Nonlinear Analysis: Modelling and Control, Online First, 1–24
https://doi.org/10.15388/namc.2025.30.42121

Vilnius University Press

A new class of integral-multipoint boundary value problems for nonlinear Hadamard fractional differential equations on a semiinfinite domain

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Received: January 2, 2025 / **Revised:** April 24, 2025 / **Published online:** May 22, 2025

Abstract. In this paper, we introduce and investigate a new class of Hadamard fractional differential equation with integral-multipoint boundary conditions on a positive semiinfinite domain. We use the contraction mapping principle and the fixed point index theorem, respectively, to prove the uniqueness and the existence of at least two positive solutions to the given problem. Our results are new and enrich the literature on Hadamard-type fractional differential equations on unbounded domains. Some examples illustrating the main results are presented.

Keywords: Hadamard fractional derivative, contraction mapping principle, fixed point index theorem, positive solution, unbounded domain.

1 Introduction

Fractional differential equations have been widely studied due to their numerous applications in a variety of areas. Examples include mathematical modeling of physical and chemical phenomena such as pseudoplastic flow [18], non-Newtonian fluids [20], heterogeneous chemical catalysts [26], infectious models [25], neural networks [27], chemostat model [31], fractal–fractional reaction diffusion models [28], fractional Monge–Ampère operators [24], financial economics [8], etc. For the theoretical aspects of fractional calculus and more applications; see, for example, [14].

In addition to the popular Riemann–Liouville and Caputo fractional derivatives, there exist other types of fractional derivatives such as Hadamard, Hilfer, Grünwald–Letnikov,

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Hilfer–Hadamard, and so on. In 1892, Hadamard [12] introduced a fractional derivative with its kernel containing a logarithmic function with an arbitrary exponent, which is now known as the Hadamard fractional derivative. For details on initial and boundary value problems involving Hadamard-type fractional differential equations and inclusions, we refer the reader to a recent text [1].

During the past few decades, many authors investigated the existence of solutions for fractional differential equations on unbounded domains; see, for instance, [4,5,16,22,23, 30] and the references cited therein. In [16], Liu applied the Schauder fixed point theorem to prove the existence of at least one positive solution for the following boundary value problem of Riemann–Liouville fractional differential equations:

$$\begin{aligned}
&D_{0+}^{\zeta_1}x(t) = f_1\bigl(t, y(t), D_{0+}^{p}y(t)\bigr), \quad t \in (0,\infty),\\
&D_{0+}^{\zeta_2}y(t) = f_2\bigl(t, x(t), D_{0+}^{q}x(t)\bigr), \quad t \in (0,\infty),\\
&\lim_{t\to 0} t^{2-\zeta_1}x(t) = a_0, \qquad \lim_{t\to 0} D_{0+}^{\zeta_1-1}x(t) = a_1,\\
&\lim_{t\to 0} t^{2-\zeta_2}y(t) = b_0, \qquad \lim_{t\to 0} D_{0+}^{\zeta_2-1}y(t) = b_1,
\end{aligned}$$

where $\zeta_1, \zeta_2 \in (1,2)$, $p \in (\zeta_2 - 1, \zeta_2)$, $q \in (\zeta_1 - 1, \zeta_1)$, $a_0, b_0, a_1, b_1 \in \mathbb{R}$, D_{0+} is the standard Riemann–Liouville fractional derivative operator, and $f_1, f_2 \in C((0,\infty)\times\mathbb{R}^2, \mathbb{R})$.

The authors in [5] applied the monotone iterative technique to establish the existence of at least two positive solutions for the following problem:

$$\begin{gathered}
D_{0+}^{\zeta}\omega(z) + h\bigl(z, \omega(z)\bigr) = 0, \quad z \in (0,\infty),\\
\lim_{t\to 0} t^{2-\zeta}\omega(z) = \lim_{t\to\infty} D_{0+}^{\zeta-1}\omega(z) = \int_0^{\infty} g(s)\omega(s)\,\mathrm{d}s,
\end{gathered}$$

where D_{0+}^{ζ} is the Riemann–Liouville fractional derivative of order $1 < \zeta < 2$, $h : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a given function, and $g \in L^1([0,\infty))$.

In [3], the authors studied the existence and uniqueness of solutions for the following nonlinear nonlocal Hadamard-type fractional boundary value problem:

$$\begin{aligned}
&{}_HD^{\zeta}\omega(z) = h\bigl(z, \omega(z)\bigr) = 0, \quad z \in [1,T],\\
&\omega(z) = 1, \qquad {}_HD^{p}\omega(T) = \sum_{i=1}^{n} \gamma_i\, {}_HD^{p}\omega(\mu_i), \quad 0 < p < 1,
\end{aligned}$$

where $n \geqslant 2$, $1 < \zeta \leqslant 2$, $\mu_i \in (1,T]$, and $\gamma_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$.

Zhang and Liu [29], by using the monotone iterative technique, studied the following Hadamard fractional differential equation with integral boundary conditions:

$$\begin{aligned}
&{}_HD^{\zeta}\omega(z) + \vartheta(z)g\bigl(z, \omega(z)\bigr) = 0, \quad z \in (1,\infty),\\
&\omega^{(j)}(1) = 0, \quad 0 \leqslant j \leqslant n-2, \qquad {}_HD^{\zeta-1}\omega(\infty) = \int_1^{\infty} g(s)\omega(s)\,\frac{\mathrm{d}s}{s},
\end{aligned}$$

where ${}_HD^{\zeta}$ is the Hadamard fractional derivative of order $n - 1 < \zeta \leqslant n$ with $n \geqslant 3$.

In [23], the authors applied the monotone iterative method to study the following nonlinear Hadamard fractional differential equation complemented with nonlocal multipoint discrete and Hadamard integral boundary conditions:

$$\begin{aligned}
&{}_HD^{\zeta}\omega(z)+\vartheta(z)g\big(z,\omega(z)\big)=0, \quad z\in(1,\infty),\ 2<\zeta\leqslant 3,\\
&\omega(1)=\omega'(1)=0, \qquad {}_HD^{\zeta-1}\omega(\infty)=a\,{}_HI^{\beta}\omega(\xi)+b\sum_{i=1}^{m-2}\gamma_i\,{}_HD^{\beta_i}\omega(\eta_i),
\end{aligned}$$

where a and b are real constants, $1<\xi<\eta_1<\eta_2<\dots<\eta_{m-2}<+\infty$, and γ_i, $i=1,2,\dots,m-2$, are positive real constants. For some more results concerning the Hadamard fractional differential equations on an unbounded domain, see, for example, [4, 30].

It is well known that boundary value problems on the half-line formulate certain situations arising from physics, chemistry, engineering, biology, and dynamical systems. For example, the following problem

$$\begin{aligned}
&-\xi''+c\xi'+\lambda\xi=f\big(t,\xi(t)\big), \quad t\in(0,\infty),\\
&\xi(0)=0, \qquad \xi(+\infty)=0
\end{aligned} \tag{1}$$

extends the classical Fisher–Kolmogorov model equation [9] when $\lambda=0$. It is imperative to mention that the generalized Fisher equation (1) arises in the modelling of the epidemiological issues [17] and wave fronts in combustion theory [2]. In [7], a boundary value problem on the half-line dealing with propagation of epidemics through given populations is discussed via the generalized Fisher equation (1) when $f(t,\xi(t))=\xi h(\xi)$. In [13], the authors used Krasnosel'skiĭ–Guo fixed point theorem in a cone to investigate the existence of positive solutions for the following second-order boundary value problem on the half-line:

$$\begin{aligned}
&y''(\omega)-\varrho^2y(\omega)+m(t)h\big(\omega,y(\omega)\big)=0, \quad \omega\in(0,\infty),\\
&y(0)=0, \qquad \lim_{\omega\to+\infty}y(\omega)=0,
\end{aligned} \tag{2}$$

where m and h are given functions, and ϱ is a positive constant. Also, Djebali and Mebarki [6] studied problem (2) by using the Krasnosel'skiĭ and Leggett–Williams fixed point theorems in cones.

In a more recent work [19], the authors investigated the existence of solutions for a Hadamard fractional differential equation equipped with integro-intial data on an unbounded domain

$$\begin{aligned}
&{}_HD^{\zeta}x(t)=g\big(t,x(t)\big), \quad t\in(1,\infty),\\
&\lim_{t\to1}(\log t)^{2-\zeta}x(t)=\lim_{t\to1}{}_HD^{\zeta-1}x(t)=\int_1^{\infty}k(s)x(s)\,\frac{\mathrm{d}s}{s},
\end{aligned}$$

where $1<\zeta\leqslant2$, ${}_HD^{\zeta}$ is the Hadamard fractional derivative of order ζ, $g:(1,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous function, and $k\in L^1(1,\infty)$.

