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Global dissipativity and quasi-Mittag-Leffler synchronization of fractional-order complex-valued neural networks with time delays and discontinuous activations*

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Abstract. This paper explores fractional-order complex-valued neural networks (FOCVNNs) with time delays and discontinuous activation functions. A novel fractional-order inequality is utilized to study this system as a whole without dividing it into different components in the complex plane. Firstly, the existence of global Filippov solutions in the complex domain is proven by using the theories of vector norms and fractional calculus. Next, some sufficient conditions are derived to ensure the global dissipativity and quasi-Mittag-Leffler synchronization of FOCVNNs through the use of nonsmooth analysis and differential inclusion theory. The error bounds of quasi-Mittag-Leffler synchronization are also estimated without relying on the initial values. Finally, some numerical simulations are conducted to demonstrate the effectiveness of the presented findings.

Keywords: dissipativity, fractional-order complex-valued neural networks, quasi-Mittag-Leffler synchronization, time delays, discontinuous activations.

1 Introduction

Fractional calculus is widely recognized as the extension of conventional integer-order differentiation to include noninteger orders [2, 17, 22]. The most crucial reward of the fractional derivative lies in its nonlocality and possession of weakly singular kernel. It has been proven to be a powerful tool for depicting the memory and hereditary properties of various substances and processes. Furthermore, we have noted that fractional differential equations exhibit infinite memory and a greater degree of freedom. Due to these advantages, fractional-order derivatives offer a more comprehensive understanding of complex systems and phenomena than their integer-order counterparts, and they possess more intricate properties [3, 12, 14, 17–19, 25]. Thus, the dynamic analysis of fractional-order systems is of great significance, and some excellent results have been reported [2, 12, 22]. In recent years, some scholars have recommended fractional calculus into neural networks

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(NNs) to form fractional-order NNs (FONNs) and discussed the dynamics of FONNs [3, 18, 25].

However, it is worth noting that parameters may not always be compatible in the actual implementation of synchronization. Consequently, the synchronization error cannot tend to zero as time passes, that is, the synchronization error is unable to be eliminated, and complete synchronization cannot be achieved in the presence of parameter mismatches. Hence, under such circumstances, it is necessary for us to analyze the bound of the synchronization error and know how to control it within a small range to realize the quasi-synchronization. Recently, the issue of quasi-synchronization has garnered considerable attention, and numerous significant findings have been reported in the literature [7, 23].

We known that most practical applications of neural networks are related to complex signals, which cannot be solved by real-valued neural networks (RVNNs), however, complex-valued neural networks (CVNNs) can be used to solve such problems [21]. So, the investigations into the dynamical characteristics of CVNNs have become more and more important in both theory and application fields [11, 24]. In additions, time delays are usually encountered in the study of linear and nonlinear systems of various real-world applications such as engineering, biological, and economical systems [10, 24]. Hence, many researchers have focused their attention and much interest to extensively look into the dynamical properties of CVNNs with time delays.

To the best of our knowledge, most excellent results of FONNs were established on the premise of Lipschitz-continuous activations. However, considering the limited channel bandwidth and the external interference, it is unrealistic to realize continuous signal output and information communication between neurons. Discontinuous activation functions are necessary to be introduced into FONNs. In fact, discontinuous activations have been proved really useful as an ideal model of activations with very-high gain, and such models have been frequently applied to solve constrained optimization problems via a sliding mode approach [4, 9]. Then more and more researchers pay more attentions to studying neural networks with discontinuous activations, e.g., [8, 16], because the fractional-order models can describe the systems more precisely than the integer-order models in practice. Therefore, it is necessary to consider discontinuous activations in the dynamic analysis of FONNs. In [7], the authors are concerned with fractional-order discontinuous complex-valued neural networks. However, time delays were not considered in their models.

Introduced in the early 1970s, dissipativity is a fundamental attribute of dynamical systems. The concept of dissipativity expands upon the idea of a Lyapunov function and has a broader applicability. It has been employed in diverse fields such as stability theory, chaos and synchronization theory, system norm estimation, and robust control. Currently, compared to the research on the dissipativity analysis of integer-order neural networks, few authors have discussed the dissipativity of FOCVNNs [5–7, 23].

Motivated by the above discussions, in this paper, we introduce a class of FOCVNNs with time delays and discontinuous activations. The contributions of this paper are as follows.

(i) A novel lemma is established by constructing a suitable fractional differential inequality. Our conclusions improve some results of the literature [7]. The advantage of the proposed lemma is that it can directly deal with the delayed terms.

(ii) Based on the above inequality, nonsmooth analysis, differential inclusion theory, and fractional Lyapunov stability theory, some sufficient criteria of the dissipativity and quasi-Mittag-Leffler synchronization for FOCVNNs are obtained.
(iii) FOCVNNs with time delays and discontinuous activations are investigated by using Lyapunov direct method rather than real decomposition method.
(iv) The error bound of quasi-Mittag-Leffler synchronization is estimated without reference to the initial values. So the results in this paper are less conservative and more general.

Notations. Throughout this paper, let $\mathbb{R}$, $\mathbb{C}$ stand for the real number set and the complex number set, respectively. $\mathbb{R}^n$ denotes the set of n-dimensional Euclidean space, and $\mathbb{C}^n$ denotes the set of n-dimensional complex space. $\overline{x}$ is the conjugate of $x = a + \mathrm{i}b \in \mathbb{C}$, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. "a.a." implies "almost all". For vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$, the norm $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2} = (\sum_{i=1}^n x_i\overline{x}_i)^{1/2}$ is defined. $C^n([t_0, +\infty), \mathbb{C})$ is a set composed of all continuous and n-order differentiable functions from $[t_0, +\infty)$ into $\mathbb{C}$. $K(M)$ denotes the closure of the convex hull of set M. Define $\|G(x(t))\|_F = \sup_{\nu \in G(x(t))} \|\nu\|_2$ for the differential inclusion ${}_{t_0}^{C}D_t^{\alpha} x(t) \in K(x(t))$.

This paper is organized as follows. In Section 2, some preliminaries are introduced, model formulation are presented. Our main results are presented in Section 3. In Section 4, two examples are given to show the rationality of theoretical results.

2 Preliminaries and system description

In this section, we introduce some necessary knowledge with respect to fractional calculation and establish a pivotal inequality, which will be useful in proving our main result.

2.1 Preliminaries

Definition 1. (See [14, 17].) For a function $f(t) : [t_0, +\infty) \to \mathbb{C}$, the Caputo fractional-order integral of $f(t)$ with order $\alpha > 0$ can be defined as

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. (See [14,17].) For a function $f(t) \in C^n([t_0, +\infty), \mathbb{C})$, the Caputo fractional-order derivative of $f(t)$ with order $n-1 < \alpha < n$ is given by

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^n(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, $$

where $t \geqslant t_0$, and n is a positive integer.

Definition 3. (See [14, 17].) The one-parameter and two-parameter Mittag-Leffler functions are defined as

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $z \in \mathbb{C}$, and $\alpha > 0, \beta > 0$.

The Laplace transform of two-parameter Mittag-Leffler function is

$$\mathcal{L}\bigl\{(t-t_0)^{\beta-1} E_{\alpha,\beta}\bigl(-\lambda(t-t_0)^\alpha\bigr)\bigr\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \operatorname{Re}(s) > |\lambda|^{1/\alpha}.$$

Particularly, when $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$.

Lemma 1. (*See* [13].) *Let* $t \geqslant t_0$, *then* $E_\alpha(\sigma(t-t_0)^\alpha)$ *is monotonically nonincreasing, and* $0 \leqslant E_\alpha(\sigma(t-t_0)^\alpha) \leqslant 1$ *for* $\sigma \leqslant 0$.

Lemma 2. (*See* [27].) *For any continuous and analytic function* $g(t) \in \mathbb{C}$ *and any complex number* θ,

$$\begin{aligned} &{}_{t_0}^{C}D_t^\alpha \bigl(g(t)-\theta\bigr)\overline{\bigl(g(t)-\theta\bigr)} \\ &\quad\leqslant \bigl(g(t)-\theta\bigr)\,{}_{t_0}^{C}D_t^\alpha \overline{g(t)} + \overline{\bigl(g(t)-\theta\bigr)}\,{}_{t_0}^{C}D_t^\alpha g(t), \quad 0<\alpha<1. \end{aligned}$$

Lemma 3. (*See* [26].) *For any two complex numbers* ξ *and* η *and any real constant* $\zeta > 0$, *the following inequality holds*:

$$\xi\overline{\eta} + \overline{\xi}\eta \leqslant \zeta\xi\overline{\xi} + \frac{1}{\zeta}\eta\overline{\eta}.$$

Lemma 4. (*See* [1].) *Given* $\alpha, \beta \in [0,1)$ *and* $\varepsilon > 0$, *there exists a positive constant* $c := c(\alpha,\beta,\varepsilon)$ *such that the following is true*: *if* $u(t) : \mathbb{J} \subset \mathbb{R} \to \mathbb{R}$ *satisfies*

$$f(t) = t^\beta u(t) \in L_{\infty,\mathrm{loc}}(\mathbb{J})$$

and

$$u(t) \leqslant a t^{-\beta} + b\int_0^t (t-\tau)^{-\alpha} u(\tau)\,\mathrm{d}\tau, \quad a.a.\ t \in \mathbb{J}\setminus\{0\},$$

where a *and* b *are positive constants, then*

$$u(t) \leqslant a t^{-\beta}\bigl(1 + cbt^{1-\alpha}\mathrm{e}^{(1+\varepsilon)\mu(a,b)t}\bigr), \quad a.a.\ t \in \mathbb{J}\setminus\{0\},$$

where $\mu(a,b) := (\Gamma(1-\alpha)b)^{1/(1-\alpha)}$, $m(\mathbb{E})$ *is the Lebesgue measure of set* $\mathbb{E}$, *and*

$$\begin{aligned} L_{\infty,\mathrm{loc}}(\mathbb{U}) = \Bigl\{ &f(x)\colon\ f(x) \textit{ is Lebesgue-measurable on set } U,\ \textit{and for an arbitrary} \\ &\textit{bounded closed set } \mathbb{K} \subset \mathbb{U},\ \inf_{\mathbb{E}\subset\mathbb{K},\, m\mathbb{E}=0}\ \sup_{x\in\mathbb{K}\setminus\mathbb{E}} \bigl|f(x)\bigr| < +\infty \Bigr\}. \end{aligned}$$

