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A class of nonlinear double-phase Dirichlet fractional differential equations*

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Abstract. In this paper, we study the existence of positive solutions for a new class of double-phase Dirichlet fractional differential equations with singular and superlinear terms. By applying the Nehari manifold method we show that for all small values of the parameter $\tau > 0$, the considered equation has at least two positive solutions.

Keywords: ψ-Hilfer fractional derivative, double-phase problem, positive solution, singularity.

1 Introduction and motivation

In this paper, we study the existence of solutions to the following singular double-phase fractional problem:

$$\begin{aligned} &\mathbf{L}_p^{\alpha,\beta;\psi}\phi = (\xi)\phi^{-\varpi} + \tau\phi^{r-1} \quad \text{in } \Omega := (0,T)\times(0,T), \\ &\phi = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P_τ}$$

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where $T > 0$, and $\mathbf{L}_p^{\alpha,\beta;\psi}\phi$ is the double-phase fractional differential operator defined by

$$\begin{aligned}\mathbf{L}_p^{\alpha,\beta;\psi}\phi := \mathfrak{D}_T^{\alpha,\beta;\psi}\big(\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^{p-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big) \\ - \mathfrak{D}_T^{\alpha,\beta;\psi}\big(\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^{q-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big)\end{aligned}$$

with $1 < q < p < r < p_\alpha^*$, $0 < \varpi < 1$, $\tau > 0$, $\alpha \in (0,1)$, $\beta \in [0,1]$, $p_\alpha^* = 2p/(2-\alpha p)$, and $\alpha p < 2$. Also, $\mathfrak{D}_T^{\alpha,\beta;\psi}(\cdot)$ and $\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}(\cdot)$ denote, respectively, the ψ-Hilfer fractional differential operators of order $0 < \alpha < 1$ and type $0 \leqslant \beta \leqslant 1$. In addition, the weight $\kappa : \Omega \to \mathbb{R}_+$ is assumed to be essentially bounded.

The energy functional $\mathcal{E}_\tau^{\alpha,\beta;\psi} : \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega) \to \mathbb{R}$ corresponding to problem (P_τ) is given by

$$\begin{aligned}\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi) &= \frac{1}{p}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + \frac{1}{q}\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q\,\mathrm{d}\xi \\ &\quad - \frac{1}{1-\varpi}\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi - \frac{\tau}{r}\|\phi\|_r^r \quad \text{for all } \phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega). \qquad (1)\end{aligned}$$

Furthermore, we say that $\phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ is a weak positive solution of (P_τ) if $\phi(\xi) \geqslant 0$ for a.a. $\xi \in \Omega$ $\phi \neq 0$ and

$$\begin{aligned}&\int_\Omega |\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi|^{p-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\,\mathrm{d}\xi + \int_\Omega \kappa(\xi)|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi|^{q-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\,\mathrm{d}\xi \\ &\quad = \int_\Omega a(\xi)\phi^{-\varpi}h\,\mathrm{d}\xi + \tau\int_\Omega \phi^{r-1}h\,\mathrm{d}\xi \quad \text{for all } \phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega).\end{aligned}$$

Let $p \in (1,\infty)$. The p-Laplace operator, Δ_p, is defined as follows:

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big). \qquad (2)$$

It is clear that if $p = 2$, then (2) reduces to the Laplacian. In terms of applications, the Laplace operator (i.e., $p = 2$) is well suited to the study of the problems with isotropic structure. Unlike the Laplace operator, the p-Laplacian is a nonlinear and nonhomogeneous operator, which can be applied to the study of non-Newtonian fluids, nonlinear elasticity theory, oil production, etc.; see [6, 11, 13, 20, 27, 28].

In order to provide effective mathematical modeling for the study of strongly anisotropic materials, in the 1980s, Zhikov [34] introduced a class of nonhomogeneous and nonlinear differential operators with nonbalance growth (which is called by double-phase operators), i.e.,

$$u \mapsto \Delta_{(p,q)}u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big), \qquad (3)$$

where $1 < p < q < N$, $N \geqslant 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, and $a : \Omega \to [0,\infty)$ is Lipschitz continuous. In recent years, problems involving differential operators (3) have gained dramatic development in the study of the existence and multiplicity of solutions via the application of variational methods; see, e.g., [3, 4, 7, 18, 19, 23]. We mention some excellent work on double-phase problems. Colombo and Mingione [8, 9] considered the integral functional driven by the double-phase differential operator (3)

$$\int_{\Omega} \left(|\nabla u|^p + a(x)\left|\nabla u|^q\right) \mathrm{d}x$$

and discussed the regularity (including L^∞ and Hölder continuity) of such operators. In 2018, Liu and Dai [17] proved the existence and multiplicity of solutions for the following double-phase problem:

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = f(x,u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geqslant 2$, $1 < p, q < N$. Motivated by problem (4), Gasiński–Winkert [12] investigated the existence and uniqueness of the solution for the following double-phase problem with convection term:

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) = f(x,u,\nabla u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $1 < p < q < N$, $\mu : \overline{\Omega} \to [0, \infty)$ is supposed to be Lipschitz-continuous, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e., $x \mapsto f(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and $(s, \xi) \mapsto f(x, s, \xi)$ is continuous for a.a. $x \in \Omega$). Meanwhile, Papageorgiou et al. [21] used variational methods along with truncation techniques to prove the existence of positive solutions for a class of double-phase problems with singular term. For more details on the research in this direction, we can refer to Liu–Dai [17], Gasiński–Winkert [12], Papageorgiou–Repovš–Vetro [21], Le [15], Arora–Fiscella–Mukherjee–Winkert [2] and Faria–Miyagaki–Motreanu [10], Zeng–Rădulescu [33], and the references therein.

On the other hand, the study of fractional operators has grown considerably in recent years; see [29–31]. This is because it has been used to describe complicated phenomena as diverse as the Lévy diffusion process, flame propagation, continuum mechanics, population dynamics studies, and even game theory; see Applebaum [1], Vazquez [32], Servadei–Valdinoci [24,25], Iannizzoto–Pereira–Squassina [14], and the references therein. We highlight some representative work: Bouabdallah et al. [5] used the Nehari manifold approach and fibre maps to prove a multiplicity result for fractional p-Laplace equations; Sun [26] considered a complex differential equation with variable exponents p-Laplace differential operator and used a fixed point argument and the extension theory of Mawhin's coincidence theory to show the existence of solutions of the differential equation.

The following are important features of problem (P_τ):

(i) The weight $\kappa(\cdot)$, which is discontinuous and not bounded by zero, leads to the invalidity of Lieberman's theory of global regularity [16] and Pucci's nonlinear strong maximum principle [22] for (P_τ). Therefore, to overcome such difficulties, our idea is to use the Nehari manifold method.

(ii) The reaction term of (P_τ) is combined with a singular term and a $(p-1)$-superlinear parametric perturbation.

