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Controllability of stochastic impulsive integro-differential systems involving nonlocal conditions and conformable derivatives

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Abstract. In this paper, the controllability of a stochastic impulsive integro-differential system involving nonlocal conditions and conformable derivatives is analyzed. The solution of the system is derived by Duhamel's formula using Laplace and inverse Laplace transforms. The controllability result for the linear system is proved by using controllability Grammian matrix, and for the nonlinear integro-differential system, fixed point techniques are used. The applicability of the system is verified by means of an example.

Keywords: controllability, impulsive stochastic system, integro-differential system, conformable derivatives, nonlocal conditions.

1 Introduction

A dynamical nonlinear system is a mathematical model, which is used to refer to a time-dependant behaviour of a system that governs the evolution of the state of the system over time [14, 27]. Nonlinear dynamical systems are used to study various problems in the field of planetary motion control, economic growth models, epidemiological models, neural networks, mechanical and aerospace systems, and so on [8]. A nonlinear system is

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one in which the input and output are not directly dependant on each other resulting in behaviours like bifurcations, chaos, and multiple equilibria that cannot be adequately described by simple linear equations. Hence, the analysis and control of nonlinear systems are often unpredictable and show intricate behaviours [24]. The integro-differential system depends on both its instantaneous rate of change and a cumulative effect of past interactions [16, 30]. A conformable derivative extends the idea of a regular derivative to noninteger orders while keeping its definition simple and similar to classical calculus. It preserves many familiar properties like the product and chain rules, making it easier to understand and apply to real-world problems [18].

Controllability is one of the most important qualitative behaviour. It is often studied as the ability of a system to steer from an initial state to a final state at certain time intervals using appropriate control functions [11, 23]. The study of controllability of linear and nonlinear system is of considerable significance among many researchers [17,28]. In [32], You et al. analysed the relative controllability of delay differential equations, and in [21], Jothilakshmi et al. examined the controllability of integro-delay impulsive differential equations by using a delayed perturbation of Mittag-Leffler functions. Controllability of second-order systems have been studied with great interest using various conditions over the recent years [4, 7, 26]. Impulsive conditions in a system are often referred to as the circumstances where instantaneous and sudden changes happen in the state of the system [3, 29]. The study of systems with impulses is inevitable as it alters the behaviour of the system causing analogous changes [6, 15]. Cabada et al. [12] have investigated a semilinear impulsive noninstantaneous delay system and proved the controllability of the same using fixed point approaches. Compared to the classical initial conditions of a system, the nonlocal initial conditions provide more specific and explicit details concerning the system [1, 13].

The stochastic systems involve random variables or processes that influence the behaviour of the system over a specific period of time that incorporates some sort of randomness or uncertainities to the state of the system [2,20,33]. The existence and controllability of a stochastic system with infinite delay involving conformable derivatives was studied by Huang et al. [19] via measures of noncompactness. The authors utilize advanced mathematical methodologies, such as fixed point theorems and stochastic analysis, to derive sufficient conditions for the controllability of system. This work extends existing theories by incorporating conformable derivatives, which generalize classical derivatives to provide a more accurate representation of real-world phenomena. This contribution is particularly relevant for developing effective control strategies in engineering and applied sciences, where managing the behavior of complex systems under uncertainty is a critical challenge. For instance, similar studies have investigated the controllability of differential systems with conformable derivatives [31], as well as the relative controllability of impulsive-delay conformable differential equations [25].

- The conformable derivatives are more compactible and consistent to functions of different orders of differentiability permitting more precise applications with accuracy.
- They can more effectively prove the principle of action in some real-world issues than the classical fractional derivative.

- It may be regularly used to describe some unique events since the conformable derivative has certain characteristics that the Riemann–Liouville derivative and Caputo derivative do not have and the resulting formula is significantly less complex.
- By focusing on systems with integro-differential equations and impulsive conditions, the study enhances the theoretical understanding and control of complex stochastic systems exhibiting impulsive nature.

Hence, it is inevitable to study the controllability behaviour of such systems using conformable derivatives. To the best of the author's knowledge, there is no prior work concerning to the controllability analysis of stochastic impulsive integro-differential system with nonlocal conditions and conformable derivatives, which is the main inspiration for this work. The structure of the paper is as follows: Section 2 formulates the integro-differential problem and states some important definitions, propositions, and lemmas used to prove the results. Section 3 contains the controllability results of the corresponding linear system and the nonlinear integro-differential system. Section 4 briefs about an example related to the integro-differential system studied. Section 5 concludes the paper.

2 Problem formulation and preliminaries

In this paper, the stochastic impulsive differential system involving conformable derivatives and nonlocal conditions is considered as follows:

$$\mathsf{D}^{\gamma}\big(\mathsf{D}^{\gamma}\big(x(t)\big)\big) = \mathsf{A}x(t) + \mathsf{B}u(t) + \tilde{\ell}\big(t, s, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t} + \tilde{g}\left(t, x(t), \int_0^t \tilde{f}\big(t, s, x(s)\big)\,\mathrm{d}s\right), \quad t \in [0, \zeta], \tag{1}$$

$$x(0) = x_0 + p(x), \tag{2}$$

$$x'(0) = x_1 + q(x), \tag{3}$$

$$x\big(t_i^+\big) - x\big(t_i^-\big) = \mathsf{M}_i\big(x(t_i)\big), \quad i = 1, 2 \dots n, \tag{4}$$

$$x'\big(t_i^+\big) - x'\big(t_i^-\big) = \mathsf{N}_i\big(x(t_i)\big), \quad i = 1, 2, \dots n, \tag{5}$$

where $\mathsf{D}^{\gamma}(t)$ denotes the conformable derivative of order $0 < \gamma \leqslant 1$. A is the infinitesimal generator of a cosine family $\{\mathsf{C}(t), \mathsf{S}(t)\}_{t\in\mathbb{R}}$ on $(\mathfrak{B}, \|\cdot\|)$. B is a bounded linear operator of $\mathsf{V} \to \mathfrak{B}$, where V is a Hilbert space. The control function $u(\cdot)$ is an element of $\mathfrak{L}^2([0, \zeta], \mathsf{V})$. The elements x_0 and x_1 are two fixed vectors in $\mathfrak{B}$. We denote a space of piecewise continuous functions $\widehat{PC}([0, \zeta], \mathfrak{B}) = \{x : [0, \zeta] \to \mathfrak{B}\}$ such that $x \in \widehat{PC}([0, \zeta], \mathfrak{B})$ with norm defined as $|x|_{\widehat{PC}} = \sup_{t\in[0,\zeta]} \|x(t)\|$. Also, the continuous functions are defined such that $\tilde{g} : [0, \zeta] \times \mathfrak{B} \to \mathfrak{B}$, $f : [0, \zeta] \times \mathfrak{B}$, $p : \widehat{PC} \to \mathfrak{B}$, $q : \widehat{PC} \to \mathfrak{B}$ with its values in $\mathfrak{B}$. The complete probability space is defined by $(\Omega, \mathfrak{F}_t, \mathsf{P})$ generated with the filtration $\mathfrak{F}_t$ and the covariance operator Q such that $\operatorname{tr}\mathsf{Q} < \infty$. $W(t)$ is a Wiener process with $\mathfrak{F}_t$ generated by $W(\zeta)$, $0 \leqslant \zeta \leqslant t$, and $\tilde{\ell}$ is a continuous function defined as $\tilde{\ell} : [0, \zeta] \times \mathfrak{B} \to \mathfrak{B}$. Here (4) and (5) denote the impulsive conditions such that the function $\varkappa(\cdot)$ is continuous on each interval $0 < t_i \leqslant t_{i+1}$ with right and left limits given by $\varkappa(t_i^+), \varkappa'(t_i^+)$ and $\varkappa(t_i^-), \varkappa'(t_i^-)$.

