varLambda^{-s}u\,\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^3}\varLambda^{-s}\left(3u^2\Delta u + 6u|\nabla u|^2\right)\cdot\varLambda^{-s}u\,\mathrm{d}x\\ &\quad\leqslant C\left\|\varLambda^{-s}u\right\|_{L^2}\left(\left\|\varLambda^{-s}(u^2\Delta u)\right\|_{L^2} + \left\|\varLambda^{-s}(u|\nabla u|^2)\right\|_{L^2}\right)\\ &\quad\leqslant C\left\|\varLambda^{-s}u\right\|_{L^2}\left(\left\|u^2\Delta u\right\|_{L^{1/(1/2+s/3)}} + \left\|u|\nabla u|^2\right\|_{L^{1/(1/2+s/3)}}\right).\end{aligned} \tag{48}$$

Simple calculation shows that

$$\begin{aligned}\left\|u^2\Delta u\right\|_{L^{1(1/2+s/3)}} &\leqslant C\|u\|_{L^\infty}\|\Delta u\|_{L^2}\|u\|_{L^{3/s}}\\ &\leqslant C\|u\|_{H^2}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{1/2+s}\|\Delta u\|_{L^2}^{1/2-s}\\ &\leqslant C\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{1/2+s}\|\Delta u\|_{L^2}^{1/2-s}\\ &\leqslant C\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2 \end{aligned}\tag{49}$$

and

$$\begin{aligned}\left\|u|\nabla u|^2\right\|_{L^{1(1/2+s/3)}} &\leqslant C\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla u\|_{L^{3/s}}\\ &\leqslant C\|u\|_{H^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{1/2+s}\|\nabla\Delta u\|_{L^2}^{1/2-s}\\ &\leqslant C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{1/2+s}\|\nabla\Delta u\|_{L^2}^{1/2-s}\\ &\leqslant C\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|\nabla\Delta u\|_{L^2}^2. \end{aligned}\tag{50}$$

Combining (48)–(50) together gives

$$\int_{\mathbb{R}^3}\Lambda^{-s}\Delta u^3\cdot\Lambda^{-s}u\,\mathrm{d}x\leqslant C\|\Lambda^{-s}u\|_{L^2}\|\nabla u\|_{H^2}^2.\tag{51}$$

Similarly,

$$3\int_{\mathbb{R}^3}\Lambda^{-s}\Delta u^2\cdot\Lambda^{-s}u\,\mathrm{d}x\leqslant C\|\Lambda^{-s}u\|_{L^2}\|\nabla u\|_{H^2}^2.\tag{52}$$

Besides, we also have

$$\begin{aligned}\lambda\int_{\mathbb{R}^3}\Lambda^{-s}\Delta|\nabla u|^2\cdot\Lambda^{-s}u\,\mathrm{d}x &\leqslant \lambda\left\|\Lambda^{-s}u\right\|_{L^2}\left\|\Lambda^{-s}\Delta|\nabla u|^2\right\|_{L^2}\\ &\leqslant C\left\|\Lambda^{-s}u\right\|_{L^2}\left\|\Delta|\nabla u|^2\right\|_{L^{1(1/2+s/3)}}\\ &\leqslant C\left\|\Lambda^{-s}u\right\|_{L^2}\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^{3/s}}\\ &\leqslant C\left\|\Lambda^{-s}u\right\|_{L^2}\|\nabla\Delta u\|_{L^2}\|\Delta u\|_{L^2}^{1/2+s}\|\nabla\Delta u\|_{L^2}^{1/2-s}\\ &\leqslant C\left\|\Lambda^{-s}u\right\|_{L^2}\|\Delta u\|_{H^1}^2. \end{aligned}\tag{53}$$

Then (45) can be obtained by plugging estimate (51)–(53) into (47).

Next, if $s\in(1/2,3/2)$, we can estimate the right-hand side term of (47) in a different way. Since $s\in(1/2,3/2)$, it is easy to see that $1/2+s/3<1$ and $2<3/s<6$. Then Hardy–Littlewood–Sobolev theorem, Kato–Ponce inequality, and Hölder's inequality imply that

$$\begin{aligned}\left\|u^2\Delta u\right\|_{L^{1(1/2+s/3)}} &\leqslant C\|u\|_{L^\infty}\|\Delta u\|_{L^2}\|u\|_{L^{3/s}}\\ &\leqslant C\|u\|_{H^2}\|\Delta u\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\\ &\leqslant C\|\Delta u\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s} \end{aligned}$$

and

$$\big\|u|\nabla u|^2\big\|_{L^{1(1/2+s/3)}} \leqslant C\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla u\|_{L^{3/s}}$$
$$\leqslant C\|u\|_{H^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{s-1/2}\|\Delta u\|_{L^2}^{3/2-s}$$
$$\leqslant C\|\nabla u\|_{L^2}^{s+1/2}\|\Delta u\|_{L^2}^{3/2-s} \leqslant C\big(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\big).$$

Hence,

$$\int_{\mathbb{R}^3} \Lambda^{-s}\Delta u^3 \cdot \Lambda^{-s}u \,\mathrm{d}x$$
$$\leqslant C\big\|\Lambda^{-s}u\big\|_{L^2}\big(\|\nabla u\|_{H^1}^2 + \|\Delta u\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\big). \tag{54}$$

Similarly,

$$3\int_{\mathbb{R}^3} \Lambda^{-s}\Delta u^2 \cdot \Lambda^{-s}u \,\mathrm{d}x$$
$$\leqslant C\big\|\Lambda^{-s}u\big\|_{L^2}\big(\|\nabla u\|_{H^1}^2 + \|\Delta u\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\big). \tag{55}$$

