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M-matrices and one-dimensional discrete Sturm–Liouville problems with nonlocal boundary conditions

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Abstract. This article is the second part of a survey dedicated to M-matrices and the application of the finite difference method to elliptic problems with nonlocal boundary conditions. Here, we examine cases in which the matrix of the resulting system of linear equations is an M-matrix. Here, we address the discrete Sturm–Liouville problem with nonlocal boundary conditions, describing its spectrum in one-dimensional case. This enables us to determine the values of the nonlocality parameters for which the finite difference scheme is represented by an M-matrix.

Keywords: M-matrices, finite difference method, Sturm–Liouville problem, nonlocal boundary conditions.

1 Introduction

Mathematical models based on differential equations with various types of nonlocal boundary conditions (NBCs) have been intensely studied in the fields of differential equation theory and numerical analysis [2, 3, 12, 24, 25, 48]. The application of numerical methods to problems involving NBCs results in non-symmetric, non-self-adjoint operators and matrices [17, 37].

M-matrices appear in many mathematical models, including those in engineering, mathematical economics, optimization theory, probability theory, statistics, and many other fields [5]. Professor Mifodijus Sapagovas is one of the researchers who uses M-matrix theory to study the properties of finite difference schemes (FDS) for problems with (NBCs) [11]. More information about M-matrices and FDS for problems with NBCs can be found in the first part of this survey [10]. There, we use a method based on regular splitting to provide necessary and sufficient conditions for finding the spectrum radius of a special matrix [4, Cor. 6.17]. Here, we use a different approach. We solve the discrete Sturm–Liouville problem (dSLP) to find the spectrum [29, 35, 38]. We conclude that the matrix of our problem is an M-matrix if the real parts of all the eigenvalues are positive [34, Thm. 2.1].

Our survey paper focuses on the study of the one-dimensional discrete Sturm–Liouville problem with nonlocal boundary conditions in relation to M-matrix theory. Section 2 briefly presents the main properties of M-matrices and their relation to other types of matrices. Section 3 discusses the algebraic eigenvalue problem. Section 4 introduces the finite difference method for the Sturm–Liouville problem, grids, grid functions, and grid operators. Section 5 describes the general solution to the Sturm–Liouville equation using Chebyshev polynomials. A large part of the section is devoted to solutions in trigonometric form. Section 6 discusses natural boundary conditions. The natural approximation of derivatives is presented. Section 7 considers examples of discrete Sturm–Liouville problems with nonlocal boundary conditions and uses the obtained results to determine whether the corresponding matrix is an M-matrix.

2 Notation. M-matrices

We will begin with the common notation, general concepts, and statements of matrix theory that will be used throughout this paper.

We denote $\mathbb{C}^+ := \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$. We denote the set of all $n \times m$ complex (or real) matrices $\mathbf{A} = (a_{ij})$, $a_{ij} \in \mathbb{C}$ (or $a_{ij} \in \mathbb{R}$) as $\mathbb{C}^{n\times m}$ (or $\mathbb{R}^{n\times m}$). To denote a row of a matrix, we often use the superscript $\mathbf{A} = (a^i_j)$. We will consider matrices $\mathbf{v} = (v_1, \dots, v_n)^\top \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ (or $\mathbf{v} = (v^1, \dots, v^n)^\top$) as vectors. We denote a matrix (or vector) consisting only of zeros by $\mathbf{O}$ (or $\mathbf{0}$) and the identity matrix by $\mathbf{I}$. If necessary for clarity, the dimension of these vectors or the identity matrix is indicated by subscripts.

In this article, we consider real matrices $\mathbf{A} \in \mathbb{R}^{n\times m}$. We use the notation $\mathbf{A} \leqslant \mathbf{B}$ (or $\mathbf{A} < \mathbf{B}$) if $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times m}$ and $a_{ij} \leqslant b_{ij}$ (or $a_{ij} < b_{ij}$) for all i, j. Following traditional notation, we will write $\mathbf{A} \geqslant 0$ (or $\mathbf{A} > 0$) instead of $\mathbf{A} \geqslant \mathbf{O}$ (or $\mathbf{A} > \mathbf{O}$) for a nonnegative (or positive) matrices and vectors.

The set of all eigenvalues $\lambda_1, \dots, \lambda_n$ of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ (spectrum of a matrix) is denoted by $\sigma(\mathbf{A})$. The spectral radius of $\mathbf{A}$ is $\rho(\mathbf{A}) := \max_{i=1,\dots,n} |\lambda_i|$.

Theorem 1. (*See* [54, *Thm.* 1.21], [4, *Thm.* 4.9].) *If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is strictly diagonally dominant or irreducibly diagonally dominant, then $\mathbf{A}$ is nonsingular. If, in addition, its diagonal entries are positive, i.e., $a_{ii} > 0$, then $\operatorname{Re}\lambda_i > 0$ for all eigenvalues λ_i of $\mathbf{A}$.*

Thus, if Theorem 1 holds (in the case $a_{ii} > 0$), then $\sigma(\mathbf{A}) \subset \mathbb{C}^+$.

The term *M-matrix* was first introduced by Ostrowski [31] in reference to the work of Herman Minkowski [28]. In this paper, Minkowski proved that for a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, under the conditions $a_{ij} < 0$, $i \neq j$, and $\sum_{j=1}^n a_{ij} > 0$, $i = 1, \dots, n$, the inequality $\det \mathbf{A} > 0$ holds. The second condition is the strict diagonal dominance condition. Ostrowski used the weaker condition $a_{ij} \leqslant 0$, $i \neq j$.

2.1 M-matrices

Definition 1 [Z-matrix]. A matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ is *a Z-matrix* if it satisfies the condition $a_{ij} \leqslant 0$, $i \neq j$.

We denote a class of square Z-matrices as $\mathcal{Z}_n := \{\mathbf{A} \in \mathbb{R}^{n\times n}\colon a_{ij} \leqslant 0,\ i \neq j\}$.

Definition 2 [Monotone matrix]. A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is called *monotone* if it is nonsingular and $\mathbf{A}^{-1} \geqslant 0$.

Definition 3 [*M*-matrix]. A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is an *M-matrix* if:

(i) the off-diagonal entries are nonpositive, $a_{ij} \leqslant 0$, $i \neq j$;
(ii) $\mathbf{A}$ is nonsingular, and $\mathbf{A}^{-1} \geqslant 0$.

Lemma 1. (*See* [34, *Thm.* 2.1].) *Let* $\mathbf{A} \in \mathcal{Z}_n$ *and* $a_{ii} > 0$, $i = 1, \dots, n$, *then each of the following statements is equivalent to "Matrix* $\mathbf{A}$ *is an* M*-matrix"*:

(i) $\mathbf{A}$ *is monotone. That is,* $\mathbf{A}^{-1}$ *exists, and* $\mathbf{A}^{-1} \geqslant 0$ *or* $\mathbf{A}\mathbf{v} \geqslant 0$ *implies* $\mathbf{v} \geqslant 0$;
(ii) $\operatorname{Re}\lambda(\mathbf{A}) > 0$, *where* $\lambda(\mathbf{A})$ *is the eigenvalue of matrix* $\mathbf{A}$, *i.e.,* $\sigma(\mathbf{A}) \subset \mathbb{C}^+$.

2.2 Linear systems

In [10], we considered FDSs for one- and two-dimensional Poisson equations with boundary conditions (BCs), writing them as linear systems in the form

$$\mathbf{A}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} = \mathbf{F}^{\mathrm{i}}, \quad \mathbf{A}^{\mathrm{i}} \in \mathbb{R}^{n\times n},\ \mathbf{A}^{\mathrm{b}} \in \mathbb{R}^{n\times m}, \tag{1}$$

$$\mathbf{U}^{\mathrm{b}} = \mathbf{B}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} + \mathbf{B}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{F}^{\mathrm{b}}, \quad \mathbf{B}^{\mathrm{i}} \in \mathbb{R}^{m\times n},\ \mathbf{B}^{\mathrm{b}} \in \mathbb{R}^{m\times m}, \tag{2}$$

where Eq. (1) is a discretization of Poisson's equation at the inner nodes, $\mathbf{U}^{\mathrm{i}}, \mathbf{F}^{\mathrm{i}} \in \mathbb{R}^{n\times 1}$, and Eq. (2) is a discretization of the BCs at the boundary nodes, $\mathbf{U}^{\mathrm{b}}, \mathbf{F}^{\mathrm{b}} \in \mathbb{R}^{m\times 1}$. Equation (2) is equivalent to

$$\mathbf{A}^{\mathrm{bi}}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{bb}}\mathbf{U}^{\mathrm{b}} = \mathbf{F}^{\mathrm{b}},$$

where $\mathbf{A}^{\mathrm{bi}} = -\mathbf{B}^{\mathrm{i}}$ and $\mathbf{A}^{\mathrm{bb}} = \mathbf{I} - \mathbf{B}^{\mathrm{b}}$.

If $\mathbf{A}^{\mathrm{bb}}$ is an M-matrix and both $\mathbf{A}^{\mathrm{bi}} \leqslant 0$ and $\mathbf{A}^{\mathrm{b}} \leqslant 0$, then the equation for the inner nodes is [10]

$$\mathbf{A}\mathbf{U}^{\mathrm{i}} = \mathbf{F}, \quad \mathbf{A} = \mathbf{A}^{\mathrm{i}} - \mathbf{C}, \quad \mathbf{C} = \mathbf{A}^{\mathrm{b}}\bigl(\mathbf{A}^{\mathrm{bb}}\bigr)^{-1}\mathbf{A}^{\mathrm{bi}} \geqslant 0.$$

If we use matrices $\tilde{\mathbf{B}}^{\mathrm{b}}$ and $\tilde{\mathbf{B}}^{\mathrm{i}}$ instead of $\mathbf{B}^{\mathrm{b}}$ and $\mathbf{B}^{\mathrm{i}}$ in (2), then we can calculate the matrices $\tilde{\mathbf{A}}^{\mathrm{bi}} = -\tilde{\mathbf{B}}^{\mathrm{i}}$, $\tilde{\mathbf{A}}^{\mathrm{bb}} = \mathbf{I} - \tilde{\mathbf{B}}^{\mathrm{b}}$, $\tilde{\mathbf{C}} = \mathbf{A}^{\mathrm{b}}(\tilde{\mathbf{A}}^{\mathrm{bb}})^{-1}\tilde{\mathbf{A}}^{\mathrm{bi}}$, and $\tilde{\mathbf{A}} = \mathbf{A}^{\mathrm{i}} - \tilde{\mathbf{C}}$.

Lemma 2. (*See* [10, *Lemma* 6].) *Let* $0 \leqslant \mathbf{B}^{\mathrm{b}} \leqslant \tilde{\mathbf{B}}^{\mathrm{b}}$, *and* $\tilde{\mathbf{A}}^{\mathrm{bb}}$ *are the* M*-matrices. Then* $\mathbf{A}^{\mathrm{bb}}$ *is an* M*-matrix, and* $0 \leqslant (\mathbf{A}^{\mathrm{bb}})^{-1} \leqslant (\tilde{\mathbf{A}}^{\mathrm{bb}})^{-1}$. *If, additionally,* $0 \leqslant \mathbf{B}^{\mathrm{i}} \leqslant \tilde{\mathbf{B}}^{\mathrm{i}}$ *and* $\tilde{\mathbf{A}}$ *is an* M*-matrix, then* $\mathbf{A}$ *is an* M*-matrix, and* $0 \leqslant \mathbf{A}^{-1} \leqslant \tilde{\mathbf{A}}^{-1}$.

3 Algebraic Sturm–Liouville problem

We consider the algebraic Sturm–Liouville Problem (aSLP) for problem (1)–(2):

$$\mathbf{A}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} = \lambda \mathbf{U}^{\mathrm{i}}, \tag{3}$$

$$\mathbf{U}^{\mathrm{b}} = \mathbf{B}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} + \mathbf{B}^{\mathrm{i}}\mathbf{U}^{\mathrm{i}}. \tag{4}$$

Equation (4) is equivalent to

$$\mathbf{A}^{\mathrm{bi}}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{bb}}\mathbf{U}^{\mathrm{b}} = \mathbf{0}. \tag{5}$$

The parameter λ is an eigenvalue of this aSLP if nontrivial solution $\mathbf{U} = (\mathbf{U}^{\mathrm{i}}, \mathbf{U}^{\mathrm{b}})^\top \neq \mathbf{0}$ exists. We rewrite Eq. (3) as

$$\mathbf{A}^{\mathrm{i}}_{\lambda}\mathbf{U}^{\mathrm{i}} = -\mathbf{A}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}}, \quad \mathbf{A}^{\mathrm{i}}_{\lambda} := \mathbf{A}^{\mathrm{i}} - \lambda \mathbf{I}_n.$$

Then there exists the system of m linearly independent solutions $\mathcal{E}_\lambda = \{\mathbf{E}_1(\lambda), \dots, \mathbf{E}_m(\lambda)\}$. If $\det \mathbf{A}^{\mathrm{i}}_{\lambda} \neq 0$, then the system $\mathcal{E}_\lambda$ is the basis of the set of the solutions. Let us define the matrix

$$\mathbf{E}_\lambda = \big(\mathbf{E}_1(\lambda), \dots, \mathbf{E}_m(\lambda)\big) = \begin{pmatrix} E_1^1(\lambda) & \cdots & E_m^1(\lambda) \\ \cdots & \cdots & \cdots \\ E_1^{n+m}(\lambda) & \cdots & E_m^{n+m}(\lambda) \end{pmatrix}.$$

Then we have

$$\mathbf{U} = \big(\mathbf{U}^{\mathrm{i}}, \mathbf{U}^{\mathrm{b}}\big)^\top = \sum_{l=1}^{m} v^l \mathbf{E}_l(\lambda) = \mathbf{E}_\lambda \mathbf{V}, \quad \mathbf{V} := \big(v^1, \dots, v^m\big)^\top, \tag{6}$$

We can rewrite the BCs (5) as

$$\mathbf{B}\mathbf{U} = \mathbf{0}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}^1 \\ \cdots \\ \mathbf{B}^m \end{pmatrix} = \begin{pmatrix} b_1^1 & \cdots & b_{n+m}^1 \\ \cdots & \cdots & \cdots \\ b_1^m & \cdots & b_{n+m}^m \end{pmatrix} := \big(\mathbf{A}^{\mathrm{bi}}, \mathbf{A}^{\mathrm{bb}}\big), \tag{7}$$

where $\mathbf{B}^l = (b_1^l, \dots, b_{n+m}^l) \in \mathbb{R}^{1\times(n+m)}$, $l = 1, \dots, m$.