Lemma 5. *Let $v(t) \in C([t_0-\tau, +\infty), [0, +\infty))$ be a differential positive function such that*

$$ {}_{t_0}^{C}D_t^{\alpha} v(t) \leqslant -av(t) + a_1 v^{1/2}(t) + bv(t-\tau) + b_1 v^{1/2}(t-\tau) + c \tag{1} $$

for any $a > 1 + b$, $t \geqslant t_0 \geqslant 0$, and nonnegative constants a_1, b, b_1, c. Then

$$ \begin{aligned} v(t) \leqslant{}& \biggl\{ \frac{(\frac{1}{2}+b)(a-1-b)^{-1/\alpha}(t-t_0)^{\alpha-1}\xi}{\Gamma(\alpha)} + \biggl(\xi - \frac{(\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c)}{a-1-b}\biggr) \\ &\times E_\alpha\bigl[-(a-1-b)(t-t_0)^\alpha\bigr] \biggr\} + \frac{\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c}{a-1-b}, \end{aligned} \tag{2} $$

where $0 < \alpha < 1$, $\xi = \max_{t\in[t_0-\tau,\, t_0]} |v(t)|$, and $0 \leqslant \tau \leqslant (a-1-b)^{-1/\alpha}$.

Proof. From (1) we have

$$ {}_{t_0}^{C}D_t^{\alpha} v(t) \leqslant -\biggl(a - \frac{1}{2}\biggr)v(t) + \biggl(\frac{1}{2} + b\biggr)v(t-\tau) + \frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c. $$

Then there exists a nonnegative function $\omega(t)$ satisfying

$$ {}_{t_0}{}^{C}D_t^{\alpha} v(t) + \omega(t) = -\biggl(a - \frac{1}{2}\biggr)v(t) + \biggl(\frac{1}{2} + b\biggr)v(t-\tau) + \frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c. $$

According to the Laplace transform,

$$ \begin{aligned} s^\alpha v(s) - s^{\alpha-1}v(t_0) + \omega(s) &= -\biggl(a - \frac{1}{2}\biggr)v(s) + \biggl(\frac{1}{2} + b\biggr)\mathcal{L}\bigl\{v(t-\tau)\bigr\} \\ &\quad + \frac{\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c}{s}, \\ \mathcal{L}\bigl\{v(t-\tau)\bigr\} &= \int_{t_0}^{+\infty} \mathrm{e}^{-st} v(t-\tau)\,\mathrm{d}t = \mathrm{e}^{-st}v(s) + \mathrm{e}^{-s\tau}\int_{t_0-\tau}^{t_0} \mathrm{e}^{-st}v(t)\,\mathrm{d}t, \end{aligned} $$

where $\mathcal{L}\{v(t)\} = v(s)$, $\mathcal{L}\{\omega(t)\} = \omega(s)$.

Then one has

$$ \begin{aligned} \bigl[s^\alpha + (a-1-b)\bigr]v(s) &= \biggl[s^{\alpha-1} + \biggl(\frac{1}{2} + b\biggr)(a-1-b)^{-1/\alpha}\biggr]\xi - \mathcal{W}(s) \\ &\quad + \frac{\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c}{s}, \end{aligned} $$

where

$$ \begin{aligned} \mathcal{W}(s) &= \omega(s) + \biggl(\frac{1}{2} + b\biggr)\biggl[(a-1-b)^{-1/\alpha}\xi - \mathrm{e}^{-s\tau}\int_{t_0-\tau}^{t_0} \mathrm{e}^{-st}v(t)\,\mathrm{d}t\biggr] \\ &\quad + \biggl(\frac{1}{2} + b\biggr)\bigl(1 - \mathrm{e}^{-s\tau}\bigr)v(s) + s^{\alpha-1}\bigl[\xi - v(t_0)\bigr]. \end{aligned} $$

Note that τ satisfies the known condition, then we have

$$(a-1-b)^{-1/\alpha}\xi - \mathrm{e}^{-s\tau}\int\limits_{t_0-\tau}^{t_0} \mathrm{e}^{-st}v(t)\,\mathrm{d}t \geqslant 0.$$

Then

$$v(s) = \frac{[s^{\alpha-1} + (\frac{1}{2}+b)(a-1-b)^{-1/\alpha}]\xi}{s^\alpha + (a-1-b)} - \frac{\mathcal{W}(s)}{s^\alpha + (a-1-b)} + \frac{(\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c)s^{-1}}{s^\alpha + (a-1-b)}.$$

By the inverse Laplace transform of the above equation

$$\begin{aligned} v(t) \leqslant{}& \biggl\{\biggl(\frac{1}{2}+b\biggr)(a-1-b)^{-1/\alpha}(t-t_0)^{\alpha-1}E_{\alpha,\alpha}\bigl[-(a-1-b)(t-t_0)^\alpha\bigr] \\ &+ E_\alpha\bigl[-(a-1-b)(t-t_0)^\alpha\bigr]\biggr\}\xi + \biggl(\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c\biggr)(t-t_0)^\alpha \\ &\times E_{\alpha,\alpha+1}\bigl[-(a-1-b)(t-t_0)^\alpha\bigr] \quad \text{for } t \geqslant t_0. \end{aligned} \tag{3}$$

On the one hand, we derive from the definition of Mittag-Leffler function that

$$\begin{aligned} &\biggl(\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c\biggr)(t-t_0)^\alpha E_{\alpha,\alpha+1}\bigl[-(a-1-b)(t-t_0)^\alpha\bigr] \\ &= \biggl(\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c\biggr)(t-t_0)^\alpha \sum_{k=0}^{+\infty} \frac{[-(a-1-b)(t-t_0)^\alpha]^k}{\Gamma(\alpha k + \alpha + 1)} \\ &= -\frac{\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c}{a-1-b} \sum_{k=1}^{+\infty} \frac{[-(a-1-b)(t-t_0)^\alpha]^k}{\Gamma(\alpha k + 1)} \\ &= -\frac{\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c}{a-1-b} E_\alpha\bigl[-(a-1-b)(t-t_0)^\alpha\bigr] + \frac{\frac{1}{2}a_1^2 + \frac{1}{2}b_1^2 + c}{a-1-b}. \end{aligned} \tag{4}$$

On the other hand, it follows from Lemma 1 that

$$\begin{aligned} &(a-1-b)^{-1/\alpha}(t-t_0)^{\alpha-1}E_{\alpha,\alpha}\bigl[-(a-1-b)(t-t_0)^\alpha\bigr] \\ &\quad \leqslant \frac{(a-1-b)^{-1/\alpha}(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \end{aligned} \tag{5}$$

for all $t \geqslant t_0$.

Substituting (4) and (5) into (3), we obtain that (2) is valid. □

Remark 1. When $b = b_1 = 0$, Lemma 5 degenerates to Lemma 5 in [7]. With the help of Lemma 5, the delay term can be dealt with directly, instead of applying the fractional Razumikhin theorem in [15, 22]. So, Lemma 5 can be used as a new fractional-order differential inequality to deal with FOCVNNs with time delays and discontinuous activation functions.

2.2 System description

In this paper, we consider the following FOCVNNs as the drive system:

$$
{}_{t_0}^{C}D_t^{\alpha} x_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(x_j(t-\tau)\big) + I_i(t), \tag{6}
$$

where $i = 1, 2, \dots, n$, and n is the number of units in neural networks, $t \geqslant t_0 \geqslant 0$, ${}_{t_0}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order α, and $0 < \alpha < 1$; $x(t) = (x_1(t), \dots, x_n(t))^{\mathrm{T}} \in \mathbb{C}^n$ is the vector of neuron states; $d_i > 0$; $a_{ij}, b_{ij} \in \mathbb{C}$ are constants, which represent the neuron interconnection weight and the delayed neuron interconnection weight, respectively; $I_i(t) \in \mathbb{C}$ denotes the external input, $|I_i(t)| \leqslant I_i$. $x_i(s) = \varphi_i(s) \in C([t_0 - \tau, t_0], \mathbb{C})$ is the initial condition of system (6), where $C([t_0 - \tau, t_0], \mathbb{C})$ is Banach space of all continuous functions, and time delay $\tau > 0$; $f_j(x_j(t)), g_j(x_j(t-\tau)) : \mathbb{C} \to \mathbb{C}$ are vector-valued activation functions.

The functions $f_j(\cdot)$ and $g_j(\cdot)$ are required to satisfy the following assumptions:

(H1) The activation function f_j (and g_j) is continuous and have at most a finite number of jump discontinuities ρ_l (and ϱ_l) in every bounded interval. Moreover, there exist finite right and left limits $f_j(\rho_l^+)$ (and $g_l(\varrho_l^-)$), respectively.

