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Finite-time stability of discrete fractional uncertain recurrent neural networks*

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Abstract. This paper investigates finite-time stability of fractional uncertain difference equations with time delay. A fractional sum inequality is obtained from uncertain initial-value conditions. A delayed discrete Gronwall's inequality is used, and sample paths are numerically illustrated. Finally, finite-time stability results are obtained for a fractional uncertain recurrent neural network model. It can be concluded that this paper provides an efficient tool for finite-time analysis of high-dimensional fractional uncertain systems with time delay.

Keywords: fractional difference equations, uncertainty distribution, discrete Gronwall's inequality, time delay.

1 Introduction

Many nonlinear phenomena hold features of discrete time, long-term interaction, and uncertainty. Since fractional-order operators hold memory effects, the uncertainty theory developed by Liu [8, 9] was recently introduced in fractional differential and difference equations [6, 10, 13, 14].

An uncertain initial-value problem can be given presented by

$$\begin{aligned} {}^{C}\Delta_a^{\beta} X(t) &= \mathscr{F}\big(t+\beta-1,\, X(t+\beta-1)\big) \\ &\qquad + \mathscr{G}(t+\beta-1, X(t+\beta-1))\xi(t+\beta-1), \\ X(a) &= X_\chi, \quad 0<\beta\leqslant 1, \end{aligned} \tag{1}$$

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where $t \in \Omega =: \{a + 1 - \beta, \ldots, a + T - \beta\}$ and $T \in \mathbb{N}_1 := \{1, 2, \ldots\}$. ${}^C\Delta_a^\beta$ denotes the Caputo difference, the real-valued functions $\mathscr{F}$ and $\mathscr{G}$ are Lipschitz continuous, and $\xi(a), \xi(a+1), \ldots, \xi(a+n)$ are $n+1$ i.i.d. uncertain variable with the linear uncertainty distribution $\mathcal{L}(c, d)$ [8].

The fractional chaotic map or fractional difference equations with time delays was proposed in [15, 17]. Rich dynamic behaviors were reported in both initial and terminal value problems. It has been used successfully in deep learning and data-driven learning [18]. Very recently, the fractional recurrent neural network was proposed in [4]:

$$\begin{aligned} &{}^C\Delta_a^\beta \mathbf{X}(t) = A\mathbf{F}\big(\mathbf{X}(t + \beta - 1)\big), \quad 0 < \beta \leqslant 1,\ t \in \Omega, \\ &\mathbf{X}(a) = \mathbf{X}_\chi, \end{aligned} \tag{2}$$

where $\mathbf{X} : \mathbb{N}_a \to \mathbb{R}^m$ is the system's state, and $\mathbf{X} = (x_1, \ldots, x_m)^{\mathrm{T}}$. $\mathbf{F} = (f(x_1), \ldots, f(x_m))^{\mathrm{T}}$ is a vector active function.

When we consider the uncertainty by using the drift term $\mathbf{X}(t + \beta - 1)\xi(t + \beta - 1)$ and $0 < \beta \leqslant 1$, the fractional uncertain recurrent neural network can be improved as a time-varying system

$$\begin{aligned} &{}^C\Delta_a^\beta \mathbf{X}(t) = A\mathbf{F}\big(\mathbf{X}(t + \beta - 1)\big) + B\mathbf{X}(t + \beta - 1)\xi(t + \beta - 1), \quad t \in \Omega, \\ &\mathbf{X}(a) = \mathbf{X}_\chi. \end{aligned} \tag{3}$$

It describes how the trajectory moves through space from an uncertain initial-value. It is also indeed an uncertain function of time t. Interval-valued fractional calculus was suggested in continuous and discrete-time systems [5,12]. Unfortunately, in high-dimensional systems, we must consider the solutions' w-monotonity problems, which complicates the analysis. This is the main purpose for which we turn to the uncertainty theory [8, 9] and consider fractional uncertain difference equations in this paper.

However, this also becomes complicated since one encounters the discretization problems of the nonlocal operators, as well as time delays and uncertainty. As is well known, the Gronwall inequality is an important tool to finite-time stability analysis (see, for example, [7, 16]). Thus, we need one in an uncertain case (3). Concerning the uncertainty initial-value problems, it is important to guarantee that the fractional uncertain system is stable in a finite time.