Inspired by the aforementioned work, in this paper, we introduce and investigate a new class of Hadamard fractional differential equations with integral-multipoint boundary conditions on a positive semiinfinite domain given by

$$\begin{aligned} &-\left(\frac{1}{p(\omega)}\,{}_HD^{\zeta+2}x(\omega)\right)+\varrho\left(\frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega)\right)+a(\omega)\mathcal{G}\big(\omega,x(\omega)\big)=0,\\ &{}_HD^{\zeta}x(1)=0,\qquad \lim_{\omega\to\infty}\frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega)=0,\qquad \lim_{\omega\to1}(\log\omega)^{2-\zeta}x(\omega)=\lambda,\\ &\lim_{\omega\to\infty}{}_HD^{\zeta-1}x(\omega)=\int_1^{\infty}k(s)x(s)\,\frac{\mathrm{d}s}{s}+\sum_{i=1}^{m}\gamma_i\,{}_HD^{\beta_i}x(\eta_i), \end{aligned} \tag{3}$$

where ${}_HD^{\zeta}$ is the Hadamard fractional derivative of order $\zeta\in(1,2]$, $\beta_i\in(0,1]$, $\gamma_i>0$, $\omega\in(1,\infty)$, $\eta_i\in(0,\infty)$, $i=1,2,\dots,m$, λ, $\varrho>0$ are two positive constants, $\mathcal{G}:(1,\infty)\times\mathbb{R}\to\mathbb{R}$, $p,a:(1,\infty)\to[1,\infty)$ are continuous functions such that $0<\int_1^{\infty}p(s)/s\,\mathrm{d}s<\infty$, $0<\int_1^{\infty}p^2(s)/s\,\mathrm{d}s<\infty$, and $k\in L^1(1,\infty)$ with

$$\begin{aligned} \varOmega_1&=\Gamma(\zeta)+\sum_{i=1}^{m}\frac{\gamma_i\Gamma(\zeta)}{\Gamma(\zeta-\beta_i)}(\log\eta_i)^{\zeta-\beta_i-1}>0,\\ \varOmega_2&=\varOmega_1-\int_1^{\infty}k(s)(\log s)^{\zeta-1}\,\frac{\mathrm{d}s}{s}>0. \end{aligned}$$

The key idea of the present work is to develop the existence theory for the proposed problem. As a first result, we establish the existence of a unique solution to problem (3), while the second one deals with the existence of its at least two positive solutions. The application of these results is also discussed. We make use of the standard tools of the fixed point theory (Banach's contraction mapping principle and the fixed point index theorem) to derive the desired results. However, we prove several subsidiary lemmas before applying the chosen fixed point theorems. It is well known that the fixed point technique is an effective and fruitful method for developing a variety of existence results for boundary value problems under different criteria.

It is imperative to mention that much of the literature on Hadamard-type fractional differential equations is concerned with bounded domains. Our objective in this study is to solve a Hadamard fractional differential equation with integral-multipoint boundary conditions on a positive semiinfinite domain. It is worthwhile to mention that the multipoint and integral boundary conditions provide a more practical platform (than the one with classical boundary conditions) to take into account the changes happening on nonlocal positions and substrips within the domain or full domain of the given problem. The coupling of integral and Hadamard-type multipoint boundary conditions in the formulation of the given problem makes our study more interesting as one can obtain the results for problem (3) subject to a purely integral condition by letting $\gamma_i=0$, $i=1,2,\dots,m$, in the results of this paper. Here we emphasize that our proposed problem in the given configuration is novel and contribute usefully to the existing literature on Hadamard-type fractional boundary value problems; see, for instance, [19, 23, 29].

In passing, we remark that the present study is motivated by the application of Hadamard fractional derivative operators in the study of Lomnitz logarithmic creep law, fractional telegraph-type equations, probability [11], fractional relaxation models with power law time-varying coefficients [10], fractional cumulative entropy [21], etc.

The rest of the manuscript is arranged as follows. Section 2 contains some basic definitions and subsidiary results, while the main results (Theorems 1 and 2) are proven in Section 3. Illustrative examples for the abstract results are presented in Section 4. Our results in the given configuration are new and useful for further study on the topic.

2 Preliminaries

In this section, we present some preliminary concepts of fractional calculus related to our work and establish an auxiliary lemma that plays a fundamental role in converting the given problem into a fixed point problem.

Definition 1. (See [14].) The Hadamard derivative of fractional order $\zeta \in (n-1,\, n]$ for a function $h : [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$ {}_H D^{\zeta} h(\omega) = \frac{1}{\Gamma(n-\zeta)} \left(\omega \frac{\mathrm{d}}{\mathrm{d}\omega}\right)^n \int_1^{\omega} \left(\log \frac{\omega}{s}\right)^{n-\zeta-1} \frac{h(s)}{s}\,\mathrm{d}s, $$

where $n = [\zeta] + 1$, $[\zeta]$ denotes the integer part of the real number ζ, and $\log(\cdot) = \log_e(\cdot)$.

Definition 2. (See [14].) The Hadamard fractional integral of order $\zeta > 0$ for a function h is defined as

$$ {}_H I^{\zeta} h(\omega) = \frac{1}{\Gamma(\zeta)} \int_1^{\omega} \left(\log \frac{\omega}{s}\right)^{\zeta-1} \frac{h(s)}{s}\,\mathrm{d}s, $$

provided the integral exists.

Lemma 1. (*See* [14].) *If* $0 < a < \infty$ *and* $\zeta, \beta > 0$*, then*

$$ \left({}_H I^{\zeta}_{a+} \left(\log \frac{\omega}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\zeta)} \left(\log \frac{x}{a}\right)^{\beta+\zeta-1}, $$
$$ \left({}_H D^{\zeta}_{a+} \left(\log \frac{\omega}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta)} \left(\log \frac{x}{a}\right)^{\beta-\zeta-1}. $$

In particular, $({}_H D^{\zeta}_{a+} (\log(t/a))^{\zeta-1})(x) = 0$ *when* $0 < \zeta < 1$.

Next, we introduce the space X related to our work as follows:

$$ X := \left\{ x \in C(1,\infty)\colon\ \frac{(\log\omega)^{2-\zeta}}{1 + (\log\omega)^{\sigma+2}} x(\omega),\ \sigma > -1,\ \text{is bounded on } (1,\infty) \right\}, $$

where $C(1,\infty)$ denotes the set of all continuous functions defined on $(1,\infty)$. For $x \in X$, we define the norm by

$$\|x\|_X := \sup_{\omega\in(1,\infty)} \left(\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} \big|x(\omega)\big| \right).$$

It is easy to show that X endowed with the above norm is a real Banach space.

Remark 1. One can notice that the Banach space $(X, \|\cdot\|_X)$ is a weighted space, which is defined to ensure the boundedness of solutions to the given problem on $[1,\infty)$. Observe that the last term in (7) is unbounded at $\omega = 1$ as $1 < \zeta \leqslant 2$. Even the solution of the form $(\log\omega)^{2-\zeta}x(\omega)$ may be unbounded on $[1,\infty)$, and thus, it needs to be scaled by a multiplicative factor like $1/(1+(\log\omega)^{\sigma+2})$ with $\sigma > -1$. The assumption $\sigma > -1$ is also required to keep the terms like $\log\omega/(1+(\log\omega)^{\sigma+2})$ bounded in our analysis for $\omega \in [1,\infty)$; see, for instance, (15).