Definition 1. (See [24].) The conformable derivative of order γ, $0 < \gamma \leqslant 1$, of the function $\varkappa(\cdot)$, where $\varkappa(\cdot) : [0, \infty) \to \mathbb{R}$, is given by

$$\mathsf{D}^\gamma \varkappa(t) = \lim_{\widehat{\Psi} \to 0} \frac{\varkappa(t + \widehat{\Psi} t^{1-\gamma}) - \varkappa(t)}{\widehat{\Psi}}$$

for $t > 0$, and if the limit exists,

$$\mathsf{D}^\gamma \varkappa(0) = \lim_{t \to 0^+_-} \mathsf{D}^\gamma \varkappa(t).$$

Any conformable integral $\mathsf{I}^\gamma(\cdot)$ for function $\varkappa(\cdot)$ is given by

$$\mathsf{I}^\gamma \varkappa(t) = \int_0^t \jmath^{\gamma-1} \varkappa(\jmath) \, \mathrm{d}\jmath$$

for $t > 0$.

Lemma 1. (*See* [24].) *If $\varkappa(\cdot)$ is a continuous and differential function in the domain $\mathsf{I}^\gamma(\cdot)$, then the following holds*:

$$\mathsf{D}^\gamma\bigl(\mathsf{I}^\gamma(t)\bigr) = \varkappa(t), \qquad \mathsf{I}^\gamma\bigl(\mathsf{D}^\gamma \varkappa(\cdot)\bigr)(t) = \varkappa(t) - \varkappa(0).$$

Proposition 1. (*See* [24].) *For a differentiable function $\varkappa(\cdot)$, the following result is true*:

$$\mathfrak{L}_\gamma(\mathsf{D}^\gamma \varkappa(t))(\rho) = \rho \mathfrak{L}_\gamma\bigl(\varkappa(t)\rho\bigr) - \varkappa(0),$$

where $\mathfrak{L}_\gamma$ denotes the Laplace transform of conformable derivatives of order γ of a function $\varkappa(\cdot)$, which can be given as follows:

$$\mathfrak{L}_\gamma(\varkappa(t))\rho = \int_0^\infty t^{\gamma-1} \mathrm{e}^{-\rho t^\gamma/\gamma} \varkappa(t) \mathrm{d}t, \quad \rho > 0.$$

Proposition 2. (*See* [24].) *For a twice differential function $\varkappa(\cdot)$, the following result is true*:

$$\mathfrak{L}_\gamma\bigl(\mathsf{D}^\gamma\bigl(\mathsf{D}^\gamma\bigl(\varkappa(t)\bigr)\bigr)\bigr)(\rho) = \rho^2 \mathfrak{L}_\gamma\bigl(\varkappa(t)\bigr)(\rho) - \rho\varkappa(0) - \varkappa'(0).$$

Definition 2. (See [10].) Consider a cosine family $(\mathsf{C}(t))_{t\in\mathbb{R}}$ of one parameter with bounded linear operators. Then $(\mathsf{C}(t))_{t\in\mathbb{R}}$ is said to be a strongly continuous cosine family iff it satisfies the following conditions:

- When I is an identity operator, $\mathsf{C}(0) = \mathsf{I}$.
- $\mathsf{C}(\jmath + t) + \mathsf{C}(\jmath - t) = 2\mathsf{C}(\jmath)\mathsf{C}(t)$ for all $\jmath, t \in \mathbb{R}$.
- For each $\varkappa \in \mathfrak{B}$, a function t, defined as $t \to \mathsf{C}(t)\varkappa$, is strongly continuous.

Let $(\mathsf{S}(t))_{t\in\mathbb{R}}$ be the sine family associated with $(\mathsf{C}(t))_{t\in\mathbb{R}}$. Then $(\mathsf{S}(t))_{t\in\mathbb{R}}$ can be defined as follows:

$$\mathsf{S}(t)z = \int_0^t \mathsf{C}(s)z\,\mathrm{d}s, \quad z \in \mathfrak{B}.$$

Let A be the infinitesimal generator of a strongly continuous cosine family $(\mathsf{C}(t), \mathsf{S}(t))_{t\in\mathbb{R}}$. Then for any $t \to \mathsf{C}(t)z$, which is continuous and twice differentiable, the domain of A can be defined by

$$\mathscr{D}(\mathsf{A}) = \{z \in \mathfrak{B}\}: \quad \mathsf{A}z = (\mathsf{D}^\gamma)^2\mathsf{C}(0)z, \quad z \in \mathscr{D}(\mathsf{A}).$$

Remark 1. (See [10].) Arbitrary functions $z(t)$ and $z_1(t)$ satisfies the following property of Laplace transforms:

$$\mathfrak{L}_\gamma \int_0^t s^{\gamma-1} z\left(\frac{t^\gamma - s^\gamma}{\gamma}\right) z_1(s)\,\mathrm{d}s(\rho) = \mathfrak{L}\big(z(t)(\rho)\big)\mathfrak{L}\big(z_1(t)(\rho)\big).$$

3 Controllability results

In this section, Duhamel's formula is used to derive the solution of system (1)–(5).

At any $t \in [0, \zeta]$, applying the Laplace transform to system (1)–(5), we get

$$\mathfrak{L}_\gamma\big[\mathsf{D}^\gamma\big(\mathsf{D}^\gamma\big(z(t)\big)\big)\big]\rho = \rho^2\mathfrak{L}_\gamma\big(z(t)\big)(\rho) - \rho\big(z_0 + p(z)\big) - \big(z_1 + q(z)\big).$$

By Proposition 2, we get

$$\begin{aligned}\mathfrak{L}_\gamma\big[\mathsf{D}^\gamma\big(z(t)\big)\big](\rho) &= \rho\big(\rho^2\mathsf{I} - \mathsf{A}\big)^{-1}\big[z_0 + p(z)\big] + \big(\rho^2\mathsf{I} - \mathsf{A}\big)^{-1}\big[z_1 + q(z)\big] \\ &\quad + \big(\rho^2\mathsf{I} - \mathsf{A}\big)^{-1}\mathfrak{L}_\gamma\left(\tilde{\ell}\big(t, s, z(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right) \\ &\quad + \big(\rho^2\mathsf{I} - \mathsf{A}\big)^{-1}\mathfrak{L}_\gamma\left(\tilde{g}\left(t, z(t), \int_0^t \tilde{f}\big(t, s, z(s)\big)\,\mathrm{d}s\right)\right) \\ &\quad + \mathsf{B}u(t)(\rho).\end{aligned}$$

By applying the inverse Laplace transform and Remark 1, we have

$$\begin{aligned}z(t) &= \mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\big[z_0 + p(z)\big] + \mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\big[z_1 + q(z)\big] \\ &\quad + \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\ell}\big(s, z(s)\big)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s \\ &\quad + \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{g}\left(s, z(s), \int_0^t \tilde{f}\big(t, r, z(r)\big)\,\mathrm{d}r + \mathsf{B}u(s)\right)\right)\mathrm{d}s,\end{aligned}$$

where $\{\mathsf{C}(t), \mathsf{S}(t)\}$ is a one parameter cosine family of bounded linear operators that satisfies the conditions given in Definition 2.