Fortunately, a delay discrete Gronwall inequality was newly proposed in discrete fractional calculus [19]. Then this paper possibly provides the finite-time stability results of (1) and fractional recurrent networks with uncertainty (3) as a high-dimensional case.

2 Preliminaries

We revisit some basics of discrete fractional calculus and uncertainty theory. $\beta \in \mathbb{R}$, and $t^{(\beta)}$ is the discrete factorial function

$$t^{(\beta)} := \frac{\Gamma(t+1)}{\Gamma(t+1-\beta)},$$

where $t \in \mathbb{N}_\beta := \{\beta, \beta + 1, \ldots\}$.

2.1 Discrete fractional calculus

Definition 1. (See [2].) Suppose $u : \mathbb{N}_a \to \mathbb{R}$ and $0 < \beta$. The β-order fractional sum is defined by

$$\Delta_a^{-\beta} u(t) := \frac{1}{\Gamma(\beta)} \sum_{s=a}^{t-\beta} (t - \sigma(s))^{(\beta-1)} u(s), \quad t \in \mathbb{N}_{a+\beta},$$

where $\sigma(s) = s + 1$.

Definition 2. (See [1].) Let $u : \mathbb{N}_a \to \mathbb{R}$, $\beta > 0$, and $N = \lceil \beta \rceil$. The Caputo difference of order β is given by

$${}^C\Delta_a^{\beta} u(t) := \begin{cases} \Delta_a^{-(N-\beta)} \Delta^N u(t), & N - 1 < \beta < N, \\ \Delta^N u(t), & \beta = N, \end{cases}$$

where $t \in \mathbb{N}_{a+N-\beta}$.

Theorem 1. (*See* [3].) *For the initial-value problem*

$$\begin{aligned} {}^C\Delta_a^{\beta} u(t) &= f\bigl(t + \beta - 1, u(t + \beta - 1)\bigr), \quad 0 < \beta \leqslant 1, \\ u(a) &= c, \end{aligned}$$

the equivalent fractional sum equation is given as

$$\begin{aligned} u(t) &= u(a) \\ &\quad + \frac{1}{\Gamma(\beta)} \sum_{s=a+1-\beta}^{t-\beta} \bigl(t - \sigma(s)\bigr)^{(\beta-1)} f\bigl(s + \beta - 1, u(s + \beta - 1)\bigr), \quad t \in \mathbb{N}_{a+1}. \end{aligned}$$

First, the uncertainty space is denoted by $(\digamma, \mathcal{L}, \mathcal{M})$, where $\mathcal{L}$ stands for a σ-algebra in a nonempty set $\digamma$, and the set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ represents an uncertain measure. A measurable function ξ defined in the uncertainty space can be called an uncertain variable; for example, the uncertainty distribution $\Psi(\ell) = \mathcal{M}\{\xi \leqslant \ell\}$ ($\ell \in \mathbb{R}$) is used to describe the incomplete information of the uncertain variable ξ. When a continuous and strictly increasing function $\Psi(\ell)$ satisfies $\Psi(\ell) + \Psi(1 - \ell) = 1$, it is called the symmetrical uncertainty distribution. For example, the linear uncertain variable $\mathcal{L}(-\psi, \psi)$ ($\psi > 0$) and standard normal uncertain variable $\mathcal{N}(0, 1)$ are also symmetric. We introduce some definitions and lemmas.

2.2 Uncertainty theory

Definition 3. (See [8].) Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables and f be a measurable function with real value. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable defined by

$$\xi(\gamma) = f\bigl(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)\bigr) \quad \forall \gamma \in \digamma.$$

Lemma 1. (*See* [8].) *Let* $\xi_1, \xi_2, \dots$ *be uncertain variables, and let* $\lim_{i\to\infty} \xi_i = \xi$ *almost surely. Then* ξ *is an uncertain variable.*

The definitions of discrete fractional calculus can be extended to the uncertain case.