Next, we consider the linear variant of (3):

$$\begin{aligned} &-\left(\frac{1}{p(\omega)}\,{}_HD^{\zeta+2}x(\omega)\right) + \varrho\left(\frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega)\right) + h(\omega) = 0, \quad \omega \in (1,\infty), \\ &{}_HD^{\zeta}x(1) = 0, \qquad \lim_{\omega\to\infty} \frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega) = 0, \end{aligned} \tag{4}$$

where $h : (1,\infty) \to \mathbb{R}^+$ is a given function such that $\int_1^\infty h(s)\,\mathrm{d}s/s < \infty$. Substituting

$$Z(\omega) = -\frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega)$$

in (4), we obtain the following problem:

$$\begin{aligned} &-Z'' + \varrho Z = h(\omega), \quad \omega \in (1,\infty), \\ &Z(1) = Z(\infty) = 0. \end{aligned} \tag{5}$$

The general solution of the homogeneous equation in problem (5) is $c_1\mathrm{e}^{\varrho(\omega-1)} + c_2\mathrm{e}^{-\varrho(\omega-1)}$. So we can take $Z_1(\omega) = \mathrm{e}^{-\varrho(\omega-1)}$ and $Z_2(\omega) = \mathrm{e}^{\varrho(\omega-1)} - \mathrm{e}^{-\varrho(\omega-1)}$ as the solutions of the homogeneous equation satisfying the boundary conditions at $\omega = \infty$ and $\omega = 1$, respectively. Hence, the Green's function of problem (5) can be written as

$$K(\omega,s) = \frac{1}{2\sqrt{\varrho}} \begin{cases} \mathrm{e}^{-\sqrt{\varrho}s}(\mathrm{e}^{\sqrt{\varrho}\omega} - \mathrm{e}^{-\sqrt{\varrho}(\omega-2)}), & 1 < \omega \leqslant s < \infty, \\ \mathrm{e}^{-\sqrt{\varrho}\omega}(\mathrm{e}^{\sqrt{\varrho}s} - \mathrm{e}^{-\sqrt{\varrho}(s-2)}), & 1 < s \leqslant \omega < \infty. \end{cases}$$

Thus, problem (5) has a unique solution given by

$$Z(\omega) = \int_1^\infty K(\omega,s)h(s)\,\mathrm{d}s.$$

Remark 2. By the straightforward calculation, we know that $K(\omega,s) \geqslant 0$ and $K(\omega,s) \leqslant 1/(2\sqrt{\varrho})$ for all $\omega, s \in [0,\infty)$.

Lemma 2. *Suppose that $Z:(1,\infty)\to\mathbb{R}$ is a given function such that $|Z(\omega)|\leqslant M$, where $M>0$ and $\delta>-1$. Then the solution of the following Hadamard fractional differential equation with logarithmic-type integro-initial conditions*

$$\begin{aligned}
&{}_HD^{\zeta}x(\omega)+Z(\omega)p(\omega)=0,\quad 1<\zeta<2,\ \omega\in(1,\infty),\\
&\lim_{\omega\to1}(\log\omega)^{2-\zeta}x(\omega)=\lambda,\\
&\lim_{\omega\to\infty}{}_HD^{\zeta-1}x(\omega)=\int_1^\infty k(s)x(s)\,\frac{\mathrm{d}s}{s}+\sum_{i=1}^m\gamma_i\,{}_HD^{\beta_i}x(\eta_i)
\end{aligned}\tag{6}$$

is given by

$$\begin{aligned}
x(\omega)={}&\int_1^\infty\Sigma(\omega,s)Z(s)p(s)\,\frac{\mathrm{d}s}{s}+\frac{\lambda(\log\omega)^{\zeta-1}}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\\
&+\lambda(\log\omega)^{\zeta-2},
\end{aligned}\tag{7}$$

where

$$\Sigma(\omega,s)=\Sigma_1(\omega,s)+\Sigma_2(\omega,s)$$

with

$$\begin{aligned}
\Sigma_1(\omega,s)&=\Lambda_1(\omega,s)+\sum_{i=1}^m\frac{\gamma_i(\log\omega)^{\zeta-1}}{\Omega_1}\Lambda_2(\eta_i,s),\\
\Sigma_2(\omega,s)&=\frac{(\log\omega)^{\zeta-1}}{\Omega_2}\int_1^\infty\Sigma_1(\omega,s)k(s)\,\frac{\mathrm{d}s}{s},
\end{aligned}$$

$$\Lambda_1(\omega,s)=\frac{1}{\Gamma(\zeta)}\begin{cases}(\log\omega)^{\zeta-1}-(\log\frac{\omega}{s})^{\zeta-1}, & 1\leqslant s\leqslant\omega\leqslant\infty,\\ (\log\omega)^{\zeta-1}, & 1\leqslant\omega\leqslant s\leqslant\infty,\end{cases}$$

and

$$\Lambda_2(\eta_i,s)=\frac{1}{\Gamma(\zeta-\beta_i)}\begin{cases}(\log\eta_i)^{\zeta-\beta_i-1}-(\log\frac{\eta_i}{s})^{\zeta-\beta_i-1}, & 1\leqslant s\leqslant\eta_i\leqslant\infty,\\ (\log\eta_i)^{\zeta-1}, & 1\leqslant\eta_i\leqslant s\leqslant\infty.\end{cases}$$

Proof. As argued in [1], the solution of the Hadamard fractional differential equation in (6) can be written as

$$x(\omega)=-{}_HI^{\zeta}Z(\omega)p(\omega)+c_1(\log\omega)^{\zeta-1}+c_2(\log\omega)^{\zeta-2}\tag{8}$$

for some arbitrary constants $c_1,c_2\in\mathbb{R}$. By Hölder's inequality, we have that

$$\begin{aligned}
&\left|(\log\omega)^{2-\zeta}\int_1^\omega\left(\log\frac{\omega}{s}\right)^{\zeta-1}\frac{Z(s)p(s)}{s}\,\mathrm{d}s\right|\\
&\quad\leqslant M(\log\omega)^{2-\zeta}\int_1^\omega\left(\log\frac{\omega}{s}\right)^{\zeta-1}\frac{p(s)}{s}\,\mathrm{d}s
\end{aligned}$$

$$\leqslant M(\log\omega)^{2-\zeta}\left(\int_1^{\omega}\left(\log\frac{\omega}{s}\right)^{2\zeta-2}\frac{\mathrm{d}s}{s}\right)^{1/2}\left(\int_1^{\omega}\frac{p^2(s)}{s}\,\mathrm{d}s\right)^{1/2}$$
$$=\frac{M}{\sqrt{2\zeta-1}}(\log\omega)^{3/2}\left(\int_1^{\omega}\frac{p^2(s)}{s}\,\mathrm{d}s\right)^{1/2}\to 0\quad\text{as } t\to 1. \tag{9}$$

Combining (8) and (9) with the condition $\lim_{\omega\to 1}(\log\omega)^{2-\zeta}x(\omega)=\lambda$, we find that $c_2=\lambda$. By Lemma 1, we have

$${}_HD^{\zeta-1}x(\omega)=-\int_1^{\omega}\frac{Z(s)p(s)}{s}\,\mathrm{d}s+c_1\Gamma(\zeta). \tag{10}$$

Using (10) in the condition

$$\lim_{\omega\to\infty}{}_HD^{\zeta-1}x(\omega)=\int_1^{\infty}k(s)x(s)\,\frac{\mathrm{d}s}{s}+\sum_{i=1}^{m}\gamma_i\,{}_HD^{\beta_i}x(\eta_i),$$

we get

$$c_1=\frac{1}{\Omega_1}\left\{\int_1^{\infty}Z(s)p(s)\,\frac{\mathrm{d}s}{s}+\int_1^{\infty}k(s)x(s)\,\frac{\mathrm{d}s}{s}\right.$$
$$\left.+\sum_{i=1}^{m}\frac{\gamma_i}{\Gamma(\zeta-\beta_i)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\zeta-\beta_i-1}Z(s)p(s)\,\frac{\mathrm{d}s}{s}\right\},$$

where we used the fact that

$$\left|\int_1^{\infty}\frac{Z(s)p(s)}{s}\,\mathrm{d}s\right|\leqslant M\int_1^{\infty}p(s)\,\frac{\mathrm{d}s}{s}.$$

Thus, by a straightforward calculation, we get

$$x(\omega)=-\frac{1}{\Gamma(\zeta)}\int_1^{\omega}\left(\log\frac{\omega}{s}\right)^{\zeta-1}Z(s)p(s)\,\frac{\mathrm{d}s}{s}$$
$$+\frac{(\log\omega)^{\zeta-1}}{\Omega_1}\left\{\int_1^{\infty}Z(s)p(s)\,\frac{\mathrm{d}s}{s}+\int_1^{\infty}k(s)x(s)\,\frac{\mathrm{d}s}{s}\right.$$
$$\left.+\sum_{i=1}^{m}\frac{\gamma_i}{\Gamma(\zeta-\beta_i)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\zeta-\beta_i-1}Z(s)p(s)\,\frac{\mathrm{d}s}{s}\right\}+\lambda(\log\omega)^{\zeta-2}$$