(iii) The energy functional $\mathcal{E}_\tau^{\alpha,\beta;\psi}$ is not C^1 due to the presence of the singular term $a(\xi)\phi^{-\varpi}$. This means that the variational method cannot be applied directly.

(iv) (P_τ) is a generalized double-phase problem, which contains a large class of possible special cases. On the other hand, it should be mentioned that $\psi(t) = \log_a(t)$ with $0 < a < 1$ cannot be used to obtain the results discussed in this paper since $\psi(t) = \log_a(t)$ is a nonincreasing function and contradicts the conditions of the function ψ as presented in Section 2.

In this paper, we assume that

(Q_κ) $\kappa \in L^\infty(\Omega)$ and $\kappa(\xi) > 0$ for a.a. $\xi \in \Omega$;
(Q_a) $a \in L^\infty(\Omega)$ and $a(\xi) \geqslant 0$ for a.a. $\xi \in \Omega$, and $a \neq 0$.

In the next sections, we will discuss some critical properties of the energy functional $\mathcal{E}_\tau^{\alpha,\beta;\psi}(\cdot)$ and the Nehari manifold Ξ_τ; see below. Then we will prove the main result of this paper, which is given by the following theorem.

Theorem 1. *If hypotheses* (Q_κ) *and* (Q_a) *hold, then there exists* $\hat{\tau}_0^* > 0$ *such that for all* $\tau \in (0, \hat{\tau}_0^*]$, (P_τ) *has at least two positive solutions* $\phi^*, v^* \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ *such that* $\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*) < 0 < \mathcal{E}_\tau^{\alpha,\beta;\psi}(v^*)$.

2 Fractional operators and variational setting

We use the symbol $\mathscr{L}^p(\Omega)$ to represent Lebesgue's function space

$$\mathscr{L}^p(\Omega) = \left\{\phi : \Omega \to \mathbb{R} \text{ is measurable} \;\middle|\; \int_\Omega |\phi|^p \,\mathrm{d}\xi < +\infty\right\}.$$

It is clear that $\mathscr{L}^p(\Omega)$ endowed with the norm $\|\phi\| = (\int_\Omega |\phi|^p \,\mathrm{d}\xi)^{\frac{1}{p}}$, becomes a reflexive and separable Banach space.

Let $0 < \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$. Set $\Lambda = I_1 \times I_2$ with $I_1 = (0, L]$ and $I_2 = (0, T]$, where T, L are two positive constants. Let $\psi(\cdot)$ be an increasing and positive function on I_2 and have a continuous derivative $\psi'(\cdot)$ on I_2. Furthermore, let $\phi, \psi \in C^n(\Lambda)$ be two functions such that ψ is increasing and $\psi'(\xi_2) \neq 0$ for all $\xi_2 \in I_2$. The left- and right-sided ψ-Hilfer fractional partial derivatives of $\phi \in AC^n(\Lambda)$ of order α and type β are defined by (see, e.g., [31])

$$\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\phi(\xi_1,\xi_2) = \mathbf{I}_{0+}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(\xi_2)}\frac{\partial}{\partial\xi_2}\right)\mathbf{I}_{0+}^{(1-\beta)(1-\alpha),\psi}\phi(\xi_1,\xi_2)$$

and

$$\mathfrak{D}_{T}^{\alpha,\beta;\psi}\phi(\xi_1,\xi_2) = \mathbf{I}_{T}^{\beta(1-\alpha),\psi}\left(-\frac{1}{\psi'(\xi_2)}\frac{\partial}{\partial\xi_2}\right)\mathbf{I}_{T}^{(1-\beta)(1-\alpha),\psi}\phi(\xi_1,\xi_2)$$

for $\xi_1 \in I_1$ and $\xi_2 \in I_2$, where $\mathbf{I}_{0+}^{\alpha,\psi}\phi(\xi_1,\xi_2)$ and $\mathbf{I}_{T}^{\alpha,\psi}\phi(\xi_1,\xi_2)$ are the left- and right-sided ψ-Riemann–Liouville fractional integrals of $\phi \in \mathscr{L}^1(\Lambda)$ of order α given by (see, e.g., [31])

$$\mathbf{I}_{0+}^{\alpha,\psi}\phi(\xi_1,\xi_2) = \frac{1}{\Gamma(\alpha)}\int_0^{\xi_2} \big(\psi(\xi_2)-\psi(s)\big)^{\alpha-1}\phi(\xi_1,s)\,\mathrm{d}s$$

and

$$\mathbf{I}_{T}^{\alpha,\psi}\phi(\xi_1,\xi_2) = \frac{1}{\Gamma(\alpha)}\int_{\xi_2}^{T} \big(\psi(s)-\psi(\xi_2)\big)^{\alpha_2-1}\phi(\xi_1,s)\,\mathrm{d}s$$

for all $\xi_1 \in I_1$ and $\xi_2 \in I_2$.

It is not difficult to see that the following identity holds for the ψ-Riemann–Liouville fractional integral:

$$\int_0^T \big(\mathbf{I}_{0+}^{\alpha;\psi}\varphi(\xi_1,\xi_2)\big)\phi(\xi_1,\xi_2)\,\mathrm{d}\xi_2 = \int_0^T \varphi(\xi_1,\xi_2)\psi'(\xi_2)\mathbf{I}_{T}^{\alpha;\psi}\left(\frac{\phi(\xi_1,\xi_2)}{\psi'(\xi_2)}\right)\mathrm{d}\xi_2.$$

On the other hand, if $\psi(\cdot)$ is an increasing and positive monotone function on $[0,T]$ such that $\psi'(\cdot) \neq 0$ is continuous on $(0,T)$, then, with $0 < \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$,

$$\int_0^T \big(\mathfrak{D}_{\theta}^{\alpha,\beta;\psi}\varphi(\xi_1,\xi_2)\big)\phi(\xi_1,\xi_2)\,\mathrm{d}\xi_2 = \int_0^T \varphi(\xi_1,\xi_2)\psi'(\xi_2)\mathfrak{D}_{T}^{\alpha,\beta;\psi}\left(\frac{\phi(\xi_1,\xi_2)}{\psi'(\xi_2)}\right)\mathrm{d}\xi_2$$

for any $\varphi \in AC^1(I_1 \times I_2)$ and $\phi \in C^1(I_1 \times I_2)$ satisfying the boundary conditions $\varphi(0,0) = 0 = \varphi(L,T)$.