Definition 4. (See [14].) Let $\beta > 0$ and $\xi(t)$ be an uncertain sequence. Then the fractional sum for uncertain sequence $\xi(t)$ is given by

$$\Delta_a^{-\beta}\xi(t) := \frac{1}{\Gamma(\beta)}\sum_{s=a}^{t-\beta}\big(t-\sigma(s)\big)^{(\beta-1)}\xi(s), \quad t \in \mathbb{N}_{a+\beta}.$$

Definition 5. (See [14].) For $0 < \beta$, $N = \lceil\beta\rceil$, and $\xi(t)$ defined on $\mathbb{N}_a$, the Caputo difference of uncertain sequence $\xi(t)$ is defined as

$$ {}^C\Delta_a^{\beta}\xi(t) := \begin{cases} \Delta_a^{-(N-\beta)}\Delta^N\xi(t), & N-1<\beta<N, \\ \Delta^N\xi(t), & \beta = N, \end{cases}$$

where $t \in \mathbb{N}_{a+N-\beta}$.

Theorem 1 can also hold for the uncertain sequence. It is useful for numerical solutions of chaotic systems and finite-time stability analysis.

3 Finite-time stability

Suppose that the real-valued functions $\mathscr{F}(t,x)$ and $\mathscr{G}(t,x)$ are Lipschitz-continuous. They satisfy the Lipschitz condition

$$\begin{aligned} &\big|\mathscr{F}(t,p)-\mathscr{F}(t,q)\big| + \big|\mathscr{G}(t,p)-\mathscr{G}(t,q)\big| \\ &\quad \leqslant \mathscr{L}|p-q|, \quad t\in\{a,\, a+1,\, \dots,\, a+T\} \quad \forall p,q\in\mathbb{R}, \end{aligned} \tag{4}$$

where

$$\mathscr{L} < \mathscr{S} = \frac{\Gamma(\beta+1)\Gamma(T-a-N+1)}{(1+\mathscr{M})\Gamma(T-a-N+\beta+1)} \quad \text{and} \quad \mathscr{M} = \max\big\{|c|, |d|\big\}.$$

Then the initial-value problem (1) has a unique solution (see [11]).

Lemma 2. (*See* [19].) *Let* x, f, *and* $g\colon \mathbb{N}_a \to \mathbb{R}$. $g(t)$ *is a nondecreasing and nonnegative function. Let* $q(t) = f(t)g(t)$ *and* q *be a nondecreasing function. If* $x(t)$ *satisfies*

$$x(t) \leqslant f(t-1) + \Delta_{a+1-\beta}^{-\beta} g(t+\beta-1)x(t+\beta-1), \quad 0<\beta\leqslant 1,$$

then $x(t)$ *is bounded:*

$$x(t) \leqslant f(t-1)e_\beta\big(g(t-1), (t-\sigma(a))^{(\beta)}\big), \quad t\in\mathbb{N}_{a+1}.$$

If $f(t) = K$, $g(t) = \lambda$ are constants and $x(t)$ satisfies

$$x(t) \leqslant K + \lambda\Delta_{a+1-\beta}^{-\beta}x(t+\beta-1), \quad t\in\mathbb{N}_{a+1},$$

then $x(t)$ is bounded:

$$x(t) \leqslant K e_\beta\bigl(\lambda, \bigl(t-\sigma(a)\bigr)^{(\beta)}\bigr).$$

Here $e_\beta(\lambda,\ (t-\sigma(a))^{(\beta)})$ is the delay discrete-time Mittag-Leffler function [1], and

$$e_\beta\bigl(\lambda, \bigl(t-\sigma(a)\bigr)^{(\beta)}\bigr) := \sum_{k=0}^{+\infty} \frac{\lambda^k (t-a+k\beta-k)^{(k\beta)}}{\Gamma(k\beta+1)}, \quad 0<\beta\leqslant 1,\ t\in\mathbb{N}_{a+1}.$$

We note that the delay discrete-time Mittag-Leffler function is a piece-wise function:

$$e_\nu\bigl(\lambda, \bigl(t-\sigma(a)\bigr)^{(\beta)}\bigr) = \begin{cases} 1+\lambda, & t=a+1,\\ 1+\lambda\frac{(1+\beta)^{(\beta)}}{\Gamma(\beta+1)}+\lambda^2, & t=a+2,\\ \dots & \\ \sum_{k=0}^{n}\frac{\lambda^k(n+k\beta-k)^{(k\beta)}}{\Gamma(k\beta+1)}, & t=a+n. \end{cases}$$

We define a finite-time set $\Omega^* = \{a+1-\beta,\ \dots,\ a+T^*-\beta\}$ with a positive integer number $T^* \leqslant T$.