If $\det \mathbf{A}^{\mathrm{bb}} \neq 0$, then this aSLP is equivalent to the problem

$$\mathbf{A}^{\mathrm{i}}_{\lambda}\mathbf{U}^{\mathrm{i}} + \mathbf{A}^{\mathrm{b}}\mathbf{U}^{\mathrm{b}} = \mathbf{0},$$

$$\mathbf{A}^{\mathrm{n}}\mathbf{U}^{\mathrm{i}} + \mathbf{U}^{\mathrm{b}} = \mathbf{0}, \quad \mathbf{A}^{\mathrm{n}} = \big(\mathbf{A}^{\mathrm{bb}}\big)^{-1}\mathbf{A}^{\mathrm{bi}}.$$

Eliminating $\mathbf{U}^{\mathrm{b}}$ gives

$$\mathbf{A}_\lambda \mathbf{U}^{\mathrm{i}} = \mathbf{0}, \quad \mathbf{A}_\lambda := \mathbf{A}^{\mathrm{i}}_{\lambda} - \mathbf{C}, \quad \mathbf{C} = \mathbf{A}^{\mathrm{b}}\big(\mathbf{A}^{\mathrm{bb}}\big)^{-1}\mathbf{A}^{\mathrm{bi}},$$

i.e., an Algebraic Eigenvalue Problem (AEP)

$$\mathbf{A}\mathbf{U}^{\mathrm{i}} = \lambda \mathbf{U}^{\mathrm{i}}, \quad \mathbf{A} := \mathbf{A}^{\mathrm{i}} - \mathbf{C} \in \mathbb{R}^{n\times n}. \tag{8}$$

This proves the following lemma.

Lemma 3. *If* $\det \mathbf{A}^{\mathrm{bb}} \neq 0$*, then* λ *is the eigenvalue of the algebraic Sturm–Liouville problem* (3)–(4) *if and only if* λ *is the eigenvalue of the algebraic eigenvalue problem* (8).

Nontrivial solutions of AEP exist if $\det \mathbf{A}_\lambda = 0$. The roots of the latter equation are the eigenvalues λ_k, $k = 1, \dots, n$, of both the AEP and the SLP. Therefore, for each λ_k, we obtain the homogeneous linear equation

$$\mathbf{A}\mathbf{U}^{\mathrm{i}} = \lambda_k \mathbf{U}^{\mathrm{i}}.$$

A nontrivial solution of this equation is the eigenvector $\mathbf{U}^{k,\mathrm{i}}$ for problem (8). Therefore, the discrete SLP has the same eigenvalue λ_k as the AEP. The corresponding eigenvector is $\mathbf{U}^k = (\mathbf{U}^{k,\mathrm{i}}, -\mathbf{A}^{\mathrm{n}}\mathbf{U}^{k,\mathrm{i}})^\top$.

If system $\mathcal{E}_\lambda$ is known, then the BCs (7) can be used to find the eigenvalues and eigenvectors. Substituting expression (6) into (7) yields the homogeneous linear system

$$\mathbf{H}_\lambda \mathbf{V} = \mathbf{0}, \quad \mathbf{H}_\lambda := \mathbf{B}\mathbf{E}_\lambda \in \mathbb{R}^{m\times m}. \tag{9}$$

If $\det \mathbf{H}_\lambda = 0$, then a nontrivial solution of (9) exists, and the set of roots of the characteristic equation (CE) forms the spectrum of the aSLP. In the case λ_k belongs to the aSLP spectrum, then we can find nontrivial solutions $\mathbf{V}^k$ of the equation $\mathbf{H}_{\lambda_k}\mathbf{V} = \mathbf{0}$.

Consider the matrices $\mathbf{H} := \mathbf{H}_{\lambda_k}$, $\mathbf{E} := \mathbf{E}_{\lambda_k}$ and the vectors $\mathbf{E}_l := \mathbf{E}_l(\lambda_k)$, where $E^i_j := E^i_j(\lambda_k)$. If $\operatorname{rank} \mathbf{H} = r < m$, then renumbering the rows and columns of matrix $\mathbf{H}$ yields

$$\mathbf{H} = \begin{pmatrix} h^1_1 & \cdots & h^1_r & h^1_{r+1} & \cdots & h^1_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h^r_1 & \cdots & h^r_r & h^r_{r+1} & \cdots & h^r_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h^m_1 & \cdots & h^m_r & h^m_{r+1} & \cdots & h^m_m \end{pmatrix}, \qquad \begin{vmatrix} h^1_1 & \cdots & h^1_r \\ \cdots & \cdots & \cdots \\ h^r_1 & \cdots & h^r_r \end{vmatrix} \neq 0,$$

where $h^i_j = \mathbf{B}^i\mathbf{E}_j = \sum_{l=1}^{n+m} b^i_l E^l_j$. Let us denote

$$\mathbf{H}^{\mathrm{r}} = \begin{pmatrix} h^1_1 & \cdots & h^1_r \\ \cdots & \cdots & \cdots \\ h^r_1 & \cdots & h^r_r \end{pmatrix}, \qquad \mathbf{H}^{\mathrm{p}} = \begin{pmatrix} h^1_{r+1} & \cdots & h^1_m \\ \cdots & \cdots & \cdots \\ h^r_{r+1} & \cdots & h^r_m \end{pmatrix},$$

$$\widetilde{\mathbf{H}} = \left(\mathbf{H}^{\mathrm{r}}\right)^{-1}\mathbf{H}^{\mathrm{p}} = \begin{pmatrix} \tilde{h}^1_{r+1} & \cdots & \tilde{h}^1_m \\ \cdots & \cdots & \cdots \\ \tilde{h}^r_{r+1} & \cdots & \tilde{h}^r_m \end{pmatrix},$$

$$\mathbf{V}^{\mathrm{r}} = \left(v^1, \dots, v^r\right), \qquad \mathbf{V}^{\mathrm{p}} = \left(v^{r+1}, \dots, v^m\right),$$
$$\mathbf{E}^{\mathrm{r}} = (\mathbf{E}_1, \dots, \mathbf{E}_r), \qquad \mathbf{E}^{\mathrm{p}} = (\mathbf{E}_{r+1}, \dots, \mathbf{E}_m).$$

The first r equations of the system $\mathbf{H}\mathbf{V} = \mathbf{0}$ give $\mathbf{V}^{\mathrm{r}} = -\widetilde{\mathbf{H}}\mathbf{V}^{\mathrm{p}}$ and

$$\mathbf{U} = \mathbf{E}\mathbf{V} = \mathbf{E}^{\mathrm{r}}\mathbf{V}^{\mathrm{r}} + \mathbf{E}^{\mathrm{p}}\mathbf{V}^{\mathrm{p}} = \left(\mathbf{E}^{\mathrm{p}} - \mathbf{E}^{\mathrm{r}}\widetilde{\mathbf{H}}\right)\mathbf{V}^{\mathrm{p}} = \sum_{l=r+1}^{m} v^l \left(\mathbf{E}_l - \sum_{s=1}^{r} \mathbf{E}_s \tilde{h}^s_l\right).$$

Thus, an eigenvalue $\lambda = \lambda_k$ has $m - r$ eigenvectors (geometric multiplicity $\gamma_{\mathbf{A}}(\lambda_k) = m - r$): $\widetilde{\mathbf{E}}_l = \mathbf{E}_{r+l} - \sum_{s=1}^{r} \mathbf{E}_s \tilde{h}^s_{r+l}$, $l = 1, \dots, m - r$. If the algebraic multiplicity $\mu_{\mathbf{A}}(\lambda_k) > m - r$, then generalized eigenvectors exist.

Example 1 [$m = 2$]. For $m = 2$, we have $\mathcal{E} = \{\mathbf{E}_1(\lambda_k), \mathbf{E}_2(\lambda_k)\}$. If $\operatorname{rank} \mathbf{H}_{\lambda_k} = r = 0$, then there are two linearly independent eigenvectors $\mathbf{U}_1 = \mathbf{E}_1(\lambda_k)$ and $\mathbf{U}_2 = \mathbf{E}_2(\lambda_k)$ for the eigenvalue λ_k.

Suppose $\operatorname{rank} \mathbf{H}_{\lambda_k} = r = 1$. Let us denote [36, 52]

$$\mathbf{D}[\mathcal{E}] = \bigl(D[\mathcal{E}]^{ij}\bigr) = \bigl(\mathbf{D}[\mathcal{E}]^1, \dots, \mathbf{D}[\mathcal{E}]^{n+2}\bigr),$$

where

$$D[\mathcal{E}]^{ij} := \begin{vmatrix} E_1^i & E_2^i \\ E_1^j & E_2^j \end{vmatrix} = \begin{vmatrix} E_1^i & E_1^j \\ E_2^i & E_2^j \end{vmatrix}, \qquad \mathbf{D}[\mathcal{E}]^j = \bigl(D[\mathcal{E}]^{1j}, \dots, D[\mathcal{E}]^{n+2,j}\bigr)^\top.$$

In the case $r = 1$ ($h_1^1 \neq 0$), the eigenvalue $\lambda = \lambda_k$ has single eigenvector $\mathbf{U} \in \mathbb{R}^{(n+2)\times 1}$:

$$\begin{aligned} U^j &= -h_1^1\bigl(E_2^j - E_1^j \tilde{h}_2^1\bigr) = h_2^1 E_1^j - h_1^1 E_2^j = -\sum_{i=1}^{n+2} b_i^1 \bigl(E_1^i E_2^j - E_2^i E_1^j\bigr) \\ &= -\sum_{i=1}^{n+2} b_i^1 D[\mathcal{E}]^{ij} = -\mathbf{B}^1 \mathbf{D}[\mathcal{E}]^j. \end{aligned}$$

The matrix $\mathbf{D}[\mathcal{E}]$ is skew-symmetric. Therefore,

$$\mathbf{U} = \mathbf{D}[\mathcal{E}]\mathbf{B}_1, \quad \mathbf{B}_1 = \bigl(\mathbf{B}^1\bigr)^\top. \tag{10}$$

In the case $h_2^1 \neq 0$, equality (10) holds, too. If $h_2^1 = h_2^1 = 0$, then the eigenvector $\mathbf{U} = \mathbf{D}[\mathcal{E}]\mathbf{B}_2$, where $\mathbf{B}_2 = (\mathbf{B}^2)^\top$.

Corollary 1. *In the case $m = 2$, the eigenvalue $\lambda = \lambda_k$ has the following eigenvectors*:

$$\mathbf{U}_1 \sim \mathbf{E}_1, \quad \mathbf{U}_2 \sim \mathbf{E}_2 \quad \textit{for } \operatorname{rank} \mathbf{H}_{\lambda_k} = 0, \tag{11}$$

$$\mathbf{U} \sim \mathbf{D}[\mathcal{E}]\mathbf{B}_1 \sim \mathbf{D}[\mathcal{E}]\mathbf{B}_2 \quad \textit{for } \operatorname{rank} \mathbf{H}_{\lambda_k} = 1. \tag{12}$$

4 Finite difference method for Sturm–Liouville problem

Let us consider one-dimensional SLP with two additional (boundary) conditions

$$-u'' = \lambda u, \quad x \in \varOmega = [0, l], \tag{13}$$

$$b^1(u) = 0, \qquad b^2(u) = 0, \tag{14}$$

where equalities b^1 and b^2 describe additional (boundary) conditions.

In the domain $\varOmega = [0, l]$, we introduce the grid $\overline{\omega}_t^h := \{t_i\colon t_i = i \cdot h,\ i = 0, \dots, N\}$ and a subgrid $\omega_t^h := \{t_i,\ i = 1, \dots, N-1\}$ with uniform steps $h = l/N$, $0 < N \in \mathbb{N}$. We can extend the grid $\overline{\omega}_t^h$ to the grid $\varOmega^h := \{t_j,\ i = -1, 0, \dots, N, N+1\}$ or to the grid $\mathbb{Z}^h := \{t_j\colon t_j = ih,\ i \in \mathbb{Z}\}$, which will be used for periodic problems. We use the standard ordering of the nodes of the grid $\mathbb{Z}^h$ ($\varOmega^h$), i.e., $t_i < t_{i+1}$.

We will use the trapezoidal approximation for the integrals in the BCs

$$[V,U] := (V_0U_0 + V_NU_N)\frac{h}{2} + \sum_{i=1}^{N-1} V_iU_ih \approx \int_0^l v(x)u(x)\,\mathrm{d}x, \quad U,V \in \mathbb{R}^{\overline{\omega}_t^h},$$

and the grid operators δ^2, $\bar{\delta}$, $\bar{s}$ for the grid functions

$$\delta^2 U_i := \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}, \quad \bar{\delta} U_i := \frac{U_{i+1} - U_{i-1}}{2h},$$
$$\bar{s} U_i := \frac{U_{i+1} + U_{i+1}}{2}, \quad t_i \in \mathbb{Z}^h.$$

These two operators approximate $u''(t)$ and $u'(t)$ with the order of $\mathcal{O}(h^2)$.