The basic function space in this paper is the ψ-fractional space given by

$$\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega) = \big\{\phi \in \mathscr{L}^p(\Omega)\colon \big|{}^{\mathrm{H}}\mathfrak{D}_{0+}^{\alpha,\beta;\psi}\phi\big| \in \mathscr{L}^p(\Omega)\big\}$$

with the norm

$$\|\phi\| = \|\phi\|_{\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)} = \|\phi\|_{\mathscr{L}^p(\Omega)} + \big\|{}^{\mathrm{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\phi\big\|_{\mathscr{L}^p(\Omega)}.$$

Proposition 1. *The spaces $\mathscr{L}^p(\Omega)$ and $\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ are separable and reflexive Banach spaces. Moreover, $C_0^\infty(\Omega)$ is dense in $\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$.*

Furthermore, for any $\tau > 0$, we consider the Nehari manifold associated with (P_τ) as follows:

$$\begin{aligned}\boldsymbol{\Xi}_\tau = \bigg\{\phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)\colon \big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p \int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q\,\mathrm{d}\xi \\ = \int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi + \tau\|\phi\|_r^r,\ \phi \neq 0\bigg\}.\end{aligned}$$

We also decompose Ξ_τ into three disjoint parts:

$$\Xi_\tau^+ = \biggl\{\phi \in \Xi_\tau\colon\ (p+\varpi-1)\bigl\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr\|_p^p + (q+\varpi-1)\int\limits_\Omega \kappa(\xi)\bigl|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr|^q \,\mathrm{d}\xi > \tau(r+\varpi-1)\|\phi\|_r^r\biggr\},$$

$$\Xi_\tau^0 = \biggl\{\phi \in \Xi_\tau\colon\ (p+\varpi-1)\bigl\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr\|_p^p + (q+\varpi-1)\int\limits_\Omega \kappa(\xi)\bigl|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr|^q \,\mathrm{d}\xi = \tau(r+\varpi-1)\|\phi\|_r^r\biggr\},$$

and

$$\Xi_\tau^- = \biggl\{\phi \in \Xi_\tau\colon\ (p+\varpi-1)\bigl\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr\|_p^p + (q+\varpi-1)\int\limits_\Omega \kappa(\xi)\bigl|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr|^q \,\mathrm{d}\xi < \tau(r+\varpi-1)\|\phi\|_r^r\biggr\}.$$

3 Main results

In this section, we give a detailed proof of our main theorem, Theorem 1. The strategy is based on the Nehari manifold and variational techniques.

To do this, we need the following propositions.

Proposition 2. *If hypotheses* (Q_κ) *and* (Q_a) *hold, then for all* $\tau > 0$*, the energy functional* $\mathcal{E}_\tau^{\alpha,\beta;\psi}(\cdot)$*, given in* (1)*, is coercive on* Ξ_τ*.*

Proof. Let $\phi \in \Xi_\tau$, we have

$$-\frac{1}{r}\bigl\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr\|_p^p - \frac{1}{r}\int\limits_\Omega \kappa(\xi)\bigl|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr|^q \,\mathrm{d}\xi + \frac{1}{r}\int\limits_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi + \frac{\tau}{r}\|\phi\|_r^r = 0.$$

It follows that

$$\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi) = \left(\frac{1}{p}-\frac{1}{r}\right)\bigl\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr\|_p^p + \left(\frac{1}{q}-\frac{1}{r}\right)\int\limits_\Omega \kappa(\xi)\bigl|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\bigr|^q \,\mathrm{d}\xi + \left(\frac{1}{r}-\frac{1}{1-\varpi}\right)\int\limits_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi.$$

Since $q < p < r$, so, we apply the Poincaré inequality and the Sobolev embedding theorem to get

$$\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi) \geqslant C_1\|\phi\|^p - C_2\|\phi\|^{1-\varpi} \quad \text{for some } C_1, C_2 > 0.$$

The latter, combined with $p > 1 > 1-\varpi$, implies that $\mathcal{E}_\tau^{\alpha,\beta;\psi}|_{\Xi_\tau}$ is coercive. □

Set $\mathfrak{M}_\tau^+ := \inf_{\Xi_\tau^+} \mathcal{E}_\tau^{\alpha,\beta;\psi}$.

Proposition 3. *Under hypotheses* (Q_κ) *and* (Q_a)*, if* $\Xi_\tau^+ \neq \emptyset$*, then* $\mathfrak{M}_\tau^+ < 0$.

Proof. For any $\phi \in \Xi_\tau^+$, one has

$$\tau\|\phi\|_r^r < \frac{p+\varpi-1}{r+\varpi-1}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + \frac{q+\varpi-1}{r+\varpi-1}\int\limits_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi. \tag{5}$$

By the fact $\Xi_\tau^+ \subset \Xi_\tau$, we get

$$-\frac{1}{1-\varpi}\int\limits_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi = -\frac{1}{1-\varpi}\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^p - \frac{1}{1-\varpi}\int\limits_\Omega \kappa(\xi)|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi|^q\,\mathrm{d}\xi + \frac{\tau}{1-\varpi}\|\phi\|_r^r. \tag{6}$$

Because of $q < p < r$, it follows from (5) and (6) that

$$\begin{aligned}
\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi) &= \frac{1}{p}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p \\
&\quad + \frac{1}{q}\int\limits_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q\,\mathrm{d}\xi - \frac{\tau}{r}\|\phi\|_r^r - \frac{1}{1-\varpi}\int\limits_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi \\
&= \left(\frac{1}{p} - \frac{1}{1-\varpi}\right)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + \left(\frac{1}{q} - \frac{1}{1-\varpi}\right)\int\limits_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q\,\mathrm{d}\xi \\
&\quad + \tau\left(\frac{1}{1-\varpi} - \frac{1}{r}\right)\|\phi\|_r^r \\
&\leqslant \left[-\frac{(p+\varpi-1)}{p(1-\varpi)} + \frac{(p+\varpi-1)(r+\varpi-1)}{(r+\varpi-1)(r(1-\varpi))}\right]\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p \\
&\quad + \left[-\frac{(q+\varpi-1)}{q(1-\varpi)} + \frac{(q+\varpi-1)(r+\varpi-1)}{(r+\varpi-1)(r(1-\varpi))}\right]\int\limits_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q\,\mathrm{d}\xi \\
&= \frac{p+\varpi-1}{1-\varpi}\left(\frac{1}{r} - \frac{1}{p}\right)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p \\
&\quad + \frac{q+\varpi-1}{1-\varpi}\left(\frac{1}{r} - \frac{1}{q}\right)\int\limits_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q\,\mathrm{d}\xi \\
&< 0.
\end{aligned}$$

This means that $\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi) < 0$ for all $\phi \in \Xi_\tau^+$, so, we get $\mathfrak{M}_\tau^+ < 0$. □

Proposition 4. *If hypotheses* (Q_κ) *and* (Q_a) *hold, then there exists* $\tau^* > 0$ *such that* $\Xi_\tau^0 = \emptyset$ *for all* $\tau \in (0, \tau^*)$.