Definition 6. For given positive real scalars ε, ϱ and T^* satisfying $\varepsilon<\varrho$ and $T^*\leqslant T$, the initial-value problem (1) is said to be finite-time stable under $(\varepsilon,\varrho,T^*)$ almost surely when $|Y_0-Z_0|<\varepsilon$ such that $\mathcal{M}\{|Y_t-Z_t|<\varrho$ for any $t\in\Omega^*\}=1$, where Y_t and Z_t are solutions of the initial-value problem (1) with the initial conditions Y_0 and Z_0.

Theorem 2. *The initial-value problem* (1) *is said to be finite-time stable under* $(\varepsilon,\varrho,T^*)$ *almost surely if* $\mathscr{F}$ *and* $\mathscr{G}$ *satisfy the Lipschitz condition* (4) *and*

$$e_\beta\bigl(\mathscr{L}(1+\mathscr{M}), \bigl(T^*-\sigma(a)\bigr)^{(\beta)}\bigr) < \frac{\varrho}{\varepsilon}.$$

Proof. Let Y_t and Z_t be two solutions of FUDE (1) with initial conditions Y_0 and Z_0, respectively. Then the initial-value problem (1) is equal to the following discrete integral equation by Theorem 1:

$$X_t = xX_0 + \frac{1}{\Gamma(\beta)}\sum_{s=a+1-\beta}^{t-\beta}\bigl(t-\sigma(s)\bigr)^{(\beta-1)}\mathscr{F}\bigl(s+\beta-1, X(s+\beta-1)\bigr) + \mathscr{G}\bigl(s+\beta-1,\ X(s+\beta-1)\bigr)\xi(s+\beta-1).$$

We give

$$\begin{aligned}&\bigl|Y_t(\gamma)-Z_t(\gamma)\bigr| \\ &\quad\leqslant |Y_0-Z_0| + \frac{1}{\Gamma(\beta)}\sum_{s=a+1-\beta}^{t-\beta}\bigl(t-\sigma(s)\bigr)^{(\beta-1)}\bigl[\bigl|\mathscr{F}(s+\beta-1,\ Y(s+\beta-1)\bigr) \\ &\qquad - \mathscr{F}\bigl(s+\beta-1,\ Z(s+\beta-1)\bigr)\bigr| + \mathscr{M}\bigl|\mathscr{G}\bigl(s+\beta-1,\ Y(s+\beta-1)\bigr) \\ &\qquad - \mathscr{G}\bigl(s+\beta-1,\ Z(s+\beta-1)\bigr)\bigr|\bigr]\end{aligned}$$

$$\begin{aligned}
&\leqslant |Y_0 - Z_0| + \frac{1+\mathscr{M}}{\Gamma(\beta)} \sum_{s=a+1-\beta}^{t-\beta} \big(t-\sigma(s)\big)^{(\beta-1)} \big[\big|\mathscr{F}\big(s+\beta-1, Y(s+\beta-1)\big) \\
&\quad - \mathscr{F}\big(s+\beta-1, Z(s+\beta-1)\big)\big| + \big|\mathscr{G}\big(s+\beta-1, Y(s+\beta-1)\big) \\
&\quad - \mathscr{G}\big(s+\beta-1, Z(s+\beta-1)\big)\big|\big] \\
&\leqslant |Y_0 - Z_0| + \frac{\mathscr{L}(1+\mathscr{M})}{\Gamma(\beta)} \sum_{s=a+1-\beta}^{t-\beta} \big(t-\sigma(s)\big)^{(\beta-1)} \big|Y(s+\beta-1) - Z(s+\beta-1)\big| \\
&= |Y_0 - Z_0| + \mathscr{L}(1+\mathscr{M}) \Delta_{a+1-\beta}^{-\beta} \big|Y(t+\beta-1) - Z(t+\beta-1)\big|.
\end{aligned}$$