(H2) For all $j = 1, 2, \dots, n$, suppose there exist constants $F_j, G_j, L_j \geqslant 0$ and $M_j \geqslant 0$ such that for all $\iota_j(t) \in K[f_j(x_j(t))]$, $\kappa_j(t) \in K[f_j(y_j(t))]$, $\iota'_j(t) \in K[g_j(x_j(t))]$, $\kappa'_j(t) \in K[g_j(y_j(t))]$, the following holds:

$$
\sup \big|\iota_j(t) - \kappa_j(t)\big| \leqslant F_j \big|x_j(t) - y_j(t)\big| + L_j;
$$
$$
\sup \big|\iota'_j(t) - \kappa'_j(t)\big| \leqslant G_j \big|x_j(t) - y_j(t)\big| + M_j.
$$

Definition 4. FOCVNNs (6) is said to be dissipative if there exist a compact set $\Omega \subset \mathbb{C}^n$ and $T > 0$ such that for all $z_0 \in \mathbb{C}^n$, $z(t, t_0, z_0) \subset \Omega$ when $t \geqslant t_0 + T$ and $z(t, t_0, z_0)$ denotes the solution of FOCVNNs from initial state z_0 and initial time t_0. In this case, Ω is called a globally attractive set.

Remark 2. In FOCVNNs (6), complex-valued activation functions are assumed to be discontinuous and satisfy a growth condition, which include the continuous activation functions as a special case when $L_j = M_j = 0$ of assumption (H2).

Notice that assumption (H1) is satisfied, that is, there exist discontinuous activations f_j and g_j ($j = 1, 2, \dots, n$), system (6) exhibits discontinuities, and as a result, a classical solution does not exist. In this context, we examine the solutions of the system within the Filippov framework. According to the conclusion about the existence of Filippov solution in [8], we know that the solution $x(t)$ of system (6) with initial condition exists.

We refer to (6) as the drive system, and the corresponding response system is

$$
{}_{t_0}^{C}D_t^{\alpha} y_i(t) = -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(y_j(t-\tau)\big) + I_i(t) + u_i(t), \tag{7}
$$

where $u_i(t)$ is the control input.

In this paper, we choose the linear controller to deal with our system:

$$u_i(t) = -k_i\big(y_i(t) - x_i(t)\big), \quad k_i \in \mathbb{C}. \tag{8}$$

We apply the theories of differential inclusions and set-valued maps [8] to deal with above systems (6) and (7). From (6)

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} x_i(t) \in{}& -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} K\big[f_j\big(x_j(t)\big)\big] + \sum_{j=1}^{n} b_{ij} K\big[g_j\big(x_j(t-\tau)\big)\big] \\ &+ I_i(t) \end{aligned} \tag{9}$$

for a.a. $t \geqslant t_0$. By the measurable selection theorem, if $x_i(t)$ is a solution of FOCVNNs (9), then there exist measurable functions $\gamma_i(t) \in K[f_j(x_j(t))]$, $\eta_j(t) \in K[g_j(x_j(t))]$ such that

$${}_{t_0}^{C}D_t^{\alpha} x_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}\gamma_j(t) + \sum_{j=1}^{n} b_{ij}\eta_j(t-\tau) + I_i(t). \tag{10}$$

Similarly, from (7) we have

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} y_i(t) \ \in{}& -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} K\big[f_j\big(y_j(t)\big)\big] + \sum_{j=1}^{n} b_{ij} K\big[g_j\big(y_j(t-\tau)\big)\big] \\ &+ I_i(t) + u_i(t), \end{aligned} \tag{11}$$

$${}_{t_0}^{C}D_t^{\alpha} y_i(t) = -d_i y_i(t) + \sum_{j=1}^{n} a_{ij}\gamma_j'(t) + \sum_{j=1}^{n} b_{ij}\eta_j'(t-\tau) + I_i(t) + u_i(t) \tag{12}$$

for a.a. $t \geqslant t_0$, and there exist $\gamma_j'(t) \in K[f_j(y_j(t))]$, $\eta_j'(t) \in K[g_j(y_j(t)]$.

3 Main results

3.1 Existence of Filippov solutions

In this section, we prove that under some conditions, there exists solution for FOCVNNs (6) in the sense of Filippov.

Theorem 1. *Suppose that assumptions* (H1) *and* (H2) *hold, then there exists at least one solution $x(t)$ with initial condition $x(s) = \varphi(s)(s \in [t_0 - \tau, t_0])$ of system* (6) *on* $[t_0, +\infty)$.

Proof. The local existence of a solution $x(t)$ of system (6) can be guaranteed in [8], so we obtain that system (6) with initial condition $x(s) = \varphi(s)(s \in [t_0 - \tau, t_0])$ has at least a local solution $x(t)$ defined on a maximal interval $[t_0, T)$ for $T > t_0$. Next, we investigate whether it exists at least one solution of system (6) on $[t_0, +\infty)$.

For simplicity, we denote $G(x(t)) = (G_1(x(t)), G_2(x(t)), \dots, G_n(x(t)))^{\mathrm{T}}$ and

$$G_i\big(x(t)\big) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} K\big[f_j\big(x_j(t)\big)\big] + \sum_{j=1}^{n} b_{ij} K\big[g_j\big(x_j(t-\tau)\big)\big] + I_i(t).$$

Next, we will prove that for the differential inclusion ${}_{t_0}^{C}D_t^{\alpha}x(t) \in G(x(t))$, there exists at least one solution $x(t)$ with initial condition $x(s) = \varphi(s) (s \in [t_0 - \tau, t_0])$ on $[t_0, +\infty)$.

For $x(t) \in \mathbb{C}^n$, there exists some $\varpi(t) \in G(x(t))$, which is equivalent to $\varpi_i \in G_i(x(t))$ $(i = 1, 2, \ldots, n)$. By the measurable selection theorem, then there exist measurable functions $\gamma_j \in K[f_j(x_j(t))]$ and $\eta_j \in K[g_j(x_j(t))]$ such that

$$\varpi_i = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}\gamma_j(t) + \sum_{j=1}^{n} b_{ij}\eta_j(t-\tau) + I_i(t) \tag{13}$$

for a.a. $t \geqslant t_0$, $i = 1, 2, \ldots, n$.

Then, basing on (13), we have

$$\begin{aligned}
&\bigl\|G\bigl(x(t)\bigr)\bigr\|_F \\
&\quad = \sup_{\varpi \in G(x(t))} \|\varpi\|_2 \leqslant \sup_{\varpi \in G(x(t))} \|\varpi\|_1 \\
&\quad \leqslant \sup_{x(t)\in\mathbb{C}^n} \sum_{i=1}^{n} \Biggl\{ d_i\bigl|x_i(t)\bigr| + \sum_{j=1}^{n} |a_{ij}|\bigl|\gamma_j(t)\bigr| + \sum_{j=1}^{n} |b_{ij}|\bigl|\eta_j(t-\tau)\bigr| + I_i \Biggr\}.
\end{aligned} \tag{14}$$

According to assumption (H2), for all $\gamma_j(t) \in K[f_j(x_j(t))]$, $\gamma_j'(t) \in K[f_j(x_j(t_0))]$, $\eta_j(t) \in K[g_j(x_j(t))]$, $\eta_j'(t) \in K[g_j(x_j(t_0))]$, the following inequalities hold:

$$\begin{aligned}
\bigl|\gamma_j(t)\bigr| &\leqslant F_j\bigl|x_j(t)\bigr| + F_j\bigl|x_j(t_0)\bigr| + L_j + \bigl|\gamma_j'(t)\bigr|; \\
\bigl|\eta_j(t-\tau)\bigr| &\leqslant G_j\bigl|x_j(t-\tau)\bigr| + G_j\bigl|x_j(t_0)\bigr| + M_j + \bigl|\eta_j'(t)\bigr|.
\end{aligned} \tag{15}$$

From (14) and (15) we obtain

$$\begin{aligned}
&\bigl\|G\bigl(x(t)\bigr)\bigr\|_F \\
&\quad \leqslant \sup_{x(t)\in\mathbb{C}^n} \sum_{i=1}^{n} \Biggl\{ d_i\bigl|x_i(t)\bigr| + \sum_{j=1}^{n} |a_{ij}|\bigl(F_j\bigl|x_j(t)\bigr| + F_j\bigl|x_j(t_0)\bigr| + L_j + \bigl|\gamma_j'(t)\bigr|\bigr) \\
&\qquad + \sum_{i=1}^{n} |b_{ij}|\bigl(G_j\bigl|x_j(t-\tau)\bigr| + G_j\bigl|x_j(t_0)\bigr| + M_j + \bigl|\eta_j'(t)\bigr|\bigr) + I_i \Biggr\} \\
&\quad \leqslant \sup_{x(t)\in\mathbb{C}^n} \sum_{i=1}^{n} \Biggl(d_i + \sum_{j=1}^{n} |a_{ji}||F_i| \Biggr)\bigl|x_i(t)\bigr| + \sum_{i=1}^{n} \Biggl(\sum_{j=1}^{n} |b_{ji}||G_i| \Biggr)\bigl|x_i(t-\tau)\bigr| \\
&\qquad + \sum_{i=1}^{n} \Biggl\{ \sum_{j=1}^{n} |a_{ij}|\bigl(L_j + \bigl|\gamma_j'(t)\bigr| + |F_j|\bigl|x_j(t_0)\bigr|\bigr) \\
&\qquad + \sum_{j=1}^{n} |b_{ij}|\bigl(M_j + \bigl|\eta_j'(t)\bigr| + |G_j|\bigl|x_j(t_0)\bigr|\bigr) + I_i \Biggr\}.
\end{aligned} \tag{16}$$

For $t \in [t_0, +\infty)$, we have

$$\begin{aligned}\left|x_i(t-\tau)\right| &\leqslant \sup_{t_0-\tau\leqslant s\leqslant t}\left|x_i(s)\right| \leqslant \sup_{t_0-\tau\leqslant s\leqslant t_0}\left|x_i(s)\right| + \sup_{t_0\leqslant s\leqslant t}\left|x_i(s)\right| \\ &= \left|\varphi_i(s)\right| + \left|x_i(t)\right|. \end{aligned} \tag{17}$$