Proof. To argue by contradiction, suppose $\Xi_\tau^0 \neq \emptyset$. So, for each $\phi \in \Xi_\tau^0$, we have

$$(p+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + (q+\varpi-1)\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi = \tau(r+\varpi-1)\|\phi\|_r^r. \tag{7}$$

Also, it holds

$$(r+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + (r+\varpi-1)\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi - (r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi = \tau(r+\varpi-1)\|\phi\|_r^r. \tag{8}$$

Using (7) and (8), one has

$$(r-p)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + (r-q)\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi = (r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi.$$

This means that $\|\phi\|^p \leqslant C_3\|\phi\|$ for some $C_3 > 0$, namely,

$$\|\phi\|^{p-1} \leqslant C_3. \tag{9}$$

Applying (7) and the Sobolev embedding theorem, we get

$$\|\phi\|^p \leqslant \tau C_4\|\phi\|^r \quad \text{for some } C_4 > 0.$$

Hence,

$$\left(\frac{1}{\tau C_4}\right)^{\frac{1}{r-p}} \leqslant \|\phi\|.$$

Letting $\tau \to 0^+$ yields $\|\phi\| \to +\infty$. This contradicts with (9). Therefore, we conclude that there exists $\tau^* > 0$ such that $\Xi_\tau^0 = \emptyset$ for all $\tau \in (0, \tau^*)$. □

Let $\phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ be fixed. We also consider the functional $\widehat{\Theta}_\phi : (0,\infty) \to \mathbb{R}$ given by

$$\widehat{\Theta}_\phi(t) = t^{p-r}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p - t^{-r-\varpi+1}\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi \quad \text{for all } t > 0.$$

Since $r-p < r+\varpi-1$, so, it is not difficult to see that there exists $\hat{t}_0 > 0$ such that

$$\widehat{\Theta}_\phi(\hat{t}_0) = \max_{t>0}\widehat{\Theta}_\phi(t).$$

So, we have $\widehat{\Theta}'_\phi(\hat{t}_0) = 0$, which means that

$$(p-r)\hat{t}_0^{p-r-1}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + (r+\varpi-1)\hat{t}_0^{-r-\varpi}\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi = 0.$$

This gives

$$\hat{t}_0 = \left(\frac{(r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi}{(r-p)\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^p} \right)^{\frac{1}{p+\varpi-1}}.$$

Therefore, we have

$$\begin{aligned}
\widehat{\Theta}_\phi(\hat{t}_0) &= \frac{((r-p)\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^p)^{\frac{r-p}{p+\varpi-1}}}{((r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi)^{\frac{r-p}{p+\varpi-1}}} \big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p \\
&\quad - \frac{((r-p)\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^p)^{\frac{r-p}{p+\varpi-1}}}{((r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi)^{\frac{r-p}{p+\varpi-1}}} \int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi \\
&= \frac{(r-p)^{\frac{r-p}{p+\varpi-1}}\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^{\frac{p(r+\varpi-1)}{p+\varpi-1}}}{((r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi)^{\frac{r-p}{p+\varpi-1}}} \\
&\quad - \frac{(r-p)^{\frac{r+\varpi-1}{p+\varpi-1}}\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^{\frac{p(r+\varpi-1)}{p+\varpi-1}}}{((r+\varpi-1)\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi)^{\frac{r-p}{p+\varpi-1}}} \\
&= \frac{p+\varpi-1}{r-p}\left(\frac{r-p}{r+\varpi-1}\right)^{\frac{r+\varpi-1}{p+\varpi-1}} \frac{\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^{\frac{p(r+\varpi-1)}{p+\varpi-1}}}{(\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi)^{\frac{r-p}{p+\varpi-1}}}.
\end{aligned}$$

Let S be the best Sobolev constant that satisfies the inequality

$$S\|\phi\|_{p_\alpha^*}^p \leqslant \big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p. \tag{10}$$

Also, by Hölder inequality, we obtain

$$\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi \leqslant C_5\|\phi\|_{p_\alpha^*}^{1-\varpi} \quad \text{for some } C_5 > 0. \tag{11}$$

On the other hand, using inequalities (10), (11) and $r < p_\alpha^*$, it follows that

$$\begin{aligned}
&\widehat{\Theta}_\phi(\hat{t}_0) - \tau\|\phi\|_r^r \\
&= \frac{p+\varpi-1}{r-p}\left(\frac{r-p}{r+\varpi-1}\right)^{\frac{r+\varpi-1}{p+\varpi-1}} \frac{\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\|_p^{\frac{p(r+\varpi-1)}{p+\varpi-1}}}{(\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi)^{\frac{r-p}{p+\varpi-1}}} - \tau\|\phi\|_r^r \\
&\geqslant \frac{p+\varpi-1}{r-p}\left(\frac{r-p}{r+\varpi-1}\right)^{\frac{r+\varpi-1}{p+\varpi-1}} \frac{S^{\frac{p(r+\varpi-1)}{p+\varpi-1}}(\|\phi\|_{p_\alpha^*}^p)^{\frac{p(r+\varpi-1)}{p+\varpi-1}}}{(C_5\|\phi\|_{p_\alpha^*}^{1-\varpi})^{\frac{r-p}{p+\varpi-1}}} - \tau C_6\|\phi\|_{p_\alpha^*}^r \\
&= [C_7 - \tau C_6]\|\phi\|_{p_\alpha^*}^r \quad \text{for some } C_6, C_7 > 0.
\end{aligned}$$

This indicates that, independently of ϕ, there exists $\hat{\tau}^* \in (0,\tau^*)$ such that

$$\widehat{\Theta}_\phi(\hat{t}_0) - \tau\|\phi\|_r^r > 0 \quad \text{for all } \tau \in (0,\hat{\tau}^*). \tag{12}$$

Proposition 5. *If hypotheses* (Q_κ) *and* (Q_a) *hold, then there exists* $\hat{\tau}^* \in (0,\tau^*)$ *such that for every* $\tau \in (0,\hat{\tau}^*)$*, we can find* $\phi^* \in \Xi_\tau^+$ *with* $\phi^*(\xi) \geqslant 0$ *for a.a.* $\xi \in \Omega$ *such that* $\mathcal{E}_p^{\alpha,\beta;\psi}(\phi^*) = \mathfrak{M}_\tau^+ < 0$.

Proof. Let $\phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ be fixed and consider the map $\Theta_\phi : (0,+\infty) \to \mathbb{R}$ given by

$$\Theta_\phi(t) = t^{p-r}\left\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\right\|_p^p + t^{q-r}\int_\Omega \kappa(\xi)\left|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\right|^q \mathrm{d}\xi - t^{-r-\varpi+1}\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi \quad \text{for all } t>0.$$

Remember that $r-p<r-q<r+\varpi-1$, and we can find $t_0>0$ such that

$$\Theta_\phi(t_0) = \max_{t>0}\Theta_\phi(t).$$

Obviously, we have $\Theta_\phi(t) \geqslant \widehat{\Theta}_\phi(t)$ for all $t>0$. So, from inequality (12) we can take $\hat{\tau}^* \in (0,\tau^*)$ such that

$$\Theta_\phi(t_0) - \tau\|\phi\|_r^r > 0 \quad \text{for all } \tau \in (0,\hat{\tau}^*).$$

So, there are $t_1, t_2 > 0$ with $t_1 < t_0 < t_2$ such that

$$\Theta_\phi(t_1) = \tau\|\phi\|_r^r = \Theta_\phi(t_2) \quad \text{and} \quad \Theta_\phi'(t_2) < 0 < \Theta_\phi'(t_1). \tag{13}$$

This indicates that $t_1\phi \in \Xi_\tau^+$ and $t_2\phi \in \Xi_\tau^-$. So, for all $\tau \in (0,\tau^*)$, we have $\Xi_\tau^\pm \neq \emptyset$, while $\Xi_\tau^0 = \emptyset$; see Proposition 4.