Later in the paper, we will use the following equalities useful to find generalized eigenfunctions:

$$\delta^2(tU)_i = t_i\delta^2(U)_i + 2\bar{\delta}U_i, \tag{15}$$

$$\bar{\delta}(tU)_i = t_i\bar{\delta}(U)_i + \bar{s}U_i. \tag{16}$$

For Neumann or Robin BCs, ghost boundary nodes t_{-1}, $t_{N+1} \in \partial\omega_g$ are used [1, Sect. 3.1.3], [23, Sect. 2.7]. Using the ghost node t_{-1} makes t_0 an inner node, and using the ghost node t_{N+1} makes t_N an inner node. There are four possible one-dimensional grids in the domain Ω (see Fig. 1(a)):

(i) $\omega = \omega_t^h$, $|\omega| = N - 1$, $\partial\widetilde{\omega} = \{t_0, t_N\}$, $\partial_g\omega = \emptyset$;

(ii) $\omega = \omega_t^h + \{t_N\}$, $|\omega| = N$, $\partial\widetilde{\omega} = \{t_0, t_{N+1}\}$, $\partial_g\omega = \{t_{N+1}\}$;

(iii) $\omega = \{t_0\} + \omega_t^h$, $|\omega| = N$, $\partial\widetilde{\omega} = \{t_{-1}, t_N\}$, $\partial_g\omega = \{t_{-1}\}$;

(iv) $\omega = \overline{\omega}_t^h$, $|\omega| = N + 1$, $\partial\widetilde{\omega} = \{t_{-1}, t_{N+1}\}$, $\partial_g\omega = \{t_{-1}, t_{N+1}\}$.

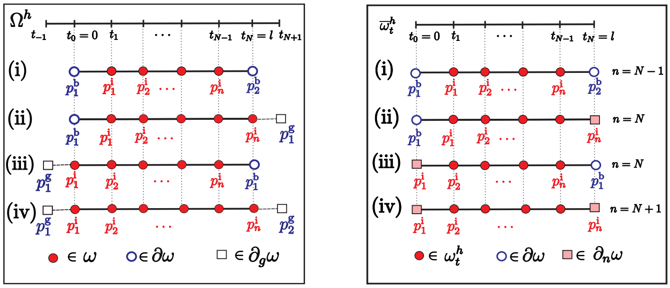
We now define $n = |\omega|$, $\partial_n\omega = \omega \setminus \omega_t^h$, $\partial\omega = \partial\widetilde{\omega} \setminus \partial_g\omega = \overline{\omega}_t^h \setminus \omega$, $\overline{\omega} = \omega + \partial\omega = \overline{\omega}_t^h$, $\widetilde{\omega} = \omega + \partial\widetilde{\omega}$. Thus, using the grid $\widetilde{\omega}$ (with ghost nodes), cases (2)–(4) have two boundary points and become similar to case (1).

We consider one-dimensional dSLP

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega, \tag{17}$$

$$\begin{aligned}\widetilde{B}^l(U) &:= \sum_{t_i\in\omega_t^h} \tilde{b}_i^l U_i + \sum_{t_i\in\partial_n\omega} \tilde{b}_i^l U_i + \sum_{t_i\in\partial\omega} \bar{b}_i^l U_i + \sum_{t_i\in\partial_g\omega} \tilde{b}_i^l U_i \\ &= \sum_{t_i\in\omega} \tilde{b}_i^l U_i + \sum_{t_i\in\partial\widetilde{\omega}} \tilde{b}_i^l U_i = \sum_{t_i\in\widetilde{\omega}} \tilde{b}_i^l U_i = 0, \quad l = 1, 2,\end{aligned} \tag{18}$$

on the grid $\widetilde{\omega}$.



(a) with the ghost nodes (b) without the ghost nodes

Figure 1. One-dimensional grids ω, $\partial\omega$, $\partial_g\omega$, ω_t^h, $\partial_n\omega$; □ – ghost nodes, ■ – nodes for the natural derivative, ○ – nodes not used by the natural derivative.

Remark 1. If eigenvalues λ_k and eigenfunctions $\mathbf{U}^k(t_j) = \mathbf{U}^k(j/N)$, $t_j = hj$, $h = N^{-1}$, for dSLP in the case $\Omega_1 = [0,1]$ $(l = 1)$ are found, then for $\Omega = [0,l]$, $\tau_j = h_l j$, $h_l = lN^{-1}$, the eigenvalues and eigenfunctions are as follows:

$$\mu_k = h^2 {h_l}^{-2}\lambda_k = l^{-2}\lambda_k, \qquad \mathbf{U}_l^k(\tau_j) = \mathbf{U}^k\left(\frac{j}{N}\right) = \mathbf{U}^k\left(\frac{\tau_j}{l}\right).$$

As a result, only the case of $l = 1$ with $h = N^{-1}$ can be considered.

Equations (17) can be written in the matrix form (3), where

$$\mathbf{A}^{\mathrm{i}} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}, \qquad \mathbf{A}^{\mathrm{b}} = -\frac{1}{h^2}\tilde{\mathbf{J}}, \quad \tilde{\mathbf{J}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{n\times 2}.$$

5 Discrete Sturm–Liouville equation and Chebyshev polynomials

The discrete Sturm–Liouville Equation (dSLE)

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \mathbb{Z}^h,$$

can be written in the form [8]

$$U_{j+1} - 2zU_j + U_{j-1} = 0, \quad z = 1 - \frac{\lambda h^2}{2}. \tag{19}$$

Equation (19) for $j \in \mathbb{Z}$ and its solutions have been studied for $z \in \mathbb{R}$ [38]. The general solution of this equation is given by

$$U_j = c^1 T_j(z) + c^2 \widetilde{T}_{j-1}(z)h, \quad j \in \mathbb{Z},$$

where

$$T_j(z) = \frac{(z + \sqrt{z^2-1})^j + (z - \sqrt{z^2-1})^j}{2}, \quad j \in \mathbb{Z},$$

are the Chebyshev polynomials of the first kind of degree j,

$$\widetilde{T}_j(z) = \frac{(z+\sqrt{z^2-1})^{j+1} - (z-\sqrt{z^2-1})^{j+1}}{2\sqrt{z^2-1}}, \quad j \in \mathbb{Z},$$

are the Chebyshev polynomial of the second kind of degree j in z. The Chebyshev polynomials can be further extended to (or initially defined as) polynomials of a complex variable z [26,46]. These Chebyshev polynomials are solutions to two Cauchy problems:

$$T_{j+1} - 2zT_j + T_{j-1} = 0, \qquad T_0 = 1, \qquad T_1 = z, \tag{20}$$

$$\widetilde{T}_{j+1} - 2z\widetilde{T}_j + \widetilde{T}_{j-1} = 0, \qquad \widetilde{T}_{-1} = 0, \qquad \widetilde{T}_0 = 1. \tag{21}$$

The recurrence equations (20)–(21) allow us to find the Chebyshev polynomials $T_j(z)$ and $\widetilde{T}_{j-1}(z)$ for all $j \in \mathbb{Z}$.

Chebyshev polynomials (T_n, $\widetilde{T}_n$, $n > 0$) can be represented explicitly as the corresponding determinants of the tridiagonal matrices of size $n \times n$ [26]:

$$T_n(z) := \begin{vmatrix} 2z & -1 & & & \\ -1 & 2z & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2z & -1 \\ & & & -1 & z \end{vmatrix}, \qquad \widetilde{T}_n(z) := \begin{vmatrix} 2z & -1 & & & \\ -1 & 2z & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2z & -1 \\ & & & -1 & 2z \end{vmatrix}. \tag{22}$$

Note that the corresponding matrices are M-matrices for $z > 0$.

The roots of T_j are $z_{j,k} = \cos(\pi(k+1/2)/j)$, $k = 0, \dots, j-1$, the roots of $\widetilde{T}_j$ are $\tilde{z}_{j,k} = \cos(\pi k/(j+1))$, $k = 1, \dots, j$. Thus, all roots of the Chebyshev polynomials belong to $(-1, 1)$. If $\deg \widetilde{T}_k > \deg \widetilde{T}_l$ (or $\deg T_k > \deg T_l$), then the similar innequality holds for the largest root, i.e., $\tilde{z}_{k,1} > \tilde{z}_{j,1}$ (or $z_{k,0} > z_{j,0}$).

We define the function $D[\mathcal{T}]_{ij}(z)$ [36]:

$$D[\mathcal{T}]_{ij}(z) := h\begin{vmatrix} T_i(z) & T_j(z) \\ \widetilde{T}_{i-1}(z) & \widetilde{T}_{j-1}(z) \end{vmatrix} = \begin{vmatrix} T_i(z) & T_j(z) \\ h\widetilde{T}_{i-1}(z) & h\widetilde{T}_{j-1}(z) \end{vmatrix}$$
$$= h\bigl(T_i(z)\widetilde{T}_{j-1}(z) - T_j(z)\widetilde{T}_{i-1}(z)\bigr) = -h\widetilde{T}_{i-j-1}(z).$$

Note that $T_j(1) = 1$, $h\widetilde{T}_j(1) = t_{j+1}$, $T_j(-1) = (-1)^j$, $h\widetilde{T}_j(1) = (-1)^j t_{j+1}$. Thus,

$$D[\mathcal{T}]_{ij}(1) = t_{j-i}, \qquad D[\mathcal{T}]_{ij}(-1) = (-1)^i(-1)^j t_{i-j}.$$

Lemma 4. *The following property of Chebyshev polynomials is true*:

$$\bar{\delta}\bigl(h\widetilde{T}_{j-1}(z)\bigr) = T_j(z), \quad t_j \in \mathbb{Z}^h.$$

Proof. According to [26, Sect. 1.2.2],

$$\bar{\delta}\widetilde{T}_{j-1}(z) = \frac{h^{-1}(\widetilde{T}_j(z) - \widetilde{T}_{j-2}(z))}{2} = h^{-1}T_j(z). \qquad \square$$

The values of the Chebyshev polynomials $T_j = \bar\delta(h\widetilde T_{j-1})$ and $h\widetilde T_{j-1}$ form a fundamental system of solutions $\mathcal{T}_z = \{\mathbf{T}(z), \widetilde{\mathbf{T}}(z)\}$ of the homogeneous equation (19), where $\mathbf{T}(z) := \{T_j(z),\, j \in \mathbb{Z}\}$, $\widetilde{\mathbf{T}}(z) := \{h\widetilde T_{j-1}(z),\, j \in \mathbb{Z}\}$.

Remark 2. As multiplier $h \to 0$, the fundamental system of solutions of the difference equation converges to the fundamental system of solutions of the differential equation $-u'' = \lambda u$ [8]:

$$\begin{aligned}\bigl\{T_j(z), h\widetilde T_{j-1}(z)\bigr\} &= \bigl\{\bar\delta\bigl(h\widetilde T_{j-1}(z)\bigr), h\widetilde T_{j-1}(z)\bigr\} \to \left\{\cos(\pi q t), \frac{\sin(\pi q t)}{\pi q}\right\}\\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sin(\pi q t)}{\pi q}, \frac{\sin(\pi q t)}{\pi q}\right\}, \quad \lambda = (\pi q)^2.\end{aligned}$$

The general solution of difference equation (19) can be written in the matrix form:

$$\mathbf{U} = v^1\mathbf{T}(z) + v^2\widetilde{\mathbf{T}}(z) = \mathbf{T}_z\mathbf{V}, \quad \mathbf{T}_z = \bigl(\mathbf{T}(z), \widetilde{\mathbf{T}}(z)\bigr), \quad \mathbf{V} = \bigl(v^1, v^2\bigr)^\top, \tag{23}$$

where v^1, v^2 are two arbitrary constants. We define

$$\mathbf{E}_z = \bigl(\mathbf{T}_z^{\mathrm{i}}, \mathbf{T}_z^{\mathrm{b}}\bigr)^\top = \begin{pmatrix} T_1(z) & \cdots & T_n(z) & T_0(z) & T_N(z)\\ h\widetilde T_0(z) & \cdots & h\widetilde T_{n-1}(z) & h\widetilde T_{-1}(z) & h\widetilde T_{N-1}(z)\end{pmatrix}^\top \in \mathbb{R}^{(n+2)\times 2}.$$

Equation (9) becomes $\mathbf{M}_z\mathbf{V} = \mathbf{0}$, where $\mathbf{M}_z = \mathbf{B}\mathbf{E}_z = \mathbf{A}^{\mathrm{bb}}\mathbf{T}_z^{\mathrm{b}} + \mathbf{A}^{\mathrm{bi}}\mathbf{T}_z^{\mathrm{i}}$. A nontrivial solution of this equation exists if $\det \mathbf{M}_z = 0$.

5.1 Domain $\mathbb{C}_q^h$

Since $\lambda = 2h^{-2}(1 - z)$, then in the case $z = \cos(\pi q h)$, the formula

$$\lambda = \lambda^h(q) = \frac{4}{h^2}\sin^2\frac{\pi q h}{2} = \frac{2}{h^2}\bigl(1 - \cos(\pi q h)\bigr), \quad q \in \mathbb{C}_q^h,$$

defines bijection $\lambda^h\colon \mathbb{C}_q^h \to \mathbb{C}_\lambda = \mathbb{C} = \{\lambda = \operatorname{Re}\lambda + \operatorname{Im}\lambda\imath\}$ (see Fig. 2), where $\mathbb{C}_q^h := \{q = x + y\imath\colon 0 \leqslant x = \operatorname{Re} q \leqslant N,\ y = \operatorname{Im} q \in \mathbb{R};\ \text{if } x = 0 \text{ or } x = N, \text{ then } y \geqslant 0\}$ [6,8].