Based on (17), inequality (16) can be turned into the following inequality:

$$\begin{aligned}&\left\|G\left(x(t)\right)\right\|_F \\ &\quad\leqslant \sup_{x(t)\in\mathbb{C}^n}\sum_{i=1}^n\left(d_i + \sum_{j=1}^n|a_{ji}||F_i| + \sum_{j=1}^n|b_{ji}||G_i|\right)|x_i(t)| \\ &\qquad + \sum_{i=1}^n\left\{\sum_{j=1}^n|a_{ij}|\left(L_j + \left|\gamma_j'(t)\right| + |F_j|\left|x_j(t_0)\right|\right)\right. \\ &\qquad \left. + \sum_{j=1}^n|b_{ij}|\left(M_j + \left|\eta_j'(t)\right| + |G_j|\left|x_j(t_0)\right|\right) + \sum_{j=1}^n|b_{ji}||G_i|\left|\varphi_i(s)\right| + I_i\right\}. \end{aligned} \tag{18}$$

We denote

$$\begin{aligned} p_1 &= \max_{1\leqslant i\leqslant n}\sum_{i=1}^n\left(d_i + \sum_{j=1}^n|a_{ji}||F_i| + \sum_{j=1}^n|b_{ji}||G_i|\right) > 0, \qquad p = \sqrt{n}p_1; \\ q &= \sum_{i=1}^n\left\{\sum_{j=1}^n|a_{ij}|\left(L_j + \left|\gamma_j'(t)\right| + |F_j|\left|x_j(t_0)\right|\right)\right. \\ &\quad \left. + \sum_{j=1}^n|b_{ij}|\left(M_j + \left|\eta_j'(t)\right| + |G_j|\left|x_j(t_0)\right|\right) + \sum_{j=1}^n|b_{ji}||G_i|\left|\varphi_i(s)\right| + I_i\right\} > 0, \end{aligned}$$

then

$$\left\|G\left(x(t)\right)\right\|_F \leqslant p_1\sum_{i=1}^n\left|x_i(t)\right| + q \leqslant p_1\sqrt{n}\left\|x(t)\right\|_2 + q = p\left\|x(t)\right\|_2 + q.$$

When fix $\varpi(t) \in \mathbb{C}^n$, ${}^C_{t_0}D^\alpha_t x(t) = \varpi(t)$, taking fractional integral on both sides of above equation, we have

$$x(t) = x(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-h)\varpi(h)\,\mathrm{d}h.$$

Then

$$\begin{aligned}\left\|x(t)\right\|_2 &= \left\|x(t_0)\right\|_2 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-h)^{\alpha-1}\left\|\varpi(h)\right\|_2\,\mathrm{d}h \\ &\leqslant \left\|x(t_0)\right\|_2 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-h)^{\alpha-1}\left(p\left\|x(h)\right\|_2 + q\right)\mathrm{d}h. \end{aligned} \tag{19}$$

Let $S(t) = p\|x(t)\|_2 + q$, from (19)

$$S(t) \leqslant p\big\|x(t_0)\big\|_2 + q + \frac{p}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-h)^{\alpha-1} \big\|S(h)\big\| \,\mathrm{d}h, \tag{20}$$

where $a = p\|x(t_0)\|_2 + q > 0$, $b = p/\Gamma(\alpha) > 0$. By applying Lemma 4, given $\alpha \in (0,1)$, $\beta = 0$, and $\varepsilon > 0$, there exists a positive constant $c := c(\alpha, \beta, \varepsilon)$ such that

$$S(t) \leqslant \big(p\big\|x(t_0)\big\|_2 + q\big)\left(1 + \frac{cp}{\Gamma(\alpha)} t^{\alpha} \mathrm{e}^{(1+\varepsilon)\mu(1-\alpha,\, p/\Gamma(\alpha))t}\right), \tag{21}$$

where

$$\mu\left(1-\alpha,\, \frac{p}{\Gamma(\alpha)}\right) := \left(\Gamma\big(1-(1-\alpha)\big)\frac{p}{\Gamma(\alpha)}\right)^{1/\alpha} = p^{1/\alpha}.$$

So, (21) can be converted to the following inequality:

$$S(t) \leqslant \big(p\big\|x(t_0)\big\|_2 + q\big)\left(1 + \frac{cp}{\Gamma(\alpha)} t^{\alpha} \mathrm{e}^{(1+\varepsilon)p^{1/\alpha}t}\right).$$

Based on above inequality, we can express $x(t)$ as

$$\big\|x(t)\big\|_2 = \frac{S(t) - q}{p} \leqslant \big\|x(t_0)\big\|_2 + \big(p\big\|x(t_0)\big\|_2 + q\big)\frac{c}{\Gamma(\alpha)} t^{\alpha} \mathrm{e}^{(1+\varepsilon)p^{1/\alpha}} t. \tag{22}$$

Suppose that there exists $t_0 < T_1 < +\infty$ such that the maximal interval of existence of the solution $x(t)$ is $[t_0, T_1)$, then

$$\big\|x(t)\big\|_2 \leqslant \big\|x(t_0)\big\|_2 + \big(p\|x(t_0)\|_2 + q\big)\frac{c}{\Gamma(\alpha)} T_1^{\alpha} \mathrm{e}^{(1+\varepsilon)p^{1/\alpha}T_1}. \tag{23}$$

Choose sufficiently small constant $\sigma > 0$ and denote

$$\mathit{\Xi} = \left\{x(t) \in \mathbb{C}^n\colon\ \big\|x(t)\big\|_2 \leqslant \big\|x(t_0)\big\|_2 + \big(p\big\|x(t_0)\big\|_2 + q\big)\frac{c}{\Gamma(\alpha)} T_1^{\alpha} \mathrm{e}^{(1+\varepsilon)p^{1/\alpha}T_1} + \sigma\right\}$$

as a bounded domain. Based on continuation theorem [8], each solution $x(t)$ can be continued on both sides up to the boundary of the domain $\mathit{\Xi}$, which contradicts with (23). Hence, $x(t)$ is bounded for any positive time, and it is defined on $[t_0, +\infty)$. That is, for FOCVNNs (6), there exists at least one solution $x(t)$ with initial condition $x(s) = \varphi(s)$ ($s \in [t_0 - \tau,\, t_0]$) on $[t_0, +\infty)$. □

3.2 Dissipativity of FOCVNNs with time delays and discontinuous activations

In this section, we will analyze the dissipativity of FOCVNNs (6).

Theorem 2. *FOCVNNs* (6) *is globally dissipative when*

$$\lambda = \min_{1\leqslant i\leqslant n}\left\{d_i + \overline{d}_i - \sum_{j=1}^{n} a_{ij}\overline{a}_{ij} - \sum_{j=1}^{n} b_{ij}\overline{b}_{ij} - 1 - \sum_{j=1}^{n}\frac{r_j}{r_i}F_i^2\right\} > \chi + 1,$$
$$0 \leqslant \tau \leqslant (\lambda - \chi - 1)^{-1/\alpha}.$$

In addition,

$$\Omega = \left\{x\colon \|x\|_2 \leqslant \left(\frac{\frac{1}{2}(\mu^2 + \zeta^2) + \nu}{\min_{1\leqslant i\leqslant n}\{r_i\}(\lambda - \chi - 1)}\right)^{1/2}\right\}$$

is a globally attractive set, where

$$\chi = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\frac{r_j}{r_i}G_i^2\right\},$$
$$\mu = \Bigl(n\max_{1\leqslant i\leqslant n}\{r_i\}\Bigr)^{1/2}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\frac{r_j}{r_i}\bigl(2F_iL_i + 2F_i^2\bigl|x_i(t_0)\bigr| + 2L_i\bigl|\gamma_i'(t)\bigr|\bigr)\right\},$$
$$\zeta = \Bigl(n\max_{1\leqslant i\leqslant n}\{r_i\}\Bigr)^{1/2}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\frac{r_j}{r_i}\bigl(2G_iM_i + 2G_i^2\bigl|x_i(t_0)\bigr| + 2G_i\bigl|\eta_i'(t)\bigr|\bigr)\right\},$$
$$\nu = \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(F_j\bigl|x_j(t_0)\bigr| + L_j + \bigl|\gamma_j'(t)\bigr|\bigr)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(G_j\bigl|x_j(t_0)\bigr| + M_j + \bigl|\eta_j'(t)\bigr|\bigr)^2 + \sum_{i=1}^{n} r_iI_i^2.$$

Proof. Consider the following nonnegative function:

$$V(t) = \sum_{i=1}^{n} r_i x_i(t)\overline{x_i(t)}, \tag{24}$$

where $r_i \in \mathbb{R}$. By means of Lemma 2

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) &\leqslant \sum_{i=1}^{n} r_i\bigl(\overline{x_i(t)}\,{}_{t_0}^{C}D_t^{\alpha}x(t) + x_i(t)\,{}_{t_0}^{C}D_t^{\alpha}\overline{x_i(t)}\bigr) \\
&= \sum_{i=1}^{n} r_i\bigl[-(d_i + \overline{d_i})x_i(t)\overline{x_i(t)}\bigr] + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[a_{ij}\overline{x_i(t)}\gamma_j(t) + \overline{a_{ij}}x_i(t)\overline{\gamma_j(t)}\bigr] \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[b_{ij}\overline{x_i(t)}\eta_j(t-\tau) + \overline{b_{ij}}x_i(t)\overline{\eta_j(t-\tau)}\bigr] \\
&\quad + \sum_{i=1}^{n} r_i\bigl[I_i(t)\overline{x_i(t)} + \overline{I_i(t)}x_i(t)\bigr].
\end{aligned} \tag{25}$$