Let $\{\phi_n\}_{n\geqslant 1} \subseteq \Xi_\tau^+$ be a minimizing sequence of $\mathfrak{M}_\tau^+ := \inf_{\Xi_\tau^+}\mathcal{E}_\tau^{\alpha,\beta;\psi}$, i.e.,

$$\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi_n) \to \mathfrak{M}_\tau^+ \quad \text{as } n\to\infty.$$

Using Proposition 2 and $\Xi_\tau^+ \subseteq \Xi_\tau$, we have that $\{\phi_n\}_{n\geqslant 1} \subseteq \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ is bounded. Then, without loss of generality, there exists $\phi^* \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ such that

$$\phi_n \rightharpoonup \phi^* \quad \text{on } \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega) \quad \text{and} \quad \phi_n \to \phi^* \quad \text{in } \mathscr{L}^r(\Omega).$$

Let Θ_{ϕ^*} and $t_1 < t_0$ be as in (13) with $\phi = \phi^*$. Then we have $t_1\phi^* \in \Xi_\tau^+$.

Claim 1. $\phi_n \to \phi^*$ on $\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ as $n\to\infty$.

If the statement is not true, then we can assume that $\phi_n \nrightarrow \phi^*$ in $\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$. Hence,

$$\liminf_{n\to\infty}\left\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi_n\right\|_p^p > \left\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\right\|_p^p. \tag{14}$$

For any fixed $\phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$, let us consider the fibering function $\mu_\phi : (0,\infty) \to \mathbb{R}$ defined by

$$\mu_\phi(t) = \mathcal{E}_\tau^{\alpha,\beta;\psi}(t\phi) \quad \text{for all } t>0.$$

Using (13) and (14), we get

$$\begin{aligned}\liminf_{n\to\infty}\mu'_{\phi_n}(t_1) &= \liminf_{n\to\infty}\bigg[t_1^{p-1}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi_n\big\|_p^p + t_1^{q-1}\int\limits_{\Omega}\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi_n\big|_q^q\,\mathrm{d}\xi \\ &\quad - t_1^{-\varpi}\int\limits_{\Omega}a(\xi)|\phi_n|^{1-\varpi} - \tau t_1^{r-1}\|\phi_n\|_r^r\bigg] \\ &> t_1^{p-1}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|_p^p + t_1^{q-1}\int\limits_{\Omega}\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big|_q^q\,\mathrm{d}\xi \\ &\quad - t_1^{-\varpi}\int\limits_{\Omega}a(\xi)|\phi^*|^{1-\varpi} - \tau t_1^{r-1}\|\phi^*\|_r^r \\ &= \mu'_{\phi^*}(t_1) = 0. \end{aligned} \tag{15}$$

On the other hand, it follows from (15) that there is $n_0 \in \mathbb{N}$ such that

$$\mu'_{\phi_n}(t_1) > 0 \quad \text{for all } n \geqslant n_0.$$

Since $\phi_n \in \boldsymbol{\Xi}_\tau^+ \subseteq \boldsymbol{\Xi}_\tau$ and $\mu'_{\phi_n}(t) = t^{r-1}(\Theta_{\phi_n}(t) - \tau\|\phi_n\|_r^r)$, we have

$$\mu'_{\phi_n}(t) < 0 \quad \text{for all } t \in (0,1) \quad \text{and} \quad \mu'_{\phi_n}(1) = 0.$$

This implies that $t_1 > 1$. But from the fact that $\mu'_{\phi^*}(t_1) = 0$ we can see that μ_{ϕ^*} is decreasing in $(1, t_1)$. So, it holds that

$$\mathcal{E}_\tau^{\alpha,\beta;\psi}(t_1\phi^*) \leqslant \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*) < \mathfrak{M}_\tau^+,$$

where the last inequality is given by (14). Note that $t_1\phi^* \in \boldsymbol{\Xi}_\tau^+$ (since $\mu'_{\phi^*}(t_1) = t_1^{r-1}(\Theta_{\phi^*}(t_1) - \tau\|\phi_n\|_r^r) = 0$ and $\phi_* \in \varXi_\tau$), we get

$$\mathfrak{M}_\tau^+ \leqslant \mathcal{E}_\tau^{\alpha,\beta;\psi}(t_1\phi^*) < \mathfrak{M}_\tau^+.$$

This leads to a contradiction. Therefore, Claim 1 is valid, and

$$\begin{gathered}\phi_n \to \phi^* \quad \text{in } \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega), \\ \lim_{n\to\infty}\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi_n) = \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*), \quad \text{and} \quad \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*) = \mathfrak{M}_\tau^+.\end{gathered}$$

Since $\phi_n \in \boldsymbol{\Xi}_\tau^+$, for all $n \in \mathbb{N}$, it follows that

$$\begin{aligned}&(p+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi_n\big\|_p^p + (q+\varpi-1)\int\limits_{\Omega}\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi_n\big|^q\,\mathrm{d}\xi \\ &\quad > \tau(r+\varpi-1)\|\phi_n\|_r^r.\end{aligned}$$

Hence,

$$(p+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|_p^p+(q+\varpi-1)\int\limits_{\Omega}\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big|^q\,\mathrm{d}\xi \\ \geqslant \tau(r+\varpi-1)\|\phi^*\|_r^r. \tag{16}$$

From (4) we know that $\Xi_\tau^0=\emptyset$. Then (16) is a strict inequality, so, $\phi^*\in\Xi_\tau^+$. Obviously, we can replace ϕ^* by $|\phi^*|$, and so, we can say that $\phi^*(\xi)\geqslant 0$ for a.a. $\xi\in\Omega$. □

For any $\varepsilon>0$, let us consider the open ball

$$B_\varepsilon(0)=\big\{w\in\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)\ \big\|\ w\|<\varepsilon\big\}.$$

Lemma 1. *If hypotheses* (Q_κ) *and* (Q_a) *hold and* $\phi\in\Xi_\tau^\pm$*, then there exists* $\varepsilon>0$ *and a continuous function* $\tilde\beta: B_\varepsilon(0)\to\mathbb{R}_+$ *such that* $\tilde\beta(0)=1$ *and* $\tilde\beta(w)(\phi+w)\in\Xi_\tau^+$ *for all* $w\in B_\varepsilon(0)$.

Proof. We will only prove the case $w\in\Xi_\tau^+$ because the same arguments can be used to prove the case $\phi\in\Xi_\tau^-$.