According to Lemma 2, we get

$$\begin{aligned}
\big|Y_t(\gamma) - Z_t(\gamma)\big| &\leqslant |Y_0 - Z_0| e_\beta\big(\mathscr{L}(1+\mathscr{M}), (t-\sigma(a))^{(\beta)}\big) \\
&\leqslant |Y_0 - Z_0| e_\beta\big(\mathscr{L}(1+\mathscr{M}), (T^*-\sigma(a))^{(\beta)}\big).
\end{aligned}$$

For positive real scalars $\varepsilon < \varrho$ and $T^* \leqslant T$, if the conditions hold $e_\beta(\mathscr{L}(1+\mathscr{M}), (T^*-\sigma(a))^{(\beta)}) < \varrho/\varepsilon$ and $|Y_0 - Z_0| < \varepsilon$, then one can obtain that $|Y_t(\gamma) - Z_t(\gamma)| < \varrho$ for any $t \in \Omega^*$. As a result, this means

$$\mathcal{M}\big\{|Y_t - Z_t| < \varrho,\ t \in \Omega^*\big\} = 1$$

from which the proof is completed. □

Example 1. Consider the initial-value problem

$$\begin{aligned}
{}^C\Delta_a^\beta X(t) &= \vartheta \sin X(t+\beta-1) + \varsigma X(t+\beta-1)\xi(t+\beta-1), \quad 0 < \beta \leqslant 1, \\
X(a) &= X_0, \quad t \in \Omega.
\end{aligned} \tag{5}$$

$\xi(t)$ is a disturbance factor. $\varsigma X(t+\beta-1)\xi(t+\beta-1)$ is a disturbance term. The uncertain variable with an uncertainty measure has an average value defined as the expected value and variance. More conveniently, the uncertainty theory also gives high-order moment estimation, which is very helpful in parameter estimation and machine learning.

By the comparison principle and the existence conditions, we can investigate the boundedness of the system's states.

First, we have the fractional sum equation

$$\begin{aligned}
X(t) &= X_0 \\
&\quad + \frac{1}{\Gamma(\beta)} \sum_{s=a+1-\beta}^{t-\beta} \big(t-\sigma(s)\big)^{(\beta-1)} \big(\vartheta \sin X(s+\beta-1) + \varsigma X(s+\beta-1)\xi(s+\beta-1)\big),
\end{aligned}$$

where $t \in \{a+1, \dots, a+T\}$. Set the parameters $a = 0$, $\vartheta = 0.05$, $\varsigma = 0.1$, $\beta = 0.95$, and $T = 4$. The uncertain variable $\xi(t)$ follows the linear uncertainty distribution $\mathcal{L}(-0.5, 0.5)$. We obtain $\mathscr{L} = 0.15 < \mathscr{S} = 0.1760$. As a result, the initial-value problem (5) has a unique solution.

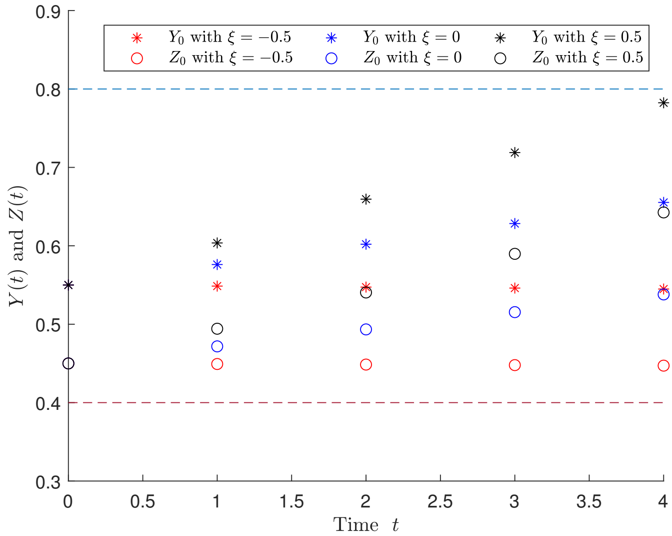

Figure 1. Sample paths of Y_t and Z_t.