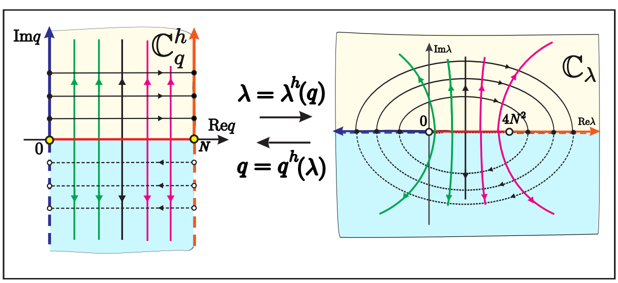


Figure 2. Domain $\mathbb{C}_q^h$; ○ – ramification points $q = 0, N$, ○ – branch points $\lambda = 0, 4N^2$.

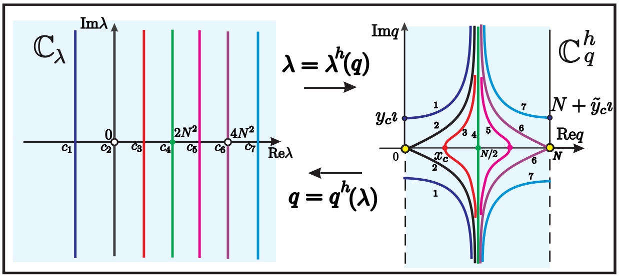


Figure 3. Images of lines $\operatorname{Re}\lambda = c_k$, $k = 1, \dots, 7$, in $\mathbb{C}_q^h$: $c_1 < 0$, $c_2 = 0$, $0 < c_3 < 2N^2$, $c_4 = 2N^2$, $2N^2 < c_5 < 4N^2$, $c_6 = 4N^2$, $c_7 > 4N^2$.

We use notation $\mathbb{R}_q^h := \{q = \imath y\colon y > 0\} \cup \{q = x\colon x \in [0, N]\} \cup \{q = N + \imath y\colon y > 0\}$. Note, if $q \in \mathbb{R}_q^h$, then $\lambda \in \mathbb{R}$. The points $\lambda = 0$ and $\lambda = 4N^2$ are the second-order Branch Points (BP) of the mapping $\lambda = \lambda^h(q)$.

Remark 3. Suppose, $b = \lambda^h(a)$, $a = 0, N$, is the second-order BP of the mapping $\lambda = \lambda^h(q)$. Consider two functions $f(\lambda)$ and $g(q) = f(\lambda^h(q))$. If function $f(\lambda) = f(b) + f'(b)(\lambda - b) + \mathcal{O}((\lambda - b)^2)$, then $g(q) = f(b) - (-1)^{\lfloor a/N \rfloor}\pi^2 f'(b)(q-a)^2 + \mathcal{O}((q-a)^4)$. Thus, if the function f has simple zero at the point $\lambda = b$, then function g has double zero at the point $q = a$ (if the function f has a zero of order k at the point $\lambda = b$, then the function g has zero of order $2k$ at the point $q = a$, $k \in \mathbb{N}$).

Horizontal lines $y = \operatorname{Im} q = b$ are mapped to ellipses (see Fig. 2)

$$\frac{(1 - \operatorname{Re}\lambda \cdot h^2/2)^2}{\cosh^2(\pi bh)} + \frac{(\operatorname{Im}\lambda \cdot h^2/2)^2}{\sinh^2(\pi bh)} = 1.$$

If $b = 0$, the ellipse becomes a segment $[0, 4N^2]$. Vertical lines $x = \operatorname{Re} q = a$ are mapped to hyperbolas

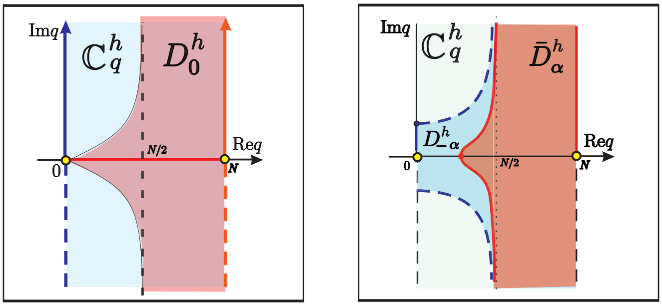
$$\frac{(1 - \operatorname{Re}\lambda \cdot h^2/2)^2}{\cos^2(\pi ah)} - \frac{(\operatorname{Im}\lambda \cdot h^2/2)^2}{\sin^2(\pi ah)} = 1.$$

If $a = 0$ ($y \in [0, +\infty)$), a branch of the hyperbola becomes a segment $(-\infty, 0]$; if $a = N$ ($y \in [0, +\infty)$), a branch of the hyperbola becomes asegment $[4N^2, +\infty)$; if $a = N/2$, the whole hyperbola becomes a line $\operatorname{Re}\lambda = 2N^2$.

Now consider the inverse mapping $q = q^h(\lambda)$ and the vertical lines $\operatorname{Re}\lambda = c$. Images of these lines are shown in Fig. 3 and are defined by the equation

$$x = \varphi_c(y) = \frac{\arccos((1 - h^2/2c)/\cosh(\pi yh))}{\pi/h}.$$

The function φ_c is even and is monotonic on the intervals $(-\infty, 0)$, $(0, +\infty)$. If $c \leqslant 0$ (or $c \geqslant 4N^2$), then there exists $y_c \geqslant 0$ ($\tilde{y}_c \geqslant 0$) such that $\varphi_c(y_c) = 0$ ($\varphi_c(\tilde{y}_c) = N$), if $0 \leqslant c \leqslant 4N^2$, then there exists $x_c = \varphi_c(0) \in [0, N]$. The point $y_c\imath$ corresponds



(a) D_0^h (b) $D_{-\alpha}^h$, $\bar{D}_\alpha^h$, $\alpha > 0$

Figure 4. Domains D_0^h, $D_{-\alpha}^h$, $\bar{D}_\alpha^h$.

to the eigenvalue $\lambda = c \leqslant 0$, the point $N + \tilde{y}_c\imath$ corresponds to the positive eigenvalue $\lambda = c \geqslant 4N^2$, the point x_c corresponds to the eigenvalue $\lambda = c \in [0, 4N^2]$.

If $\lambda \in \mathbb{C}_\lambda$ is an eigenvalue of dSLP (17)–(18), then the corresponding value of $q \in \mathbb{C}_q^h$ is called an *Eigenvalue Point* (EP).

For M-matrices, all EPs belong to $D_0^h := \{q \in \mathbb{C}_q^h\colon\ \operatorname{Re}\lambda > 0\}$ (see Fig. 4(a)). Additionally, for $\alpha > 0$, we define two domains $\bar{D}_\alpha^h := \{q \in \mathbb{C}_q^h\colon\ \operatorname{Re}\lambda \geqslant \alpha\}$, $D_{-\alpha}^h := \{q \in \mathbb{C}_q^h\colon\ \operatorname{Re}\lambda > -\alpha\}$ (see Fig. 4(b), the case $\alpha < 2N^2$).

We denoted by $\sigma_q^h(\mathbf{A})$ the image of the spectrum of the matrix $\mathbf{A}$ in $\mathbb{C}_q^h$, that is, $\lambda^h(\sigma_q^h(\mathbf{A})) = \sigma(\mathbf{A})$.

Lemma 5. *A matrix* $\mathbf{A}$ *spectrum* $\sigma(\mathbf{A}) \subset \mathbb{C}^+$ *if and only if* $\sigma_q^h(\mathbf{A}) \subset D_0^h$. *If, in addition,* $\mathbf{A} \in \mathcal{Z}_n$, *then* $\mathbf{A}$ *is an* M*-matrix.*

Proof. It follows from Lemma 1(ii). □

Lemma 6. *If* $\sigma_q^{h_1}(\mathbf{A}_1) \subset \bar{D}_\alpha^{h_1} \subset \mathbb{C}_q^{h_1}$ *and* $\sigma_q^{h_2}(\mathbf{A}_2) \subset D_{-\alpha}^{h_2} \subset \mathbb{C}_q^{h_2}$, *then* $\sigma(\mathbf{A}_1) + \sigma(\mathbf{A}_2) \subset \mathbb{C}^+$.

Proof. If $q_1 \in \bar{D}_\alpha^{h_1}$, $q_2 \in D_{-\alpha}^{h_2}$, then $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda(q_1) \geqslant \alpha \geqslant 0$, $\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda(q_2) > -\alpha$. Therefore, $\operatorname{Re}(\lambda_1 + \lambda_2) > 0$. □

We will rewrite dSLE (19) using the parameter $q \in \mathbb{C}_q^h$ instead of $\lambda \in \mathbb{C}_\lambda$. Since $z = \cos(\pi q h)$, Eq. (19) becomes [8]

$$U_{j-1} - 2\cos(\pi q h)U_j + U_{j+1} = 0, \quad t_j \in \mathbb{Z}^h,\ q \in \mathbb{C}_q^h. \tag{24}$$

We have a new basis function instead of Chebyshev polynomials:

$$E_j(q) = T_j\big(z(q)\big) = \cos(\pi q t_j),$$
$$\widetilde{E}_j(q) = h\widetilde{T}_{j-1}\big(z(q)\big) = h\frac{\sin(\pi q t_j)}{\sin(\pi q h)}, \quad t_j \in \mathbb{Z}^h,$$

and $\mathcal{E}_q = \{\mathbf{E}(q), \widetilde{\mathbf{E}}(q)\}$ is the fundamental system of solutions of homogeneous equation (24), where $\mathbf{E}(q) = (E_j(q),\, t_j \in \mathbb{Z}^h)$, $\widetilde{\mathbf{E}}(q) = (\widetilde{E}_j(q),\, t_j \in \mathbb{Z}^h)$. Therefore,

$$D[\mathcal{E}]_{ij}(q) = -h\frac{\sin\left(\pi q(t_i - t_j)\right)}{\sin(\pi q h)}, \quad t_i, t_j \in \mathbb{Z}^h. \tag{25}$$

Lemma 7. *The following equalities hold*:

$$\bar\delta E_i(q) = -h^{-2}\sin^2(\pi q h)\widetilde{E}_i(q), \quad \bar\delta \widetilde{E}_i(q) = E_i(q), \quad t_i \in \mathbb{Z}^h, \tag{26}$$

$$\delta^2 E_i(q) = -\lambda E_i(q), \quad \delta^2 \widetilde{E}_i(q) = -\lambda \widetilde{E}_i(q), \quad t_i \in \mathbb{Z}^h. \tag{27}$$

Proof. Using definitions of the grid operators, we get:

$$\begin{aligned}
\bar\delta E_i(q) &= \frac{E_{i+1}(q) - E_{i-1}(q)}{2h} = \frac{\cos(\pi q(t_i + h)) - \cos(\pi q(t_i - h))}{2h} \\
&= -\frac{\sin(\pi q t_i)\sin(\pi q h)}{h}, \\
\bar\delta \widetilde{E}_i(q) &= \frac{\widetilde{E}_{i+1}(q) - \widetilde{E}_{i-1}(q)}{2h} = \frac{\sin(\pi q(t_i + h)) - \sin(\pi q(t_i - h))}{2\sin(\pi q h)} \\
&= \cos(\pi q t_i), \\
\delta^2 E_i(q) &= \frac{\cos(\pi q(t_i + h)) + \cos(\pi q(t_i - h)) - 2\cos(\pi q t_i)}{h^2} \\
&= -\frac{2 - 2\cos(\pi q h)}{h^2}\cos(\pi q t_i), \\
\delta^2 \widetilde{E}_i(q) &= \frac{\sin(\pi q(t_i + h)) + \sin(\pi q(t_i - h)) - 2\sin(\pi q t_i)}{h\sin(\pi q h)} \\
&= -\frac{2 - 2\cos(\pi q h)}{h}\frac{\sin(\pi q t_i)}{\sin(\pi q h)}.
\end{aligned}$$

□

Corollary 2. *At the nodes $t_0 = 0$ and $t_N = 1$ the following is true*:

$$E_0 = 1, \quad E_N = \cos(\pi q), \qquad \bar\delta E_0 = 0, \quad \bar\delta E_N = -\frac{\sin(\pi q)\sin(\pi q h)}{h}, \tag{28}$$

$$\widetilde{E}_0 = 0, \quad \widetilde{E}_N = \frac{h\sin(\pi q)}{\sin(\pi q h)}, \qquad \bar\delta \widetilde{E}_0 = 1, \quad \bar\delta \widetilde{E}_N = \cos(\pi q). \tag{29}$$

The general solution of difference equation (24) is given by [8]

$$\mathbf{U} = v^1\mathbf{E}(q) + v^2\widetilde{\mathbf{E}}(q) = \mathbf{E}_q\mathbf{V}, \quad \mathbf{E}_q = \left(\mathbf{E}(q), \widetilde{\mathbf{E}}(q)\right), \quad \mathbf{V} = \left(v^1, v^2\right)^\top, \tag{30}$$

and

$$\mathbf{E}_q = \left(\mathbf{E}_q^{\mathrm{i}}, \mathbf{E}_q^{\mathrm{b}}\right)^\top = \begin{pmatrix} E_1(q) & \cdots & E_n(q) & E_0(q) & E_N(q) \\ \widetilde{E}_1(q) & \cdots & \widetilde{E}_n(q) & \widetilde{E}_0(q) & \widetilde{E}_N(q) \end{pmatrix}^\top \in \mathbb{R}^{(n+2)\times 2},$$

$$\mathbf{E}_q^{\mathrm{i}} = \begin{pmatrix} \cos(\pi q t_1) & \cdots & \cos(\pi q t_n) \\ h\frac{\sin(\pi q t_1)}{\sin(\pi q h)} & \cdots & h\frac{\sin(\pi q t_n)}{\sin(\pi q h)} \end{pmatrix}^\top, \qquad \mathbf{E}_q^{\mathrm{b}} = \begin{pmatrix} 1 & \cos(\pi q) \\ 0 & h\frac{\sin(\pi q)}{\sin(\pi q h)} \end{pmatrix}^\top.$$

We rewrite Eq. (9) as

$$\mathbf{M}_q\mathbf{V} = \mathbf{0}, \quad \mathbf{M}_q = \mathbf{B}\mathbf{E}_q = \mathbf{A}^{\mathrm{bb}}\mathbf{E}_q^{\mathrm{b}} + \mathbf{A}^{\mathrm{bi}}\mathbf{E}_q^{\mathrm{i}}.$$

Nontrivial solution of this equation exists if $\det \mathbf{M}_q = 0$. Roots of it are the EPs $q \in \mathbb{C}_q^h$. If q_k is an EP, then we can find a nontrivial solution $\mathbf{V}^k$ of the equation $\mathbf{M}\mathbf{V} = \mathbf{0}$, where $\mathbf{M} = \mathbf{M}_{q_k}$, and obtain the eigenfunction $\mathbf{U}^k = \mathbf{E}_{q_k}\mathbf{V}^k$, corresponding to the eigenvalue

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi q_k h}{2}. \tag{31}$$

Remark 4. In the case $\Omega = [0, l]$, the eigenvalues are given by:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi q_k h}{2l} = \frac{4}{h^2}\sin^2\frac{\pi q_k}{2N} = \frac{4N^2}{l^2}\sin^2\frac{\pi q_k}{2N}.$$

Remark 5. If EP $q = 0$ or $q = N$, i.e., if the eigenvalues are $\lambda = 0$ or $\lambda = 4N^2$, then to find eigenfunctions, we must use expression (23) with $z = \pm 1$:

$$U_j = v^1 + v^2 t_j, \quad \bar{\delta}U_j = v^2 \quad \text{for } z = 1,$$
$$U_j = v^1(-1)^j + v^2(-1)^{j-1}t_j, \quad \bar{\delta}U_j = (-1)^j v^2 \quad \text{for } z = -1,$$

where $t_j \in \overline{\omega}$. The reason is that, when calculating the rank of $\mathbf{M}_z = \mathbf{B}\mathbf{E}_z$, the algebraic multiplicity of an eigenvalue must be taken into account (see Remark 3).