The following definition provides the solution of the considered system.

Definition 3. A stochastic process $x \in \widehat{PC}([0,\zeta], \mathfrak{B})$ is a mild solution of (1)–(5) if for each $u \in \mathfrak{L}^2([0,\zeta], \mathsf{V})$, $x(t)$ is $\mathfrak{F}_t$-adapted and measurable for each $t \geqslant 0$ and it satisfies the integral equation

$$\begin{aligned} x(t) = {} & \mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\left[x_0 + p(x)\right] + \mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\left[x_1 + q(x)\right] \\ & + \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\sigma}\left(s, x(s)\right)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s \\ & + \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{g}\left(s, x(s), \int_0^t \tilde{f}\left(t, r, x(r)\right)\mathrm{d}r\right) + \mathsf{B}u(s)\right)\mathrm{d}s \\ & + \sum_{i=1}^m \mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\left(x(t_i)\right) + \sum_{i=1}^m \mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\left(x(t_i)\right). \end{aligned}$$

Definition 4. For any $\tilde{x} \in \widehat{PC}([0,\zeta], \mathfrak{B})$, system (1)–(5) is controllable on $[0,\zeta]$ if a control u exists such that $u \in \mathfrak{L}^2([0,\zeta], \mathsf{V})$ and the mild solution $x(t)$ of (1)–(5) should satisfy the condition $x(\zeta) = \tilde{x}$.

3.1 Linear system

Consider a stochastic linear impulsive differential system as follows:

$$\mathsf{D}^\gamma\left(\mathsf{D}^\gamma\left(x(t)\right)\right) = \mathsf{A}x(t) + \mathsf{B}u(t) + \tilde{\sigma}(t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in [0,\zeta], \tag{6}$$

$$x(0) = x_0 + p(x), \tag{7}$$

$$x'(0) = x_1 + q(x), \tag{8}$$

$$x\left(t_i^+\right) - x\left(t_i^-\right) = \mathsf{M}_i\left(x(t_i)\right), \quad i = 1, 2 \ldots, n, \tag{9}$$

$$x'\left(t_i^+\right) - x'\left(t_i^-\right) = \mathsf{N}_i\left(x(t_i)\right), \quad i = 1, 2, \ldots, n. \tag{10}$$

Here D^γ, A, B, $\tilde{\sigma}$, $W(t)$ are defined as in Section 2.

Theorem 1. *The stochastic linear impulsive differential system* (6)–(10) *is controllable on* $t \in [0,\zeta]$ *iff the controllability Grammian matrix*

$$\widetilde{\mathbb{W}}(t) = \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\mathsf{B}\mathsf{B}^*\mathsf{S}^*\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\mathrm{d}s$$

is positive definite for some $\zeta > 0$.

Proof. The positive definite Grammian matrix is not singular. Hence, the inverse can be defined as $\widetilde{\mathbb{W}}^{-1}$. The control function can be defined as follows:

$$\begin{aligned}u(t) = {}& s^{\gamma-1}\mathsf{B}^*\mathsf{S}^*\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\widetilde{\mathbb{W}}^{-1}\Bigg[\tilde{z} - \mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_0 + p(z)\big] \\ & - \mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_1 + q(z)\big] - \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\sigma}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s \\ & - \sum_{i=1}^m \mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) - \sum_{i=1}^m \mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big)\Bigg].\end{aligned}$$

Here $*$ indicates the transpose of matrix. The system gets steered from 0 to $\tilde{z}$ by the control $u(t)$ at $t = \zeta$ based on starting points $z_0 = z_1 = \dots = 0$ and ending point $\tilde{z} = y$. The solution of the linear system is

$$\begin{aligned}z(t) = {}& \mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\big[z_0 + p(z)\big] + \mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\big[z_1 + q(z)\big] \\ & + \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\sigma}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s} + \mathsf{B}u(s)\right)\mathrm{d}s \\ & + \sum_{i=1}^m \mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) + \sum_{i=1}^m \mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big). \qquad (11)\end{aligned}$$

Substituting $t = \zeta$ in (11), we get

$$\begin{aligned}z(\zeta) = {}& \mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_0 + p(z)\big] + \mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_1 + q(z)\big] \\ & + \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\sigma}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s} + \mathsf{B}u(s)\right)\mathrm{d}s \\ & + \sum_{i=1}^m \mathsf{C}\left(\frac{\zeta^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) + \sum_{i=1}^m \mathsf{S}\left(\frac{\zeta^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big).\end{aligned}$$

Substituting $u(t)$ in above equation, we get

$$\begin{aligned}z(\zeta) = {}& \mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_0 + p(z)\big] + \mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_1 + q(z)\big] \\ & + \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\sigma}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s \\ & + \mathsf{B}\Bigg[s^{\gamma-1}\mathsf{B}^*\mathsf{S}^*\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\widetilde{\mathbb{W}}^{-1}\Bigg[\tilde{z} - \mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[z_0 + p(z)\big]\end{aligned}$$

$$
\begin{aligned}
&- \mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\left[z_1 + q(z)\right] - \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\ell}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s \\
&- \sum_{i=1}^{m}\mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) - \sum_{i=1}^{m}\mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big)\Bigg]\Bigg] \\
&+ \sum_{i=1}^{m}\mathsf{C}\left(\frac{\zeta^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) + \sum_{i=1}^{m}\mathsf{S}\left(\frac{\zeta^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big), \\
= {}& \tilde{z}.
\end{aligned}
$$

Next, to prove that $\widetilde{\mathbb{W}}$ is positive definite, consider some $y \neq 0$ such that $y^*\widetilde{\mathbb{W}}y = 0$.

$$
y^*\int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\mathsf{B}\mathsf{B}^*\mathsf{S}^*\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\mathrm{d}s\, y = 0
$$

on $[0,\zeta]$. Then

$$
\begin{aligned}
z(t) = y = {}& \mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\left[z_0 + p(z)\right] + \mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\left[z_1 + q(z)\right] \\
&+ \int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\ell}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s} + \mathsf{B}u\big(s\big)\right)\mathrm{d}s \\
&+ \sum_{i=1}^{m}\mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) + \sum_{i=1}^{m}\mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big).
\end{aligned}
$$

Then

$$
\begin{aligned}
y^*y = {}& y^*\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\left[z_0 + p(z)\right] + y^*\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\left[z_1 + q(z)\right] \\
&+ y^*\int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\ell}(s)\frac{\mathrm{d}W(s)}{\mathrm{d}s} + \mathsf{B}u(s)\right)\mathrm{d}s \\
&+ y^*\sum_{i=1}^{m}\mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(z(t_i)\big) + y^*\sum_{i=1}^{m}\mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(z(t_i)\big).
\end{aligned}
\tag{12}
$$

