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Analysis and exact solutions for reaction–diffusion predator–prey system with prey-taxis by ϕ^6 method

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Abstract. This article analyses the biomathematical model that describes the reaction–diffusion predator–prey system with prey-taxis. The analysis includes addressing the question of whether the solution exists and is unique. The next goal is to obtain the exact solutions. The ϕ^6 method has been utilized for this purpose. The simulations of the obtained solutions are also incorporated, providing a broader understanding of the solutions.

Keywords: nonlinear fractional-order partial differential equations, exact solutions, ϕ^6 method.

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1 Introduction

The mathematical analysis of biomathematical systems, such as the reaction–diffusion predator–prey system with prey-taxis, is an active and represents a novel area of exploration. The word "taxis", originated from the Greek language, refers to the response of a living organism towards the environment in which it lives. It can be a motion in response to an external stimulus, which can involve either approaching or distancing from it. Thus the term "prey-taxis" describes a situation where a predator moves directly in response to changes in the variation of the prey [1, 18].

Classical predator–prey models assume instantaneous interactions, but real-world populations exhibit memory effects such as past food availability influencing predator behavior. Fractional-order derivatives provide a more accurate framework for modeling these effects, offering greater flexibility and a more realistic description of population dynamics. A model that describes the reaction–diffusion predator–prey system is given in [18]. In this article, the first-order partial derivatives of the model have been replaced by fractional-order derivatives. It involves two unknown functions, u and v, that denote the population density of prey and predator, respectively. The vector x lies in the bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. This model also includes parameters related to the intrinsic growth and death rates of the predator as well as a bifurcation parameter χ. The $-\chi\nabla(v\nabla u)$ denotes the prey-taxis term. Crandall–Rabinowitz's bifurcation theory suggests that all of the remaining coefficients are positive constants. This model was first established by [1], and its properties have been investigated by many researchers; see, for example, [6, 17]. There are numerous readings available that analyze the nonlinear mathematical model describing the reaction–diffusion predator–prey system; see [4, 5, 7, 12, 13, 19, 23, 24, 29, 30, 37]. However, there is little research available on the aforementioned model incorporating prey-taxis. From a biological perspective, fractional derivatives are particularly useful for constructing mathematical models. By incorporating fractional-order derivatives, dynamic systems can effectively capture hereditary properties and memory effects, which are fundamental features of many biological processes; see [3, 16, 26, 33].

The primary goal is to obtain the exact solutions for the mentioned model. These kinds of nonlinear mathematical models can be solved using various techniques. For instance, one can see [8–10, 14, 15, 20, 22, 25, 27, 28, 31, 32, 34, 35] for more information. Here we apply the ϕ^6-model expansion method for finding the exact solutions. This method has been chosen because the obtained solutions are Jacobi elliptic functions whose modulus approaches 0 or 1, resulting in hyperbolic or trigonometric solutions.

Our paper is organised as follows: Section 2 discusses the problem statement as well as the existence and uniqueness of its solution. Many important results of fixed point theory have been implemented for this kind of analysis. Section 3 describes the methodology. The methodology given in this section has been applied to the said model, and solutions have been obtained, which are presented in Section 4. The interpretation and simulations of the solutions are presented and discussed in Section 5. These graphical depictions are elucidated as periodic kink solutions, periodic solitary, symmetric periodic and periodic waves, and dark solitons. These types of solutions frequently arise in mathematical models of biological processes, including neural activity, cellular transport, population dynamics, and cardiovascular flows.

2 Problem statement

The reaction–diffusion predator–prey model [18] involving fractional-order derivative $(0 < \alpha \leqslant 1)$ is given by

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= d_1 \Delta u + r\left(1 - \frac{u}{K}\right)u - \frac{cuv}{m + bu},\\ \frac{\partial^\alpha v}{\partial t^\alpha} &= d_2 \Delta v - \chi \vec{\nabla} \cdot (v \vec{\nabla} u) - av + \frac{\beta cuv}{m + bu}.\end{aligned} \tag{1}$$

Since $\vec{\nabla} \cdot \vec{\nabla} = \Delta$, it can be written as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= d_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + r\left(1 - \frac{u}{K}\right)u - \frac{cuv}{m + bu},\\ \frac{\partial^\alpha v}{\partial t^\alpha} &= d_2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \chi\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right)\\ &\quad - av + \frac{\beta cuv}{m + bu}.\end{aligned} \tag{2}$$

In the above model, $x \in \Omega \subset \mathbb{R}^n$ and $t > 0$. The population density of prey and predator is denoted by u and v, respectively. The diffusion rates of the prey and predator are denoted by d_1 and d_2, respectively. The intrinsic growth rate of the prey and the mortality rate of the predator are denoted by r and a, respectively.

The main goal is to acquire exact solutions and the simulations of the model described above. We will apply the ϕ^6 methodology to find its exact solutions.

Existence results. The existence and uniqueness of the solutions of model (1) (without fractional derivative) and the initial conditions $u(0, x) = u_0(x)$, $v(0, x) = v_0(x)$ have been investigated in [1, 21]. The only difference is that our purposed model is a fractional order with $\alpha \in (0, 1]$. Since $1 - \alpha < n$ $(= 2)$, a weak singularity exists and thus the integral is bounded. Therefore, the prior knowledge and weak singularity imply that the solutions exist and are unique.

The fixed point reduction and unique existence results related to fractional-order nonlinear PDEs; one can see the literature [2, 11].