Consider the functional $\mathcal{E}:\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)\times\mathbb{R}\to\mathbb{R}$ given by

$$\mathcal{E}(w,t)=t^{p+\varpi-1}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}(\phi+w)\big\|_p^p+t^{q+\varpi-1}\int\limits_{\Omega}\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}(\phi+w)\big|^q\,\mathrm{d}\xi \\ -\int\limits_{\Omega}a(\xi)|\phi+w|^{1-\varpi}\,\mathrm{d}\xi-\tau t^{r+\varpi-1}\|\phi+w\|_r^r \quad \text{for all } w\in\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega).$$

Since $\Xi_\tau^+\subseteq\Xi$, we have $\mathcal{E}(0,1)=0$. Furthermore, for any fixed $\phi\in\Xi_\tau^+$, we have

$$\mathcal{E}_t'(0,1)=(p+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|_p^p+(q+\varpi-1)\int\limits_{\Omega}\kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big|^q\,\mathrm{d}\xi \\ -\tau(r+\varpi-1)\|\phi^*\|_r^r>0.$$

We are now able to use the implicit function theorem to find $\varepsilon>0$ and a continuous function $\tilde\beta: B_\varepsilon(0)\to\mathbb{R}_+$ such that

$$\tilde\beta(0)=1,\quad \tilde\beta(w)(w+\phi)\in\Xi_\tau\quad\text{for all } w\in B_\varepsilon(0).$$

Furthermore, we can make $\varepsilon>0$ small enough so that

$$\tilde\beta(w)(w+\phi)\in\Xi_\tau^+\quad\text{for all } w\in B_\varepsilon(0).$$

This completes the proof of the lemma. □

Proposition 6. *If hypotheses* (Q_κ) *and* (Q_a) *hold,* $\tau\in(0,\hat\tau^*]$*, and* $h\in\mathcal{H}_p^{\alpha,\beta,;\psi}(\Omega)$*, then we can find* $b>0$ *such that* $\mathcal{E}_p^{\alpha,\beta,;\psi}(\phi^*)\leqslant\mathcal{E}_p^{\alpha,\beta,;\psi}(\phi^*+th)$ *for all* $t\in[0,b]$.

Proof. Consider the function $\eta_h : [0, \infty) \to \mathbb{R}$ given by

$$\eta_h(t) = (p-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^* + t\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\big\|_p^p + (q-1)\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^* + t\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\big|^q\,\mathrm{d}\xi + \varpi\int_\Omega a(\xi)|\phi^* + th|^{1-\varpi}\,\mathrm{d}\xi - \tau(r-1)\|\phi^* + th\|_r^r. \tag{17}$$

Due to $\phi^* \in \Xi_\tau^+ \subseteq \Xi_\tau$ (see (5)), we have

$$\varpi\int_\Omega a(\xi)|\phi^*|^{1-\varpi}\,\mathrm{d}\xi = \varpi\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|_p^p + \varpi\int_\Omega \kappa(\xi)|\phi^*|^q\,\mathrm{d}\xi - \tau\varpi\|\phi^*\|_r^r \tag{18}$$

and

$$(p+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|_p^p + (q+\varpi-1)\int_\Omega \kappa(\xi)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|^q\,\mathrm{d}\xi - \tau(r+\varpi-1)\|\phi^*\| > 0. \tag{19}$$

Taking into account (17)–(19), this implies $\eta_h(0) > 0$. We can also use the continuity of η_h to find $b_0 > 0$ such that

$$\eta_h(t) > 0 \quad \text{for all } t \in [0, b_0].$$

By (1) we can find $\nu(t) > 0$ for $t \in [0, b_0]$ such that

$$\nu(t)(\phi^* + th) \in \Xi_\tau^+, \quad \nu(t) \to 1 \quad \text{as } t \to 0^+. \tag{20}$$

Therefore, we derive

$$\mathfrak{M}_\tau^+ = \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*) \leqslant \mathcal{E}_\tau^{\alpha,\beta;\psi}\big(\nu(t)(\phi^* th)\big) \quad \text{for all } t \in [0, b_0].$$

Hence, we concluded that $\mathfrak{M}_\tau^+ \leqslant \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*) \leqslant \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^* + th)$ for all $t \in [0, b]$ with $0 < b \leqslant b_0$; see (20). □

Proposition 7. *If hypotheses* (Q_κ) *and* (Q_a) *hold and* $\tau \in (0, \hat{\tau}^*)$*, then* ϕ^* *is a weak solution of problem* (P_τ)*.*

Proof. Let $\phi \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$, and let $b > 0$ be the constant given in Proposition 6. For $0 \leqslant t \leqslant b$, using Proposition 6, we get

$$0 \leqslant \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^* + th) - \mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi^*).$$

This implies that

$$\begin{aligned}
&\frac{1}{1-\varpi}\int\limits_{\Omega} a(\xi)\big(|\phi^* + th|^{1-\varpi} - |\phi^*|^{1-\varpi}\big)\,\mathrm{d}\xi \\
&\quad\leqslant \frac{1}{p}\big(\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^* + t\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\big\|_p^p - \big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big\|_p^p\big) \\
&\qquad + \frac{1}{q}\bigg[\int\limits_{\Omega}\kappa(\xi)\big(\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^* + t\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\big|^q - \big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big|^q\big)\,\mathrm{d}\xi\bigg] \\
&\qquad - \frac{\tau}{r}\big(\|\phi^* + th\|_r^r - \|\phi\|_r^r\big).
\end{aligned}$$

If we divide the above inequality by $t > 0$ and let $t = 0^+$, we get

$$\begin{aligned}
&\int\limits_{\Omega} a(\xi)|\phi^*|^{-\varpi}h\,\mathrm{d}\xi \\
&\quad\leqslant \int\limits_{\Omega}\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big|^{p-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\,\mathrm{d}\xi \\
&\qquad + \int\limits_{\Omega}\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\big|^{q-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\,\mathrm{d}\xi - \tau\int\limits_{\Omega}(\phi^*)^{r-1}h\,\mathrm{d}\xi.
\end{aligned}$$

The arbitrariness of $h \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ shows that for all $h \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$,

$$\begin{aligned}
&\int\limits_{\Omega} a(\xi)|\phi^*|^{-\varpi}h\,\mathrm{d}\xi \\
&\quad= \int\limits_{\Omega}|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*|^{p-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\,\mathrm{d}\xi \\
&\qquad + \int\limits_{\Omega}|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*|^{q-2}\;\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi^*\;\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}h\,\mathrm{d}\xi - \tau\int\limits_{\Omega}(\phi^*)^{r-1}h\,\mathrm{d}\xi.
\end{aligned}$$

Therefore, ϕ^* is a weak solution of (P_τ) for $\tau \in (0, \hat{\tau}^*)$. □

Following Propositions 5 and 7, we have the following result, which shows the existence of a positive solution of (P_τ) for $\tau \in (0, \hat{\tau}^*)$.

Proposition 8. *If hypotheses* (Q_κ) *and* (Q_a) *hold and* $\tau \in (0, \hat{\tau}^*)$, *then problem* (P_τ) *admits a positive solution* $\phi^* \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ *such that* $\mathcal{E}_p^{\alpha,\beta;\psi}(\phi^*) < 0$ *and* $\phi^*(\xi) \geqslant 0$ *for a.a.* $\xi \in \Omega$, $\phi^* \neq 0$.