But (12) tends to zero. Hence,

$$
y^*y = 0 \quad\Longrightarrow\quad y = 0,
$$

which is a contradiction. Hence, the linear system (6)–(10) is controllable. □

3.2 Nonlinear system

For the controllability of the impulsive stochastic integro-differential system, the following assumptions are to be taken into account:

($\mathfrak{H}_1$) There exist continuous functions $\tilde{g}(t)$, $\tilde{f}(t)$ such that $\tilde{g} : [0, x] \to \mathfrak{B}$ and $\tilde{f} : [0, x] \to \mathfrak{B}$ for all $x \in \mathfrak{B}$. Then for some positive constants G_1, G_2, F_1, F_2 and for some $x, x_1 \in \mathfrak{B}$,

$$\mathbf{E}\big\|\tilde{g}(t, x)\big\| \leqslant G_1\|1 + x\|, \qquad \mathbf{E}\big\|\tilde{g}(t, x) - \tilde{g}(t, x_1)\big\| \leqslant G_2\|x - x_1\|$$

and

$$\mathbf{E}\Bigg\|\int_0^t \tilde{f}\big(t, s, x(s)\big)\,\mathrm{d}s\Bigg\| \leqslant F_1\|1 + x\|,$$

$$\mathbf{E}\Bigg\|\int_0^t \tilde{f}\big(t, s, x(s)\big)\,\mathrm{d}s - \int_0^t \tilde{f}\big(t, s, x_1(s)\big)\,\mathrm{d}s\Bigg\| \leqslant F_2\|x - x_1\|.$$

($\mathfrak{H}_2$) The function $\tilde{\ell}$ is continuous, and there exist constants B_1 and B_2 such that

$$\mathbf{E}\big\|\tilde{\ell}(t, x)\big\| \leqslant B_1\|1 + x\| \quad \text{and} \quad \mathbf{E}\big\|\tilde{\ell}(t, x) - \tilde{\ell}(t, x_1)\big\| \leqslant B_2\|x - x_1\|.$$

($\mathfrak{H}_3$) The functions $p : \widehat{PC} \to \mathfrak{B}$ and $q : \widehat{PC} \to \mathfrak{B}$ are continuous such that for any positive constants P_1, P_2 and Q_1, Q_2,

$$\mathbf{E}\big\|p(x)\big\| \leqslant P_1|1 + x|_{\widehat{PC}}, \qquad \mathbf{E}\big\|p(x) - p(x_1)\big\| \leqslant P_2|x - x_1|_{\widehat{PC}},$$
$$\mathbf{E}\big\|q(x)\big\| \leqslant Q_1|1 + x|_{\widehat{PC}}, \qquad \mathbf{E}\big\|q(x) - q(x_1)\big\| \leqslant Q_2|x - x_1|_{\widehat{PC}}$$

for every $x, x_1 \in \widehat{PC}$.

($\mathfrak{H}_4$) For any constants M_1, M_2, N_1, $N_2 > 0$,

$$\mathbf{E}\big\|\mathsf{M}_i\big(x(t_i)\big)\big\| \leqslant M_1|1 + x|_{\widehat{PC}}$$
$$\mathbf{E}\big\|\mathsf{M}_i\big(x(t_i)\big) - \mathsf{M}_i\big(x_1(t_i)\big)\big\| \leqslant M_2|x - x_1|_{\widehat{PC}}$$

and

$$\mathbf{E}\big\|\mathsf{N}_i\big(x(t_i)\big)\big\| \leqslant N_1|1 + x|_{\widehat{PC}}$$
$$\mathbf{E}\big\|\mathsf{N}_i\big(x(t_i)\big) - \mathsf{N}_i\big(x_1(t_i)\big)\big\| \leqslant N_2|x - x_1|_{\widehat{PC}}$$

for all $x, x_1 \in \widehat{PC}$.

($\mathfrak{H}_5$) Consider a bounded linear operator $\widetilde{\mathbb{W}} : \mathfrak{L}^2([0, \zeta], \mathsf{V}) \to \mathfrak{B}$ defined by $\widetilde{\mathbb{W}}(u) = \int_0^\zeta s^{\gamma-1}\mathsf{S}((\zeta^\gamma - s^\gamma)/\gamma)\mathsf{B}u(s)\,\mathrm{d}s$, which has an operator $\widetilde{\mathbb{W}}^{-1}$ that is induced and inverse with the values in $c = \mathfrak{L}^2([0, \zeta], \mathsf{V})/\ker(\widetilde{\mathbb{W}})$ and for any positive constants $\mathbb{Z}_1$ and $\mathbb{Z}_2$,

$$\|\mathsf{B}\| \leqslant \mathbb{Z}_1 \quad \text{and} \quad \|\widetilde{\mathbb{W}}^{-1}\| \leqslant \mathbb{Z}_2.$$

Theorem 2. *The stochastic impulsive integro-differential system* (1)–(5) *is controllable on* $[0,\zeta]$ *if hypotheses* ($\mathfrak{H}_1$)–($\mathfrak{H}_5$) *hold, provided that*

$$\begin{aligned}&\left[1+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\right]\left(7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\right.\\ &\qquad\times\max\left(P_1+\sum_{t_i\in(0,t)}M_1,P_2+\sum_{t_i\in(0,t)}M_2\right)+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\\ &\qquad\left.\times\max\left(Q_1+\sum_{t_i\in(0,t)}N_1+\frac{\zeta^\gamma}{\gamma}(B_1+G_1F_1),\ Q_2+\sum_{t_i\in(0,t)}N_2+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)\right)\right)\\ &\quad<1.\end{aligned}$$

Proof. For any arbitrary $x(\cdot)$, a control function $u_x(t)$ can be defined by using hypotheses ($\mathfrak{H}_4$), ($\mathfrak{H}_5$) as follows:

$$\begin{aligned}u_x(t)=\widetilde{\mathbb{W}}(u)^{-1}&\left(\tilde{x}-\mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\left[x_0+p(x)\right]-\mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\left[x_1+q(x)\right]\right.\\ &-\int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-s^\gamma}{\gamma}\right)\left(\tilde{b}\left(s,x(s)\right)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s\\ &-\int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-s^\gamma}{\gamma}\right)\left(\tilde{g}\left(s,x(s),\int_0^t\tilde{f}\left(t,r,x(r)\right)\mathrm{d}r\right)\right)\mathrm{d}s\\ &\left.-\sum_{i=1}^m\mathsf{C}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\mathsf{M}_i\left(x(t_i)\right)-\sum_{i=1}^m\mathsf{S}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\mathsf{N}_i\left(x(t_i)\right)\right)(t).\end{aligned}\tag{13}$$

For using this control, it is necessary to define an operator $\widehat{\Psi}(x)(t)$ as follows:

$$\begin{aligned}\widehat{\Psi}(x)(t)=\mathsf{C}&\left(\frac{t^\gamma}{\gamma}\right)\left[x_0+p(x)\right]+\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\left[x_1+q(x)\right]\\ &+\int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-s^\gamma}{\gamma}\right)\left(\tilde{b}\left(s,x(s)\right)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s\\ &+\int_0^t s^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-s^\gamma}{\gamma}\right)\left(\tilde{g}\left(s,x(s),\int_0^t\tilde{f}\left(t,r,x(r)\right)\mathrm{d}r\right)+\mathsf{B}u(s)\right)\mathrm{d}s\\ &+\sum_{i=1}^m\mathsf{C}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\mathsf{M}_i\left(x(t_i)\right)+\sum_{i=1}^m\mathsf{S}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\mathsf{N}_i\left(x(t_i)\right).\end{aligned}\tag{14}$$

Now, introduce $B_r=\{x\in\widehat{PC}([0,\zeta],\mathfrak{B}),\ |x|_{\widehat{PC}}\leqslant r\}$ with a positive radius r, and we denote $|\cdot|$ as the norm in the space of bounded operators defined by $\mathfrak{B}$ into itself. First, we prove that $\widehat{\Psi}\varkappa(t)$ has a fixed point, which acts as a mild solution of system (1)–(5). For this, it is necessary to show that there exists a positive radius β such that $\beta(\varkappa)(t)$ maps $B_\beta \to B_\beta$.