Then, for $\varepsilon = 0.11$, $\varrho = 0.4$ and $T^* = 4$, since $e_\beta(\lambda,\ (t-\sigma(a))^{(\beta)})$ is nondecreasing with $\lambda > 0$, we give

$$e_\beta\bigl(\mathscr{L}(1+\mathscr{M}),\bigl(T^*-\sigma(a)\bigr)^{(\beta)}\bigr) = 2.2125 < \frac{\varrho}{\varepsilon} = 3.6364.$$

Thus, the initial-value problem (5) is finite-time stable almost surely by Theorem 2.

We give several possible sample paths for the initial-value problem (5) and illustrate the finite-time stability. Specifically, for $\varepsilon = 0.11$, the initial conditions are set to $Y_0 = 0.45$ and $Z_0 = 0.55$, respectively. Suppose the uncertain variable $\xi(t)$ follows the linear uncertainty distribution $\mathcal{L}(-0.5, 0.5)$. For simplicity, we only choose ξ as -0.5, 0, and 0.5 for numerical simulations.

Finally, these sample paths are numerically illustrated in Fig. 1. All of the paths are located in the interval $[0.4, 0.8]$, which means that the finite-time stability is guaranteed almost surely.

Now we can extend the idea to a high-dimensional case. The fractional recurrent neural network (3) is a system of fractional difference equations in form of

$$\begin{aligned} {}^C\Delta_a^\beta \mathbf{X}(t) &= \bar{\mathscr{F}}\bigl(t+\beta-1,\ \mathbf{X}(t+\beta-1)\bigr) \\ &\quad + \bar{\mathscr{G}}\bigl(t+\beta-1,\ \mathbf{X}(t+\beta-1)\bigr)\xi(t+\beta-1), \\ \mathbf{X}(a) &= \mathbf{X}_\chi, \quad 0 < \beta \leqslant 1,\ t \in \Omega. \end{aligned} \tag{6}$$

Definition 7. For given positive real scalars ε, ϱ, and T^* satisfying $\varepsilon < \varrho$ and $T^* \leqslant T$, the initial-value problem (1) is said to be finite-time stable under $(\varepsilon, \varrho, T^*)$ almost surely when $\|\mathbf{Y}_0 - \mathbf{Z}_0\| < \varepsilon$ such that $\mathcal{M}\{\|\mathbf{Y}_t - \mathbf{Z}_t\| < \varrho$ for any $t \in \Omega^*\} = 1$, where $\mathbf{Y}_t$ and $\mathbf{Z}_t$ are solutions of the initial-value problem (1) with the initial conditions $\mathbf{Y}_0$ and $\mathbf{Z}_0$.

In this paper, the symbol $\|\cdot\|$ means the infinite norm. We set the n-dimensional vector $\rho = (\rho_1, \rho_2, \ldots, \rho_n)^{\mathrm{T}}$ and $n \times n$ matrix $A = (a_{ij})$, then

$$\|\rho\| = \max_{1 \leqslant i \leqslant n} |\rho_i| \quad \text{and} \quad \|A\| = \max_{1 \leqslant i \leqslant n} \sum_{j}^{n} |a_{ij}|.$$

Suppose the real-valued vector functions $\bar{\mathscr{F}}(t, x)$ and $\bar{\mathscr{G}}(t, x)$ are Lipschitz-continuous and satisfy

$$\left\|\bar{\mathscr{F}}(t, \mathbf{p}) - \bar{\mathscr{F}}(t, \mathbf{q})\right\| + \left\|\bar{\mathscr{G}}(t, \mathbf{p}) - \bar{\mathscr{G}}(t, \mathbf{q})\right\| \leqslant \mathscr{L}\|\mathbf{p} - \mathbf{q}\| \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^n \tag{7}$$

with the positive Lipschitz constant

$$\mathscr{L} < \frac{\Gamma(\beta + 1)\Gamma(T - a - N + 1)}{(1 + \mathscr{M})\Gamma(T - a - N + \beta + 1)} \quad \text{and} \quad \mathscr{M} = \max\bigl\{|c|, |d|\bigr\}.$$

Then system (6) with initial conditions almost surely has a unique solution almost surely.

We straightforwardly give the following theorems. The proofs are similar to those of Theorem 2.