$$
\begin{aligned}
&= -\frac{1}{\Gamma(\zeta)} \int_1^{\omega} \left(\log \frac{\omega}{s}\right)^{\zeta-1} Z(s)p(s)\,\frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log \omega)^{\zeta-1}}{\Omega_1} \Biggl\{ \int_1^{\infty} Z(s)p(s)(s)\,\frac{\mathrm{d}s}{s} + \int_1^{\infty} k(s)x(s)\,\frac{\mathrm{d}s}{s} \\
&\quad + \sum_{i=1}^{m} \frac{\gamma_i}{\Gamma(\zeta-\beta_i)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\zeta-\beta_i-1} Z(s)p(s)\,\frac{\mathrm{d}s}{s} \Biggr\} + \lambda(\log \omega)^{\zeta-2} \\
&\quad \pm \frac{(\log \omega)^{\zeta-1}}{\Gamma(\zeta)} \int_1^{\infty} Z(s)p(s)\,\frac{\mathrm{d}s}{s} \\
&= \int_1^{\infty} \Sigma_1(\omega,s)Z(s)p(s)\,\frac{\mathrm{d}s}{s} + \frac{(\log \omega)^{\zeta-1}}{\Omega_1} \int_1^{\infty} k(s)x(s)\,\frac{\mathrm{d}s}{s} + \lambda(\log \omega)^{\zeta-2}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\int_1^{\infty} k(s)x(s)\,\frac{\mathrm{d}s}{s} &= \int_1^{\infty} k(\omega) \int_1^{\infty} \Sigma_1(t,s)Z(s)p(s)\,\frac{\mathrm{d}s}{s}\frac{\mathrm{d}\omega}{\omega} \\
&\quad + \frac{1}{\Omega_1} \int_1^{\infty} k(s)(\log s)^{\zeta-1}\,\frac{\mathrm{d}s}{s} \int_1^{\infty} k(s)x(s)\,\frac{\mathrm{d}s}{s} \\
&\quad + \lambda \int_1^{\infty} k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x(\omega) &= \int_1^{\infty} \Sigma_1(\omega,s)Z(s)p(s)\,\frac{\mathrm{d}s}{s} + \int_1^{\infty} \Sigma_2(\omega,s)Z(s)p(s)\,\frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log \omega)^{\zeta-1}}{\Omega_1} \int_1^{\infty} k(s)x(s)\,\frac{\mathrm{d}s}{s} + \lambda(\log \omega)^{\zeta-2},
\end{aligned}
$$

which leads to the desired conclusion. □

We need the following lemmas to prove our main results.

Lemma 3. *The function $\Sigma(\omega,s)$ satisfies the following properties*:

(i) $\Sigma(\omega,s) \geqslant 0$ *is a continuous function for* $\omega, s \in (1,\infty)$;

(ii) *For* $\omega, s \in (1,\infty)$,

$$
\frac{(\log \omega)^{2-\zeta}}{1+(\log \omega)^{\sigma+2}} \Sigma(\omega,s) \leqslant \frac{1}{\Omega_2};
$$

(iii) *For any* $\vartheta > 1$ *and* $s \in (1, \infty)$,

$$\min_{\eta_1 \leqslant \omega \leqslant \vartheta\eta_1} \frac{(\log\omega)^{2-\zeta}}{1 + (\log\omega)^{\sigma+2}} \Sigma(\omega, s) \geqslant \sum_{i=1}^{m} \frac{\gamma_i(\log\eta_1)}{\Omega_1(1 + (\log\vartheta\eta_1)^{\sigma+2})} \Lambda_2(\eta_i, s).$$

Proof. We do not provide the proofs for (i) and (ii) as one can obtain them in a straightforward manner. To prove (iii), for $\omega, s \in (1, \infty)$, we have

$$\begin{aligned}
&\min_{\eta_1 \leqslant \omega \leqslant \vartheta\eta_1} \frac{(\log\omega)^{2-\zeta}}{1 + (\log\omega)^{\sigma+2}} \Sigma(\omega, s) \\
&= \min_{\eta_1 \leqslant \omega \leqslant \vartheta\eta_1} \frac{(\log\omega)^{2-\zeta}}{1 + (\log\omega)^{\sigma+2}} \Biggl\{ \Lambda_1(\omega, s) + \sum_{i=1}^{m} \frac{\gamma_i(\log\omega)^{\zeta-1}}{\Omega_1} \Lambda_2(\eta_i, s) \\
&\quad + \frac{(\log\omega)^{\zeta-1}}{\Omega_2} \int_1^{\infty} \Sigma_1(\omega, s) k(s) \,\frac{\mathrm{d}s}{s} \Biggr\} \\
&\geqslant \min_{\eta_1 \leqslant \omega \leqslant \vartheta\eta_1} \frac{(\log\omega)^{2-\zeta}}{1 + (\log\omega)^{\sigma+2}} \sum_{i=1}^{m} \frac{\gamma_i(\log\omega)^{\zeta-1}}{\Omega_1} \Lambda_2(\eta_i, s) \\
&\geqslant \sum_{i=1}^{m} \frac{\gamma_i(\log\eta_1)}{\Omega_1(1 + (\log\vartheta\eta_1)^{\sigma+2})} \Lambda_2(\eta_i, s).
\end{aligned}$$

□

Lemma 4. *The functions* $K(\omega, s)$ *satisfies the following property*:

$$K(\omega, s) \geqslant \Pi_0 K(s, s) \mathrm{e}^{-\sqrt{\varrho}s}, \quad \omega \in [\eta_1, \vartheta\eta_1],\ s \in (1, \infty),$$
$$\Pi_0 = \min\bigl\{\mathrm{e}^{-\vartheta\eta_1\sqrt{\varrho}},\ \mathrm{e}^{\eta_1\sqrt{\varrho}} - \mathrm{e}^{-\sqrt{\varrho}(\eta_1-2)}\bigr\}.$$

Proof. By the definition of $K(\omega, s)$, for the case $0 < \omega \leqslant s < \infty$, since $\varrho_1 < 0 < \varrho_2$ and $\eta_1 \leqslant \omega \leqslant \vartheta\eta_1$, we get

$$0 < 1 - \mathrm{e}^{-2\sqrt{\varrho}(s-1)} \leqslant 1, \quad 0 < \mathrm{e}^{-2\sqrt{\varrho}(s-1)} \leqslant 1, \tag{11}$$
$$\mathrm{e}^{\sqrt{\varrho}\omega} - \mathrm{e}^{-\sqrt{\varrho}(\omega-2)} \geqslant \mathrm{e}^{\eta_1\sqrt{\varrho}} - \mathrm{e}^{-\sqrt{\varrho}(\eta_1-2)},$$

and

$$\frac{K(\omega, s)}{K(s, s)} = \mathrm{e}^{-\sqrt{\varrho}s} \frac{\mathrm{e}^{\sqrt{\varrho}\omega} - \mathrm{e}^{-\sqrt{\varrho}(\omega-2)}}{1 - \mathrm{e}^{-2\sqrt{\varrho}(s-1)}} \geqslant \mathrm{e}^{-\sqrt{\varrho}s}\bigl(\mathrm{e}^{\eta_1\sqrt{\varrho}} - \mathrm{e}^{-\sqrt{\varrho}(\eta_1-2)}\bigr). \tag{12}$$

For the case $0 < s < \omega < \infty$, it follows by (11) that

$$\frac{K(\omega, s)}{K(s, s)} = \frac{\mathrm{e}^{-\sqrt{\varrho}\omega}}{\mathrm{e}^{-\sqrt{\varrho}s}} = \frac{\mathrm{e}^{-\sqrt{\varrho}\omega}\mathrm{e}^{-\sqrt{\varrho}s}}{\mathrm{e}^{-2\sqrt{\varrho}s}} \geqslant \mathrm{e}^{-\vartheta\eta_1\sqrt{\varrho}}\mathrm{e}^{-\sqrt{\varrho}s}. \tag{13}$$

Hence, from (12) and (13) we have the conclusion. □

3 Main results

This section is concerned with the existence results for problem (3). In view of Lemma 2, we define an operator $\mathcal{F}: \mathcal{P} \to X$ by

$$\begin{aligned}\mathcal{F}x(\omega) &= \int_1^\infty \varSigma(\omega, s)p(s) \int_1^\infty K(s,r)a(r)\mathcal{G}\bigl(r, x(r)\bigr)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\ &\quad + \frac{\lambda(\log\omega)^{\zeta-1}}{\varOmega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s} + \lambda(\log\omega)^{\zeta-2},\end{aligned}$$

where $\mathcal{P} \subset X$ is a cone given by $\mathcal{P} := \{\xi \in X\colon\ \xi(\omega) \geqslant 0,\ \omega \in (1,+\infty)\}$. We know that the existence of fixed points of $\mathcal{F}$ in $\mathcal{P}$ are equivalent to the existence of positive solutions to problem (3).