The next proposition gives the existence of the second positive solution to (P_τ) when $\tau > 0$ is small. This solution belongs to $\boldsymbol{\Xi}_\tau^-$.

Proposition 9. *If hypotheses* (Q_κ) *and* (Q_a) *hold, then we can find* $\hat{\tau_0}^* > 0$ *such that for all* $\tau \in (0, \hat{\tau_0}^*]$*, functional* $\mathcal{E}_\tau^{\alpha,\beta;\psi}$ *is nonnegative on* $\boldsymbol{\Xi}_\tau^-$.

Proof. Let $\phi \in \boldsymbol{\Xi}_\tau^-$, then one has

$$(p+\varpi-1)\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + (q+\varpi-1)\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi < \tau(r+\varpi-1)\|\phi\|_r^r.$$

The continuity of the embedding $\mathcal{H}_p^{\alpha,\beta;\psi}(\Omega) \hookrightarrow \mathscr{L}^r(\Omega)$ implies

$$(p+\varpi-1)C_8\|\phi\|_r^p < \tau(r+\varpi-1)\|\phi\|_r^r \quad \text{for some } C_8 > 0,$$

i.e.,

$$\left[\frac{(p+\varpi-1)C_8}{\tau(r+\varpi-1)}\right]^{\frac{1}{r-p}} \leqslant \|\phi\|_r. \tag{21}$$

Suppose that the conclusion of this proposition is not true. Then we can find $\phi \in \boldsymbol{\Xi}_\tau^-$ such that $\mathcal{E}_\tau^{\alpha,\beta;\psi}(\phi) < 0$. Hence,

$$\frac{1}{p}\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p + \frac{1}{q}\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi - \frac{1}{1-\varpi}\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi - \frac{\tau}{r}\|\phi\|_r^r < 0. \tag{22}$$

Keeping in mind $\phi \in \boldsymbol{\Xi}_\tau$, namely,

$$\big\|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big\|_p^p = \int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi + \tau\|\phi\|_r^r - \int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi, \tag{23}$$

we use (22) and (23) to obtain

$$\left[\frac{1}{p} - \frac{1}{1-\varpi}\right]\int_\Omega a(\xi)|\phi|^{1-\varpi}\,\mathrm{d}\xi + \left[\frac{1}{q} - \frac{1}{p}\right]\int_\Omega \kappa(\xi)\big|\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\phi\big|^q \,\mathrm{d}\xi + \tau\left(\frac{1}{p} - \frac{1}{r}\right)\|\phi\| < 0.$$

This implies

$$\tau\left(\frac{1}{p} - \frac{1}{r}\right)\|\phi\|_r^r < \frac{p+\varpi-1}{p(1-\varpi)}C_9\|\phi\|_r^{1-\varpi} \quad \text{for some } C_9 > 0.$$

Recall that $q < p < r$, this leads to

$$\|\phi\|_r^{r+\varpi-1} \leqslant \frac{(p+\varpi-1)rc_9}{\tau(r-p)(1-\varpi)}.$$

Hence,

$$\|\phi\|_r \leqslant C_{10}(\frac{1}{\tau})^{\frac{1}{r+\varpi-1}} \quad \text{for some } C_{10} > 0. \tag{24}$$

Using inequalities (24) and (21), we have

$$C_{11}\left(\frac{1}{\tau}\right)^{\frac{1}{r-p}} \leqslant C_{10}\left(\frac{1}{\tau}\right)^{\frac{1}{r+\varpi-1}} \quad \text{with } C_{11} = \left(\frac{(p+\varpi-1)C_8}{r+\varpi-1}\right)^{\frac{1}{r-p}} > 0.$$

But the facts $1 < p < r$ and $\varpi \in (0,1)$ imply that

$$C_{12} \leqslant \tau^{\frac{1}{r-p} - \frac{1}{r+\varpi-1}} \to 0$$

as $\tau \to 0^+$ with $C_{12} = C_{11}/C_{10}$. This generates a contradiction. So, we can find $\hat{\tau_0}^* \in (0, \hat{\tau}^*]$ such that $\mathcal{E}_\tau^{\alpha,\beta;\psi}|_{\boldsymbol{\Xi}_\tau^-} \geqslant 0$ for all $\tau \in (0, \hat{\tau_0}^*]$. □

Proposition 10. *If hypotheses* (Q_κ) *and* (Q_a) *hold and* $\tau \in (0, \hat{\tau_0}^*]$*, then there exists* $v^* \in \boldsymbol{\Xi}$ *with* $v^* \geqslant 0$ *such that*

$$\mathfrak{M}_\tau^- = \inf_{\boldsymbol{\Xi}_\tau^-} \mathcal{E}_\tau^{\alpha,\beta;\psi} = \mathcal{E}_\tau^{\alpha,\beta;\psi}(v^*).$$

Proof. The reasoning is similar to the proof of Proposition 5. If $\{v_n\}_{n\geqslant 1} \subseteq \boldsymbol{\Xi}_\tau^-$ is a minimizing sequence of $\mathfrak{M}_\tau^- = \inf_{\boldsymbol{\Xi}_\tau^-} \mathcal{E}_\tau^{\alpha,\beta;\psi}$, then by Proposition 2 we have that $\{v_n\}_{n\geqslant 1} \subseteq \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$ is bounded. So, we can assume that for some $v^* \in \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega)$,

$$v_n \rightharpoonup v^* \quad \text{in } \mathcal{H}_p^{\alpha,\beta;\psi}(\Omega) \quad \text{and} \quad v_n \to v^* \quad \text{in } L^r(\Omega) \quad \text{as } n \to \infty.$$

Using the same idea as in the proof of Proposition 5, we can find $t_0 < t_2$ such that

$$\Theta'_{v^*}(t_2) < 0 \quad \text{and} \quad \Theta'_{v^*}(t_2) = \tau\|v^*\|_r^r \tag{25}$$

(see (13)), where $t_0 > 0$ is the maximizer of Θ_{v^*}. Furthermore, we argue as in the proof of Proposition 8 and use inequality (25) to obtain that $v^* \in \boldsymbol{\Xi}_\tau^-$, $v^* \geqslant 0$ and $\mathfrak{M}_\tau^- = \mathcal{E}_\tau^{\alpha,\beta;\psi}(v^*)$. □

Proposition 11. *If hypotheses* (Q_κ) *and* (Q_a) *hold and* $\tau \in (0, \hat{\tau}^*)$*, then* v^* *is a weak solution of problem* (P_τ).

Following the above results, we are now in a position to give the proof of Theorem 1.