Theorem 3. *System* (6) *is said to be finite-time stable under* $(\varepsilon, \varrho, T^*)$ *almost surely if* $\bar{\mathscr{F}}$ *and* $\bar{\mathscr{G}}$ *satisfy the Lipschitz condition* (7) *and*

$$e_\beta\bigl(\mathscr{L}(1 + \mathscr{M}),\ \bigl(T^* - \sigma(a)\bigr)^{(\beta)}\bigr) < \frac{\varrho}{\varepsilon}.$$

By Theorem 3, we can use it in the following two real applications.

Example 2. The fractional uncertain recurrent neural network with uncertainty reads

$$\begin{aligned} {}^{C}\Delta_a^{\beta}\mathbf{X}(t) &= A \tanh \mathbf{X}(t + \beta - 1) + B\mathbf{X}(t + \beta - 1)\xi(t + \beta - 1), \quad 0 < \beta \leqslant 1, \\ X(a) &= X_0, \quad t \in \Omega, \end{aligned}$$

where $\mathbf{X} = (x_1, x_2)^{\mathrm{T}}$, $\tanh \mathbf{X} = (\tanh x_1, \tanh x_2)^{\mathrm{T}}$,

$$A = \begin{pmatrix} 0.05 & 0.04 \\ 0.03 & 0.03 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.08 \end{pmatrix}.$$

First, if the parameters $a = 0$, $\beta = 0.95$, $T = 4$, $\varepsilon = 0.11$, and $\varrho = 0.4$, we have $\mathscr{L} = 0.17 < 0.1760$, and the fractional uncertain recurrent neural network (2) has a unique solution almost surely. Assume that $\mathbf{Y}(t)$ and $\mathbf{Z}(t)$ are the solutions with initial conditions $\mathbf{Y}_0$ and $\mathbf{Z}_0$, respectively.

Now, we set the initial conditions $\mathbf{Y}_0 = (0.45, 0.4)^{\mathrm{T}}$ and $\mathbf{Z}_0 = (0.38, 0.5)^{\mathrm{T}}$ such that $\|\mathbf{Y}_0 - \mathbf{Z}_0\| = 0.1 < \varepsilon$. From Theorem 3 it can be concluded that system (2) is finite-time stable almost surely. That is, within the given time domain $\{a + 1, \ldots, a + T^*\}$, the distance $\|\mathbf{Y}_t - \mathbf{Z}_t\|$ (starting from $\mathbf{Y}_0$ and $\mathbf{Z}_0$ with $\|\mathbf{Y}_0 - \mathbf{Z}_0\| < \varepsilon = 0.11$) does not exceed $\rho = 0.4$. As a result, the finite-time stability is verified in Figs. 2 and 3.

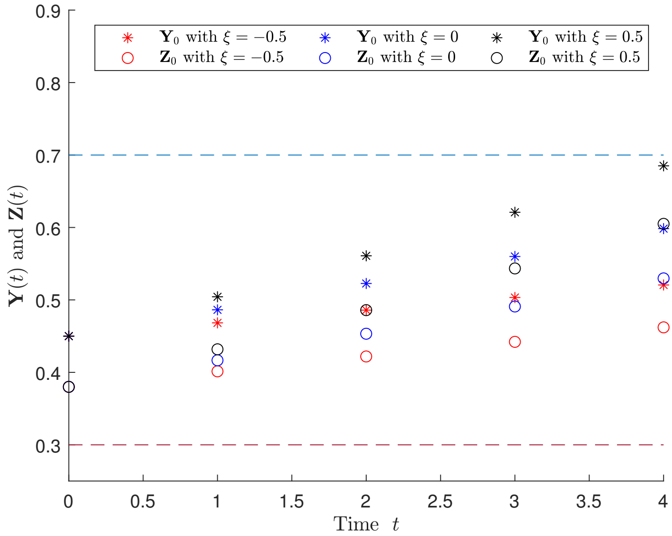

Figure 2. Sample paths of Example 2: $y_1(t)$ and $z_1(t)$.