Then from (14), for any $\varkappa \in \widehat{PC}([0,\zeta],\mathfrak{B})$ and $t \in [0,\zeta]$, we have

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)\big\| \\
&\quad\leqslant 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\big\|\varkappa_0+\wp(\varkappa)\big\| + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\Bigg[\big\|\varkappa_1+q(\varkappa)\big\| \\
&\qquad + \int_0^t \jmath^{\gamma-1}\Bigg\|\Bigg(\tilde{b}\big(\jmath,\varkappa(\jmath)\big)\frac{\mathrm{d}W(\jmath)}{\mathrm{d}\jmath} + \tilde{g}\Bigg(\jmath,\varkappa(\jmath),\int_0^t \tilde{f}\big(t,r,\varkappa(r)\big)\,\mathrm{d}r\Bigg) \\
&\qquad\qquad + \mathsf{B}u_\varkappa(\jmath)\Bigg)\Bigg\|\Bigg]\,\mathrm{d}\jmath \\
&\qquad + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\sum_{i=1}^m\big\|\mathsf{M}_i\big(\varkappa(t_i)\big)\big\| + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\sum_{i=1}^m\big\|\mathsf{N}_i\big(\varkappa(t_i)\big)\big\|.
\end{aligned}$$

From hypotheses $(\mathfrak{H}_1)$–$(\mathfrak{H}_5)$

$$\begin{aligned}
&\mathbf{E}\|\widehat{\Psi}(\varkappa)(t)\| \\
&\quad\leqslant 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\Bigg[\|\varkappa_0\| + P_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} M_1|1+\varkappa|_{\widehat{PC}}\Bigg] \\
&\qquad + 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\Bigg[\|\varkappa_1\| + Q_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} N_1|1+\varkappa|_{\widehat{PC}} \\
&\qquad + \frac{\zeta^\gamma}{\gamma}(B_1+G_1F_1)|1+\varkappa|_{\widehat{PC}}\Bigg] + 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left(\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\big\|u_\varkappa(\jmath)\big\|_c\right).
\end{aligned}$$

Now, for the control function using Eq. (13), we obtain

$$\begin{aligned}
&\mathbf{E}\big\|u_\varkappa(t)\big\|_c \\
&\quad\leqslant 7\mathbf{E}\Bigg[\|\widetilde{\mathbb{W}}(u)^{-1}\|\Bigg\|\Bigg(\tilde{\varkappa} - \mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[\varkappa_0+\wp(\varkappa)\big] - \mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[\varkappa_1+q(\varkappa)\big] \\
&\qquad - \int_0^t \jmath^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-\jmath^\gamma}{\gamma}\right)\left(\tilde{b}\big(\jmath,\varkappa(\jmath)\big)\frac{\mathrm{d}W(\jmath)}{\mathrm{d}\jmath}\right)\mathrm{d}\jmath \\
&\qquad - \int_0^t \jmath^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-\jmath^\gamma}{\gamma}\right)\tilde{g}\Bigg(\jmath,\varkappa(\jmath),\int_0^t \tilde{f}\big(t,r,\varkappa(r)\big)\,\mathrm{d}r\Bigg)\,\mathrm{d}\jmath \\
&\qquad - \sum_{i=1}^m \mathsf{C}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\mathsf{M}_i\big(\varkappa(t_i)\big) - \sum_{i=1}^m \mathsf{S}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\mathsf{N}_i\big(\varkappa(t_i)\big)\Bigg)(t)\Bigg\|\Bigg].
\end{aligned}$$

From hypotheses $(\mathfrak{H}_1)$–$(\mathfrak{H}_5)$ we have

$$\begin{aligned}
&\mathbf{E}\big\|u_{\varkappa}(t)\big\|_{c} \\
&\quad\leqslant 7\mathbb{Z}_2\biggl[\|\tilde{\varkappa}\| + \sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{C}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl(\|\varkappa_0\| + P_1|1+\varkappa|_{\widehat{PC}} \\
&\qquad + \sum_{t_i\in(0,t)} M_1|1+\varkappa|_{\widehat{PC}}\biggr) + \sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl(\|\varkappa_1\| + Q_1|1+\varkappa|_{\widehat{PC}} \\
&\qquad + \sum_{t_i\in(0,t)} N_1|1+\varkappa|_{\widehat{PC}} + (B_1+G_1F_1)|1+\varkappa|_{\widehat{PC}}\frac{\zeta^\gamma}{\gamma}\biggr)\biggr].
\end{aligned}$$

Substituting the control function $\|u_{\varkappa}(t)\|_{c}$ in (18), we get

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)\big\| \\
&\quad\leqslant 7\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{C}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl[\|\varkappa_0\| + P_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} M_1|1+\varkappa|_{\widehat{PC}}\biggr] \\
&\qquad + 7\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl[\|\varkappa_1\| + Q_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} N_1|1+\varkappa|_{\widehat{PC}} \\
&\qquad + \frac{\zeta^\gamma}{\gamma}(B_1+G_1F_1)|1+\varkappa|_{\widehat{PC}}\biggr] + 49\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl(\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\biggl[\|\tilde{\varkappa}\| \\
&\qquad + \sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{C}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl(\|\varkappa_0\| + P_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} M_1|1+\varkappa|_{\widehat{PC}}\biggr) \\
&\qquad + \sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl(\|\varkappa_1\| + Q_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} N_1|1+\varkappa|_{\widehat{PC}} \\
&\qquad + (B_1+G_1F_1)|1+\varkappa|_{\widehat{PC}}\frac{\zeta^\gamma}{\gamma}\biggr)\biggr)\biggr].
\end{aligned}$$

By simple computations

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)\big\| \\
&\quad\leqslant 7\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{C}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl[\|\varkappa_0\| + P_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} M_1|1+\varkappa|_{\widehat{PC}}\biggr] \\
&\qquad \times\biggl(1 + 7\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggr) + 7\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggl[\|\varkappa_1\| \\
&\qquad + Q_1|1+\varkappa|_{\widehat{PC}} + \sum_{t_i\in(0,t)} N_1|1+\varkappa|_{\widehat{PC}} + \frac{\zeta^\gamma}{\gamma}(B_1+G_1F_1)|1+\varkappa|_{\widehat{PC}}\biggr] \\
&\qquad \times\biggl(1 + 7\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\biggr) + 49\sup_{0\leqslant t\leqslant\zeta}\biggl|\mathsf{S}\biggl(\frac{t^\gamma}{\gamma}\biggr)\biggr|\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\|\tilde{\varkappa}\|.
\end{aligned}$$