6 Natural boundary conditions

Classical BCs for differential SLE (13) can be of Dirichlet, Neumann or Robin type. The latter two types use the value of the derivative at the endpoint of the interval. To approximate with second-order accuracy, the standard method is to use ghost nodes (see (18)). Thus, we obtain two equations for U_i at the nodes in $\widetilde{\omega}$ that use ghost nodes. For example, for the Neumann condition $u'(0) = 0$ we have the equation $U_{-1} = U_1$. This equation can be written in the form (5) with a nonzero matrix $\mathbf{B}^{\mathrm{i}}$.

If the node $t_j = 0$ ($t_j = 1$) is the inner node, then we have the equation $U_{-1} = 2zU_0 - U_1$ ($U_{N+1} = 2zU_N - U_{N-1}$) [8]. Therefore, in these cases we get the following approximations of the first order derivative:

$$\bar{\delta}U_0 = \frac{U_1 - U_{-1}}{2h} = \frac{U_1 - zU_0}{h}, \qquad \bar{\delta}U_N = \frac{U_{N+1} - U_{N-1}}{2h} = \frac{zU_N - U_{N-1}}{h}. \tag{32}$$

Using the variable $q \in \mathbb{C}_q^h$ or λ instead of z, we get

$$\bar{\delta}U_0 = \frac{U_1 - \cos(\pi q h)U_0}{h}, \qquad \bar{\delta}U_N = \frac{\cos(\pi q h)U_N - U_{N-1}}{h}; \tag{33}$$

$$\bar{\delta}U_0 = \frac{U_1 - U_0}{h} + \lambda U_0\frac{h}{2}, \qquad \bar{\delta}U_N = \frac{U_N - U_{N-1}}{h} - \lambda U_N\frac{h}{2}. \tag{34}$$

Corollary 3.

$$\bar{\delta}U_0 = \frac{U_1 - U_0}{h}, \quad \bar{\delta}U_N = \frac{U_N - U_{N-1}}{h} \quad \textit{for } \lambda = 0,\ z = 1,\ q = 0,$$
$$\bar{\delta}U_0 = \frac{U_1 + U_0}{h}, \quad \bar{\delta}U_N = -\frac{U_N + U_{N-1}}{h} \quad \textit{for } \lambda = 4N^2,\ z = -1,\ q = N.$$

Formulas (32)–(34) are said to be natural approximations of the first-order derivative [8, 38]. Note that these approximations do not include ghost nodes. BCs that use such discrete approximations of the derivatives are called natural BCs. They contain the eigenparameter q (or λ, or z). Although they are defined at the inner nodes, we traditionally call them BCs for $t_j \in \partial_n\omega$ (see Fig. 1(b)). Discrete SLPs with an eigenparameter in BCs were studied in [15, 16]. The differential SLP with an eigenparameter in BCs was studied in [14, 20].

Discrete SLP (17)–(18) can be rewritten as (equivalent) system

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega_t^h, \tag{35}$$
$$B^l(U) := \sum_{t_i \in \omega_t^h} b_i^l U_i + \sum_{t_i \in \partial_n\omega} b_i^l(\lambda) U_i + \sum_{t_i \in \partial\omega} b_i^l U_i = 0, \quad l = 1, 2, \tag{36}$$
$$U_{-1} = 2zU_0 - U_1 \quad \text{if } t_0 \in \partial_n\omega, \tag{37}$$
$$U_{N+1} = 2zU_N - U_{N-1} \quad \text{if } t_N \in \partial_n\omega, \tag{38}$$

where coefficients b_i^l depend on $\tilde{b}_i^l$, and b_i^l, $t_i \in \partial\omega_n$, depend on λ. Such a dSLP is defined on the grid $\overline{\omega}$.

Lemma 8. *The dSLP* (17)–(18) *is equivalent to the dSLP* (35)–(38).

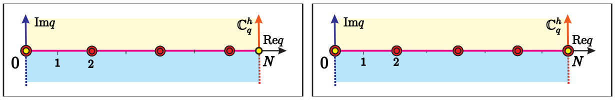
Remark 6. Problem (35)–(36) is dSLP on $\overline{\omega}$ with an eigenparameter λ in BCs, which can be solved separately from (37)–(38). Then we use (37)–(38) to find U_{-1} or U_{N+1}. On the other hand, Eqs. (37)–(38) have the form (35), and, for simplicity, we write them as a single equation $-\delta^2 U_i = \lambda U_i$, $t_i \in \omega = \omega_t^h + \partial_n\omega$, keeping in mind that for $t_i \in \partial_n\omega$, (37)–(38) hold.

7 Discrete Sturm–Liouville problems with nonlocal boundary conditions

In this section, we will present several dSLPs with NBCs to demonstrate the diversity of the spectrum and find expressions for discrete eigenfunctions. We will begin with classical periodic BCs, which are the oldest example of NBCs [13, 14].

Problem 1. Consider one-dimensional dSLP with periodic BCs [38, Chap. 1, Sect. 1.5.4] on the grid $\overline{\omega}$:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega = \omega_t^h + \{t_N\}, \tag{39}$$
$$U_0 = U_N, \quad \bar{\delta}U_0 = \bar{\delta}U_N. \tag{40}$$



(a) N is odd ($N = 7$) (b) N is even ($N = 8$)

Figure 5. EPs for SLP (39)–(40); ○ – RP, ◎ – simple EP at RP, ◉ – double EP.

We can rewrite this dSLP on the grid $\widetilde{\omega}$ with a single ghost node t_{N+1} and two BCs:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega,\ n = N, \tag{41}$$

$$U_0 = U_N, \qquad U_{N+1} = U_1. \tag{42}$$

For SLP (41)–(42), the matrix $\mathbf{A}$ of AEP (8) belongs to $\mathcal{Z}_n$, and

$$\mathbf{A} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{43}$$

For the problem with periodic BCs $\mathbf{B}^{\mathrm{b}} = \mathbf{O}$,

$$\mathbf{A}^{\mathrm{bb}} = \mathbf{I}, \qquad \mathbf{A}^{\mathrm{bi}} = -\mathbf{B}^{\mathrm{i}}, \quad \mathbf{B}^{\mathrm{i}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times n}.$$

Then we have

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 & 1 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times(n+2)},$$

$$\mathbf{E}_q = \begin{pmatrix} \cos(\pi q h) & \cdots & \cos(\pi q) & 1 & \cos\bigl(\pi q(1+h)\bigr) \\ h & \cdots & h\frac{\sin(\pi q)}{\sin(\pi q h)} & 0 & h\frac{\sin(\pi q(1+h))}{\sin(\pi q h)} \end{pmatrix}^{\top}.$$

Thus,

$$\mathbf{M}_q = \begin{pmatrix} 1 - \cos(\pi q) & -h\frac{\sin(\pi q)}{\sin(\pi q h)} \\ \cos\bigl(\pi q(1+h)\bigr) - \cos(\pi q h) & h\frac{\sin(\pi q(1+h))}{\sin(\pi q h)} - h \end{pmatrix}$$

and

$$\det \mathbf{M}_q = h\begin{vmatrix} 1 - \cos(\pi q) & -\frac{\sin(\pi q)}{\sin(\pi q h)} \\ -\sin(\pi q)\sin(\pi q h) & \cos(\pi q) - 1 \end{vmatrix} = 2h\bigl(\cos(\pi q) - 1\bigr) = -4h\sin^2\frac{\pi q}{2}.$$

We find EPs $q_{2k} = q_{2k+1} = 2k$, $k = 0, \dots, \lfloor N/2 \rfloor$. Double EP $q_0 = q_1$ coincides with the Ramification Point (RP) $q = 0$ (see Fig. 5) and corresponds to the simple eigenvalue $\lambda_1 = 0$. We assume that the point $q_1 = 0$ is the simple eigenvalue point, i.e., $q_0 \sim q_1$. The same situation is true for even N. Since $q_{N+1} \sim q_N$, then $q_N = N$ is a simple EP (see Fig. 5(b)). Finally, for any N, there are exactly N EPs q_k, $k = 1, \dots, N$ (taking multiplicity into account), and the corresponding eigenvalues are given by (31). Note that $\mathbf{M}_{q_k} = \mathbf{O}$ for all k. Therefore, for double EPs $q_{2k} = q_{2k+1} = 2k \in (0, N)$, the eigenfunctions form a two-dimensional space:

$$\lambda_{2k,2k+1} = \frac{4}{h^2}\sin^2(\pi k h), \qquad U_j^{2k} = \cos(2\pi k t_j), \quad U_j^{2k+1} = \sin(2\pi k t_j), \quad t_j \in \overline{\omega}.$$

The eigenvalue $\lambda_1 = 0$ is simple with only one corresponding eigenfunction. Since $\lambda = 0$ is BP, the matrix $\mathbf{M}_q = \mathbf{O}$ is not useful for finding this eigenfunction (see Remark 5). Instead, we use the matrix

$$\mathbf{M}_z = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \quad \operatorname{rank}\mathbf{M}_z = 1,$$

to get $v^2 = 0$, i.e., $U_j^1 \equiv 1$. For the simple eigenvalue $\lambda_N = 4N^2$ (N is even), we have $U_j^N = (-1)^j$. Note that this problem has a zero eigenvalue.

Corollary 4. *The matrix* (43) *is not an* M*-matrix.*

Problem 2. Let us consider one-dimensional dSLP with two NBCs [10,30,40–42] on the grid $\overline{\omega} = \omega + \partial\omega = \overline{\omega}_t^h$, $\omega = \omega_t^h$, $\partial\omega = \{t_0, t_N\}$, $n = N - 1$, without ghost nodes:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega, \tag{44}$$

$$U_0 = [\alpha, U], \qquad U_N = [\beta, U], \tag{45}$$

where $\alpha := (\alpha_0, \dots, \alpha_N)$, $\beta := (\beta_0, \dots, \beta_N)$, U are functions on the grid $\overline{\omega}$. Notation $[\alpha, U]$ and $[\beta, U]$ is approximation of integrals $\int_0^l \alpha(x)u(x)\mathrm{d}x$ and $\int_0^l \beta(x)u(x)\mathrm{d}x$ by trapezoidal rule.

For SLP (44)–(45), the matrix $\mathbf{A}$ in AEP (8) is given by [10,41]:

$$\mathbf{A} = \frac{1}{h^2}\begin{pmatrix} 2-\tilde{\alpha}_1 & -1-\tilde{\alpha}_2 & \cdots & -\tilde{\alpha}_{n-1} & -\tilde{\alpha}_n \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -\tilde{\beta}_1 & -\tilde{\beta}_2 & \cdots & -1-\tilde{\beta}_{n-1} & 2-\tilde{\beta}_n \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{46}$$

$$\left(\mathbf{A}^{\mathrm{bb}}\right)^{-1} = \begin{pmatrix} \tilde{\beta}_N & \tilde{\alpha}_N \\ \tilde{\beta}_0 & \tilde{\alpha}_0 \end{pmatrix} = \begin{pmatrix} d^{-1}(1-\beta_N h/2) & d^{-1}\alpha_N h/2 \\ d^{-1}\beta_0 h/2 & d^{-1}(1-\alpha_0 h/2) \end{pmatrix},$$

$$\tilde{\alpha}_i = \alpha_i h \tilde{\beta}_N + \beta_i h \tilde{\alpha}_N = \alpha_i d^{-1} h + \begin{vmatrix} \alpha_N & \alpha_i \\ \beta_N & \beta_i \end{vmatrix} \frac{d^{-1}h^2}{2},$$

$$\tilde{\beta}_i = \beta_i h \tilde{\alpha}_0 + \alpha_i h \tilde{\beta}_0 = \beta_i d^{-1} h + \begin{vmatrix} \alpha_i & \alpha_0 \\ \beta_i & \beta_0 \end{vmatrix} \frac{d^{-1}h^2}{2}, \quad i = 1, \dots, n,$$

where $d = \det(\mathbf{A}^{\mathrm{bb}}) = 1 - (\alpha_0 + \beta_N)h/2 + (\alpha_0\beta_N - \alpha_N\beta_0)h^2/4$.