3 Methodology

Consider the two unknown functions as $u(x, y; t)$ and $v(x, y; t)$. The partial derivatives with respect to the independent variables x, y, and t, including both first and higher-order derivatives, are: u_x, u_y, u_t, v_x, v_y, v_t, u_{xx}, v_{xx}, u_{yy}, v_{yy}, u_{xy}, v_{xy}, and so on.

Let $F(u, v, u_x, v_x, u_y, v_y, u_t, v_t, u_{xx}, v_{xx}, \dots)$ be a polynomial. We obtain a nonlinear partial differential equation if we set this polynomial equal to zero.

The following steps describe the ϕ^6 methodology.

By assuming $\xi = k_1 x + k_2 y - (\gamma/\alpha)t^\alpha$, $u(x, y; t) = U(\xi)$, and $v(x, y; t) = V(\xi)$, we obtain a nonlinear ordinary differential equation

$$G(U, V, U', V', U'', V'', \dots) = 0.$$

Let

$$U(\xi) = \sum_{i=0}^{2N} s_i g^i(\xi), \qquad V(\xi) = \sum_{i=0}^{2N} t_i g^i(\xi),$$

where s_i and t_i are the constants that need to be evaluated first. The function $g^i(\xi)$ satisfies the following equations for real numbers f_0, f_2, f_4, f_6:

$$\begin{aligned} g'^2(\xi) &= f_0 + f_2 g^2(\xi) + f_4 g^4(\xi) + f_6 g^6(\xi), \\ g''(\xi) &= f_2 g(\xi) + 2 f_4 g^3(\xi) + 3 f_6 g^5(\xi). \end{aligned} \tag{3}$$

Equation (3) has the solution

$$g(\xi) = \frac{q(\xi)}{\sqrt{h q^2(\xi) + l}},$$

provided that the denominator is a real number. The function $q(\xi)$ is the solution of the Jacobian elliptic equation; for details, see [36].

$$q'^2 = j_1 + j_2 q^2(\xi) + j_3 q^4(\xi),$$

where j_i, $i = 1, 2, 3$, are constants with their specific values provided in reference [34]. The values for the constants h and l are provided as follows:

$$\begin{aligned} h &= \frac{f_4(j_2 - f_2)}{(j_2 - f_2)^2 + 3 j_1 j_3 - 2 j_2 (j_2 - f_2)}, \\ l &= \frac{3 j_1 f_4}{(j_2 - f_2)^2 + 3 j_1 j_3 - 2 j_2 (j_2 - f_2)} \end{aligned}$$

under the constraint conditions[2]

$$f_4^2 (j_2 - f_2)\big[9 j_0 j_4 - (j_2 - f_2)(2 j_2 + f_2)\big] + 3 f_6 \big[3 j_0 j_4 - \big(j_2^2 - f_2^2\big)\big]^2 = 0.$$

A lot of literature exists that discusses limiting values and multiple exact solutions using ϕ^6 method: for instance, one can refer to [34].

4 Utilization of the approach and obtained solutions

By introducing the transformation $\xi = k_1 x + k_2 y - (\gamma/\alpha)t^\alpha$, we express $u(x, y; t)$ as $U(\xi)$ and $v(x, y; t)$ as $V(\xi)$.

[2] The solution of a partial differential equation must satisfy the PDE and the additional conditions as well.

Since $\partial^\alpha u/\partial t^\alpha = -\gamma\Gamma(\alpha)\mathrm{d}U\mathrm{d}\xi$, the ordinary differential equations corresponding to (2) are

$$\begin{aligned}&\frac{rb}{K}U^3 - r\left(b-\frac{m}{K}\right)U^2 - rmU + cUV - m\gamma\Gamma(\alpha)U' - \gamma\Gamma(\alpha)bUU'\\&\quad - md_1\left(k_1^2+k_2^2\right)U'' - bd_1\left(k_1^2+k_2^2\right)UU'' = 0, \qquad (4)\\&amV - m\gamma\Gamma(\alpha)V' - d_2m\left(k_1^2+k_2^2\right)V'' - b\gamma\Gamma(\alpha)V'U - bd_2\left(k_1^2+k_2^2\right)V''U\\&\quad + m\chi\left(k_1^2+k_2^2\right)U'V' + b\chi\left(k_1^2+k_2^2\right)U'V'U + m\chi\left(k_1^2+k_2^2\right)VU''\\&\quad + \chi b\left(k_1^2+k_2^2\right)VUU'' + abUV - \beta cUV = 0. \qquad (5)\end{aligned}$$

Using the homogeneous balance principle, we get $N = 1$ for both equations. Equations (4) and (5) have the following solutions:

$$U(\xi) = s_0 + s_1 g(\xi) + s_2 g^2(\xi), \qquad V(\xi) = t_0 + t_1 g(\xi) + t_2 g^2(\xi). \qquad (6)$$

After completing tedious calculations, we have $s_0 = t_0 = s_1 = t_1 = 0$. Thus (6) reduces to

$$U(\xi) = s_2\frac{q^2(\xi)}{hq^2(\xi)+l}, \qquad V(\xi) = t_2\frac{q^2(\xi)}{hq^2(\xi)+l}.$$

By choosing $r = 1$, $\gamma = 0.5$, $K = 0.5$, $c = 0.8$, $a = 0.005$, $m = 1$, $b = 0.003$, $\beta = 0.3$, $d_1 = d_2 = 1$, $f_4 = 8$, $f_6 = 5$, $k_1 = 4$, $k_2 = 1$, $f_0 = 22$, $f_2 = 12$, we got

$$s_2 = 348.3437368, -111.0064432.$$

For $s_2 = 348.3437368$, we get $t_2 = 3351.897195 - 0.4360230274\Gamma^2(\alpha)$, and for $s_2 = -111.0064432$, we get $t_2 = 1226.898862 + 0.01994957765\Gamma^2(\alpha)$. We simulate the family of solutions by choosing $\alpha = 0.4, 0.7$, or 0.9.