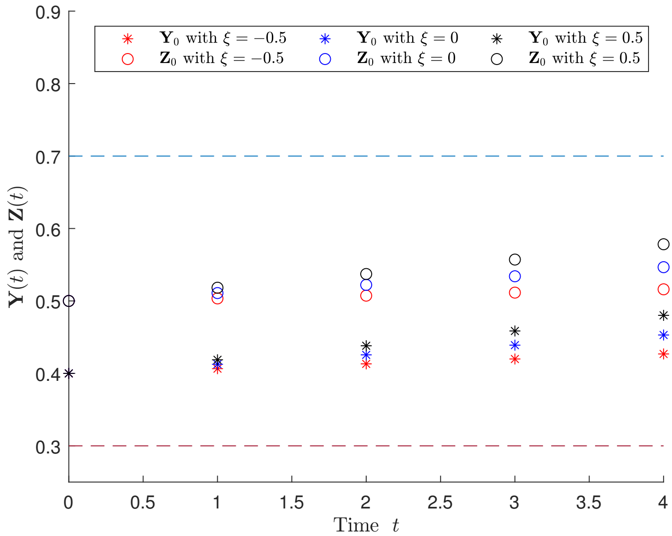

Figure 3. Sample paths of Example 2: $y_2(t)$ and $z_2(t)$.

The Lipschitz condition can be generally given as

$$\begin{aligned} &\left\|\mathscr{F}(t,p) - \mathscr{F}(t,q)\right\| + \left\|\mathscr{G}(t,p) - \mathscr{G}(t,q)\right\| \\ &\quad \leqslant \mathscr{L}(t)\|p-q\| \quad \forall p,q \in \mathbb{R},\ t \in \mathbb{N}_{a+1}, \end{aligned}$$

where the Lipschitz constant becomes a function with respect to t, and

$$\mathscr{L}(t) < \mathscr{S} = \frac{\Gamma(\beta+1)\Gamma(T-a-N+1)}{(1+\mathscr{M})\Gamma(T-a-N+\beta+1)}, \quad t \in \Omega.$$

Theorem 4. *System* (3) *is finite-time stable under* $(\varepsilon, \varrho, T^*)$ *almost surely if there exists*

$$\begin{aligned} T^* = \max\bigl\{t-a\colon\ & e_\beta\bigl(\mathscr{L}(t)(1+\mathscr{M}),\ \bigl(t-\sigma(a)\bigr)^{(\beta)}\bigr) \\ & < \frac{\varrho}{\varepsilon}, \quad t \in \{a+1,\, a+2,\, \dots,\, a+T\}\bigr\}. \end{aligned}$$

Example 3. The fractional uncertain recurrent neural network with time-variable coefficients is given as

$$\begin{aligned} {}^{C}\Delta_a^\beta \mathbf{X}(t) &= A(t+\beta-1)\tanh \mathbf{X}(t+\beta-1) \\ &\quad + B(t+\beta-1)\mathbf{X}(t+\beta-1)\xi(t+\beta-1), \\ \mathbf{X}(a) &= \mathbf{X}_0, \quad 0<\beta\leqslant 1,\ t\in\Omega, \end{aligned}$$

where $\mathbf{X} = (x_1, x_2, \dots, x_m)^{\mathrm{T}}$.

Suppose

$$A(t) = \frac{1}{2}\begin{pmatrix} 0.05 & 0.04 \\ 0.03 & 0.03 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and

$$B(t) = \begin{pmatrix} 0.04\cos t & 0.04\sin t \\ -0.04\sin t & 0.04\cos t \end{pmatrix}.$$

Other parameters are kept the same as those of Example 2. If we set $\rho = 0.4$, we can determine the finite time $T^* = 4$ by $\|Y_t - Z_t\| < \rho/\varepsilon$ and Theorem 4.

4 Conclusions

The fractional discrete recurrent neural networks are a class of fractional difference equations with time delay. We consider the model with uncertain initial-value conditions and provide the numerical schemes. We give the finite-time stability analysis method and determine the finite time through the delay discrete Gronwall inequality.

We note that the fractional uncertain difference equations have a recurrent formula that is more general than the famous time series model

$$x_{n+1} = c_0 x_n + c_1 x_{n-1} + \dots + c_p x_{n-p} + r\varepsilon, \quad 1 \leqslant p < n,$$

which is a linear one. $c_0, \dots, c_p$, and r are unknown parameters to be determined. ε is an uncertain term. It is popularly used as a prediction model. Our nonlinear model holds long memory effects with only fractional-order and system parameters. It also has discrete fractional calculus and uncertainty theories to support it. This new feature can explore new theories and applications when it meets machine learning. We will consider this application in future work.

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