In the subsequent analysis, we need the following assumptions:

(G1) There exists a real positive function ϕ on $(1,\infty)$ with $\int_1^\infty a(r)\phi(r)\,\mathrm{d}r < \infty$ such that for all $x, y \in \mathbb{R}$, $\omega \in (1,\infty)$,

$$\begin{aligned}&\left|\mathcal{G}\left(\omega, \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}x(\omega)\right) - \mathcal{G}\left(\omega, \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}y(\omega)\right)\right| \\ &\quad\leqslant \phi(\omega)|x-y|;\end{aligned}$$

(G2) There exists a number $\varUpsilon$ such that $\varrho_0 \leqslant \varUpsilon < 1$, $\omega \in (1,\infty)$, where

$$\varrho_0 = \frac{1}{2\varOmega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\phi(r)\,\mathrm{d}r.$$

Theorem 1. *Suppose that $\mathcal{G} \in C((1,\infty)\times\mathbb{R},\mathbb{R})$ satisfies conditions* (G1) *and* (G2) *and that there exists a number $M > 0$ such that $|\mathcal{G}(\omega, x(\omega))| \leqslant M$. Then problem* (3) *has a unique solution when $\varrho_0 < 1$ (ϱ_0 is defined in* (G2)).

Proof. Let us set $\sup_{\omega\in(1,\infty)}\|\mathcal{G}(\omega,0)\| = \varLambda$,

$$\begin{aligned}\mathcal{Q}_1 &= \frac{1}{2\varOmega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\,\mathrm{d}r, \\ \mathcal{Q}_2 &= \frac{\lambda}{\varOmega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\sup_{\omega\in(1,\infty)}\frac{\log\omega}{1+(\log\omega)^{\sigma+2}}, \\ \mathcal{Q}_3 &= \lambda\sup_{\omega\in(1,\infty)}\frac{1}{1+(\log\omega)^{\sigma+2}},\end{aligned}$$

and choose $\rho \geqslant |\omega_1\varLambda + \omega_2 + \omega_3|/(1-\varUpsilon)$, where $\varrho_0 \leqslant \varUpsilon < 1$. Introduce $B_\rho = \{x \in X\colon \|x\|_X \leqslant \rho\}$. For any $x \in B_\rho$ and $\omega \in (1,\infty)$, by the triangle inequality and (G1), we obtain

$$\begin{aligned}
\left|\mathcal{G}\big(\omega, x(\omega)\big)\right| &= \left|\mathcal{G}\left(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}\,\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}x(\omega)\right)\right| \\
&\leqslant \left|\mathcal{G}\left(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}\,\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}x(\omega)\right) - \mathcal{G}(\omega,0)\right| \\
&\quad + \left|\mathcal{G}(\omega,0)\right| \\
&\leqslant \phi(\omega)\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}|x| + \varLambda \leqslant \phi(\omega)\|x\|_X + \varLambda. \qquad (14)
\end{aligned}$$

Now, we will show that $\mathcal{F}B_\rho \subset B_\rho$. For any $x \in B_\rho$, by Remark 2, Lemma 3(ii), (G1), (G2), and (14), we have

$$\begin{aligned}
\big\|(\mathcal{F}x)\big\| &\leqslant \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} \\
&\quad \times \left[\int_1^\infty \varSigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\big[\phi(r)\rho+\varLambda\big]\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}\right. \\
&\quad \left. + \frac{\lambda(\log\omega)^{\zeta-1}}{\varOmega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s} + \lambda(\log\omega)^{\zeta-2}\right] \\
&\leqslant \frac{1}{2\varOmega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\phi(r)\,\mathrm{d}r\cdot\rho + \frac{1}{2\varOmega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\,\mathrm{d}r\cdot\varLambda \\
&\quad + \frac{\lambda}{\varOmega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\sup_{\omega\in(1,\infty)}\frac{\log\omega}{1+(\log\omega)^{\sigma+2}}\,\lambda\sup_{\omega\in(1,\infty)}\frac{1}{1+(\log\omega)^{\sigma+2}} \\
&\leqslant \frac{1}{2\varOmega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\phi(r)\,\mathrm{d}r\cdot\rho + \mathcal{Q}_1\varLambda + \mathcal{Q}_2 + \mathcal{Q}_3 \\
&\leqslant \varrho_0\rho + (1-\varUpsilon)\rho \leqslant \rho. \qquad (15)
\end{aligned}$$

Therefore, $\|(\mathcal{F}x)\| \leqslant \rho$.

Next, we show that $\mathcal{F}$ is a contraction. For $x, y \in X$ and $\omega \in (1,\infty)$, it follows by (G1) and (G2) that

$$\begin{aligned}
&\big\|(\mathcal{F}x)-(\mathcal{F}y)\big\| \\
&\quad \leqslant \sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} \\
&\qquad \times \int_1^\infty \varSigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\big|\mathcal{G}\big(r,x(r)\big)-\mathcal{G}\big(r,y(r)\big)\big|\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$\leqslant \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} \int_1^\infty \Sigma(\omega,s)p(s) \int_1^\infty K(s,r)a(r)\phi(r)\|x-y\|_X \,\mathrm{d}r \,\frac{\mathrm{d}s}{s}$$

$$\leqslant \frac{1}{2\Omega_2\sqrt{\varrho}} \int_1^\infty p(s)\,\frac{\mathrm{d}s}{s} \int_1^\infty a(r)\phi(r)\,\mathrm{d}r\, t\|x-y\|_X = \varrho_0\|x-y\|_X,$$

where ϱ_0 is given in (G2). Since $\varrho_0 < 1$, therefore $\mathcal{F}$ is a contraction. Hence, the assumptions of the contraction mapping principle are satisfied. This leads to the conclusion. □

Lemma 5. *Suppose that $\mathcal{Q} \subset X$ is a bounded set. Then $\mathcal{Q}$ is relatively compact in X if the following conditions hold*:

(i) $(\log\omega)^{2-\zeta}x(\omega)/(1+(\log\omega)^{\sigma+2})$ *is equicontinuous on any compact interval of* $(1,\infty)$ *for any* $x \in \mathcal{Q}$;

(ii) *For any $\varepsilon > 0$, there exists a positive constant $M_{\mathcal{Q}} > 0$ such that*

$$\left|\frac{(\log\omega_1)^{2-\zeta}x(\omega_1)}{1+(\log\omega_1)^{\sigma+2}} - \frac{(\log\omega_2)^{2-\zeta}x(\omega_2)}{1+(\log\omega_2)^{\sigma+2}}\right| < \varepsilon \quad \forall t_1, t_2 \geqslant M_{\mathcal{G}},\ x\in\mathcal{Q}.$$

Proof. The proof is similar to that of [9, Lemma 3.1] and is omitted. □

We need the following lemmas to establish our next main result (Theorem 2).

Lemma 6. *Suppose that the following condition hold*:

(G3) $\mathcal{G} : (1,\infty)\times\mathbb{R} \to \mathbb{R}$ *is a continuous function with* $\mathcal{G}(\omega,x) \in X$. *Also, if x is bounded, then* $\mathcal{G}(\omega,\ (1+(\log\omega)^{\sigma+2})x/(\log\omega)^{2-\zeta})$ *be bounded on* $(1,\infty)$.

Then the operator $\mathcal{F} : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Proof. We first establish that $\mathcal{F}$ is uniformly bounded in $\mathcal{P}$. Let $\Delta \subset X$ be bounded subset, then by (G3) there exists a plosive constant ς_0 such that $\|x\| \leqslant \varsigma_0$ for all $x \in \Delta$. Using (G3), we have

$$\widetilde{M} = \sup\left\{\mathcal{G}\left(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}x\right):\ (\omega,x)\in(1,\infty)\times[0,\varsigma_0]\right\} < \infty.$$

For any $x \in \Delta$, we get

$$\mathcal{G}(\omega,x) = \left|\mathcal{G}\left(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}x\right)\right| \leqslant \widetilde{M}.$$

Consequently, by Remark 2, Lemma 3(ii), we have

$$\|\mathcal{F}x\|_X \leqslant \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\left[\widetilde{M}\int_1^\infty \Sigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}\right.$$
$$\left. + \frac{\lambda(\log\omega)^{\zeta-1}}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s} + \lambda(\log\omega)^{\zeta-2}\right]$$

$$\leqslant \widetilde{M}\frac{1}{2\Omega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\,\mathrm{d}r + \frac{\lambda}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}$$
$$\times \sup_{\omega\in(1,\infty)}\frac{\log\omega}{1+(\log\omega)^{\sigma+2}} + \lambda\sup_{\omega\in(1,\infty)}\frac{1}{1+(\log\omega)^{\sigma+2}}$$
$$:= K.$$

Thus, $\|\mathcal{F}x\|_X \leqslant K$, and hence, $\mathcal{F}$ is uniformly bounded.