By considering the different values of j_1, j_2, and j_3 and evaluating the $q(\xi)$ at limiting values 0 and 1, the following family of solutions has been obtained.

Case 1. The following possibilities arise if $j_1 = 1$, $j_2 = -(1+n^2)$, and $j_3 = n^2$.

(a) $q(\xi) = \mathrm{sn}(\xi)$. By considering $n \to 0$ and $n \to 1$, we obtain the following solutions, respectively:

$$U_{(1,0)}(\xi) = s_2\frac{\sin^2(\xi)}{h\sin^2(\xi)+l}, \qquad V_{(1,0)}(\xi) = t_2\frac{\sin^2(\xi)}{h\sin^2(\xi)+l}$$

and

$$U_{(1,1)}(\xi) = s_2\frac{\tanh^2(\xi)}{h\tanh^2(\xi)+l}, \qquad V_{(1,1)}(\xi) = t_2\frac{\tanh^2(\xi)}{h\tanh^2(\xi)+l}.$$

(b) $q(\xi) = \mathrm{cd}(\xi)$. By letting $m \to 0$, we obtain the following solutions:

$$U_{(2,0)}(\xi) = s_2\frac{\cos^2(\xi)}{h\cos^2(\xi)+l}, \qquad V_{(2,0)}(\xi) = t_2\frac{\cos^2(\xi)}{h\cos^2(\xi)+l}.$$

Case 2. The following possibilities arise if we consider $j_1 = 1 - n^2$, $j_2 = 2n^2 - 1$, and $j_3 = -n^2$. It yields $q(\xi) = \mathrm{cn}(\xi)$. Now, by taking the limiting values $n \to 0$ and $n \to 1$, respectively, We obtain the following solutions:

$$U_{(3,0)}(\xi) = s_2 \frac{\cos^2(\xi)}{h\cos^2(\xi) + l}, \qquad V_{(3,0)}(\xi) = t_2 \frac{\cos^2(\xi)}{h\cos^2(\xi) + l}$$

and

$$U_{(3,1)}(\xi) = s_2 \frac{\operatorname{sech}^2(\xi)}{h\operatorname{sech}^2(\xi) + l}, \qquad V_{(3,1)}(\xi) = t_2 \frac{\operatorname{sech}^2(\xi)}{h\operatorname{sech}^2(\xi) + l}.$$

Case 3. If $j_1 = n^2 - 1$, $j_2 = 2 - n^2$, and $j_3 = -1$, we can express $q(\xi)$ as $q(\xi) = \mathrm{dn}(\xi)$. Upon taking the limit as $n \to 1$, we derive the following results:

$$U_{(4,1)}(\xi) = s_2 \frac{\operatorname{sech}^2(\xi)}{h\operatorname{sech}^2(\xi) + l}, \qquad V_{(4,1)}(\xi) = t_2 \frac{\operatorname{sech}^2(\xi)}{h\operatorname{sech}^2(\xi) + l}.$$

Case 4. Assuming that $j_1 = n^2$, $j_2 = -1 - n^2$, and $j_3 = 1$, we can express $q(\xi)$ as $q(\xi) = \mathrm{ns}(\xi)$. Now, by considering $n \to 0$ and $n \to 1$, respectively, solutions are as follows:

$$U_{(5,0)}(\xi) = s_2 \frac{\csc^2(\xi)}{h\csc^2(\xi) + l}, \qquad V_{(5,0)}(\xi) = t_2 \frac{\csc^2(\xi)}{h\csc^2(\xi) + l}$$

and

$$U_{(5,1)}(\xi) = s_2 \frac{\coth^2(\xi)}{h\coth^2(\xi) + l}, \qquad V_{(5,1)}(\xi) = t_2 \frac{\coth^2(\xi)}{h\coth^2(\xi) + l}.$$

Case 5. Consider $j_1 = -n^2$, $j_2 = 2n^2 - 1$, and $j_3 = 1 - n^2$. The function $q(\xi)$ can be expressed as $\mathrm{nc}(\xi)$. Now, by considering $n \to 0$ and $n \to 1$, respectively, we derive the following outcomes:

$$U_{(6,0)}(\xi) = s_2 \frac{\sec^2(\xi)}{h\sec^2(\xi) + l}, \qquad V_{(6,0)}(\xi) = t_2 \frac{\sec^2(\xi)}{h\sec^2(\xi) + l}$$

and

$$U_{(6,1)}(\xi) = s_2 \frac{\cosh^2(\xi)}{h\cosh^2(\xi) + l}, \qquad V_{(6,1)}(\xi) = t_2 \frac{\cosh^2(\xi)}{h\cosh^2(\xi) + l}.$$

Case 6. Assume that $j_1 = -1$, $j_2 = 2 - n^2$, and $j_3 = -(1 - n^2)$. In this case, it follows that $q(\xi)$ equals $\mathrm{nd}(\xi)$. Now, taking the limiting cases as $n \to 0$, we obtain $\mathrm{nd}(\xi) = 1$, and as $n \to 1$, we obtain $\mathrm{nd}(\xi) = \cosh(\xi)$, respectively.