For all $\gamma_j'(t) \in K[f_j(x_j(t_0))]$, $\eta_j'(t) \in K[g_j(x_j(t_0))]$, we can obtain the following inequality based on Lemma 3 and assumption (H2):

$$\begin{aligned}
&a_{ij}\overline{x_i(t)}\gamma_j(t) + \overline{a_{ij}}x_i(t)\overline{\gamma_j(t)} \\
&\quad\leqslant a_{ij}\overline{a_{ij}}x_i(t)\overline{x_i(t)} + \gamma_j(t)\overline{\gamma_j(t)} \\
&\quad\leqslant a_{ij}\overline{a_{ij}}x_i(t)\overline{x_i(t)} + \bigl(F_j\bigl|x_j(t)\bigr| + F_j\bigl|x_j(t_0)\bigr| + L_j + \bigl|\gamma_j'(t)\bigr|\bigr)^2 \\
&\quad\leqslant a_{ij}\overline{a_{ij}}x_i(t)\overline{x_i(t)} + F_j^2 x_j(t)\overline{x_j(t)} \\
&\qquad + \bigl(2F_jL_j + 2F_j^2\bigl|x_j(t_0)\bigr| + 2F_j\bigl|\gamma_j'(t)\bigr|\bigr)\bigl|x_j(t)\bigr| \\
&\qquad + \bigl(F_j\bigl|x_j(t_0)\bigr| + L_j + \bigl|\gamma_j'(t)\bigr|\bigr)^2.
\end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned}
&b_{ij}\overline{x_i(t)}\eta_j(t-\tau) + \overline{b_{ij}}x_i(t)\overline{\eta_j(t-\tau)} \\
&\quad\leqslant b_{ij}\overline{b_{ij}}x_i(t)\overline{x_i(t)} + \eta_j(t-\tau)\overline{\eta_j(t-\tau)} \\
&\quad\leqslant b_{ij}\overline{b_{ij}}x_i(t)\overline{x_i(t)} + \bigl(G_j\bigl|x_j(t-\tau)\bigr| + G_j\bigl|x_j(t_0)\bigr| + M_j + \bigl|\eta_j'(t)\bigr|\bigr)^2 \\
&\quad= b_{ij}\overline{b_{ij}}x_i(t)\overline{x_i(t)} + G_j^2 x_j(t-\tau)\overline{x_j(t-\tau)} \\
&\qquad + \bigl(2G_jM_j + 2G_j^2\bigl|x_j(t_0)\bigr| + 2G_j\bigl|\eta_j'(t)\bigr|\bigr)\bigl|x_j(t-\tau)\bigr| \\
&\qquad + \bigl(G_j\bigl|x_j(t_0)\bigr| + M_j + \bigl|\eta_j'(t)\bigr|\bigr)^2
\end{aligned} \tag{27}$$

and

$$I_i(t)\overline{x_i(t)} + \overline{I_i(t)}x_i(t) \leqslant x_i(t)\overline{x_i(t)} + I_i(t)\overline{I_i(t)} \leqslant x_i(t)\overline{x_i(t)} + I_i^2. \tag{28}$$

Submitting (26)–(28) into (25), we have

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) \leqslant{}& \sum_{i=1}^{n} r_i\Biggl[-(d_i + \overline{d_i}) + \sum_{j=1}^{n}(a_{ij}\overline{a_{ij}} + b_{ij}\overline{b_{ij}}) + 1\Biggr]x_i(t)\overline{x_i(t)} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} r_iF_j^2 x_j(t)\overline{x_j(t)} + \sum_{i=1}^{n}\sum_{j=1}^{n} r_iG_j^2 x_j(t-\tau)\overline{x_j(t-\tau)} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(2F_jL_j + 2F_j^2\bigl|x_j(t_0)\bigr| + 2L_j\bigl|\gamma_j'(t)\bigr|\bigr)\bigl|x_j(t)\bigr| \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(2G_jM_j + 2G_j^2\bigl|x_j(t_0)\bigr| + 2G_j\bigl|\eta_j'(t)\bigr|\bigr)\bigl|x_j(t-\tau)\bigr| \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(F_j\bigl|x_j(t_0)\bigr| + L_j + \bigl|\gamma_j'(t)\bigr|\bigr)^2 \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(G_j\bigl|x_j(t_0)\bigr| + M_j + \bigl|\eta_j'(t)\bigr|\bigr)^2 + \sum_{i=1}^{n} r_iI_i^2
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n} r_i \Bigg[d_i + \overline{d_i} - \sum_{j=1}^{n} (a_{ij}\overline{a_{ij}} + b_{ij}\overline{b_{ij}}) - \sum_{j=1}^{n} \frac{r_j}{r_i} F_i^2 - 1 \Bigg] \\
&\qquad \times x_i(t)\overline{x_i(t)} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i \left(\frac{r_j}{r_i} G_i^2 \right) x_i(t-\tau)\overline{x_i(t-\tau)} + \sum_{i=1}^{n} r_i I_i^2 \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i \left[\frac{r_j}{r_i} \big(2F_iL_i + 2F_i^2 \big|x_i(t_0)\big| + 2L_i \big|\gamma_i'(t)\big|\big) \big|x_i(t)\big| \right] \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i \left[\frac{r_j}{r_i} \big(2G_iM_i + 2G_i^2 \big|x_i(t_0)\big| + 2G_i \big|\eta_i'(t)\big|\big) \big|x_i(t-\tau)\big| \right] \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i \big(F_j \big|x_j(t_0)\big| + L_j + \big|\gamma_j'(t)\big|\big)^2 \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i \big(G_j \big|x_j(t_0)\big| + M_j + \big|\eta_j'(t)\big|\big)^2. \qquad (29)
\end{aligned}
$$

Let

$$
\begin{aligned}
\mu_1 &= \max_{1\leqslant i\leqslant n} \left\{ \sum_{j=1}^{n} \frac{r_j}{r_i} \big(2F_iL_i + 2F_i^2 \big|x_i(t_0)\big| + 2L_i\big|\gamma_i'(t)\big|\big) \right\}, \\
\zeta_1 &= \max_{1\leqslant i\leqslant n} \left\{ \sum_{j=1}^{n} \frac{r_j}{r_i} \big(2G_iM_i + 2G_i^2 \big|x_i(t_0)\big| + 2G_i\big|\eta_i'(t)\big|\big) \right\}, \\
\mu &= \mu_1 \Big(n \max_{1\leqslant i\leqslant n}\{r_i\} \Big)^{1/2}, \qquad \zeta = \zeta_1 \Big(n \max_{1\leqslant i\leqslant n}\{r_i\} \Big)^{1/2}.
\end{aligned}
$$

From (29)

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha} V(t) &\leqslant -\sum_{i=1}^{n} r_i \lambda x_i(t)\overline{x_i(t)} + \sum_{i=1}^{n} r_i \chi x_i(t-\tau)\overline{x_i(t-\tau)} \\
&\quad + \sum_{i=1}^{n} r_i \mu_1 \big|x_i(t)\big| + \sum_{i=1}^{n} r_i \zeta_1 \big|x_i(t-\tau)\big| + \nu \\
&\leqslant -\sum_{i=1}^{n} r_i \lambda x_i(t)\overline{x_i(t)} + \sum_{i=1}^{n} r_i \chi x_i(t-\tau)\overline{x_i(t-\tau)} \\
&\quad + \mu_1 \left(n \sum_{i=1}^{n} r_i^2 x_i(t)\overline{x_i(t)} \right)^{1/2} + \zeta_1 \left(n \sum_{i=1}^{n} r_i^2 x_i(t-\tau)\overline{x_i(t-\tau)} \right)^{1/2} + \nu \\
&\leqslant -\sum_{i=1}^{n} r_i \lambda x_i(t)\overline{x_i(t)} + \sum_{i=1}^{n} r_i \chi x_i(t-\tau)\overline{x_i(t-\tau)} + \nu
\end{aligned}
$$

$$
\begin{aligned}
&+ \mu_1 \Big(n \max_{1 \leqslant i \leqslant n} \{r_i\} \Big)^{1/2} \Bigg(\sum_{i=1}^{n} r_i x_i(t) \overline{x_i(t)} \Bigg)^{1/2} \\
&+ \zeta_1 \Big(n \max_{1 \leqslant i \leqslant n} \{r_i\} \Big)^{1/2} \Bigg(\sum_{i=1}^{n} r_i x_i(t-\tau) \overline{x_i(t-\tau)} \Bigg)^{1/2} \\
= {}& -\lambda V(t) + \chi V(t-\tau) + \mu V^{1/2}(t) + \zeta V^{1/2}(t-\tau) + \nu. \qquad (30)
\end{aligned}
$$

Since $\lambda > \chi + 1$ and $\chi, \mu, \zeta, \nu > 0$, the following inequality can be obtained with the help of Lemma 5:

$$
\begin{aligned}
&\min_{1 \leqslant i \leqslant n} \{r_i\} \sum_{i=1}^{n} x_i(t) \overline{x_i(t)} \\
&\leqslant V(t) \leqslant E_\alpha \big[-(\lambda - \chi - 1)(t-t_0)^\alpha \big] \bigg(\xi - \frac{\frac{1}{2}\mu^2 + \frac{1}{2}\zeta^2 + \nu}{\lambda - \chi - 1} \bigg) \\
&\quad + \frac{(\frac{1}{2} + \chi)(\lambda - \chi - 1)^{-1/\alpha}(t-t_0)^{\alpha-1}\xi}{\Gamma(\alpha)} + \frac{\frac{\mu^2}{2} + \frac{\zeta^2}{2} + \nu}{\lambda - \chi - 1}, \qquad (31)
\end{aligned}
$$

where $\xi = \max_{t \in [t_0 - \tau,\, t_0]} |x(t)|$.