Next, we show that $\mathcal{F}$ is equicontinuous. Let $\omega_1, \omega_2 \in (1,\infty)$ with $\omega_1 < \omega_2$ and $x \in \Delta$. Then we obtain

$$\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}(\mathcal{F}x(\omega_2) - \frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}(\mathcal{F}x(\omega_1)\right|$$
$$\leqslant \widetilde{M}\int_1^\infty\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\Sigma(\omega_2,s) - \frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\Sigma(\omega_1,s)\right|$$
$$\times p(s)\int_1^\infty K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}$$
$$+ \frac{\lambda}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\left[\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}} - \frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right]$$
$$+ \lambda\left[\frac{1}{1+(\log\omega_2)^{\sigma+2}} - \frac{1}{1+(\log\omega_1)^{\sigma+2}}\right]$$
$$\leqslant \widetilde{M}\int_1^\infty\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\Sigma(\omega_2,s) - \frac{(\log t_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\Sigma(\omega_1,s)\right|$$
$$\times p(s)\int_1^\infty K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}$$
$$+ \frac{\lambda}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\left[\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}} - \frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right]$$
$$+ \lambda\left[\frac{1}{1+(\log\omega_2)^{\sigma+2}} - \frac{1}{1+(\log\omega_1)^{\sigma+2}}\right]$$
$$\leqslant \widetilde{M}\int_1^{\omega_2}\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\Sigma(\omega_2,s) - \frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\Sigma(\omega_1,s)\right|$$
$$\times p(s)\int_1^\infty K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}$$

$$
\begin{aligned}
&+ \widetilde{M}\int_{\omega_2}^{\infty}\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\Sigma(\omega_2,s)-\frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\Sigma(\omega_1,s)\right| \\
&\qquad\times p(s)\int_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&+ \frac{\lambda}{\Omega_2}\int_1^{\infty}k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\left[\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}}-\frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right] \\
&+ \lambda\left[\frac{1}{1+(\log\omega_2)^{\sigma+2}}-\frac{1}{1+(\log\omega_1)^{\sigma+2}}\right] \\
\leqslant{}& \widetilde{M}\int_1^{\omega_2}\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\Sigma(\omega_2,s)-\frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\Sigma(\omega_1,s)\right| \\
&\qquad\times p(s)\int_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&+ \widetilde{M}\int_{\omega_2}^{\infty}\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\left\{\Lambda_1(\omega_2,s)+\sum_{i=1}^{m}\frac{\gamma_i(\log\omega_2)^{\zeta-1}}{\Omega_1}\Lambda_2(\eta_i,s)\right.\right. \\
&\qquad\left.+\frac{(\log\omega_2)^{\zeta-1}}{\Omega_2}\int_1^{\infty}\Sigma_1(\omega_2,s)k(s)\,\frac{\mathrm{d}s}{s}\right\} \\
&- \frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\left\{\Lambda_1(\omega_1,s)+\sum_{i=1}^{m}\frac{\gamma_i(\log\omega_1)^{\zeta-1}}{\Omega_1}\Lambda_2(\eta_i,s)\right. \\
&+ \left.\left.\frac{(\log\omega_1)^{\zeta-1}}{\Omega_2}\int_1^{\infty}\Sigma_1(\omega_1,s)k(s)\,\frac{\mathrm{d}s}{s}\right\}\right|p(s)\int_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&+ \frac{\lambda}{\Omega_2}\int_1^{\infty}k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\left[\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}}-\frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right] \\
&+ \lambda\left[\frac{1}{1+(\log\omega_2)^{\sigma+2}}-\frac{1}{1+(\log\omega_1)^{\sigma+2}}\right] \\
\leqslant{}& \widetilde{M}\int_1^{\omega_2}\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\Sigma(\omega_2,s)-\frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\Sigma(\omega_1,s)\right| \\
&\qquad\times p(s)\int_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\widetilde{M}}{\Omega_2}\int\limits_{\omega_2}^{\infty}\left|\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}}-\frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right|p(s)\int\limits_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}\\
&+\frac{\lambda}{\Omega_2}\int\limits_1^{\infty}k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\left[\frac{\log\omega_2}{1+(\log\omega_2)^{\sigma+2}}-\frac{\log\omega_1}{1+(\log\omega_1)^{\sigma+2}}\right]\\
&+\lambda\left[\frac{1}{1+(\log\omega_2)^{\sigma+2}}-\frac{1}{1+(\log\omega_1)^{\sigma+2}}\right].
\end{aligned}
$$

Since $((\log\omega)^{2-\zeta}/(1+(\log\omega)^{\sigma+2}))\Sigma(\omega,s)$ and $\log\omega/(1+(\log\omega)^{\sigma+2})$ are uniformly continuous on any compact set $[1,a_0]$, so we get

$$
\left|\frac{(\log\omega_2)^{2-\zeta}}{1+(\log\omega_2)^{\sigma+2}}\mathcal{F}x(\omega_2)-\frac{(\log\omega_1)^{2-\zeta}}{1+(\log\omega_1)^{\sigma+2}}\mathcal{F}x(\omega_1)\right|\to 0\quad\text{as }\omega_1\to\omega_2.
$$

Hence, $\mathcal{F}$ is equicontinuous on $[1,a_0]$.

We now show that $\mathcal{F}$ is equiconvergence at ∞. For ant $x\in\mathcal{P}$, we obtain

$$
\lim_{\omega\to\infty}\left|\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\mathcal{F}x(\omega)\right|\leqslant\frac{\widetilde{M}}{\Omega_2}\lim_{\omega\to\infty}\int\limits_1^{\infty}p(s)\int\limits_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}<\infty.
$$

Finally, we show that $\mathcal{F}$ is continuous. Let $x_n\to x$ as $n\to\infty$ in $\mathcal{P}$. Hence, $\{x\}$ is bounded in $\mathcal{P}$. Therefore, there exists a plosive constant ς_1 such that $\|x_n\|\leqslant\varsigma_1$ for all $x\in\mathcal{P}$. Using (G3), we have

$$
\widetilde{M_1}=\sup\left\{\mathcal{G}\left(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}x\right):\ (\omega,x)\in(1,\infty)\times[0,\varsigma_1]\right\}<\infty.
$$

So, by the argument similar to the one employed in (15), we get

$$
\begin{aligned}
\left|\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\mathcal{F}x_n(\omega)\right|&\leqslant\frac{\widetilde{M_1}}{\Omega_2}\lim_{\omega\to\infty}\int\limits_1^{\infty}p(s)\int\limits_1^{\infty}K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\lambda}{\Omega_2}\int\limits_1^{\infty}k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}\sup_{\omega\in(1,\infty)}\frac{\log\omega}{1+(\log\omega)^{\sigma+2}}\\
&\quad+\lambda\sup_{\omega\in(1,\infty)}\frac{1}{1+(\log\omega)^{\sigma+2}}\\
&<\infty.
\end{aligned}
$$

Consequently, it follows by the Lebesgue dominated convergence theorem and continuity of $\mathcal{G}$, K, and Σ that

$$\int_1^\infty \Sigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\mathcal{G}\big(r,x_n(r)\big)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\ \to \int_1^\infty \Sigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\mathcal{G}\big(r,x(r)\big)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \quad \text{as } n\to\infty.$$

Thus, we obtain

$$\begin{aligned}
&\big\|(\mathcal{F}x_n)-(\mathcal{F}x)\big\| \\
&\quad\leqslant \sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} \\
&\qquad\times\int_1^\infty \Sigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\big|\mathcal{G}\big(r,x_n(r)\big)-\mathcal{G}\big(r,x(r)\big)\big|\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\int_1^\infty \Sigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\phi(r)\|x_n-x\|_X\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{1}{2\Omega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\phi(r)\,\mathrm{d}r\,\|x_n-x\|_X\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

So, $\mathcal{F}$ is continuous, and by Lemma 5 we deduce that $\mathcal{F}$ is completely continuous. □

Our next main result (Theorem 2) is based on a well-known fixed point index result [15], which is stated below.

Lemma 7. (*See* [15].) *Let X be a Banach space and $\mathcal{P}$ a cone in X. For $\xi>0$, let $B_\xi=\{x\in X\colon \|x\|_X<\xi\}$. Suppose that $\mathcal{F}:\overline{B}_\xi\to\mathcal{P}$ is completely continuous such that $\mathcal{F}x\neq 0$ for $x\in\partial B_\xi=\{x\in X\colon \|x\|_X=\xi\}$.*

(i) *If $\|\mathcal{F}x\|_X\geqslant\|x\|_X$ for $x\in\partial B_\xi$, then $i(\mathcal{F},B_\xi,\mathcal{P})=0$;*
(ii) *If $\|\mathcal{F}x\|_X\leqslant\|x\|_X$ for $x\in\partial B_\xi$, then $i(\mathcal{F},B_\xi,\mathcal{P})=1$.*

Now, we present our next main result.