Case 7. Assume that $j_1 = 1$, $j_2 = 2 - n^2$, and $j_3 = 1 - n^2$. The function $q(\xi)$ can be expressed as (ξ). Now, by considering $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(7,0)}(\xi) = s_2 \frac{\tan^2(\xi)}{h\tan^2(\xi) + l}, \qquad V_{(7,0)}(\xi) = t_2 \frac{\tan^2(\xi)}{h\tan^2(\xi) + l}$$

and

$$U_{(7,1)}(\xi) = s_2 \frac{\sinh^2(\xi)}{h\sinh^2(\xi) + l}, \qquad V_{(7,1)}(\xi) = t_2 \frac{\sinh^2(\xi)}{h\sinh^2(\xi) + l}.$$

Case 8. Suppose $j_1 = 1$, $j_2 = 2n^2 - 1$, and $j_3 = -n^2(1 - n^2)$. Then we have $q(\xi) = \mathrm{sd}(\xi)$. Now, by taking the limiting cases $n \to 0$ and $n \to 1$, respectively, we derive the following outcomes:

$$U_{(8,0)}(\xi) = s_2 \frac{\sin^2(\xi)}{h\sin^2(\xi) + l}, \qquad V_{(8,0)}(\xi) = t_2 \frac{\sin^2(\xi)}{h\sin^2(\xi) + l}$$

and

$$U_{(8,1)}(\xi) = s_2 \frac{\sinh^2(\xi)}{h\sinh^2(\xi) + l}, \qquad V_{(8,1)}(\xi) = t_2 \frac{\sinh^2(\xi)}{h\sinh^2(\xi) + l}.$$

Case 9. Assuming that $j_1 = 1 - n^2$, $j_2 = 2 - n^2$, and $j_3 = 1$, we can express $q(\xi)$ as $\mathrm{cs}(\xi)$. Now, by considering $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(9,0)}(\xi) = s_2 \frac{\cot^2(\xi)}{h\cot^2(\xi) + l}, \qquad V_{(9,0)}(\xi) = t_2 \frac{\cot^2(\xi)}{h\cot^2(\xi) + l}$$

and

$$U_{(9,1)}(\xi) = s_2 \frac{\operatorname{csch}^2(\xi)}{h\operatorname{csch}^2(\xi) + l}, \qquad V_{(9,1)}(\xi) = t_2 \frac{\operatorname{csch}^2(\xi)}{h\operatorname{csch}^2(\xi) + l}.$$

Case 10. Consider $j_1 = -n^2(1 - n^2)$, $j_2 = 2n^2 - 1$, and $j_3 = 1$. The function $q(\xi)$ can be expressed as $\mathrm{ds}(\xi)$. Now, by taking the limiting cases $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(10,0)}(\xi) = s_2 \frac{\csc^2(\xi)}{h\csc^2(\xi) + l}, \qquad V_{(10,0)}(\xi) = t_2 \frac{\csc^2(\xi)}{h\csc^2(\xi) + l}$$

and

$$U_{(10,1)}(\xi) = s_2 \frac{\operatorname{csch}^2(\xi)}{h\operatorname{csch}^2(\xi) + l}, \qquad V_{(10,1)}(\xi) = t_2 \frac{\operatorname{csch}^2(\xi)}{h\operatorname{csch}^2(\xi) + l}.$$

Case 11. Consider $j_1 = (1 - n^2)/4$, $j_2 = (1 + n^2)/2$, and $j_3 = (1 - n^2)/4$. The function $q(\xi)$ can be expressed as $\mathrm{nc}(\xi) \pm sc(\xi)$. Alternatively, we can write it as $\mathrm{cn}(\xi)/(1 \pm \mathrm{sn}(\xi))$. Now, by taking the limiting cases $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(11,0)}(\xi) = s_2\phi_1(\xi), \qquad V_{(11,0)}(\xi) = t_2\phi_1(\xi)$$

or

$$U_{(12,0)}(\xi) = s_2\phi_2(\xi), \qquad V_{(12,0)}(\xi) = t_2\phi_2(\xi),$$

where

$$\phi_1(\xi) = \frac{1 + 2\tan(\xi)(\tan(\xi) \pm \sec(\xi))}{h + 2h\tan(\xi)(\tan(\xi) \pm \sec(\xi)) + l}$$

and

$$\phi_2(\xi) = \frac{\cos^2(\xi)}{h\cos^2(\xi) + l(1 \pm \sin(\xi))^2}.$$

Case 12. Assume that $j_1 = (1-n^2)^2/4$, $j_2 = (1+n^2)/2$, and $j_3 = -1/4$. We can express $q(\xi)$ as $m\mathrm{cn}(\xi) \pm \mathrm{dn}(\xi)$. Now, by considering $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(13,0)}(\xi) = s_2 \frac{(m\cos(\xi) \pm 1)^2}{h(m\cos(\xi) \pm 1)^2 + l},$$
$$V_{(13,0)}(\xi) = t_2 \frac{(m\cos(\xi) \pm 1)^2}{h(m\cos(\xi) \pm 1)^2 + l}$$

and

$$U_{(13,1)}(\xi) = s_2\phi_3(\xi), \qquad V_{(13,1)}(\xi) = t_2\phi_3(\xi),$$

where

$$\phi_3(\xi) = \frac{(m\,\mathrm{sech}(\xi) \pm \mathrm{sech}(\xi))^2}{h(m\,\mathrm{sech}(\xi) \pm \mathrm{sech}(\xi))^2 + l}.$$