Separating the terms, we obtain

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)\big\| \\
&\leqslant \left(1+7\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\right)\left[\left[7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left(P_1+\sum_{t_i\in(0,t)}M_1\right)\right.\right. \\
&\quad\left.+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left(Q_1+\sum_{t_i\in(0,t)}N_1+(B_1+G_1F_1)\right)\right]|1+\varkappa|_{\widehat{PC}} \\
&\quad\left.+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\|\varkappa_1\|+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\|\varkappa_0\|\right] \\
&\quad+49\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\|\tilde{\varkappa}\|.
\end{aligned}$$

Hence, we can choose β such that it suffices to consider β as a solution in radius r of the following inequality:

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)\big\| \\
&\leqslant \left(1+7\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\right)\left[\left[7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left(P_1+\sum_{t_i\in(0,t)}M_1\right)\right.\right. \\
&\quad\left.+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left(Q_1+\sum_{t_i\in(0,t)}N_1+(B_1+G_1F_1)\right)\right]|1+\varkappa|_{\widehat{PC}} \\
&\quad\left.+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\|\varkappa_1\|+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\|\varkappa_0\|\right] \\
&\quad+49\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\|\tilde{\varkappa}\| \\
&\leqslant r.
\end{aligned}$$

Therefore B_β maps into itself, that is $B_\beta\to B_\beta$. Next, we prove that the operator $\widehat{\Psi}(\varkappa)(t)$ is a contraction operator on B_β. Consider some $\varkappa,\varkappa_1\in\widehat{PC}([0,\zeta],\mathfrak{B})$, then

$$\begin{aligned}
&\widehat{\Psi}(\varkappa)(t)-\widehat{\Psi}(\varkappa_1)(t) \\
&\quad=\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\big[\wp(\varkappa)-\wp(\varkappa_1)\big]+\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\big[q(\varkappa)-q(\varkappa_1)\big] \\
&\qquad+\int_0^t \jmath^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-\jmath^\gamma}{\gamma}\right)\left(\big(\tilde{\ell}\big(\jmath,\varkappa(\jmath)\big)-\tilde{\ell}\big(\jmath,\varkappa_1(\jmath)\big)\big)\frac{\mathrm{d}W(\jmath)}{\mathrm{d}\jmath}\right)\mathrm{d}\jmath \\
&\qquad+\int_0^t \jmath^{\gamma-1}\mathsf{S}\left(\frac{t^\gamma-\jmath^\gamma}{\gamma}\right)\left(\tilde{g}\left(\jmath,\varkappa(\jmath),\int_0^t\tilde{f}\big(t,r,\varkappa(r)\big)\,\mathrm{d}r\right)\right.
\end{aligned}$$

$$- \tilde{g}\left(s, x_1(s), \int_0^t \tilde{f}\big(t, r, x_1(r)\big)\,\mathrm{d}r\right) + \mathsf{B}\big(u_x(s) - u_{x_1}(s)\big)\Bigg)\,\mathrm{d}s$$
$$+ \sum_{i=1}^m \mathsf{C}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\big(\mathsf{M}_i\big(x(t_i)\big) - \mathsf{M}_i\big(x_1(t_i)\big)\big)$$
$$+ \sum_{i=1}^m \mathsf{S}\left(\frac{t^\gamma - t_i^\gamma}{\gamma}\right)\big(\mathsf{N}_i\big(x(t_i)\big) - \mathsf{N}_i\big(x_1(t_i)\big)\big).$$

Now

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(x)(t) - \widehat{\Psi}(x_1)(t)\big\| \\
&\quad\leqslant 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\big\|p(x) - p(x_1)\big\| + \sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\big\|q(x) - q(x_1)\big\| \\
&\qquad + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\int_0^t s^{\gamma-1}\left\|\big(\tilde{\sigma}\big(s, x(s)\big) - \tilde{\sigma}\big(s, x_1(s)\big)\big)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right\|\mathrm{d}s \\
&\qquad + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\int_0^t\left(s^{\gamma-1}\left\|\tilde{g}\left(s, x(s), \int_0^t \tilde{f}\big(t, r, x(r)\big)\,\mathrm{d}r\right)\right.\right. \\
&\qquad \left.\left. - \tilde{g}\left(s, x_1(s), \int_0^t \tilde{f}\big(t, r, x_1(r)\big)\,\mathrm{d}r\right)\right\| + \big\|\mathsf{B}\big(u_x(s) - u_{x_1}(s)\big)\big\|\right)\mathrm{d}s \\
&\qquad + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\sum_{i=1}^m\big[\big\|\mathsf{M}_i\big(x(t_i)\big) - \mathsf{M}_i\big(x_1(t_i)\big)\big\|\big] \\
&\qquad + 7\mathbf{E}\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\sum_{i=1}^m\big[\big\|\mathsf{N}_i\big(x(t_i)\big) - \mathsf{N}_i\big(x_1(t_i)\big)\big\|\big].
\end{aligned}$$

From hypotheses $(\mathfrak{H}_1)$–$(\mathfrak{H}_5)$ we have

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(x)(t) - \widehat{\Psi}(x_1)(t)\big\| \\
&\quad\leqslant 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2|x - x_1|_{\widehat{PC}} + \sum_{t_i\in(0,t)} M_2|x - x_1|_{\widehat{PC}}\right] \\
&\qquad + 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[Q_2|x - x_1|_{\widehat{PC}}\right. \\
&\qquad + \int_0^t s^{\gamma-1}\big(B_2|x - x_1|_{\widehat{PC}} + G_2F_2|x - x_1|_{\widehat{PC}} + \mathbb{Z}_1\big\|u_x(s) - u_{x_1}(s)\big\|_c\big)\,\mathrm{d}s \\
&\qquad \left. + \sum_{t_i\in(0,t)} N_2|x - x_1|_{\widehat{PC}}\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)-\widehat{\Psi}(\varkappa_1)(t)\big\| \\
&\quad\leqslant 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}M_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right] \\
&\qquad+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[Q_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}N_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right. \\
&\qquad\left.+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)|\varkappa-\varkappa_1|_{\widehat{PC}}+\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\|u_\varkappa-u_{\varkappa_1}\|_c\right].
\end{aligned}\tag{15}$$

From Eq. (13) we have

$$\begin{aligned}
&u_\varkappa(t)-u_{\varkappa_1}(t) \\
&\quad=\widetilde{\mathbb{W}}(u)^{-1}\Bigg(-\mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[p(\varkappa)-p(\varkappa_1)\big]-\mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\big[q(\varkappa)-q(\varkappa_1)\big] \\
&\qquad-\int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma-s^\gamma}{\gamma}\right)\left(\big(\tilde{\sigma}\big(s,\varkappa(s)\big)-\tilde{\sigma}\big(s,\varkappa_1(s)\big)\big)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s \\
&\qquad-\int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma-s^\gamma}{\gamma}\right)\left(\tilde{g}\left(s,\varkappa(s),\int_0^t\tilde{f}\big(t,r,\varkappa(r)\big)\,\mathrm{d}r\right)\right. \\
&\qquad\left.-\tilde{g}\left(s,\varkappa_1(s),\int_0^t\tilde{f}\big(t,r,\varkappa_1(r)\big)\,\mathrm{d}r\right)\right)\mathrm{d}s \\
&\qquad-\sum_{i=1}^m\mathsf{C}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\big(\mathsf{M}_i\big(\varkappa(t_i)\big)-\mathsf{M}_i\big(\varkappa_1(t_i)\big)\big) \\
&\qquad-\sum_{i=1}^m\mathsf{S}\left(\frac{t^\gamma-t_i^\gamma}{\gamma}\right)\big(\mathsf{N}_i\big(\varkappa(t_i)\big)-\mathsf{N}_i\big(\varkappa_1(t_i)\big)\big)\Bigg).
\end{aligned}$$