Theorem 2. *Let condition* (G3) *and the following conditions hold*:

(G4) $$\lim_{x\to 0}\min_{\omega\in(1,\infty)}\frac{\mathcal{G}(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}x)}{|x|}=\infty;$$

(G5) *There exists a positive constant* $\varrho_1 > 0$ *such that*

$$\mathcal{G}\biggl(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}x\biggr) < m_0\varrho_1 \quad \forall\omega\in(1,\infty),\ x\in[0,\varrho_1],$$

where

$$\Theta_0 := m_0\frac{1}{2\Omega_2\sqrt{\varrho}}\int\limits_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int\limits_1^\infty a(r)\,\mathrm{d}r < 1;$$

(G6) *There exists a positive constant* $\varrho_2 > \varrho_1 > 0$ *such that*

$$\mathcal{G}\biggl(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}x\biggr) > \varpi\varrho_2 \quad \forall\omega\in(1,\infty),\ x\in[\varrho_1,\varrho_2],$$

where

$$\begin{aligned}\varpi > \Biggl(&\sum_{i=1}^m \frac{\gamma_i(\log\eta_1)}{\Omega_1(1+(\log\vartheta\eta_1)^{\sigma+2})}\\ &\times\int\limits_{\eta_1}^{\vartheta\eta_1}\Lambda_2(\eta_i,s)p(s)\,\frac{\mathrm{d}s}{s}\int\limits_{\eta_1}^{\vartheta\eta_1}\Pi_0K(r,r)\mathrm{e}^{-\varrho_2 r}a(r)\,\mathrm{d}r\Biggr)^{-1}.\end{aligned} \tag{16}$$

Then problem (3) *has at least two positive solutions* $\widehat{x}_1$ *and* $\widehat{x}_2$ *such that* $0 < \|\widehat{x}_1\|_X < \varrho_1 < \|\widehat{x}_2\|_X$.

Proof. By Lemma 6, we have that $\mathcal{F}:\mathcal{P}\to\mathcal{P}$ is completely continuous. By condition (G4), for any ϖ satisfying inequality (16), there exists $R_1\in(0,\varrho_1)$ such that

$$\mathcal{G}\biggl(\omega,\frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}x\biggr) \geqslant \varpi|x|,\quad \omega\in(1,\infty),\ 0<|x|\leqslant R_1.$$

Consequently, for $\omega\in[1,\infty)$, we get

$$\mathcal{G}(\omega,x) = \mathcal{G}\biggl(\omega,\ \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}}\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}x\biggr) \geqslant \varpi\|x\|_X, \tag{17}$$
$$0<\|x\|_X = \sup_{\omega\in(1,\infty)}\biggl(\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\bigl|x(\omega)\bigr|\biggr) \leqslant |x| \leqslant R_1.$$

Let $B_{R_1} = \{x\in\mathcal{P}\colon \|x\|_X < R_1\}$. Then, for any $x\in\partial B_{R_1}$, from (17), Lemma 3(iii), Lemma 4, and the fact $\Pi_0K(r,r)\mathrm{e}^{-\varrho_2 r}a(r)\geqslant 0$ we obtain

$$\begin{aligned}&\bigl\|\mathcal{F}x(\omega)\bigr\|_X\\ &\quad= \sup_{\omega\in(1,\infty)}\frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\Biggl[\int\limits_1^\infty \Sigma(\omega,s)p(s)\int\limits_1^\infty K(s,r)a(r)\mathcal{G}\bigl(r,x(r)\bigr)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s}\\ &\qquad+\frac{\lambda(\log\omega)^{\zeta-1}}{\Omega_2}\int\limits_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s}+\lambda(\log\omega)^{\zeta-2}\Biggr]\end{aligned}$$

$$
\begin{aligned}
&\geqslant \min_{\eta_1 \leqslant \omega \leqslant \vartheta\eta_1} \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} \Biggl[\int_1^\infty \Sigma(t,s)p(s) \int_1^\infty K(s,r)a(r)\mathcal{G}\bigl(r,x(r)\bigr)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\lambda(\log\omega)^{\zeta-1}}{\Omega_2} \int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s} + \lambda(\log\omega)^{\zeta-2} \Biggr] \\
&\geqslant \varpi R_1 \sum_{i=1}^m \frac{\gamma_i(\log\eta_1)}{\Omega_1(1+(\log\vartheta\eta_1)^{\sigma+2})} \\
&\quad \times \int_{\eta_1}^{\vartheta\eta_1} \Lambda_2(\eta_i,s)p(s)\,\frac{\mathrm{d}s}{s} \int_1^\infty \Pi_0 K(r,r)\mathrm{e}^{-\varrho_2 r}a(r)\,\mathrm{d}r \\
&\geqslant \varpi R_1 \sum_{i=1}^m \frac{\gamma_i(\log\eta_1)}{\Omega_1(1+(\log\vartheta\eta_1)^{\sigma+2})} \\
&\quad \times \int_{\eta_1}^{\vartheta\eta_1} \Lambda_2(\eta_i,s)p(s)\,\frac{\mathrm{d}s}{s} \int_{\eta_1}^{\vartheta\eta_1} \Pi_0 K(r,r)\mathrm{e}^{-\varrho_2 r}a(r)\,\mathrm{d}r \\
&> R_1 = \|x\|_X.
\end{aligned} \tag{18}
$$

So, Lemma 7 implies that

$$
i(\mathcal{F}, B_{R_1}, \mathcal{P}) = 0. \tag{19}
$$

Let $\rho_1 < R_2 < \varrho_2$ and assume that $B_{R_2} = \{x \in \mathcal{P}\colon\ \|x\|_X < R_2\}$. So, for any $x \in B_{R_2}$, we obtain

$$
0 \leqslant \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} x(\omega) \leqslant R_2
$$

for $\omega \in (1,\infty)$. Hence, assumption (G6) implies that

$$
\mathcal{G}(\omega,x) = \mathcal{G}\biggl(\omega, \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta}} \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}} x\biggr) \geqslant \varpi R_2 \tag{20}
$$

for $\omega \in (1,\infty)$. So, for any $x \in \partial B_{R_2}$, by argument in (18), (17), Lemma 3(iii), and Lemma 4, we obtain

$$
\begin{aligned}
\bigl\|\mathcal{F}x(\omega)\bigr\|_X &\geqslant \varpi R_2 \sum_{i=1}^m \frac{\gamma_i(\log\eta_1)}{\Omega_1(1+(\log\vartheta\eta_1)^{\sigma+2})} \\
&\quad \times \int_{\eta_1}^{\vartheta\eta_1} \Lambda_2(\eta_i,s)p(s)\,\frac{\mathrm{d}s}{s} \int_{\eta_1}^{\vartheta\eta_1} \Pi_0 K(r,r)\mathrm{e}^{-\varrho_2 r}a(r)\,\mathrm{d}r \\
&> R_2 = \|x\|_X.
\end{aligned}
$$

Thus, Lemma 7 implies that

$$
i(\mathcal{F}, B_{R_2}, \mathcal{P}) = 0. \tag{21}
$$

Finally, set $B_{\rho_1} = \{x \in \mathcal{P}\colon \|x\|_X < \rho_1\}$, where $\rho_1 < \varrho_1$. So, for any $x \in \partial B_{\rho_1}$, it follows by (G5) that

$$\begin{aligned}
\|\mathcal{F}x\|_X &< \sup_{\omega\in(1,\infty)} \frac{(\log\omega)^{2-\zeta}}{1+(\log\omega)^{\sigma+2}}\Bigg[m_0\varrho_1\int_1^\infty \Sigma(\omega,s)p(s)\int_1^\infty K(s,r)a(r)\,\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\lambda(\log\omega)^{\zeta-1}}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s} + \lambda(\log\omega)^{\zeta-2}\Bigg] \\
&< m_0\varrho_1\frac{1}{2\Omega_2\sqrt{\varrho}}\int_1^\infty p(s)\,\frac{\mathrm{d}s}{s}\int_1^\infty a(r)\,\mathrm{d}r + \frac{\lambda}{\Omega_2}\int_1^\infty k(s)(\log s)^{\zeta-2}\,\frac{\mathrm{d}s}{s} \\
&\quad \times \sup_{\omega\in(1,\infty)}\frac{\log\omega}{1+(\log\omega)^{\sigma+2}} + \lambda\sup_{\omega\in(1,\infty)}\frac{1}{1+(\log\omega)^{\sigma+2}} \\
&\leqslant \Theta_0\rho_1 + \mathcal{Q}_2 + \mathcal{Q}_3,
\end{aligned}$$

where $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are the same as defined in Theorem 1. Letting $\rho_1 \geqslant |\mathcal{Q}_2+\mathcal{Q}_3|/(1-\widetilde{\Upsilon})$, with $\Theta_0 \leqslant \widetilde{\Upsilon} < 1$, we have

$$\|\mathcal{F}x\|_X < \Theta_0\varrho_1 + (1-\widetilde{\Upsilon})\rho_1 \leqslant \rho_1 = \|x\|_X.$$