Case 13. Assume that $j_1 = 1/4$, $j_2 = (1-2n^2)/2$, and $j_3 = 1/4$. Then we have $q(\xi) = \mathrm{sn}(\xi)/(1 \pm \mathrm{cn}(\xi))$. Now, by considering $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(14,0)} = s_2 \frac{\sin^2(\xi)}{h\sin^2(\xi) + l(1 \pm \cos(\xi))^2},$$
$$V_{(14,0)}(\xi) = t_2 \frac{\sin^2(\xi)}{h\sin^2(\xi) + l(1 \pm \cos(\xi))^2}$$

and

$$U_{(14,1)}(\xi) = s_2\phi_4(\xi), \qquad V_{(14,1)}(\xi) = t_2\phi_4(\xi),$$

where

$$\phi_4(\xi) = \frac{\tanh^2(\xi)}{h\tanh^2(\xi) + l(1 \pm \mathrm{sech}(\xi))^2}.$$

Case 14. Consider $j_1 = 1/4$, $j_2 = (1+n^2)/2$, $j_3 = (1-n^2)^2/4$. Then we have $q(\xi) = \mathrm{sn}(\xi)/(\mathrm{cn}(\xi) \pm \mathrm{dn}(\xi))$. Now, by taking the limiting cases $n \to 0$ and $n \to 1$, respectively, we obtain the following solutions:

$$U_{(15,0)}(\xi) = s_2 \frac{\sin^2(\xi)}{h\sin^2(\xi) + l(1 \pm \cos(\xi))^2},$$
$$V_{(15,0)}(\xi) = t_2 \frac{\sin^2(\xi)}{h\sin^2(\xi) + l(1 \pm \cos(\xi))^2}$$

and

$$U_{(15,1)}(\xi) = s_2\phi_5(\xi), \qquad V_{(15,1)}(\xi) = t_2\phi_5(\xi),$$

where

$$\phi_5(\xi) = \frac{\tanh^2(\xi)}{h\tanh^2(\xi) + l(\operatorname{sech}(\xi) \pm \operatorname{sech}(\xi))^2}.$$

Remark. In the limit as $n \to 0$, periodic wave solutions emerge, while as $n \to 1$, solitary wave solutions can be observed.

5 Simulations

The following section showcases simulations of the derived solutions. The y-dimension is maintained at a constant value, and the specifics are outlined as follows:

- Figure 1, $y = 0.1$. It emerges the periodic kink solution.
- Figure 2, $y = 1$. The figure shows the periodic solitary wave for $\alpha = 0.4$, 0.7 and the periodic wave solutions for $\alpha = 0.9$.
- Figure 3, $y = 1$. Dark solitons are depicted in the provided figure.
- Figure 4, $y = 1$. In this plot, we have displayed a periodic wave for $\alpha = 0.4$ and symmetric periodic waves for $\alpha = 0.7$ and 0.9.
- Figure 5, $y = 0.5$. It represents the periodic kink solution.

6 Conclusion

In this research work, we have focused on a system of nonlinear partial differential equations, which is drawn from existing literature. Our investigation primarily revolves around establishing the existence, uniqueness, and exact solutions for this system. Specifically, the partial differential equations we have chosen belong to the realm of reaction–diffusion models, which describe predator–prey dynamics in the presence of prey-taxis. Importantly, we have introduced a fractional-order element into this model by employing the linear operator D^α, where $0 < \alpha \leqslant 1$. To unravel exact solutions, we have effectively applied the ϕ^6 technique.

Upon conducting simulations and analyzing the results, our study has yielded a diverse range of graphical representations. These graphical depictions are elucidated below:

- *Periodic kink solutions*: Figs. 1 and 5 exhibit compelling evidence of periodic kink solutions, which are prominent patterns in our findings.
- *Periodic solitary, symmetric periodic, and periodic waves*: Fig. 2 showcases the presence of periodic solitary waves and other periodic wave solutions, which offer valuable insights into the system's behavior. Symmetric periodic wave solutions have been displayed in Fig. 4.
- *Dark solitons*: in Fig. 3, we have visually depicted dark solitons.

This diverse set of graphical representations underscores the richness and complexity of solutions in our fractional-order predator–prey model with prey-taxis.

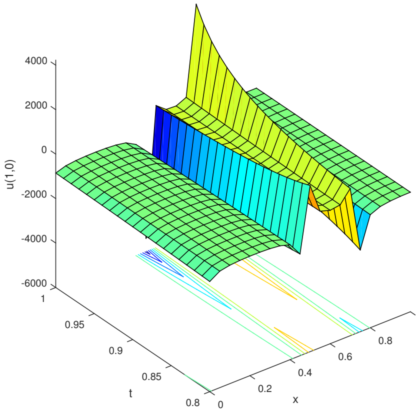
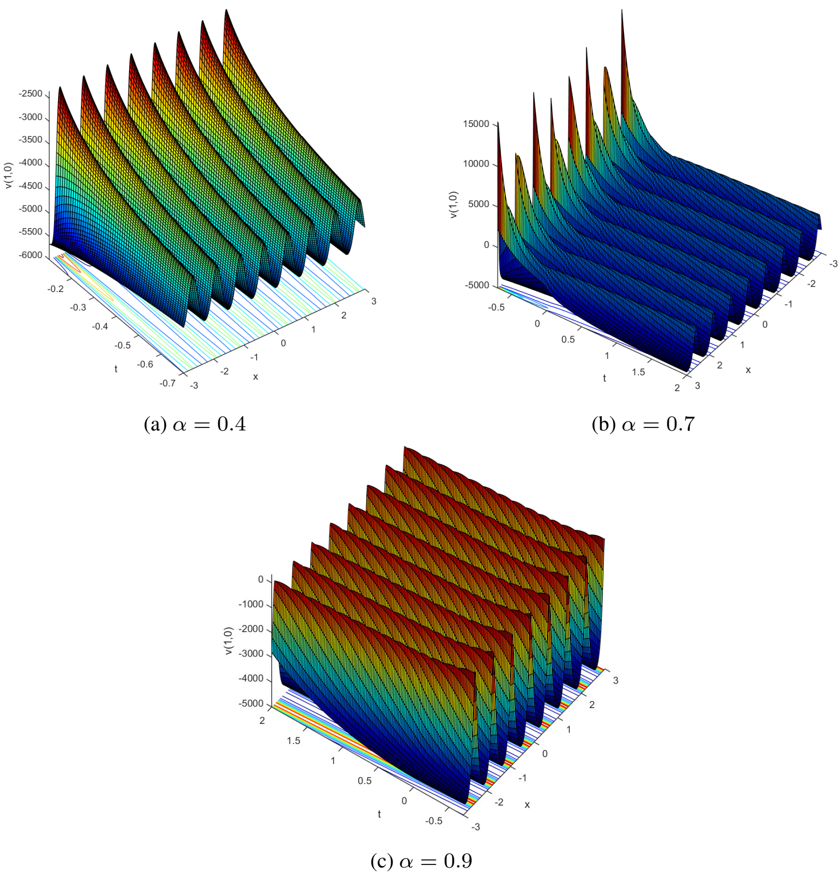