Consider the grid functions

$$\cos(\pi qt)|_i = \cos(\pi qt_i), \quad \sin\bigl(\pi q(t-a)\bigr)\big|_i = \sin\bigl(\pi q(t_i-a)\bigr), \quad t_i \in \overline{\omega}.$$

Therefore,

$$\mathbf{M}_q = \begin{pmatrix} 1-[\alpha,\cos(\pi qt)] & -[\alpha,\sin(\pi qt)]h/\sin(\pi qh) \\ \cos(\pi q)-[\beta,\cos(\pi qt)] & (\sin(\pi q)-[\beta,\sin(\pi qt)])h/\sin(\pi qh) \end{pmatrix},$$

and CE for $q \in \mathbb{C}_q^h$ is

$$\frac{\sin(\pi q)}{\sin(\pi qh)} + \left[\alpha, \frac{\sin(\pi q(t-1))}{\sin(\pi qh)}\right] - \left[\beta, \frac{\sin(\pi qt)}{\sin(\pi qh)}\right] + \begin{vmatrix} [\alpha,\cos(\pi qt)] & [\alpha, \frac{\sin(\pi qt)}{\sin(\pi qh)}] \\ [\beta,\cos(\pi qt)] & [\beta, \frac{\sin(\pi qt)}{\sin(\pi qh)}] \end{vmatrix} = 0. \tag{47}$$

Example 2. If $\alpha_i \equiv 0$, $\beta_i \equiv 0$, then CE is $\sin(\pi q)/\sin(\pi qh) = 0$. In this case, the roots of this equation are $q_k = k$, $k = 1,\dots,n$. Therefore, eigenvalues and eigenfunctions are as follows:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi kh}{2}, \quad U_j^k = \sin(\pi kt_j), \quad t_j \in \overline{\omega},\ k = 1,\dots,n.$$

We see that $\sigma(\mathbf{A}) \subset [4h^{-2}\sin^2(\pi h/2), 4h^{-2}\sin^2(\pi(N-1)h)/2)] \subset (0, 4h^{-2})$.

Example 3. If $\alpha_i \equiv \gamma_0$, $\beta_i \equiv \gamma_1$, then the determinant in (47) is equal to zero. Additionally, from [6, 30] we have

$$-\bigl[1, \sin\bigl(\pi q(t-1)\bigr)\bigr] = \bigl[1, \sin(\pi qt)\bigr] = h\frac{\sin^2(\pi q/2)\cos(\pi qh/2)}{\sin(\pi qh/2)}.$$

Thus, CE in this case is

$$P(q)\bigl(P(q)\gamma - Z(q)\bigr) = 0, \tag{48}$$

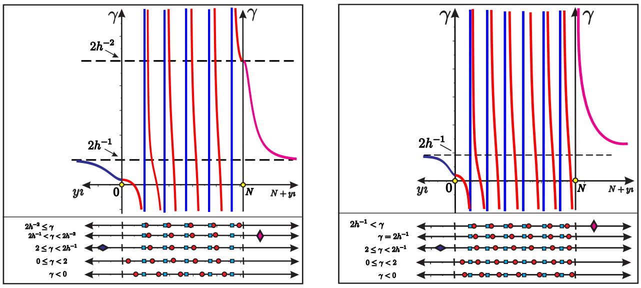
where

$$\gamma := \gamma_0 + \gamma_1, \qquad P(q) := \frac{h\sin(\pi q/2)}{2\sin(\pi qh/2)}, \qquad Z(q) := \frac{\cos(\pi q/2)}{\cos(\pi qh/2)}.$$

The equation $P(q) = 0$ has roots (Constant Eigenvalue Points (CEP) [10, 53]) $q_{2k} = c_{2k} = 2k$, $k = 1,\dots,\lfloor n/2\rfloor$. If $\gamma \neq 2h^{-1}$, then γ-values of the Characteristic Function (CF) [10, 53]

$$\gamma(q) = \frac{Z(q)}{P(q)} = \frac{\tan(\pi qh/2)}{h/2}\cdot\frac{\cos(\pi q/2)}{\sin(\pi q/2)} \tag{49}$$

give EPs $q_{2k-1} = q_{2k-1}(\gamma)$, $k = 1,\dots,K$, $K = \lfloor N/2\rfloor$, that depend on the parameter γ. Graphs of real CFs (restriction of CF on the $\mathbb{R}_q^h$) are shown in Fig. 6. The CEP are the poles $p_k = q_{2k} = 2k$, $k = 1,\dots,K$, of the CF. If N is even, then $p_{K+1} = N$ is



(a) N is odd (b) N is even

Figure 6. Real CFs (49) and EPs q_k; ■ – CEP $q_k = c_k$, ◆ – large positive EP $q_1 = N + y_1\imath$, $\lambda_1 > 4N^2$, ◆ – negative EP $q_1 = y_1\imath$, $\lambda_1 < 0$, ● – positive EP $q_k = x_k$, $\lambda_k \in (0, 4N^2)$.

also a pole. There is at least one EP between two poles. Thus, there are at least $N-2$ real eigenvalues $\lambda_k = \lambda^h(q_k) \in (0, 4N^2)$, $k = 2, \dots, n$. The remaining eigenvalue $\lambda_1 \in (0, 4N^2)$ for $\gamma < 2$; $\lambda_1 = 0$ for $\gamma = 2$; $\lambda_1 < 0$ for $2 < \gamma < 2h^{-1}$; $\lambda_1 > 0$ for $\gamma > 2h^{-1}$ (if N is even, then $\lambda_1 > 4N^2$; if N is odd, then $\lambda_1 > 4N^2$ for $\gamma < 2h^{-2}$, $\lambda_1 = 4N^2$ for $\gamma = 2h^{-2}$; $\lambda_1 \in (\lambda_n, 4N^2)$ for $\gamma > 2h^{-2}$). If $\gamma = 2h^{-1}$, then there are only $N-2$ positive eigenvalues $\lambda_2, \dots, \lambda_n \in (0, 4N^2)$. The corresponding eigenvalue points are $q_2, \dots, q_n$ and $q_1 = \infty$ [30].

We can consider the case $\gamma_0 = 0$, since the CE (48) depend on $\gamma = \gamma_0 + \gamma_1$. The spectrum for such dSLP was studied in [6, 30] (see [44, 45, 49] for the differential SLP).

All eigenvalues in this example are simple, the eigenfunctions are linearly independent and form an eigenbasis $\{\mathbf{U}^1, \dots, \mathbf{U}^n\}$ (except for the case of $\gamma = 2/h$, where there are $N-2$ eigenfunctions $\{\mathbf{U}^2, \dots, \mathbf{U}^n\}$). These eigenfunctions can be expressed by means of (12) (see (25), too):

$$\mathbf{U}^k \sim \mathbf{D}[\mathcal{E}]\mathbf{B}_1, \quad U_j^k = \sin\bigl(\pi q_k(-t_j)\bigr) - \gamma_0\bigl[1, \sin\bigl(\pi q_k(t - t_j)\bigr)\bigr],\ t_j \in \overline{\omega},$$

or

$$\mathbf{U}^k \sim \mathbf{D}[\mathcal{E}]\mathbf{B}_2, \quad U_j^k = \sin\bigl(\pi q_k(1 - t_j)\bigr) - \gamma_1\bigl[1, \sin\bigl(\pi q_k(t - t_j)\bigr)\bigr],\ t_j \in \overline{\omega},$$

in the case $\lambda_k \neq 0, 4N^2$ ($q_k \neq 0, N$), and $\mathbf{U}^1 \sim \mathbf{D}[\mathcal{T}]\mathbf{B}_1 \sim \mathbf{D}[\mathcal{T}]\mathbf{B}_2$ (see Remark 5 and [30]):

$$\begin{aligned} U_j^1 &= \frac{\gamma_0}{2} + (1 - \gamma_0)t_j, \quad t_j \in \overline{\omega}, \qquad \lambda_1 = 0, \\ U_j^1 &= (-1)^j\bigl(\gamma_0 - (\gamma_0 + \gamma_1)t_j\bigr), \quad t_j \in \overline{\omega}, \qquad \lambda_1 = 4N^2, \quad N \text{ is odd}, \end{aligned}$$

For this example the entries of the matrix $\mathbf{A}$ (see (46)) are as follows: $\tilde{\alpha}_i = \gamma_0 d^{-1}h$, $\tilde{\beta}_i = \gamma_1 d^{-1}h$, $d = 1 - \gamma h/2$. If $\gamma \in (-\infty, 2) \cup (2h^{-1}, +\infty)$, then all eigenvalues are positive. If $0 \leqslant \gamma < 2h^{-1}$, then the matrix $\mathbf{A}$ belongs to $\mathcal{Z}_n$.

Thus, using Lemma 1 (see (ii)) and Lemma 2 we prove the next lemma.

Lemma 9. (*See* [10, *Lemma* 7].) *If* $0 \leqslant \alpha_i \leqslant \gamma_0$, $0 \leqslant \beta_i \leqslant \gamma_1$, *and* $\gamma_0 + \gamma_1 < 2$, *then the matrix* $\mathbf{A}$ *for problem* (44)–(45) *is an* M*-matrix.*

Problem 3. Consider Eq. (44) with two Bitsadze–Samarskii BCs [9, 47] on the grid $\overline{\omega} = \omega + \partial\omega = \overline{\omega}^h$, $\omega = \omega_t^h$, $\partial\omega = \{t_0, t_N\}$, $n = N-1$, without ghost nodes:

$$U_0 = \gamma_0 U_{s_0}, \quad s_0 \in \{1, \dots, n\}, \qquad U_N = \gamma_1 U_{s_1}, \quad s_1 \in \{1, \dots, n\}, \tag{50}$$

and $\xi_0 = s_0 h = s_0 N^{-1}$, $\xi_1 = s_1 h = s_1 N^{-1}$.

Remark 7. If $\alpha_i = \gamma_0 \delta_{is_0} h^{-1}$, $\beta_i = \gamma_1 \delta_{is_1} h^{-1}$, then the BCs (50) are the special case of BCs (45). In this case $[\alpha, U] = \gamma_0 U_{s_0}$, $[\beta, U] = \gamma_1 U_{s_1}$.

The SLP (44), (50) has the matrix (46), where $\tilde{\alpha}_i = \gamma_0 \delta_{is_0}$, $\tilde{\beta}_i = \gamma_1 \delta_{is_1}$. In the case of BCs (50), the matrix

$$\mathbf{M}_q = \begin{pmatrix} 1 - \gamma_0 \cos(\pi q \xi_0) & -\gamma_0 \sin(\pi q \xi_0) h / \sin(\pi q h) \\ \cos(\pi q) - \gamma_1 \cos(\pi q \xi_1) & (\sin(\pi q) - \gamma_1 \sin(\pi q \xi_1)) h / \sin(\pi q h) \end{pmatrix},$$

and CE for $q \in \mathbb{C}_q^h$ is

$$\frac{\sin(\pi q)}{\sin(\pi q h)} - \frac{\sin(\pi q (1-\xi_0))}{\sin(\pi q h)} \gamma_0 - \frac{\sin(\pi q \xi_1)}{\sin(\pi q h)} \gamma_1 + \begin{vmatrix} \cos(\pi q \xi_0) & \frac{\sin(\pi q \xi_0)}{\sin(\pi q h)} \\ \cos(\pi q \xi_1) & \frac{\sin(\pi q \xi_1)}{\sin(\pi q h)} \end{vmatrix} \gamma_0 \gamma_1 = 0.$$

Example 4. If $\gamma_0 = 0$, $\gamma_1 = \gamma$, $\xi_1 = \xi$, then the CE is

$$\frac{\sin(\pi q)}{\sin(\pi q h)} = \frac{\sin(\pi q \xi)}{\sin(\pi q h)} \gamma. \tag{51}$$

Equation (51) was studied in [8, 32, 33, 43, 50, 51, 53]. Complex eigenvalues exist for $|\gamma| > 1$ only. According to Lemma 9, the condition $0 \leqslant \gamma < 2h$ holds. However, the real parts of the eigenvalues are positive for a large interval of γ [10, Lemma 8, Remark 5].

Problem 4. Consider the following discrete SLP: Eq. (44) with two BCs of the Samarskii–Ionkin type [21, 22, 47] on the grid $\overline{\omega}$:

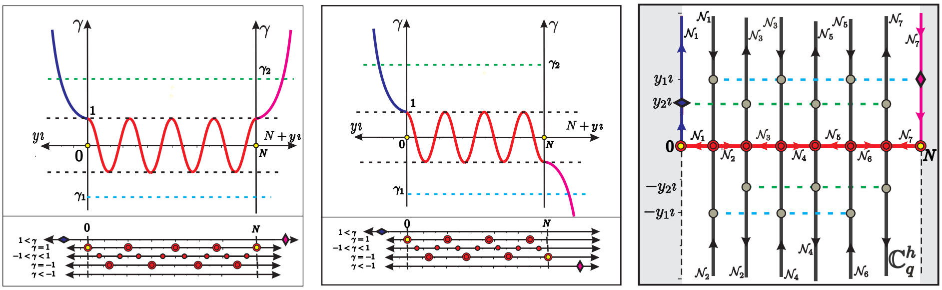
$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega = \{t_0\} + \omega_t^h, \tag{52}$$
$$\bar{\delta} U_0 = 0, \qquad U_N = \gamma U_0. \tag{53}$$

The following dSLP on the grid $\widetilde{\omega}$ with the ghost node t_{-1} is equivalent to the previous problem:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega = \{t_0\} + \omega_t^h, \ n = N, \tag{54}$$
$$U_{-1} = U_1, \qquad U_N = \gamma U_0. \tag{55}$$

For SLP (54)–(55), the matrix $\mathbf{A} =$ of AEP (8) becomes [10]

$$\mathbf{A} = \frac{1}{h^2} \begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -\gamma & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n}. \tag{56}$$



(a) N is even ($N = 8$) (b) N is odd ($N = 7$) (c) Spectrum Curves ($N = 7$)

Figure 7. (a), (b) Real CFs (58) and real EPs; (c) EPs in the cases $\cosh(\pi y_1) = |\gamma_1|$, $\gamma_1 < -1$, and $\cosh(\pi y_2) = \gamma_2 > 1$, ● – complex EP.