Then

$$
\begin{aligned}
\big\| x(t) \big\|_2 \leqslant \Bigg\{ & \frac{E_\alpha[-(\lambda - \chi - 1)(t-t_0)^\alpha]}{\min_{1 \leqslant i \leqslant n}\{r_i\}} \bigg(\xi - \frac{\frac{1}{2}\mu^2 + \frac{1}{2}\zeta^2 + \nu}{\lambda - \chi - 1} \bigg) \\
& + \frac{(\frac{1}{2} + \chi)(\lambda - \chi - 1)^{-1/\alpha}(t-t_0)^{\alpha-1}\xi}{\min_{1 \leqslant i \leqslant n}\{r_i\}\Gamma(\alpha)} \\
& + \frac{\frac{\mu^2}{2} + \frac{\zeta^2}{2} + \nu}{\min_{1 \leqslant i \leqslant n}\{r_i\}(\lambda - \chi - 1)} \Bigg\}^{1/2}. \qquad (32)
\end{aligned}
$$

Then, by using the monotonicity of $(t-t_0)^{\alpha-1}$ and Lemma 1,

$$
\limsup_{t \to +\infty} \big\| x(t) \big\|_2 \leqslant \bigg(\frac{\frac{\mu^2}{2} + \frac{\zeta^2}{2} + \nu}{\min_{1 \leqslant i \leqslant n}\{r_i\}(\lambda - \chi - 1)} \bigg)^{1/2}.
$$

Therefore, for any sufficiently small number $\epsilon > 0$, there is $T > 0$ so that

$$
\big\| x(t) \big\|_2 \leqslant \bigg(\frac{\frac{\mu^2}{2} + \frac{\zeta^2}{2} + \nu}{\min_{1 \leqslant i \leqslant n}\{r_i\}(\lambda - \chi - 1)} \bigg)^{1/2} + \epsilon.
$$

Therefor, for $x_0 \in \mathbb{C}^n$, there exists $T > 0$ such that $x(t, t_0, x_0) \subset \Omega$, which implies that system (6) is dissipative. □

The conditions of Theorem 2 can be simplified when $r_i = 1$, the following result is derived.

Corollary 1. *FOCVNNs* (6) *is globally dissipative when*

$$\lambda_1 = \min_{1\leqslant i\leqslant n}\left\{d_i + \overline{d}_i - \sum_{j=1}^n a_{ij}\overline{a}_{ij} - \sum_{j=1}^n b_{ij}\overline{b}_{ij} - 1 - \sum_{j=1}^n F_i^2\right\} > \chi_1 + 1,$$
$$0 \leqslant \tau \leqslant (\lambda_1 - \chi_1 - 1)^{-1/\alpha}.$$

Besides,

$$\Omega_1 = \left\{x\colon \|x\|_2 \leqslant \left(\frac{\frac{1}{2}(\mu'^2 + \zeta'^2) + \nu_1}{\lambda_1 - \chi_1 - 1}\right)^{1/2}\right\}$$

is a globally attractive set, where

$$\chi_1 = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n G_i^2\right\},$$
$$\mu' = \sqrt{n}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \left(2F_iL_i + 2F_i^2\left|x_i(t_0)\right| + 2L_i\left|\gamma_i'(t)\right|\right)\right\},$$
$$\zeta' = \sqrt{n}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \left(2G_iM_i + 2G_i^2\left|x_i(t_0)\right| + 2G_i\left|\eta_i'(t)\right|\right)\right\},$$
$$\nu_1 = \sum_{i=1}^n\sum_{j=1}^n \left(F_j\left|x_j(t_0)\right| + L_j + \left|\gamma_j'(t)\right|\right)^2 + \sum_{i=1}^n\sum_{j=1}^n \left(G_j\left|x_j(t_0)\right| + M_j + \left|\eta_j'(t)\right|\right)^2 + \sum_{i=1}^n I_i^2.$$

When the activation functions f_i, g_i are Lipschitz-continuous, assumption (H2) can be replaced by the following condition.

(H3) For all $j = 1, 2, \ldots, n$, suppose there exist constants $F_j, G_j > 0$ such that

$$\left|f_j(x_1) - f_j(x_2)\right| \leqslant F_j|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{C};$$
$$\left|g_j(x_1) - g_j(x_2)\right| \leqslant G_j|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{C}.$$

Corollary 2. *FOCVNNs* (6) *is global asymptotically stable if assumption* (H3) *holds and*

$$\lambda_2 = \min_{1\leqslant i\leqslant n}\left\{d_i + \overline{d}_i - \sum_{j=1}^n a_{ij}\overline{a}_{ij} - \sum_{j=1}^n b_{ij}\overline{b}_{ij} - \sum_{j=1}^n \frac{r_j}{r_i}F_i^2\right\} > \chi_2,$$
$$\chi_2 = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \left(\frac{r_j}{r_i}G_i^2\right)\right\}, \quad 0 \leqslant \tau \leqslant (\lambda_2 - \chi_2)^{-1/\alpha}.$$

Proof. Assume that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ and $\widetilde{y}(t) = (\widetilde{y}_1(t), \widetilde{y}_2(t), \ldots, \widetilde{y}_n(t))^{\mathrm{T}}$ are any two solutions of system (6) with the different initial condition $x_i(s) = \varphi_i(s)$ ($s \in [t_0 - \tau, t_0]$), $\widetilde{y}_i(s) = \phi_i(s)$ ($s \in [t_0 - \tau, t_0]$) for $i = 1, \ldots, n$.

Denote $z_i(t) = \widetilde{y}_i(t) - x_i(t)$, and we have

$$\begin{aligned}{}^{C}_{t_0}D^{\alpha}_{t} z_i(t) = -d_i z_i(t) + \sum_{j=1}^{n} a_{ij}\bigl(f_j\bigl(\widetilde{y}_j(t)\bigr) - f_j\bigl(x_j(t)\bigr)\bigr) \\ + \sum_{j=1}^{n} b_{ij}\bigl(g_j\bigl(\widetilde{y}_j(t-\tau)\bigr) - g_j\bigl(x_j(t-\tau)\bigr)\bigr).\end{aligned}$$

Let us construct the function $V(t) = \sum_{j=1}^{n} r_i z_i(t)\overline{z_i(t)}$, then

$$\begin{aligned}{}^{C}_{t_0}D^{\alpha}_{t} V(t) &\leqslant \sum_{j=1}^{n} r_i\bigl(\overline{z_i(t)}\,{}^{C}_{t_0}D^{\alpha}_{t} z_i(t) + z_i(t)\,{}^{C}_{t_0}D^{\alpha}_{t}\overline{z_i(t)}\bigr) \\ &= -\sum_{i=1}^{n} r_i\left(d_i + \overline{d_i} - \sum_{j=1}^{n}\bigl(a_{ij}\overline{a_{ij}} + b_{ij}\overline{b_{ij}}\bigr) - \sum_{j=1}^{n}\frac{r_j}{r_i}F_i^2\right) z_i(t)\overline{z_i(t)} \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\left(\frac{r_j}{r_i}G_i^2\right) z_i(t-\tau)\overline{z_i(t-\tau)} \\ &\leqslant -\lambda_2 V(t) + \chi_2 V(t-\tau).\end{aligned}$$

Obviously, similar to Theorem 2, $\|z(t)\|_2 = \|\widetilde{y}(t) - x(t)\|_2 \to 0$ ($t \to +\infty$) when $\mu = \zeta = \nu = 0$, and the solution of system (6) is globally asymptotically stable. In addition, the existence and uniqueness of equilibrium point for system (6) can be proved by contraction mapping principle. Thus, all the solutions of system (6) converge to the unique equilibrium point. □

3.3 Quasi-Mittag-Leffler synchronization

Define $e_i(t) = y_i(t) - x_i(t)$ as the synchronization error, from systems (6) and (7), we obtain the error system

$$\begin{aligned}{}^{C}_{t_0}D^{\alpha}_{t} e_i(t) = -(d_i + k_i)e_i(t) + \sum_{j=1}^{n} a_{ij}\bigl(\gamma_j(t) - \gamma_j'(t)\bigr) \\ + \sum_{j=1}^{n} b_{ij}\bigl(\eta_j\bigl(t-\tau\bigr) - \eta_j'(t-\tau)\bigr)\end{aligned} \tag{33}$$

for a.a. $t \geqslant t_0$.

Theorem 3. *System* (6) *and* (7) *are quasi-Mittag-Leffler synchronization with controller* (8) *if*

$$\lambda_3 = \min_{1\leqslant i\leqslant n}\left[d_i + \overline{d_i} + k_i + \overline{k_i} - \sum_{j=1}^{n} a_{ij}\overline{a_{ij}} - \sum_{j=1}^{n} b_{ij}\overline{b_{ij}} - \sum_{j=1}^{n}\frac{r_j}{r_i}F_j^2\right] > 1 + \chi_3,$$
$$0 \leqslant \tau \leqslant (\lambda_3 - \chi_3 - 1)^{-1/\alpha}.$$

Moreover, the error bound is

$$\limsup_{t\to+\infty}\left\|e(t)\right\|_2 \leqslant \left(\frac{\frac{1}{2}\mu_3^2+\frac{1}{2}\zeta_3^2+\nu_3}{\min_{1\leqslant i\leqslant n}\{r_i\}(\lambda_3-1-\chi_3)}\right)^{1/2},$$

where

$$\chi_3=\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n\frac{r_j}{r_i}G_i^2\right\},\qquad \nu_3=\sum_{i=1}^n\sum_{j=1}^n r_i\left(L_j^2+M_j^2\right),$$

$$\mu_3=\left(n\max_{1\leqslant i\leqslant n}\{r_i\}\right)^{1/2}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n\frac{r_j}{r_i}2F_iL_i\right\},$$

$$\zeta_3=\left(n\max_{1\leqslant i\leqslant n}\{r_i\}\right)^{1/2}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n\frac{r_j}{r_i}2G_iM_i\right\}.$$