Figure 1. Surface and contour representations of $u(x, y; t)$ at $\alpha = 0.7$ in Case 1.

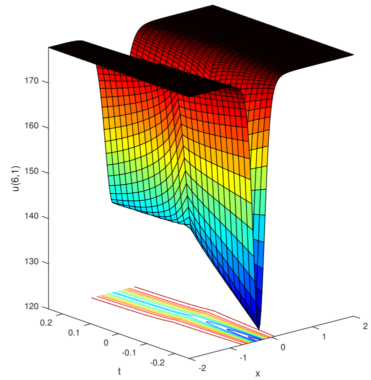


(a) $\alpha = 0.4$

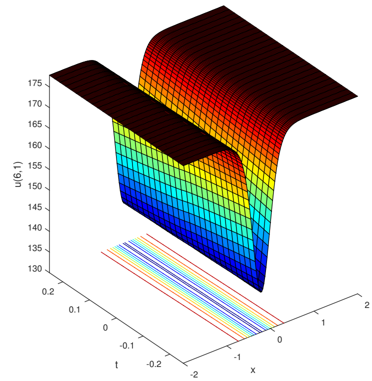
(b) $\alpha = 0.7$

(c) $\alpha = 0.9$

Figure 2. Surface and contour representations of $v(x, y; t)$ at $\alpha = 0.4,\ 0.7,\ 0.9$ in Case 1.

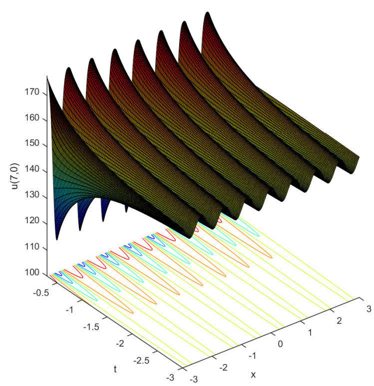


(a) $\alpha = 0.4$

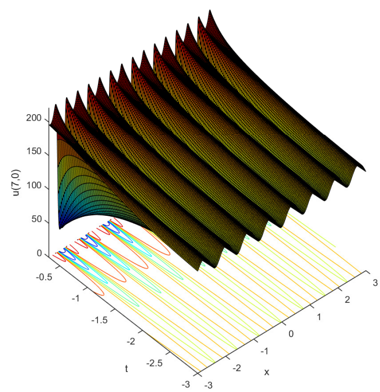


(b) $\alpha = 0.9$

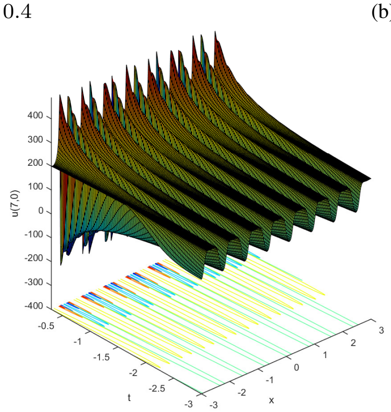
Figure 3. Surface and contour representations of $u(x, y; t)$ at $\alpha = 0.4,\ 0.9$ in Case 5.



(a) $\alpha = 0.4$

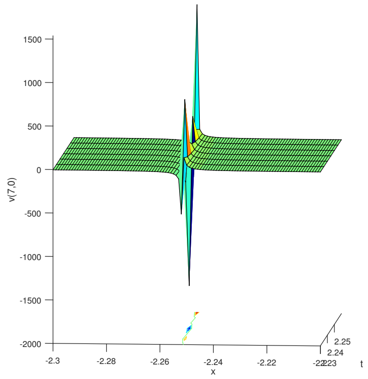


(b) $\alpha = 0.7$

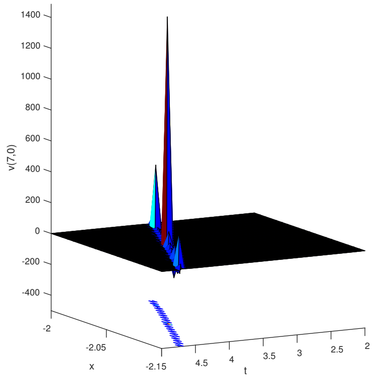


(c) $\alpha = 0.9$

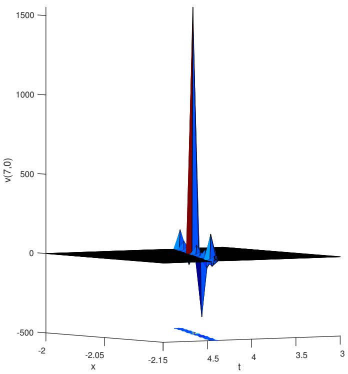
Figure 4. Surface and contour representations of $u(x, y; t)$ at $\alpha = 0.4,\ 0.7,\ 0.9$ in Case 7.