Proof of Theorem 1. The desired results can be obtained directly by using Propositions 8 and 11. □

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References

1. D. Applebaum, Lévy processes-from probability to finance and quantum groups, *Notices Am. Math. Soc.*, **51**(11):1336–1347, 2004, `https://www.ams.org/journals/notices/200411/fea-applebaum.pdf`.
2. R. Arora, A. Fiscella, T. Mukherjee, P. Winkert, On double phase Kirchhoff problems with singular nonlinearity, *Adv. Nonlinear Anal.*, **12**(1):20220312, 2023, `https://doi.org/10.1515/anona-2022-0312/html`.
3. P. Baroni, M. Colombo, G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Anal., Theory Methods Appl.*, **121**:206–222, 2015, `https://doi.org/10.1016/j.na.2014.11.001`.
4. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Vari. Partial Differ. Equ.*, **57**:1–48, 2018, `10.1007/s00526-018-1332-z`.
5. M. Bouabdallah, M. El Ahmadi, A. Lamaizi, Homoclinic solutions for discrete fractional p-Laplacian equation via the Nehari manifold method, *Rend. Circ. Mat. Palermo (2)*, 2024, `https://doi.org/10.1007/s12215-024-01110-7`.
6. T. Brander, M. Kar, M. Salo, Enclosure method for the p-Laplace equation, *Inverse Probl.*, **31**(4):045001, 2015, `https://doi.org/10.1088/0266-5611/31/4/045001`.
7. F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pura Appl. (4)*, **195**(6):1917–1959, 2016, `10.1007/s10231-015-0542-7`.
8. M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.*, **218**(1):219–273, 2015, `10.1007/s00205-015-0859-9`.
9. M. Colombo, G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.*, **215**(2):443–496, 2015, `10.1007/s00205-014-0785-2`.
10. L.F.O. Faria, O.H. Miyagaki, D. Motreanu, Comparison and positive solutions for problems with the (p,q)-Laplacian and a convection term, *Proc. Edinb. Math. Soc., II Ser.*, **57**(3):687–698, 2014, `https://doi.org/10.1017/S0013091513000576`.
11. R. Folino, R.G. Plaza, M. Strani, Long time dynamics of solutions to p-Laplacian diffusion problems with bistable reaction terms, *Discrete Contin. Dyn. Syst.*, **41**(7):3211–3240, 2021, `https://doi.org/10.3934/dcds.2020403`.
12. L. Gasiński, P. Winkert, Existence and uniqueness results for double phase problems with convection term, *J. Differ. Equations*, **268**(8):4183–4193, 2020, `https://doi.org/10.1016/j.jde.2019.10.022`.
13. R. Glowinski, J. Rappaz, Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology, *E2AN. Math. Model. Numer. Anal.*, **37**(1):175–186, 2003, `https://doi.org/10.1051/m2an:2003012`.
14. A. Iannizzotto, S. Liu, K. Perera, M. Squassina, Existence results for fractional p-Laplacian problems via Morse theory, *Adv. Calc. Var.*, **9**(2):101–125, 2016, `https://doi.org/10.1515/acv-2014-0024`.

15. P. Le, Liouville results for double phase problems in $\mathbb{R}^n$, *Qual. Theory Dyn. Syst.*, **21**(3):59, 2022, https://doi.org/10.1007/s12346-022-00596-9.
16. G.M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Commun. Partial Differ. Equations*, **16**(2–3):311–361, 1991, https://doi.org/10.1080/03605309108820761.
17. W. Liu, G. Dai, N.S. Papageorgiou, P. Winkert, Existence of solutions for singular double phase problems via the Nehari manifold method, *Anal. Math. Phys.*, **12**(3):75, 2022, https://doi.org/10.1007/s13324-022-00686-6.
18. P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, *Arch. Ration. Mech. Anal.*, **105**(3):267–284, 1989, https://doi.org/10.1007/BF00251503.
19. P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differ. Equations*, **90**(1):1–30, 1991, 10.1016/0022-0396(91)90158-6.
20. N.E. Mastorakis, H. Fathabadi, On the solution of p-Laplacian for non-Newtonian fluid flow, *WSEAS Trans. Math.*, **8**(6):238–245, 2009, https://doi.org/10.5555/1639396.1639399.
21. N.S. Papageorgiou, D.D. Repovš, C. Vetro, Positive solutions for singular double phase problems, *J. Math. Anal. Appl.*, **501**(1):123896, 2021, https://doi.org/10.1016/j.jmaa.2020.123896.
22. P. Pucci, J.B. Serrin, *The Maximum Principle*, Volume 73, Birkhäuser, Basel, 2007, https://doi.org/10.1007/978-3-7643-8145-5.
23. M.A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.*, **9**(1):710–728, 2019, https://doi.org/10.1515/anona-2020-0022.
24. R. Servadei, E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Am. Math. Soc.*, **367**(1):67–102, 2015, 10.1090/S0002-9947-2014-05884-4.
25. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discete Contin. Dyn. Syst.*, **33**(5):2105–2137, 2013, https://doi.org/10.3934/dcds.2013.33.2105.
26. B. Sun, Existence of solutions for nonlinear problems involving mixed fractional derivatives with $p(x)$-Laplacian operator, *Demonstr. Math.*, **57**:20240045, 2024, https://doi.org/10.1515/dema-2024-0045/html.
27. W. Tao, Y. Li, Boundedness of weak solutions of a chemotaxis-Stokes system with slow p-Laplacian diffusion, *J. Differ. Equations*, **268**(11):6872–6919, 2020, https://doi.org/10.1016/j.jde.2019.11.078.
28. S. ur Rahman, J.L. Díaz Palencia, The propagation of a flame front with a p-Laplacian operator, *Waves Random Complex Media*, pp. 1–19, 2023, https://doi.org/10.1080/17455030.2023.2205957.
29. J. Vanterler da C. Sousa, Nehari manifold and bifurcation for a ψ-Hilfer fractional p-Laplacian, *Math. Methods Appl. Sci.*, **44**(11):9616–9628, 2021, https://doi.org/abs/10.1002/mma.7296.
30. J. Vanterler da C. Sousa, Existence and uniqueness of solutions for the fractional differential equations with p-laplacian in $\mathbb{H}_p^{\nu,\eta;\psi}$, *J. Appl. Anal. Comput.*, **12**(2):622–661, 2022, https://doi.org/10.11948/20210258.

31. J. Vanterler da C. Sousa, J. Zuo, D. O'Regan, The Nehari manifold for a ψ-Hilfer fractional p-Laplacian, *Appl. Anal.*, **101**(14):5076–5106, 2022, https://doi.org/10.1080/00036811.2021.1880569.

32. J.-L. Vázquez, Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators, *Discrete Contin. Dyn. Syst., Ser. S*, **7**(4):857–885, 2014, https://doi.org/10.3934/dcdss.2014.7.857.

33. S. Zeng, Y. Bai, N.S. Papageorgiou, V.D. Rădulescu, Double phase implicit obstacle problems with convection term and multivalued operator, *Anal. Appl., Singap.*, **21**(4):1013–1038, 2023, https://doi.org/10.1142/S0219530523500033.

34. V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR, Izv.*, **29**(1):33–66, 1987, 10.1070/IM1987v029n01ABEH000958.