Proof. Define the following nonnegative function:

$$V(t)=\sum_{j=1}^n r_ie_i(t)\overline{e_i(t)}, \tag{34}$$

where $r_i\in\mathbb{R}$. By virtue of Lemma 2, we can get

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) &\leqslant \sum_{i=1}^n r_i\left(\overline{e_i(t)}\,{}_{t_0}^{C}D_t^{\alpha}e_i(t)+e_i(t)\,{}_{t_0}^{C}D_t^{\alpha}\overline{e_i(t)}\right)\\
&=\sum_{i=1}^n r_i\left[-(d_i+\overline{d_i}+k_i+\overline{k_i})e_i(t)\overline{e_i(t)}\right]\\
&\quad+\sum_{i=1}^n\sum_{j=1}^n r_i\left[a_{ij}\overline{e_i(t)}\left(\gamma_j(t)-\gamma_j'(t)\right)+\overline{a_{ij}}e_i(t)\left(\overline{\gamma_j(t)}-\overline{\gamma_j'(t)}\right)\right]\\
&\quad+\sum_{i=1}^n\sum_{j=1}^n r_i\left[b_{ij}\overline{e_i(t)}\left(\eta_j(t-\tau)-\eta_j'(t-\tau)\right)\right.\\
&\qquad\qquad\left.+\overline{b_{ij}}e_i(t)\left(\overline{\eta_j(t-\tau)}-\overline{\eta_j'(t-\tau)}\right)\right].
\end{aligned} \tag{35}$$

According to Lemma 3 and assumption (H2),

$$\begin{aligned}
&\sum_{i=1}^n\sum_{j=1}^n r_i\left[a_{ij}\overline{e_i(t)}\left(\gamma_j(t)-\gamma_j'(t)\right)+\overline{a_{ij}}e_i(t)\left(\overline{\gamma_j(t)}-\overline{\gamma_j'(t)}\right)\right]\\
&\quad\leqslant\sum_{i=1}^n\sum_{j=1}^n r_i\left[a_{ij}\overline{a_{ij}}e_i(t)\overline{e_i(t)}+\left(\gamma_j(t)-\gamma_j'(t)\right)\left(\overline{\gamma_j(t)}-\overline{\gamma_j'(t)}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[a_{ij}\overline{a_{ij}}e_i(t)\overline{e_i(t)} + \bigl(F_j\bigl|e_j(t)\bigr| + L_j\bigr)^2\bigr] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[a_{ij}\overline{a_{ij}}e_i(t)\overline{e_i(t)} + F_j^2 e_j(t)\overline{e_j(t)} + 2F_jL_j\bigl|e_j(t)\bigr| + L_j^2\bigr].
\end{aligned} \tag{36}$$

Similarly, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[b_{ij}\overline{e_i(t)}\bigl(\eta_j(t-\tau) - \eta_j'(t-\tau)\bigr) + \overline{b_{ij}}e_i(t)\bigl(\overline{\eta_j(t-\tau)} - \overline{\eta_j'(t-\tau)}\bigr)\bigr] \\
&\quad\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[b_{ij}\overline{b_{ij}}e_i(t)\overline{e_i(t)} + \bigl(\eta_j(t-\tau) - \eta_j'(t-\tau)\bigr)\bigl(\overline{\eta_j(t-\tau)} - \overline{\eta_j'(t-\tau)}\bigr)\bigr] \\
&\quad\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[b_{ij}\overline{b_{ij}}e_i(t)\overline{e_i(t)} + \bigl(G_j\bigl|\eta_j(t-\tau)\bigr| + M_j\bigr)^2\bigr] \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[b_{ij}\overline{b_{ij}}e_i(t)\overline{e_i(t)} + G_j^2\eta_j(t-\tau)\overline{\eta_j(t-\tau)} \\
&\qquad\qquad + 2G_jM_j\bigl|\eta_j(t-\tau)\bigr| + M_j^2\bigr].
\end{aligned} \tag{37}$$

From (35), (36), and (37) we have

$$\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha}V(t) \\
&\quad\leqslant \sum_{i=1}^{n} r_i\bigl[-(d_i + \overline{d_i} + k_i + \overline{k_i})e_i(t)\overline{e_i(t)}\bigr] \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[a_{ij}\overline{a_{ij}}e_i(t)\overline{e_i(t)} + F_j^2e_j(t)\overline{e_j(t)} + 2F_jL_j\bigl|e_j(t)\bigr| + L_j^2\bigr] \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl[b_{ij}\overline{b_{ij}}e_i(t)\overline{e_i(t)} + G_j^2\eta_j(t-\tau)\overline{\eta_j(t-\tau)} + 2G_jM_j\bigl|\eta_j(t-\tau)\bigr| + M_j^2\bigr] \\
&\quad\leqslant -\sum_{i=1}^{n} r_i\Biggl[d_i + \overline{d_i} + k_i + \overline{k_i} - \sum_{j=1}^{n} a_{ij}\overline{a_{ij}} - \sum_{j=1}^{n} b_{ij}\overline{b_{ij}} - \sum_{j=1}^{n}\frac{r_j}{r_i}F_j^2\Biggr]e_i(t)\overline{e_i(t)} \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\biggl(\frac{r_j}{r_i}2F_iL_i\biggr)\bigl|e_i(t)\bigr| + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\biggl(\frac{r_j}{r_i}G_i^2\biggr)e_i(t-\tau)\overline{e_i(t-\tau)} \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\biggl(\frac{r_j}{r_i}2G_iM_i\biggr)\bigl|e_i(t-\tau)\bigr| + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bigl(L_j^2 + M_j^2\bigr).
\end{aligned} \tag{38}$$

Let

$$\mu_3' = \max_{1\leqslant i\leqslant n}\Biggl\{\sum_{j=1}^{n}\frac{r_j}{r_i}2F_iL_i\Biggr\}, \qquad \zeta_3' = \max_{1\leqslant i\leqslant n}\Biggl\{\sum_{j=1}^{n}\frac{r_j}{r_i}2G_iM_i\Biggr\}.$$

From (38) we have

$$\begin{aligned}
&{}^{C}_{t_0}D^{\alpha}_{t}V(t) \\
&\quad\leqslant -\sum_{i=1}^{n} r_i\lambda_3 e_i(t)\overline{e_i(t)} + \sum_{i=1}^{n} r_i\chi_3 e_i(t-\tau)\overline{e_i(t-\tau)} \\
&\qquad + \sum_{i=1}^{n} r_i\mu_3'\big|e_i(t)\big| + \sum_{i=1}^{n} r_i\zeta_3'\big|e_i(t-\tau)\big| + \nu_3 \\
&\quad\leqslant -\sum_{i=1}^{n} r_i\lambda_3 e_i(t)\overline{e_i(t)} + \sum_{i=1}^{n} r_i\chi_3 e_i(t-\tau)\overline{e_i(t-\tau)} \\
&\qquad + \mu_3'\Bigg(n\sum_{i=1}^{n} r_i^2 e_i(t)\overline{e_i(t)}\Bigg)^{1/2} + \zeta_3'\Bigg(n\sum_{i=1}^{n} r_i^2 e_i(t-\tau)\overline{e_i(t-\tau)}\Bigg)^{1/2} + \nu_3. \qquad (39)
\end{aligned}$$

Denote

$$\mu_3 = \mu_3'\Big(n\max_{1\leqslant i\leqslant n}\{r_i\}\Big)^{1/2}, \qquad \zeta_3 = \zeta_3'\Big(n\max_{1\leqslant i\leqslant n}\{r_i\}\Big)^{1/2}.$$

The following inequality is given from (39):

$${}^{C}_{t_0}D^{\alpha}_{t}V(t) \leqslant -\lambda_3 V(t) + \chi_3 V(t-\tau) + \mu_3 V^{1/2}(t) + \zeta_3 V^{1/2}(t-\tau) + \nu_3. \qquad (40)$$

Since $\lambda_3 > (1+\chi_3)$ and $\chi_3, \mu_3, \zeta_3, \nu_3 > 0$, according to Lemma 5, we have

$$\begin{aligned}
&\min_{1\leqslant i\leqslant n}\{r_i\}\sum_{j=1}^{n} e_i(t)\overline{e_i(t)} \\
&\quad\leqslant V(t) \\
&\quad\leqslant \frac{(\frac{1}{2}+\chi_3)(\lambda_3-1-\chi_3)^{-1/\alpha}(t-t_0)^{\alpha-1}\xi}{\Gamma(\alpha)} \\
&\qquad + \bigg(\xi - \frac{\frac{1}{2}\mu_3^2 + \frac{1}{2}\zeta_3^2 + \nu_3}{\lambda_3-1-\chi_3}\bigg)E_\alpha\big[-(\lambda_3-1-\chi_3)(t-t_0)^\alpha\big] \\
&\qquad + \frac{\frac{1}{2}\mu_3^2 + \frac{1}{2}\zeta_3^2 + \nu_3}{\lambda_3-1-\chi_3}, \qquad (41)
\end{aligned}$$

where $\xi = \max_{t\in[t_0-\tau,\, t_0]}$.

Then

$$\begin{aligned}
\big\|e(t)\big\|_2 \leqslant \Bigg\{ &\frac{(\frac{1}{2}+\chi_3)(\lambda_3-1-\chi_3)^{-1/\alpha}(t-t_0)^{\alpha-1}\xi}{\min_{1\leqslant i\leqslant n}\{r_i\}\Gamma(\alpha)} \\
&+ \bigg(\xi - \frac{\frac{1}{2}\mu_3^2 + \frac{1}{2}\zeta_3^2 + \nu_3}{\min_{1\leqslant i\leqslant n}\{r_i\}(\lambda_3-1-\chi_3)}\bigg)E_\alpha\big[-(\lambda_3-1-\chi_3)(t-t_0)^\alpha\big] \\
&+ \frac{\frac{1}{2}\mu_3^2 + \frac{1}{2}\zeta_3^2 + \nu_3}{\min_{1\leqslant i\leqslant n}\{r_i\}(\lambda_3-1-\chi_3)}\Bigg\}^{1/2}.
\end{aligned}$$

Similarly deal with (32),

$$\limsup_{t\to+\infty}\|y(t)-x(t)\|_2=\limsup_{t\to+\infty}\|e(t)\|_2\leqslant\left(\frac{\frac{1}{2}\mu_3^2+\frac{1}{2}\zeta_3^2+\nu_3}{\min_{1\leqslant i\leqslant n}\{r_i\}(\lambda_3-1-\chi_3)}\right)^{1/2},$$

which implies that systems (6) and (7) are quasi-Mittag-Leffler synchronized. □

Remark 3. In Theorem 3, the sufficient conditions are only related to the parameters of the system and the controller. In particular, the error bounds of quasi-Mittag-Leffler synchronization have nothing with initial values. No matter what the initial values of the drive-response systems (6) and (7) are, the upper bound of the synchronization error can be obtained by adjusting the controller parameters. This not only provides a research framework for synchronization, but also extends earlier findings in fractional-order systems [6, 7]. Thus, our results are less conservative and can be considered a meaningful extension of previous synchronization work.