(a) $\alpha = 0.4$



(b) $\alpha = 0.7$



(c) $\alpha = 0.9$

Figure 5. Surface and contour representations of $v(x, y; t)$ at $\alpha = 0.4,\ 0.7,\ 0.9$ in Case 7.

Conflicts of interest. The authors declare no conflicts of interest.

References

1. B. E. Ainseba, A. Noussair M. Bendahmane, A reaction–diffusion system modeling predator–prey with prey-taxis, *Nonlinear Anal., Real World Appl.*, **9**(5):2086–2105, 2008, https://doi.org/10.1016/j.nonrwa.2007.06.017.

2. A. Akgül, U. Fatima, M.S. Iqbal, N. Ahmed, A. Raza, Z. Iqbal, M. Rafiq, A fractal fractional model for computer virus dynamics, *Chaos Solitons Fractals*, **147**:110947, 2021, https://doi.org/10.1016/j.chaos.2021.110947.

3. M.M. Amirian, I.N. Towers, Z. Jovanoski, A.J. Irwin, Memory and mutualism in species sustainability: A time-fractional Lotka-Volterra model with harvesting, *Heliyon*, **6**(9), 2020, https://doi.org/10.1016/j.heliyon.2020.e04816.

4. L. Chen, F. Chen, L. Chen, Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a constant prey refuge, *Nonlinear Anal., Real World Appl.*, **11**(1):246–252, 2010, https://doi.org/10.1016/j.nonrwa.2008.10.056.

5. F. Conforto, L. Desvillettes, C. Soresina, About reaction–diffusion systems involving the Holling-type II and the Beddington–DeAngelis functional responses for predator–prey models, *NoDEA, Nonlinear Differ. Equ. Appl.*, **25**(3):24, 2018, https://doi.org/10.1007/s00030-018-0515-9.

6. X. He, S. Zheng, Global boundedness of solutions in a reaction–diffusion system of predator–prey model with prey-taxis, *Appl. Math. Lett.*, **49**:73–77, 2015, https://doi.org/10.1016/j.aml.2015.04.017.

7. C.S. Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entomol. Soc. Can.*, **97**(S45):5–60, 1965, https://doi.org/10.4039/entm9745fv.

8. M. Inc, S. Hussain, A.H. Ali, M.S. Iqbal, R. Ashraf, M.A. Tarar, M. Adnan, Analyzing solitary wave solutions of the nonlinear Murray equation for blood flow in vessels with non-uniform wall properties, *Sci. Rep.*, **14**(1):10588, 2024, https://doi.org/10.1038/s41598-024-61276-z.

9. M. Inc, M.S. Iqbal, A.H. Ali, Z. Manzoor, F. Ashraf, Solitary wave effects of Woods–Saxon potential in Schrodinger equation with 3d cubic nonlinearity, *Vestn. Yuzhno-Ural. Gos. Univ., Ser. Mat. Model. Program.*, **17**(2):68–82, 2024, https://doi.org/10.14529/mmp240206.

10. M. Inc, M.S. Iqbal, M.Z. Baber, M. Qasim, Z. Iqbal, M.A. Tarar, A.H. Ali, Exploring the solitary wave solutions of Einstein's vacuum field equation in the context of ambitious experiments and space missions, *Alexandria Eng. J.*, **82**:186–194, 2023, https://doi.org/10.1016/j.aej.2023.09.071.

11. Z. Iqbal, N. Ahmed, D. Baleanu, M. Rafiq, M.S. Iqbal, M.A. Rehman, Structure preserving computational technique for fractional order Schnakenberg model, *Comput. Appl. Math.*, **39**:61, 2020, https://doi.org/10.1007/s40314-020-1068-1.

12. Y. Kao, S. Ma, H. Xia, C. Wang, Y. Liu, Integral sliding mode control for a kind of impulsive uncertain reaction-diffusion systems, *IEEE Trans. Autom. Control*, **68**(2):1154–1160, 2022, https://doi.org/10.1109/TAC.2022.3149865.

13. Y. Kao, C. Wang, H. Xia, Y. Cao, *Analysis and Control for Fractional-order Systems*, Springer, Singapore, 2024, https://doi.org/10.1007/978-981-99-6054-5.

14. K. Khan, M.A. Akbar, Study of analytical method to seek for exact solutions of variant Boussinesq equations, *SpringerPlus*, **3**:324, 2014, https://doi.org/10.1186/2193-1801-3-324.

15. K. Khan, M.A. Akbar, Traveling wave solutions of the $(2+1)$-dimensional Zoomeron equation and the Burgers equations via the MSE method and the Exp-function method, *Ain Shams Eng. J.*, **5**(1):247–256, 2014, https://doi.org/10.1016/j.asej.2013.07.007.