The matrix (56) is the special case of the matrix (46) with $\tilde{\alpha}_i = \delta_{i2}$, $\tilde{\beta}_i = \gamma\delta_{i1}$, $i = 1, \dots, n$.

From the first BC $\overline{\delta}U_0 = 0$ and from equalities (28)–(29) it follows that $0 = \overline{\delta}U_0 = v^1\overline{\delta}E_0 + v^2\overline{\delta}\widetilde{E}_0 = v^2$. Therefore, solutions of Eq. (52) with the first BC are of the form

$$U_i = v^1 E_i = v^1 \cos(\pi q t_i), \quad t_i \in \overline{\omega}. \tag{57}$$

Then from BC $U_N = \gamma U_0$ and (28) we get $v^1 \cos(\pi q) = v^1\gamma$. Thus, CE and CF are

$$\cos(\pi q) = \gamma, \quad \gamma(q) = \cos(\pi q), \quad q \in \mathbb{C}_q^h. \tag{58}$$

The roots q_k of this CE are EPs, and $\lambda_k = 4/h^2 \sin^2(\pi q_k h)$, $k = 1, \dots, N$, are eigenvalues of dSLP:

(i) If $|\gamma| < 1$, then all eigenvalues are simple and positive, $q_k = (-1)^k \arcsin\gamma/\pi + (k - 1/2)$.

(ii) If $\gamma = -1$, then all eigenvalues are positive, $\tilde{q}_l = q_{2l-1} = q_{2l} = 2l - 1$, $l = 1, \dots, L$, where $L = N/2$ for even N; $L = (N-1)/2$ for odd N; $q_N = N$ for odd N (in the case $q_N = N$, there is a simple EP, the other EPs are double).

(iii) If $\gamma = 1$, then $q_1 = 0$, $\tilde{q}_l = q_{2l} = q_{2l+1} = 2l$, $l = 1, \dots, L$, $L = \lfloor (N-1)/2 \rfloor$, $q_N = N$ for even N (in the case $q_1 = 0$ or $q_N = N$, we have simple EPs, the other EPs are double).

(iv) If $\gamma < -1$, then $q_{2l-1}, q_{2l} = \tilde{q}_l \pm \imath y$, $l = 1, \dots, L$ for even N; $q_{2l-1} = q_{2l} = \tilde{q}_l \pm \imath y$, $l = 1, \dots, L$ ($\tilde{q}_l$ are defined in (ii)); $q_N = N + \imath y$ for odd N (positive eigenvalue), $\cosh(\pi y) = |\gamma|$.

(v) If $\gamma > 1$, then $q_{2l}, q_{2l+1} = \tilde{q}_l \pm \imath y$, $l = 1, \dots, L$ ($\tilde{q}_l$ are defined in (iii)); $q_N = N + \imath y$ for even N (positive eigenvalue), and $q_1 = \imath y$ for all N (negative eigenvalue), $\cosh(\pi y) = \gamma$.

Complex EPs belong to Spectrum Curves (see Fig. 7(c)). The definition of spectrum curves and examples are given in [6–8, 53]. If we compare placement of the spectrum curves (see Fig. 7(c)) in the domain D_0^h (see Fig. 4(a)), we see that the EP q_1 leaves the domain D_0^h first because all complex EPs are $x_k \pm y_k\imath$ with the same y_k for a fixed γ ($|\gamma| > 1$). The graphs of the real CFs (see Figs. 7(a)–7(b)) show that $\gamma = 1$ is such a value.

If $\gamma \in [-1, 1)$, then all eigenvalues of the matrix (56) are positive. If $\gamma \geqslant 0$, then this matrix belongs to $\mathcal{Z}_n$. Thus, using Lemma 1 (see (ii)), we proved the following lemma.

Lemma 10. (*See* [10, *Lemma* 9].) *If* $0 \leqslant \gamma < 1$, *then the matrix* (56) *is an* M*-matrix.*

The eigenfunctions for dSLP (52)–(53) are (see (57) and Remark 5):

$$\begin{aligned}
|\gamma| \neq 1:&\quad U_i^k = \cos(\pi q_k t_i), \quad k = 1, \dots, N;\\
\gamma = -1:&\quad U_i^1 = 1, \quad U_i^{2l-1} = \cos(\pi q_{2l-1} t_i), \quad l = 1, \dots, L,\\
&\quad U_i^N = (-1)^i \quad \text{for odd } N;\\
\gamma = 1:&\quad U_i^1 = 1, \quad U_i^{2l+1} = \cos(\pi q_{2l+1} t_i), \quad l = 1, \dots, L,\\
&\quad U_i^N = (-1)^i \quad \text{for even } N;
\end{aligned}$$

$t_i \in \overline{\omega}$. Therefore,

$$\mathbf{M}_q = \begin{pmatrix} 0 & -2h \\ \cos(\pi q) - \gamma & h\frac{\sin(\pi q)}{\sin(\pi q h)} \end{pmatrix}, \qquad \mathbf{M}_{q_k} = \begin{pmatrix} 0 & -2h \\ 0 & h\frac{\sin(\pi q_k)}{\sin(\pi q_k h)} \end{pmatrix}.$$

Since $\operatorname{rank} \mathbf{M}_{q_k} = 1$ in the case of $|\gamma| = 1$, all of the generalized eigenfunctions are of rank 2: U_i^{2l}, $t_i \in \overline{\omega}$, for EPs $\tilde{q}_l = q_{2l}$, $l = 1, \dots, L$ ($q_{2l} = 2l - 1$ for $\gamma = -1$, $q_{2l} = 2l$ for $\gamma = 1$). These eigenfunctions satisfy the equations

$$\begin{aligned}
\gamma = -1:&\quad U_{i+1}^{2l} - 2z_{2l} U_i^{2l} + U_{i+1}^{2l} = -h^2 U_i^{2l-1},\\
\gamma = 1:&\quad U_{i+1}^{2l} - 2z_{2l} U_i^{2l} + U_{i+1}^{2l} = -h^2 U_i^{2l+1},
\end{aligned}$$

$t_i \in \omega = \omega_t^h + \{t_N\}$, $z_{2l} = \cos(\pi q_{2l} h)$, with the conditions

$$\bar{\delta} U_0^{2l} = 0, \qquad U_N^{2l} = \gamma U_0^{2l}.$$

These generalized eigenfunctions ($i = 1, \dots, N-1$) are

$$U_i^{2l} = -\frac{t_i}{2\pi q_{2l}} \sin(\pi q_{2l} t_i) \sim t_i \widetilde{E}_i(q_{2l}), \quad t_i \in \overline{\omega},\ l = 1, \dots, L.$$

For the proof, we use formulas (15)–(16) and (26)–(27).

Problem 5. Consider Eq. (44) with two BCs of the Samarskii–Ionkin type [18, 19, 21, 22, 47] on the grid $\overline{\omega}$:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega = \{t_0\} + \omega_t^h + \{t_N\}, \tag{59}$$

$$U_0 = 0, \qquad \bar{\delta} U_N = \gamma \bar{\delta} U_0. \tag{60}$$

Note that if the first BC holds, then (59) for t_0 is $U_{-1} = -U_1$, the second BC is $\bar{\delta} U_N = \gamma h^{-1} U_1$. Therefore, it is possible to remove node t_0 from the inner nodes and assign it to the boundary nodes:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega = \omega_t^h + \{t_N\},$$
$$U_0 = 0, \qquad \bar{\delta} U_N = \gamma h^{-1} U_1.$$

Thus, we have an equivalent dSLP on the grid $\widetilde{\omega}$ with a single ghost node t_{N+1}:

$$-\delta^2 U_i = \lambda U_i, \quad t_i \in \omega = \omega_t^h + \{t_N\},\ n = N, \tag{61}$$
$$U_0 = 0, \qquad U_{N+1} = 2\gamma U_1 + U_{N-1}. \tag{62}$$

For SLP (61)–(62), the matrix $\mathbf{A}$ of AEP (8) becomes [10, 39]:

$$\mathbf{A} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -2\gamma & & & -2 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{63}$$

The matrix (63) is the special case of the matrix (46) with $\tilde{\alpha}_i = 0$, $\tilde{\beta}_i = 2\gamma\delta_{i1} + \delta_{i,n-1}$, $i = 1, \ldots, n$.

From the first BC $U_0 = 0$ and equalities (28)–(30) it follows that $0 = U_0 = v^1 E_0 + v^2 \widetilde{E}_0 = v^1$. Therefore, solutions of Eq. (59) with the first BC are of the form

$$U_i = v^2 \widetilde{E}_i = v^2 h \frac{\sin(\pi q t_i)}{\sin(\pi q h)}, \quad t_i \in \overline{\omega}. \tag{64}$$

Then from the BC $\bar{\delta} U_N = \gamma \bar{\delta} U_0$ and (29) we get $v^2 \cos(\pi q) = v^2 \gamma$. Thus, we have CE and CF (58) as in the previous problem.

Lemma 11. (*See* [10, *Lemma* 9].) *If* $0 \leqslant \gamma < 1$*, then the matrix* (63) *is an* M*-matrix.*

The eigenfunctions for dSLP (59)–(60) are (see (64) and Remark 5):

$$\begin{aligned} |\gamma| \neq 1:&\quad U_i^k = \sin(\pi q_k t_i), \quad k = 1, \ldots, N; \\ \gamma = -1:&\quad U_i^1 = t_i, \quad U_i^{2l-1} = \sin(\pi q_{2l-1} t_i), \quad l = 1, \ldots, L, \\ &\quad U_i^N = (-1)^i t_i \quad \text{for odd } N; \\ \gamma = 1:&\quad U_i^1 = t_i,\ U_i^{2l+1} = \sin(\pi q_{2l+1} t_i), \quad l = 1, \ldots, L, \\ &\quad U_i^N = (-1)^i t_i \quad \text{for even } N; \end{aligned}$$

$t_i \in \overline{\omega}$. Therefore,

$$\mathbf{M}_q = \begin{pmatrix} 1 & 0 \\ -2\gamma\cos(\pi q h) - 2\sin(\pi q)\sin(\pi q h) & -2h\bigl(\gamma - \cos(\pi q)\bigr) \end{pmatrix},$$
$$\mathbf{M}_{q_k} = \begin{pmatrix} 1 & 0 \\ -2\cos\bigl(\pi q_k(1-h)\bigr) & 0 \end{pmatrix}.$$

Since $\operatorname{rank}\mathbf{M}_{q_k} = 1$ in the case of $|\gamma| = 1$, all of the generalized eigenfunctions are of rank 2: U_i^{2l}, $t_i \in \overline{\omega}$. These eigenfunctions satisfy equations

$$\begin{aligned} \gamma = -1: \quad & U_{i+1}^{2l} - 2z_{2l}U_i^{2l} + U_{i+1}^{2l} = -h^2 U_i^{2l-1}, \\ \gamma = 1: \quad & U_{i+1}^{2l} - 2z_{2l}U_i^{2l} + U_{i+1}^{2l} = -h^2 U_i^{2l+1}, \end{aligned}$$

$t_i \in \omega = \{t_0\} + \omega_t^h$, $z_{2l} = \cos(\pi q_{2l} h)$, with conditions

$$U_0^{2l} = 0, \qquad \bar\delta U_N^{2l} = \gamma \bar\delta U_0^{2l}.$$

These generalized eigenfunctions ($i = 1, \dots, N-1$) are

$$U_i^{2l} = \frac{t_i}{2\pi q_{2l}} \cos(\pi q_{2l} t_i) \sim t_i E_i(q_{2l}), \quad t_i \in \overline{\omega},\ l = 1, \dots, L.$$

7.1 The case of more general Sturm–Liouville operator

Consider a matrix

$$\mathbf{A}(z) = \begin{pmatrix} 2z & -1 & & & \\ -1 & 2z & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2z & -1 \\ & & & -1 & 2z \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad z \in \mathbb{R}, \tag{65}$$

for the equation

$$-U_{i+1} + 2zU_i - U_{i-1} = h^2 F_i, \quad t_i \in \omega_t^h,\ h = N^{-1},\ n = N-1.$$

For example, we obtain such a matrix when considering the difference equation [27]

$$-\delta^2 U_i + qU_i = F_i, \quad q \equiv \mathrm{const},$$

with Dirichlet BCs $U_0 = U_N = 0$. In the case of this equation, $z = 1 + h^2 q/2$. If $z > 0$, then $\mathbf{A}(z) \in \mathcal{Z}_n$. The matrix $\mathbf{A}(z) = \mathbf{A}(1) + 2(z-1)\mathbf{I}$. In Example 2, we find the eigenvalues of the matrix $h^{-2}\mathbf{A}(1)$, and $\sigma(\mathbf{A}(1)) \subset [4\sin^2(\pi h/2), 4)$. If $z > \tilde z_{n,1} = \cos(\pi h)$, then $4\sin^2(\pi h/2) + 2(z-1) > 0$, i.e., $\operatorname{Re}\lambda(\mathbf{A}(z)) > 0$. Thus, $\mathbf{A}(z)$ is an M-matrix for $z > \tilde z_{n,1}$. Note that $\tilde z_{n,1}$ is the rightmost zero of the Chebyshev polynomial $\widetilde{T}_n$.