In consequence, Lemma 7 implies that

$$i(\mathcal{F}, B_{\rho_1}, \mathcal{P}) = 1. \tag{22}$$

Since $R_1 < \rho_1 < R_2$, therefore, it follows by the fixed point index (Lemma 7) and (19)–(22) that

$$\begin{aligned}
i(\mathcal{F}, B_{\rho_1}\setminus\overline{B}_{R_1}, \mathcal{P}) &= i(\mathcal{F}, B_{\rho_1}, \mathcal{P}) - i(\mathcal{F}, B_{R_1}, \mathcal{P}) = 1, \\
i(\mathcal{F}, B_{R_2}\setminus\overline{B}_{\rho_1}, \mathcal{P}) &= i(\mathcal{F}, B_{R_2}, \mathcal{P}) - i(\mathcal{F}, B_{\rho_1}, \mathcal{P}) = -1.
\end{aligned}$$

Thus, $\mathcal{F}$ has two fixed points $\widehat{x}_1 \in B_{\rho_1}\setminus\overline{B}_{R_1}$ and $\widehat{x}_2 \in B_{R_2}\setminus\overline{B}_{\rho_1}$, which are the distinct positive solutions to problem (3). □

4 Application

Example 1. Consider the following Hadamard-type fractional boundary value problem:

$$\begin{gathered}
-\left(\frac{1}{p(\omega)}\,{}_HD^{\zeta+2}x(\omega)\right) + 2\left(\frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega)\right) = \frac{\mathrm{e}^{-2\omega}(\log\omega)^{1/2}}{8(1+(\log\omega)^2)}\sin\frac{|x(\omega)|}{1+|x(\omega)|}, \\
{}_HD^{\zeta}x(1) = 0, \qquad \lim_{\omega\to\infty}\frac{1}{p(\omega)}\,{}_HD^{\zeta}x(\omega) = 0, \qquad \lim_{\omega\to 1}(\log\omega)^{2-\zeta}x(\omega) = \lambda, \\
\lim_{\omega\to\infty}{}_HD^{\zeta-1}x(\omega) = \int_1^\infty \frac{s\mathrm{e}^{-s}}{30\sqrt{\log s}}x(s)\,\frac{\mathrm{d}s}{s} + \frac{1}{2}\,{}_HD^{1/2}x(\mathrm{e}^2) + \frac{1}{2}\,{}_HD^{3/4}x(\mathrm{e}^2).
\end{gathered} \tag{23}$$

Here $\omega \in (1,\infty)$, $\zeta = 3/2$, $a(\omega) = \mathrm{e}^{-\omega}$, $p(\omega) = \omega\mathrm{e}^{-\omega}$, $k(\omega) = \omega\mathrm{e}^{-\omega}/(30\sqrt{\log\omega})$, $\gamma_1 = \gamma_2 = 1/10$, $\beta_1 = 1/2$, $\beta_2 = 3/4$, $\eta_1 = \eta_2 = \mathrm{e}^2$, $m = 2$, and $\varrho = 2$. Choosing $\sigma = 0$, it follows by direct calculations that

$$\left|\mathcal{G}\left(\omega,\, \frac{1+(\log t)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}x(\omega)\right) - \mathcal{G}\left(\omega,\, \frac{1+(\log\omega)^{\sigma+2}}{(\log\omega)^{2-\zeta_1}}y(\omega)\right)\right| \leqslant \frac{\mathrm{e}^{-\omega}}{8}|x-y|$$

for all $x,y \in \mathbb{R}$, $\omega \in (1,\infty)$ with $\phi(\omega) = \mathrm{e}^{-\omega}/8$. Moreover, we find that $\Omega_1 \approx 0.14943$, $\Omega_2 \approx 0.116102$, and $\varrho_0 \approx 0.011755 < 1$. Clearly, all the conditions of Theorem 1 are satisfied. So, Theorem 1 yields that problem (23) has a unique solution on $(1,\infty)$.

Example 2. Consider the following boundary value problem for Hadamard fractional differential equations:

$$\begin{gathered}
-\left(\frac{1}{p(\omega)}{}_HD^{\zeta+2}x(\omega)\right)+2\left(\frac{1}{p(\omega)}{}_HD^{\zeta}x(\omega)\right)=\frac{\mathrm{e}^{-\omega}\exp\{\frac{(\log\omega)^{1/2}(x(\omega))^2}{1+(\log\omega)^2}\}(\log\omega)^{1/2}}{8(1+(\log\omega)^2)(1+|x|)},\\
{}_HD^{\zeta}x(1)=0,\qquad \lim_{\omega\to\infty}\frac{1}{p(\omega)}{}_HD^{\zeta}x(\omega)=0,\qquad \lim_{\omega\to1}(\log\omega)^{2-\zeta}x(\omega)=\lambda,\\
\lim_{\omega\to\infty}{}_HD^{\zeta-1}x(\omega)=\int_1^\infty\frac{s\mathrm{e}^{-s}}{30\sqrt{\log s}}x(s)\,\frac{\mathrm{d}s}{s}+\frac{1}{2}{}_HD^{1/2}x(\mathrm{e}^2)+\frac{1}{2}{}_HD^{3/4}x\big(\mathrm{e}^2\big),
\end{gathered}\tag{24}$$

where $\omega \in (1,\infty)$, $\zeta = 3/2$, $a(\omega) = \mathrm{e}^{-\omega}$, $p(\omega) = \omega\mathrm{e}^{-\omega}$, $k(\omega) = \omega\mathrm{e}^{-\omega}/(30\sqrt{\log\omega})$, $\gamma_1 = \gamma_2 = 1/10$, $\beta_1 = 1/2$, $\beta_2 = 3/4$, $\eta_1 = \eta_2 = \mathrm{e}^2$, $m = 2$, and $\varrho = 2$. Letting $\sigma = 0$ and $\varrho_1 = 1$, $m_0 = 1$, and $\varrho_1 = 2$, we find by direct calculation that $\Theta_0 \approx 0.17372 < 1$ and

$$\lim_{x\to0}\min_{\omega\in(1,\infty)}\frac{\mathcal{G}(\omega,\, \frac{1+(\log\omega)^2}{(\log\omega)^{1/2}}x)}{|x|} = \lim_{x\to0}\min_{t\in(1,\infty)}\frac{\mathrm{e}^{x^2}}{8|x|(1+|x|)} = \infty.$$

Also,

$$\mathcal{G}\left(\omega,\, \frac{1+(\log\omega)^2}{(\log\omega)^{1/2}}x\right) \approx 0.33978 < m_0\varrho_1 \quad \forall\omega\in(1,\infty),\ x\in[0,\varrho_1].$$

Moreover,

$$\mathcal{G}\left(\omega,\, \frac{1+(\log\omega)^2}{(\log\omega)^{1/2}}x\right) > 0.11326174 \quad \forall\omega\in(1,\infty),\ x\in[\varrho_1,\varrho_2],$$

and $\varpi \approx 0.5663 > 0.009665$. Thus, the hypothesis of Theorem 2 holds true. Hence, by the conclusion of Theorem 2, problem (24) has at least two positive solutions $\widehat{x}_1$ and $\widehat{x}_2$ such that $0 < \|\widehat{x}_1\|_X < 1 < \|\widehat{x}_2\|_X$.

5 Conclusions

We discussed the existence of a unique solution and at least two positive solutions for a new class of Hadamard fractional differential equations on an unbounded domain complemented with integral-multipoint boundary conditions. The main tools of our study include the contraction mapping principle and the fixed point index theorem. The results established in this study are new, useful and enrich the existing material on the topic. In our future work, we plan to study a coupled system of Hadamard-type fractional differential equations of different orders on a half-line equipped with coupled and uncoupled integral-multipoint boundary conditions.

Author contributions. All authors (N.N., B.A., and S.K.N.) have contributed as follows: methodology, N.N., B.A., and S.K.N.; formal analysis, N.N., B.A., and S.K.N.; validation, N.N., B.A., and S.K.N.; writing – original draft preparation, N.N., B.A., and S.K.N.; writing – review and editing, N.N., B.A., and S.K.N. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

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