Let $r_i=1$ in Theorem 3, we can obtain the following corollary.

Corollary 3. *Based on controller* (8)*, system* (6) *and* (7) *are quasi-Mittag-Leffler synchronization if assumptions* (H1) *and* (H2) *hold and*

$$\lambda_4=\min_{1\leqslant i\leqslant n}\left[d_i+\overline{d_i}+k_i+\overline{k_i}-\sum_{j=1}^{n}a_{ij}\overline{a_{ij}}-\sum_{j=1}^{n}b_{ij}\overline{b_{ij}}-\sum_{j=1}^{n}F_j^2\right]>1+\chi_4,$$

$0\leqslant\tau\leqslant(\lambda_4-\chi_4-1)^{-1/\alpha}$. *Moreover, the error bound is*

$$\limsup_{t\to+\infty}\|e(t)\|_2\leqslant\left(\frac{\frac{1}{2}\mu_4^2+\frac{1}{2}\zeta_4^2+\nu_4}{(\lambda_4-1-\chi_4)}\right)^{1/2},$$

where

$$\chi_4=\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}G_i^2\right\},\qquad \mu_4=\sqrt{n}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}2F_iL_i\right\},$$

$$\zeta_4=\sqrt{n}\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}2G_iM_i\right\},\qquad \nu_4=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(L_j^2+M_j^2\right).$$

Remark 4. Note that in [3, 20, 23], the authors studied the dynamic behaviors of FOCVNNs by dividing the complex-valued systems into their real parts and imaginary parts. Unlike those methods, in this paper, we deal with such system as a compact entirety without any decomposition in complex domain, which better reflects the characteristics of complex-valued systems, this makes our method more efficient and compact. If assume that the imaginary part of FOCVNNs (6) is zero, then FOCVNNs (6) can be deduced to fractional-order RVNNs with discontinuous activation functions, the results for such system in [5] could be extended to the complex domain. Li et al. [15] investigated quasi-projective synchronization of FOCVNNs, whose activation functions have not only the bounded modulus, but also satisfy Lipschitz condition. However, such conditions are not needed and our activation functions are discontinuous, and the Lipschitz-continuous activation functions can be as a special case when $L_j = 0$ and $M_j = 0$ of assumption (H2). In [7], the authors investigated global Mittag-Leffler synchronization of FONNs, note that time delay is not considered in their model. Thus our results extent and complement the existing literature [5, 7, 15], to some extent, and achieve a valuable improvement.

When the activation functions satisfy assumption (H3), the globally Mittag-Leffler synchronization result is given as special case of Theorem 3.

Corollary 4. *System* (6) *and* (7) *are globally Mittag-Leffler synchronization with the controller* (8) *if assumption* (H3) *holds and*

$$\lambda_5 = \min_{1\leqslant i\leqslant n}\left[d_i + \overline{d_i} + k_i + \overline{k_i} - \sum_{j=1}^{n} a_{ij}\overline{a_{ij}} - \sum_{j=1}^{n} b_{ij}\overline{b_{ij}} - \sum_{j=1}^{n}\frac{r_j}{r_i}F_j^2\right] > 1 + \chi_5,$$
$$0 \leqslant \tau \leqslant (\lambda_5 - \chi_5 - 1)^{-1/\alpha},$$

where

$$\chi_5 = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\frac{r_j}{r_i}G_i^2\right\}.$$

4 Numerical simulations

In this section, we give two numerical examples to show the effectiveness of our main results.

Example 1. Consider the two-dimensional FOCVNNs (6) with time delays and discontinuous activations, where

$$A = \begin{pmatrix} 0.5 + \mathrm{i}0.3 & -0.5 + \mathrm{i}0.5 \\ 2.2 + \mathrm{i}0.7 & 0.5 + \mathrm{i}0.2 \end{pmatrix}, \qquad D = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix},$$
$$B = \begin{pmatrix} 0.4 + \mathrm{i}1.5 & -0.5 + \mathrm{i}1.5 \\ 0.5 - \mathrm{i}0.8 & 0.3 + \mathrm{i}0.6 \end{pmatrix}, \qquad I = \begin{pmatrix} 0.6 - \mathrm{i}0.4 \\ -0.5 + \mathrm{i}0.3 \end{pmatrix},$$

$\alpha = 0.9$, $\tau = 0.1$, the discontinuous activation function is taken as

$$f_j\bigl(x_j(t)\bigr) = g_j\bigl(x_j(t)\bigr) = \tanh\bigl(x_j^R(t)\bigr) + 0.05\,\mathrm{sign}\bigl(x_j^R(t)\bigr) + \mathrm{i}\bigl[\sin\bigl(x_j^I(t)\bigr) + 0.05\,\mathrm{sign}\bigl(x_j^I(t)\bigr)\bigr].$$

Then from assumption (H2) we have $F_j = G_j = 1$ and $L_j = M_j = 0.1$ for $j = 1, 2$. From Corollary 1 $\lambda_1 = 5.04$, $\mu' = 1.8278$, $\zeta' = 2.8825$, $\chi_1 = 1$, $\nu_1 = 8.2524$, $0 \leqslant \tau = 0.1 \leqslant (\lambda_1 - \chi_1 - 1)^{-1/\alpha} = 0.2907$, and

$$\|S\|_2 \leqslant \left(\frac{\frac{1}{2}(\zeta'^2 + \mu'^2) + \nu_1}{\lambda_1 - \chi_1 - 1}\right)^{1/2} \leqslant 2.1519.$$

Thus, the attractive set is $\Omega = \{x\colon \|S\|_2 \leqslant 2.1519\}$.

Example 2. Consider the two-dimensional FOCVNNs (6) with time delays and discontinuous activations, where

$$A = \begin{pmatrix} 0.05 + \mathrm{i}0.3 & 0.5 + \mathrm{i}0.1 \\ 0.1 + \mathrm{i}0.7 & 0.5 + \mathrm{i}0.2 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},$$
$$B = \begin{pmatrix} 0.6 + \mathrm{i}0.5 & 0.9 + \mathrm{i}0.5 \\ 0.5 - \mathrm{i}0.8 & 0.9 + \mathrm{i}0.6 \end{pmatrix}, \qquad I = \begin{pmatrix} 0.6 - \mathrm{i}0.4 \\ 0.5 + \mathrm{i}0.3 \end{pmatrix},$$

$\alpha = 0.96$, $\tau = 0.01$, the discontinuous activation function is taken as

$$f_j\big(x_j(t)\big) = \tanh\big(x_j^R(t)\big) + 0.3\,\mathrm{sign}\big(x_j^R(t)\big) + \mathrm{i}\big[\tanh\big(x_j^I(t)\big) + 0.2\,\mathrm{sign}\big(x_j^I(t)\big)\big],$$
$$g_j\big(x_j(t)\big) = \sin\big(x_j^R(t)\big) + 0.3\,\mathrm{sign}\big(x_j^R(t)\big) + \mathrm{i}\big[\sin\big(x_j^I(t)\big) + 0.2\,\mathrm{sign}\big(x_j^I(t)\big)\big].$$

Then from assumption (H2) we have $F_j = 1$, $L_j = 0.6$, $G_j = 1$, and $M_j = 0.4$ for $j = 1, 2$. Choose the control gains $k_1 = 11 + \mathrm{i}$, $k_2 = 8 + \mathrm{i}$. Under the control scheme (8), we can obtain $\lambda_4 = 17.15$, $\mu_4 = 1.69705$, $\zeta_4 = 0.2828$, $\chi_4 = 1$, $\nu_4 = 1.0417$, $0 \leqslant \tau = 0.01 \leqslant (\lambda_1 - \chi_1 - 1)^{-1/\alpha} = 0.0589$. From Corollary 3 the error is bounded:

$$\limsup_{t\to+\infty} \|e\|_2 \leqslant \left(\frac{\frac{1}{2}(\mu_4^2 + \zeta_4^2) + \nu_4}{\lambda_1 - \chi_4 - 1}\right)^{1/2} = \left(\frac{\frac{1}{2}(1.6975^2 + 0.2828^2) + 1.0417}{17.15 - 1 - 1}\right)^{1/2}$$
$$= 0.4080.$$

Systems (6) and (7) are quasi-Mittag-Leffler synchronization.

5 Conclusions and future works

In this paper, the authors have considered the global dissipativity and quasi-Mittag-Leffler synchronization of FOCVNNs with time delays and discontinuous activations. Some new sufficient conditions are obtained with the use differential inclusions. Moreover, without decomposing the system into two real-valued systems, we utilize a novel fractional-order inequality to derive criteria that ensure dissipativity and quasi-Mittag-Leffler synchronization of FOCVNNs. Two numerical examples are presented to demonstrate the effectiveness and usefulness of our main results. In future work, we aim to analyze diverse synchronization behaviors of FOCVNNs through the design of various control strategies and to explore their applications in pattern recognition and signal processing.

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