The determinant of matrix (65) is $\det \mathbf{A} = \widetilde{T}_n(z)$ (see (22)), and

$$\mathbf{A}^{-1} = \frac{\widetilde{\mathbf{D}}(z)}{\widetilde{T}_n(z)}, \quad \widetilde{\mathbf{D}}(z) = \begin{pmatrix} \widetilde{T}_{n-1}(z) & \widetilde{T}_{n-2}(z) & \cdots & \widetilde{T}_1(z) & \widetilde{T}_0(z) \\ \widetilde{T}_{n-2}(z) & & & & \widetilde{T}_1(z) \\ \vdots & \tilde d_{i,i-1} & \tilde d_{ii} & \tilde d_{i,i+1} & \vdots \\ \widetilde{T}_1(z) & & & & \widetilde{T}_{n-2}(z) \\ \widetilde{T}_0(z) & \widetilde{T}_1(z) & \cdots & \widetilde{T}_{n-2}(z) & \widetilde{T}_{n-1}(z) \end{pmatrix},$$

where matrix $\widetilde{\mathbf{D}}(z) = (\tilde d_{ij})$ is symmetric, and $\tilde d_{ij} = \tilde d_{i1}\tilde d_{nj} = \widetilde{T}_{n-i}\widetilde{T}_{j-1}$ for $j \leqslant i$. If $z > \tilde z_{n,1}$, then all $\widetilde{T}_k(z) > 0$, $k = 0, \dots, n$. Therefore, $\mathbf{A}^{-1}(z) > 0$.

Corollary 5. *If $z > \tilde z_{n,1}$, then $\mathbf{A}(z)$ is an M-matrix.*

8 Conclusions

This article studies one-dimensional, discrete Sturm–Liouville problems with nonlocal boundary conditions. These problems can be transformed into algebraic eigenvalue problems. If the spectrum of an algebraic eigenvalue problem is found and the real parts of all the eigenvalues are positive, then the matrix of the linear system is an M-matrix. The advantage of this method is that it allows us to determine the necessary and sufficient conditions for a matrix to be an M-matrix. However, the main disadvantage is that nonlocal boundary conditions usually depend on several parameters, so finding the spectrum of such a problem can be very difficult. This paper provides examples of problems for which all the eigenvalues and eigenfunctions can be found. Additionally, it provides a thorough review of studies on the eigenvalues of difference problems with nonlocal boundary conditions.

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References

1. U.M. Ascher, *Numerical Methods for Evolutionary Differential Equations*, Computational Science and Engineering, SIAM, 2008, https://doi.org/10.1137/1.9780898718911.bm.
2. F.V. Atkinson, *Discrete and Continuous Boundary Value Problems*, Academic Press, New York, 1964.
3. K.E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley & Sons, New York, 1988.
4. O. Axelsson, *Iterative Solution Methods*, Cambridge Univ. Press, Cambridge, 1994, https://doi.org/10.1017/CBO9780511624100.
5. A. Berman, m Applications of M-Matrices, in F.Y. Phillips, J.J. Rousseau (Eds.), *Systems and Management Science by Extremal Methods*, Springer, Boston, MA, 1992, pp. 115–126, https://doi.org/10.1007/978-1-4615-3600-0_8.
6. K. Bingelė, A. Bankauskienė, A. Štikonas, Spectrum curves for a discrete Sturm–Liouville problem with one integral boundary condition, *Nonlinear Anal. Model. Control*, **24**(5):755–774, 2019, https://doi.org/10.15388/NA.2019.5.5.
7. K. Bingelė, A. Bankauskienė, A. Štikonas, Investigation of spectrum curves for a Sturm–Liouville problem with two-point nonlocal boundary conditions, *Math. Model. Anal.*, **25**(1): 53–70, 2020, https://doi.org/10.3846/mma.2020.10787.
8. K. Bingelė, A. Štikonas, Investigation of a discrete Sturm–Liouville problem with two-point nonlocal boundary condition and natural approximation of a derivative in boundary condition, *Math. Model. Anal.*, **29**(2):309–330, 2024, https://doi.org/10.3846/mma.2024.19829.

9. A.V. Bitsadze, A.A. Samarskii, Some elementary generalizations of linear elliptic boundary value problems, *Dokl. Akad. Nauk SSSR*, **185**:739–740, 1969, https://www.mathnet.ru/eng/dan34529.
10. V. Būda, M. Sapagovas, O. Štikonienė, A. Štikonas, M-matrices and discrete problems with nonlocal boundary conditions, *Lith. Math. J.*, **65**(2):179–200, 2025, https://doi.org/10.1007/s10986-025-09676-0.
11. V. Būda, A. Štikonas, Along with the numerical analysis: An interview with Mifodijus Sapagovas, *Lith. Math. J.*, **64**(4):395–404, 2024, https://doi.org/10.1007/s10986-024-09656-w.
12. M. Dehghan, Efficient techniques for the second-order parabolic equation subject to nonlocal specifications, *Appl. Numer. Math.*, **52**(1):39–62, 2005, https://doi.org/10.1016/j.apnum.2004.02.002.
13. M.S.P. Eastham, *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh, 1973.
14. C.T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Edinb. Math. Soc. A*, **77**(3–4):293–308, 1977, https://doi.org/10.1017/S030821050002521X.
15. C. Gao, R. Ma, Eigenvalues of discrete Sturm–Liouville problems with eigenparameter dependent boundary conditions, *Linear Algebra Appl.*, **503**:100–119, 2016, https://doi.org/10.1016/j.laa.2016.03.043.
16. C. Gao, Y. Wang, L. Lv, Spectral properties of discrete Sturm–Liouville problems with two squared eigenparameter-dependent boundary conditions, *Acta Math. Sci.*, **40**:755–781, 2020, https://doi.org/10.1007/s10473-020-0312-5.
17. J. Gao, D. Sun, M. Zhang, Structure of eigenvalues of multi-point boundary value problems, *Adv. Difference Equ.*, **2010**:381932, 2010, https://doi.org/10.1155/2010/381932.
18. A. Gulin, Stability criteria for non-self-adjoint finite differences schemes in the subspace, *Appl. Numer. Math.*, **93**(1):107–113, 2015, https://doi.org/10.1016/j.apnum.2014.05.005.
19. A.V. Gulin, On the spectral stability in subspaces for difference schemes with nonlocal boundary conditions, *Differ. Equ.*, **49**(7):815–823, 2013, https://doi.org/10.1134/S0012266113070045.
20. N.J. Guliyev, Essentially isospectral transformations and their applications, *Ann. Mat. Pura Appl. (4)*, **199**(4):1621–1648, 2020, https://doi.org/10.1007/s10231-019-00934-w.
21. N.I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differ. Uravn.*, **13**(2):294–304, 1977, https://www.mathnet.ru/eng/de2993.
22. N.I. Ionkin, Difference schemes for one nonclassical problem, *Vestn. Mosk. Univ., Ser. XV*, **2**: 20–32, 1977 (in Russian).
23. Z. Li, Z. Qiao, T. Tang, *Numerical Solution of Differential Equations: Introduction to Finite Difference and Finite Element Methods*, Cambridge Univ. Press, Cambridge, 2017, https://doi.org/10.1017/9781316678725.

24. R.Y. Ma, A survey on nonlocal boundary value problems, *Appl. Math. E-Notes*, **7**:257–279, 2007.

25. J. Martín-Vaquero, J. Vigo-Aguiar, On the numerical solution of the heat conduction equations subject to nonlocal conditions, *Appl. Numer. Math.*, **59**(10):2507–2514, 2009, https://doi.org/10.1016/j.apnum.2009.05.007.

26. J.C. Mason, D.C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC, Basel, Boston, Berlin, 2003, https://doi.org/10.1201/9781420036114.

27. R.M.M. Mattheij, S.W. Rienstra, J.H.M. ten Thije Boonkkamp, *Partial Differential Equations: Modeling, Analysis, Computation*, SIAM, Philadelphia, 2005.

28. H. Minkowski, Zur Theorie der Einheiten in den algebraischen Zahlkörpern, *Gött. Nachr.*, **1900**:90–93, 1900, http://eudml.org/doc/58467.

29. E.M. Nikishin, Discrete Sturm–Liouville operators and some problems of function theory, *J. Math. Sci.*, **35**(5):2679–2744, 1986, https://doi.org/10.1007/BF01119188.

30. J. Novickij, A. Štikonas, On the stability of a weighted finite difference scheme for wave equation with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):460–475, 2014, https://doi.org/10.15388/NA.2014.3.10.

31. A. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale, *Comment. Math. Helv.*, **10**:69–96, 1937, http://eudml.org/doc/138693.

32. S. Pečiulytė, A. Štikonas, On positive eigenfunctions of Sturm–Liouville problem with nonlocal two-point boundary condition, *Math. Model. Anal.*, **12**(2):215–226, 2007, https://doi.org/10.3846/1392-6292.2007.12.215-226.

33. S. Pečiulytė, O. Štikonienė, A. Štikonas, Investigation of negative critical points of the characteristic function for problems with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **13**(4):467–490, 2008, https://doi.org/10.15388/NA.2008.13.4.14552.

34. G. Poole, T. Boullion, A survey on M-matrices, *SIAM Rev.*, **16**(4):419–427, 1974, https://doi.org/10.1137/1016079.

35. J.D. Pryce, *Numerical Solution of Sturm–Liouville Problems*, Oxford Univ. Press, Oxford, 1993.

36. S. Roman, A. Štikonas, Green's function for discrete second-order problems with nonlocal boundary conditions, *Bound. Value Probl.*, **2011**:767024, 2011, https://doi.org/10.1155/2011/767024.

37. B.P. Rynne, Spectral properties and nodal solutions for second-order, m-point boundary value problems, *Nonlinear Anal. Theory Methods Appl., Ser. A, Theory Methods*, **67**(12):3318–3327, 2007, https://doi.org/10.1016/j.na.2006.10.014.

38. A.A. Samarskii, E.S. Nikolaev, *Numerical Methods for Grid Equations, Vol. I. Direct Methods*, Birkhäuser, Basel, Boston, Berlin, 1989, https://doi.org/10.1007/978-3-0348-9272-8.

39. M. Sapagovas, J. Novickij, On stability in the maximum norm of difference scheme for nonlinear parabolic equation with nonlocal condition, *Nonlinear Anal. Model. Control*, **28**(2):365–376, 2023, https://doi.org/10.15388/namc.2023.28.31562.

40. M. Sapagovas, K. Pupalaigė, R. Čiupaila, T. Meškauskas, On the spectrum structure for one difference eigenvalue problem with nonlocal boundary conditions, *Math. Model. Anal.*, **28**(3): 522–541, 2023, https://doi.org/10.3846/mma.2023.17503.

41. M. Sapagovas, A. Štikonas, O. Štikonienė, Alternating direction method for the Poisson equation with variable weight coefficients in an integral condition, *Differ. Equ.*, **47**(8):1176–1187, 2011, https://doi.org/10.1134/S0012266111080118.

42. M. Sapagovas, A. Štikonas, O. Štikonienė, ADI method for pseudo-parabolic equation with nonlocal boundary conditions, *Mathematics*, **11**(6):1303, 2023, https://doi.org/10.3390/math11061303.

43. E. Şen, A. Štikonas, Asymptotic distribution of eigenvalues and eigenfunctions of a nonlocal boundary value problem, *Math. Model. Anal.*, **26**(2):253–266, 2021, https://doi.org/10.3846/mma.2021.13056.

44. A. Skučaitė, K. Skučaitė-Bingelė, S. Pečiulytė, A. Štikonas, Investigation of the spectrum for the Sturm–Liouville problem with one integral boundary condition, *Nonlinear Anal., Model. Control*, **15**(4):501–512, 2010, https://doi.org/10.15388/NA.15.4.14321.

45. A. Skučaitė, A. Štikonas, Spectrum curves for Sturm–Liouville problem with integral boundary condition, *Math. Model. Anal.*, **20**(6):802–818, 2015, https://doi.org/10.3846/13926292.2015.1116470.

46. M.A. Snyder, *Chebyshev Methods in Numerical Approximation*, Prentice-Hall, Englewood Cliffs, NJ, 1966.

47. A. Štikonas, Investigation of characteristic curve for Sturm–Liouville problem with nonlocal boundary conditions on torus, *Math. Model. Anal.*, **16**(1):1–22, 2011, https://doi.org/10.3846/13926292.2011.552260.

48. A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3): 301–334, 2014, https://doi.org/10.15388/NA.2014.3.1.

49. A. Štikonas, E. Şen, Asymptotic analysis of Sturm–Liouville problem with nonlocal integral-type boundary condition, *Nonlinear Anal. Model. Control*, **26**(5):969–991, 2021, https://doi.org/10.15388/namc.2021.26.24299.

50. A. Štikonas, E. Şen, Asymptotic analysis of Sturm–Liouville problem with Neumann and nonlocal two-point boundary conditions, *Lith. Math. J.*, **62**(4):519–541, 2022, https://doi.org/10.1007/s10986-022-09577-6.

51. A. Štikonas, E. Şen, Asymptotic analysis of Sturm–Liouville problem with Dirichlet and nonlocal two-point boundary conditions, *Math. Model. Anal.*, **28**(2):308–329, 2023, https://doi.org/10.3846/mma.2023.17617.

52. A. Štikonas, S. Roman, Green's functions for discrete mth-order problems, *Lith. Math. J.*, **52**(3):334–351, 2012, https://doi.org/10.1007/s10986-012-9177-1.

53. A. Štikonas, O. Štikonienė, Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions, *Math. Model. Anal.*, **14**(2):229–246, 2009, https://doi.org/10.3846/1392-6292.2009.14.229-246.

54. R.S. Varga, *Matrix Iterative Analysis,* 2nd ed., Springer Ser. Comput. Math., Vol. 27, Springer, Berlin, Heidelberg, 2000, https://doi.org/10.1007/978-3-642-05156-2.