Taking norm and using hypotheses $(\mathfrak{H}_1)$–$(\mathfrak{H}_5)$, we get

$$\begin{aligned}
&\mathbf{E}\big\|u_\varkappa(t)-u_{\varkappa_1}(t)\big\|_c \\
&\quad\leqslant 7\Bigg[\mathbb{Z}_2\Bigg(\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}M_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right] \\
&\qquad+\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\Big[Q_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}N_2|\varkappa-\varkappa_1|_{\widehat{PC}} \\
&\qquad+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)|\varkappa-\varkappa_1|_{\widehat{PC}}\Big]\Bigg)\Bigg].
\end{aligned}\tag{16}$$

Substituting (16) in (15), we have

$$\begin{aligned}
&\mathbf{E}\|\widehat{\Psi}(\varkappa)(t)-\widehat{\Psi}(\varkappa_1)(t)\| \\
&\quad\leqslant 7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}M_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right] \\
&\qquad+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[Q_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}N_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right. \\
&\qquad\left.+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)|\varkappa-\varkappa_1|_{\widehat{PC}}\right]+49\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\right. \\
&\qquad\times\left(\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}M_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right]\right. \\
&\qquad+\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[Q_2|\varkappa-\varkappa_1|_{\widehat{PC}}+\sum_{t_i\in(0,t)}N_2|\varkappa-\varkappa_1|_{\widehat{PC}}\right. \\
&\qquad\left.\left.\left.+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)|\varkappa-\varkappa_1|_{\widehat{PC}}\right]\right)\right].
\end{aligned}$$

By simple computations

$$\begin{aligned}
&\mathbf{E}\big\|\widehat{\Psi}(\varkappa)(t)-\widehat{\Psi}(\varkappa_1)(t)\big\| \\
&\quad\leqslant\left[1+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\right]\left(7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2+\sum_{t_i\in(0,t)}M_2\right]\right. \\
&\qquad\left.+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[Q_2+\sum_{t_i\in(0,t)}N_2+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)\right]\right)|\varkappa-\varkappa_1|_{\widehat{PC}}.
\end{aligned}$$

Taking supremum, we get

$$\begin{aligned}
&\mathbf{E}\big|\widehat{\Psi}\big(\varkappa(t)\big)-\widehat{\Psi}(\varkappa_1)(t)\big|_{\widehat{PC}} \\
&\quad\leqslant\left[1+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\frac{\zeta^\gamma}{\gamma}\mathbb{Z}_1\mathbb{Z}_2\right]\left(7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{C}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[P_2+\sum_{t_i\in(0,t)}M_2\right]\right. \\
&\qquad\left.+7\sup_{0\leqslant t\leqslant\zeta}\left|\mathsf{S}\left(\frac{t^\gamma}{\gamma}\right)\right|\left[Q_2+\sum_{t_i\in(0,t)}N_2+\frac{\zeta^\gamma}{\gamma}(B_2+G_2F_2)\right]\right)|\varkappa-\varkappa_1|_{\widehat{PC}} \\
&\quad<1.
\end{aligned}$$

Hence, $|\widehat{\Psi}(\varkappa(t))-\widehat{\Psi}(\varkappa_1)(t)|_{\widehat{PC}}<1$, which implies that $\widehat{\Psi}$ is a contraction on B_β.

Therefore there exists an unique element $\varkappa_\beta(t)\in B_\beta$ such that

$$\widehat{\Psi}\varkappa_\beta(t)=\varkappa_\beta(t)\quad\forall t\in[0,\zeta].$$

To show that $\varkappa_\beta(\zeta) = \tilde{\varkappa}$, consider $\varkappa_\beta(\zeta) = \widehat{\Psi}\varkappa_\beta(\zeta)$:

$$\begin{aligned}\widehat{\Psi}\varkappa_\beta(\zeta) &= \mathsf{C}\left(\frac{\zeta^\gamma}{\gamma}\right)\left[\varkappa_0 + p(\varkappa_\beta)\right] + \mathsf{S}\left(\frac{\zeta^\gamma}{\gamma}\right)\left[\varkappa_1 + q(\varkappa_\beta)\right]\\ &\quad + \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\left(\tilde{\sigma}\big(s, \varkappa(s)\big)\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right)\mathrm{d}s\\ &\quad + \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\tilde{g}\left(s, \varkappa(s), \int_0^t \tilde{f}\big(t, r, \varkappa(r)\big)\,\mathrm{d}r\right)\mathrm{d}s\\ &\quad + \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\mathsf{B}u_{\varkappa\beta}(s)\,\mathrm{d}s\\ &\quad + \sum_{t_i\in(0,t)}\mathsf{C}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\mathsf{M}_i\big(\varkappa_\beta(t_i)\big) + \sum_{t_i\in(0,t)}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\mathsf{N}_i\big(\varkappa_\beta(t_i)\big)\\ &= -\widetilde{\mathbb{W}}(u_{\varkappa\beta}) + \tilde{\varkappa} + \int_0^\zeta s^{\gamma-1}\mathsf{S}\left(\frac{\zeta^\gamma - s^\gamma}{\gamma}\right)\mathsf{B}u_{\varkappa\beta}(s)\,\mathrm{d}s\\ &= -\widetilde{\mathbb{W}}(u_{\varkappa\beta}) + \tilde{\varkappa} + \widetilde{\mathbb{W}}(u_{\varkappa\beta}) = \tilde{\varkappa}.\end{aligned}$$

Hence, $\varkappa_\beta(\zeta) = \widehat{\Psi}\varkappa_\beta(\zeta) = \tilde{\varkappa}$, and system (1)–(5) is controllable on $[0, \zeta]$. □

Remark 2. Khalil et al. [22] introduced the conformable derivative and stated certain properties relating to the same. Later on new properties of conformable derivatives was given by the authors in [5]. Since then, the study of system with conformable derivatives has been a topic of interest for many authors worldwide. Bouaouid et al. [9] have studied the nonlocal behaviour of conformable equations with a measure of noncompactness in Banach spaces. For a second-order system, the sequential evolution using conformable derivatives has been proposed by the authors in [10]. The relative controllability of impulsive-delay conformable differential equations was investigated by Luo et al. [25]. On comparing the earlier works, controllability study on a second-order system involving conformable derivatives with impulsive and nonlocal conditions is studied firstly. Hence, it is vital to consider the study of controllability of conformable integro-differential system exhibiting stochastic behaviour with nonlocal and impulsive conditions.