16. G.R. Kumar, K. Ramesh, A. Khan, K. Lakshminarayan, T. Abdeljawad, Dynamical study of fractional order Leslie-Gower model of predator-prey with fear, Allee effect, and inter-species rivalry, *Results Control Optim.*, **14**:100403, 2024, https://doi.org/10.1016/j.rico.2024.100403.

17. C. Li, X. Wang, Y. Shao, Steady states of a predator–prey model with prey-taxis, *Nonlinear Anal., Theory Methods Appl.*, **97**:155–168, 2014, https://doi.org/10.1016/j.na.2013.11.022.
18. D. Luo, Global bifurcation for a reaction–diffusion predator–prey model with Holling-II functional response and prey–taxis, *Chaos Solitons Fractals*, **147**:110975, 2021, https://doi.org/10.1016/j.chaos.2021.110975.
19. Q. Meng, L. Yang, Steady state in a cross-diffusion predator–prey model with the Beddington–DeAngelis functional response, *Nonlinear Anal., Real World Appl.*, **45**:401–413, 2019, https://doi.org/10.1016/j.nonrwa.2018.07.012.
20. M. Nadeem, O.A. Arqub, A.H. Ali, H.A. Neamah, Bifurcation, chaotic analysis and soliton solutions to the $(3+1)$-dimensional p-type model, *Alexandria Eng. J.*, **107**:245–253, 2024, https://doi.org/10.1016/j.aej.2024.07.032.
21. Y. Tao, Global existence of classical solutions to a predator–prey model with nonlinear prey-taxis, *Nonlinear Anal., Real World Appl.*, **11**(3):2056–2064, 2010, https://doi.org/10.1016/j.nonrwa.2009.05.005.
22. M. Wang, X. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, *Chaos Solitons Fractals*, **24**(5):1257–1268, 2005, https://doi.org/10.1016/j.chaos.2004.09.044.
23. M. Wang, X. Li, J. Zhang, The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A*, **372**(4):417–423, 2008, https://doi.org/10.1016/j.physleta.2007.07.051.
24. Q. Wang, Y. Song, L. Shao, Nonconstant positive steady states and pattern formation of 1D prey-taxis systems, *J. Nonlinear Sci.*, **27**(1):71–97, 2017, https://doi.org/10.1007/s00332-016-9326-5.
25. A.M. Wazwaz, The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations, *Appl. Math. Comput.*, **167**(2):1196–1210, 2005, https://doi.org/10.1016/j.amc.2004.08.005.
26. Z. Wu, Z. Wang, Y. Cai, H. Yin, W. Wang, Pattern formation in a fractional-order reaction-diffusion predator-prey model with Holling-III functional response, *Adv. Contin. Discrete Models*, **2025**(1):1–13, 2025, https://doi.org/10.1186/s13662-025-03891-2.
27. S.-L. Xu, J.-C. Liang, Y. Lin, Exact soliton solutions to a generalized nonlinear Schrödinger equation, *Commun. Theor. Phys.*, **53**(1):159, 2010, https://doi.org/10.1088/0253-6102/53/1/33.
28. Z. Yan, Abundant families of Jacobi elliptic function solutions of the $(2+1)$-dimensional integrable Davey–Stewartson-type equation via a new method, *Chaos Solitons Fractals*, **18**(2):299–309, 2003, https://doi.org/10.1016/S0960-0779(02)00653-7.
29. P. Yang, Hopf bifurcation of an age-structured prey–predator model with Holling type II functional response incorporating a prey refuge, *Nonlinear Anal., Real World Appl.*, **49**:368–385, 2019, https://doi.org/10.1016/j.nonrwa.2019.03.014.
30. P. Yang, Y. Wang, Existence and properties of Hopf bifurcation in an age-dependent predation system with prey harvesting, *Commun. Nonlinear Sci. Numer. Simul.*, **91**:105395, 2020, https://doi.org/10.1016/j.cnsns.2020.105395.

31. X.-F. Yang, Z.-C. Deng, Y. Wei, A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application, *Adv. Difference Equ.*, **2015**:117, 2015, https://doi.org/10.1186/s13662-015-0452-4.

32. X.F. Yang, Y. Wei, Bilinear equation of the nonlinear partial differential equation and its application, *J. Funct. Spaces*, **2020**(1):4912159, 2020, https://doi.org/10.1155/2020/4912159.

33. Y. Yu, W. Deng, Y. Wu, Positivity and boundedness preserving schemes for space–time fractional predator–prey reaction–diffusion model, *Comput. Math. Appl.*, **69**(8):743–759, 2015, https://doi.org/10.1016/j.camwa.2015.02.024.

34. E.M.E. Zayed, A.-G. Al-Nowehy, M.E.M. Elshater, New ϕ^6-model expansion method and its applications to the resonant nonlinear Schrödinger equation with parabolic law nonlinearity, *Eur. Phys. J. Plus*, **133**(10):417, 2018, https://doi.org/10.1140/epjp/i2018-12288-2.

35. H. Zhang, New application of the $(\frac{G'}{G})$-expansion method, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(8):3220–3225, 2009, https://doi.org/10.1016/j.cnsns.2009.01.006.

36. Q. Zhou, Q. Zhu, A. Biswas, Optical solitons in birefringent fibers with parabolic law nonlinearity, *Opt. Appl.*, **44**(3), 2014, https://doi.org/10.5277/oa140305.

37. Y. Zhu, K. Wang, Existence and global attractivity of positive periodic solutions for a predator–prey model with modified Leslie–Gower Holling-type II schemes, *J. Math. Anal. Appl.*, **384**(2):400–408, 2011, https://doi.org/10.1016/j.jmaa.2011.05.081.