Moreover,

$$\lambda\int_{\mathbb{R}^3} \Lambda^{-s}\Delta|\nabla u|^2 \cdot \Lambda^{-s}u \,\mathrm{d}x \leqslant C\big\|\Lambda^{-s}u\big\|_{L^2}\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^{3/s}}$$
$$\leqslant C\big\|\Lambda^{-s}u\big\|_{L^2}\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{s-1/2}|\Delta u\|_{L^2}^{3/2-s}$$
$$\leqslant C\big\|\Lambda^{-s}u\big\|_{L^2}\|\nabla u\|_{H^2}^2. \tag{56}$$

Plugging estimate (54)–(56) into (47), we obtain (46). Hence, the proof is complete. □

Next, we give the proof of Theorem 3.

Proof of Theorem 3. First, we prove the decay rate of solutions for $s \in [0, 1/2]$. Define

$$\mathcal{E}_{-s}(t) = \big\|\Lambda^{-s}u\big\|_{L^2}^2, \qquad \mathcal{E}_l(t) = \big\|\Lambda^l u\big\|_{L^2}^2.$$

Then, integrating in time (45), by the bound (8), we derive that

$$\mathcal{E}_{-s}(t) \leqslant \mathcal{E}_{-s}(0) + C\int_0^t \|\nabla u\|_{H^2}^2\sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau \leqslant C\Big(1 + \sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{-s}(\tau)}\Big),$$

which implies (9) for $s \in [0, 1/2]$, that is,

$$\big\|\Lambda^{-s}u(t)\big\|_{L^2}^2 \leqslant C. \tag{57}$$

Moreover, if $l = 1, 2, \ldots, N$, we may use Lemma 5 to have

$$\big\|\nabla^{l+1}f\big\|_{L^2} \geqslant C\big\|\Lambda^{-s}f\big\|_{L^2}^{-1/(l+s)}\big\|\nabla^l f\big\|_{L^2}^{1+1/(l+s)}.$$

Then, by this facts and (57), we get

$$\big\|\nabla^{l+1}u\big\|_{L^2}^2 \geqslant C\big(\big\|\nabla^l u\big\|_{L^2}^2\big)^{1+1/(k+s)}.$$

Thus, for $l = 1, 2, \dots, N$,

$$\|\nabla^{l+1}u\|_{H^{N-l-1}}^2 \geqslant C\left(\left\|\nabla^l u\right\|_{H^{N-l}}^2\right)^{1+1/(l+s)}.$$

We deduce from (44) the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l + C_0(\mathcal{E}_l)^{1+1/(l+s)} \leqslant 0 \quad \text{for } l = 1, 2, \dots, N,$$

which implies

$$\mathcal{E}_l(t) \leqslant C_0(1+t)^{-l-s} \quad \text{for } l = 1, 2, \dots, N.$$

Hence, (9) holds.

On the other hand, the arguments for $s \in [0, 1/2]$ cannot be applied to $s \in (1/2, 3/2)$. However, observing that $u_0 \in \dot{H}^{-1/2}$ holds since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$, we can deduce from what we have proved for (9)–(10) with $s = 1/2$ that the following estimate holds for $l = 0, 1, \dots, N$:

$$\left\|\nabla^l u\right\|_{L^2}^2 \leqslant C_0(1+t)^{-1/2-l}.$$

Therefore, we deduce from (46) that for $s \in (1/2, 3/2)$,

$$\begin{aligned}
\mathcal{E}_{-s}(t) &\leqslant \mathcal{E}_{-s}(0) + C\int_0^t \|\nabla u\|_{H^2}^2 \sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau \\
&\quad + C\int_0^t \|u\|_{L^2}^{s-1/2}\|\nabla u\|_{H^1}^{5/2-s}\sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau \\
&\leqslant C + C\sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{-s}(\tau)} + C\int_0^t (1+\tau)^{-(ns+n-s)/2}\,\mathrm{d}\tau \sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)} \\
&\leqslant C + C\sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)},
\end{aligned}$$

which implies that (9) holds for $s \in (1/2, 3/2)$, i.e.,

$$\left\|\Lambda^{-s}u(t)\right\|_{L^2}^2 \leqslant C. \tag{58}$$

Since we have proved (58), we may repeat the arguments leading to (10) for $s \in [0, 1/2]$ to prove that they also hold for $s \in (1/2, 3/2)$. Therefore, the proof of Theorem 3 is complete. $\square$

6 Conclusion

The classical Cahn–Hilliard equation arises in the study of phase separation on cooling binary solutions such as glasses, alloys, and polymer mixtures. During the past years, there are many classical results on the well-posedness and the large time behavior of its initial–boundary value problem and Cauchy problem. Mathematically, Active model B can be seen as one of its modifications. Due to the existence of the higher-order nonlinear term $\lambda\Delta|\nabla\varphi|^2$ in (4), its mathematical analysis is more difficult than the classical Cahn–Hilliard equation (5). The main purpose of this paper is to study the local well-psoedness, small initial data global well-posedness, and decay estimates of the Cauchy problem of Active model B. Assuming that the initial data is sufficiently small, using pure energy method together with the negative Sobolev norm estimates, one overcome the difficulties caused by the nonlinear terms and obtains the main results in critical Sobolev space. We believe our results will attract the readers in the related fields.

Conflicts of interest. All authors have read and approved the published version of the manuscript.

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