4 Example

Consider the impulsive stochastic integro-differential system as follows:

$$\begin{aligned}\mathsf{D}^{4/5}\big(\mathsf{D}^{4/5}\big(\varkappa(t)\big)\big) &= \mathsf{A}\varkappa(t) + \mathsf{B}u(t) + \tilde{\sigma}\big(t, s, \varkappa(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\\ &\quad + \tilde{g}\left(t, \varkappa(t), \int_0^\pi \tilde{f}\big(t, s, \varkappa(s)\big)\,\mathrm{d}s\right), \quad t \in [0, \pi], \qquad (17)\end{aligned}$$

$$\varkappa(t,0)=\varkappa(t,\pi)=0, \tag{18}$$

$$\varkappa(0)=\varkappa_0+\sum_{j=1}^{m}p_j\varkappa(t_j), \qquad \varkappa'(0)=\varkappa_1+\sum_{j=1}^{m}q_j\varkappa(t_j), \tag{19}$$

$$\mathsf{M}_j\bigl(\varkappa(t_j)\bigr)=\sum_{t_j\in(0,t)}\mathsf{C}\biggl(\frac{5}{4}\bigl(t^{4/5}-t_j^{4/5}\bigr)\biggr)\mathsf{M}_j\bigl(\varkappa(t_j)\bigr), \quad j=1,2\dots,n, \tag{20}$$

$$\mathsf{N}_j\bigl(\varkappa(t_j)\bigr)=\sum_{t_j\in(0,t)}\mathsf{S}\biggl(\frac{5}{4}\bigl(t^{4/5}-t_j^{4/5}\bigr)\biggr)\mathsf{N}_j\bigl(\varkappa(t_j)\bigr), \quad j=1,2\dots,n. \tag{21}$$

The infinitesimal operator is defined as $\mathsf{A}_\varkappa=(\mathsf{D}^{4/5})^2\mathsf{C}(0)\varkappa$ in the domain $\mathfrak{D}(\mathsf{A})=\{\varkappa\in\mathfrak{B}(0,\pi)\colon\ \varkappa(t,0)=y(t,\pi)=0\}$, and A is given by $\mathsf{A}:\mathfrak{B}\to\mathfrak{B}$. For every $\varkappa\in\mathfrak{D}(\mathsf{A})$, $\mathsf{A}(\varkappa)=-\sum_{n=1}^{\infty}n^2\langle\varkappa,\varkappa_n\rangle\varkappa_n$. Also, A is a cosine family generator that is strongly continuous and defined on $\mathfrak{B}$ by $(\mathsf{C}(t))_{t\in\mathbb{R}}$, which has a sequence of eigenvalues $\{-n^2\}$, where $n\in\mathbb{N}$, as it has eigenfunctions that are normalised as $\varkappa_n=(2\sin(n\varkappa)/\pi)^{1/2}$ and is a discrete spectrum.

For every $\varkappa\in\mathfrak{B}$,

$$\mathsf{C}(t)(\varkappa)=\sum_{n=1}^{\infty}\cos(nt)\langle\varkappa,\varkappa_n\rangle\varkappa_n, \qquad \mathsf{D}(t)(\varkappa)=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle\varkappa,\varkappa_n\rangle\varkappa_n.$$

Let $\mathfrak{B}=\mathsf{V}=\mathfrak{L}^2([0,1])$ be equipped with inner product defined by

$$\langle u,\varkappa_n\rangle^2=\int_0^1 u(r)\varkappa_n(r)\,\mathrm{d}r.$$

For any bounded linear control operator B and control function u, we have

$$\bigl\|\mathsf{B}(u)\bigr\|=\sum_{n=1}^{\infty}\mathrm{e}^{-2/(n^2+1)}\bigl\langle u(r),\varkappa_n(r)\bigr\rangle^2\leqslant\|u\|\leqslant 1,$$

where $\|u\|=(\int_0^1|u|^2dr)^{1/2}$.

Consider

$$\begin{aligned}\widetilde{\mathbb{W}}(u)&=\int_0^1 r^{\gamma-1}\mathsf{C}\biggl(\frac{\zeta^\gamma-r^\gamma}{\gamma}\biggr)\mathsf{B}u(r)\,\mathrm{d}r\\&=\sum_{n=1}^{\infty}\frac{1-\sin(n\gamma)}{n^2}\,\mathrm{e}^{-1/(n^2+1)}\langle u,\varkappa_n\rangle\varkappa_n.\end{aligned}$$

The inverse of the control operator $\widetilde{\mathbb{W}}^{-1}(u):\mathfrak{D}(\mathsf{A})\to\mathfrak{L}^2([0,1],\mathfrak{L}^2[0,1])$ is taken as

$$\widetilde{\mathbb{W}}^{-1}(u)=\sum_{n=1}^{\infty}\frac{n^2\mathrm{e}^{1/(n^2+1)}}{1-\sin(n\gamma)}\langle u,\varkappa_n\rangle\varkappa_n.$$

Then

$$\left\|\widetilde{\mathbb{W}}^{-1}(u)\right\| = \frac{e^{1/2}}{\sqrt{3\pi}}\|u\|, \quad u \in \mathfrak{D}(\mathsf{A}).$$

Now define the integro-differential function

$$\tilde{g}\left(t, z(t), \int_0^t \tilde{f}\big(t, s, z(s)\big)\, \mathrm{d}s\right) = \frac{4|z(t)|e^{-t}}{1 + |z(t)|(50 + e^t)}.$$

Then

$$\begin{aligned}
&\tilde{g}\left(t, z(t), \int_0^t \tilde{f}\big(t, s, z(s)\big)\, \mathrm{d}s\right) - \tilde{g}\left(t, z_1(t), \int_0^t \tilde{f}\big(t, s, z_1(s)\big)\, \mathrm{d}s\right)\\
&\quad = \frac{4}{50 + e^t}\left(\frac{|z(t)|}{1 + |z(t)|} - \frac{|z_1(t)|}{1 + |z_1(t)|}\right),\\
&\left\|\tilde{g}\left(t, z(t), \int_0^t \tilde{f}\big(t, s, z(s)\big)\, \mathrm{d}s\right) - \tilde{g}\left(t, z_1(t), \int_0^t \tilde{f}\big(t, s, z_1(s)\big)\, \mathrm{d}s\right)\right\|\\
&\quad \leqslant \frac{4}{50}\|z - z_1\|.
\end{aligned}$$

Now $W(t)$ is the Wiener process, and the function $\tilde{\sigma}$ is given by

$$\tilde{\sigma} = \big(2t^2 + 1\big)e^{-2t} \quad \Longrightarrow \quad \|\tilde{\sigma}\| = -\frac{3}{2}e^{-4}.$$

Define the functions p and q as

$$p(z) = \sum_{j=1}^{n} p_j z(t_j), \quad q(z) = \sum_{j=1}^{n} q_j z(t_j), \quad t_j \in [0, 1],$$

where

$$\sum_{j=1}^{n} |p_j| = \sum_{j=1}^{n} |q_j| \leqslant \frac{4}{50}.$$

Now, with the assumptions and hypotheses under Theorem 2, it can be easily concluded that the considered system (17)–(21) is controllable.

5 Conclusion

Controllability results of stochastic impulsive nonlinear integro-differential system involving nonlocal conditions and conformable derivatives have been proved by using contraction principle and fixed point techniques. Duhamel's formula has been used to derive the solution of the considered systems. The controllability of the linear stochastic system is studied by using controllability Grammian matrix. An example that shows the applicability of the derived result is given. Further, the proposed result can be extended to system with various delays for both state and control function.

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