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A revisit to tail risk measures in the presence of bivariate regularly varying tailed insurance and financial risks*

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Abstract. Consider a discrete-time insurance risk model in which the one-period insurance and financial risks are assumed to be independent and identically distributed random pairs, but a strong dependence structure is allowed to exist between each pair. Recently, Q. Tang and Y. Yang employed a framework of bivariate regular variation to model the heavy tails and the dependence of the insurance and financial risks, and they also established an asymptotic formula for the finite-time ruin probability [Interplay of insurance and financial risks in a stochastic environment, *Scand. Actuar. J.*, 2019(5):432–451, 2019]. In this paper, by adopting a different approach, we study the asymptotic behavior of some tail risk measures for the aggregate discounted net loss, including the tail probability and the conditional loss-based tail expectation. We show both analytically and numerically how the heavy tailedness and the dependence of each pair of insurance and financial risks affect the tail risk measures.

Keywords: asymptotics, aggregate discounted net loss, insurance and financial risks, tail risk measure, bivariate regular variation.

1 Introduction

Assessing right tail risks of aggregate losses of an insurer has become an extremely important task in risk management since it can provide insurers and regulators with insightful guidance on risk capital calculation and the pricing of insurance products, among

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others. For example, to meet regulatory capital requirements such as Solvency II, insurers are required to maintain adequate risk capital to prevent bankruptcy. As investments have become more and more significant in insurance business nowadays[2], two types of fundamental risks should be carefully addressed in conducting solvency assessment: the insurance risk that is caused by insurance claims and the financial risk that is due to risky investments. A discrete-time insurance risk model serves as an effective platform for accommodating these two risks.

Consider a discrete-time risk model in which, within each period $i \in \mathbb{N}$, the insurance risk is quantified as the insurer's net loss X_i equal to the total amount of claims plus expenses minus premiums over the period, and the financial risk is quantified as the stochastic present value factor Y_i equal to the reciprocal of the stochastic accumulation factor calculated according to overall returns on investments over the same period. In this way, the randomly weighted sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n \in \mathbb{N}, \tag{1}$$

represents the stochastic present value of aggregate net insurance losses up to time n.

Such a discrete-time risk model (1) is first introduced by [19,20,26,27], who study the asymptotic behavior of the probability of ruin by time n, defined by $\mathbf{P}(\max_{1\leqslant k\leqslant n} S_k > x)$, as the initial capital x of the insurer becomes large. In the majority of works on this study, it is usually assumed that the insurance and financial risk vectors (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of independent and identically distributed (i.i.d.) copies of a generic pair (X, Y), while some specific dependence may exist between X and Y. In the presence of heavy-tailed insurance risks, [4] and [5] obtain some asymptotic formulas for the finite-time ruin probability under the condition that each pair of the insurance and financial risks follows a common bivariate Farlie–Gumbel–Morgenstern (FGM) distribution. Fruitful works have extended the FGM dependence between X and Y to various asymptotic independence structures; see, e.g., [6, 14, 32, 33], among others.

As a lesson from the global financial crisis of 2007–2009, the key issue, which is attractive to all practitioners, academics, and regulators, lies in understanding the stronger tail dependence between the insurance and financial risks and subsequently modeling it mathematically. This calls for investigation into the concept of multivariate regular variation (MRV), which provides a rather flexible framework for allowing a variety of strong tail dependence structures between variables. In recent literature, many concerns have been addressed in the MRV structure. [12, 31] and [28] carry on their studies under the MRV framework for (X, Y) and find that the decay rate of the finite-time ruin probability is much slower than that in the case of asymptotic dependence.

Motivated by [28], this paper studies the asymptotic behavior of the tail probability of the aggregate discounted net losses defined in (1), rather than the finite-time ruin

[2]The Annual Report 2023 of Allianz Group stated that as of December 31, 2023, the total assets held for investment amounted to 736.8 billion euros, increased by 33.5 billion euros compared to year-end 2022, mainly in the debt instruments. Available at `https://www.allianz.com/en/investor_relations/results-reports/annual-reports.html`.

probability, and address the interplay of the insurance and financial risks in terms of the MRV framework. The obtained result shows in a transparent way how the tail probability is affected by the tail dependence between each pair of insurance and financial risks. We remark that our main result cannot be deduced from Theorem 1(i) of [28], and our proof employs a new approach developed by [32] in random difference equations and mathematical induction.

A second goal of this paper is to study the conditional tail risk measures in the context of extreme risks and to highlight the application potential of such measures. Precisely speaking, we shall apply the result on tail probability to the assessment of a conditional tail expectation-type risk measure of the aggregate discounted net loss, given extreme scenarios of this aggregate loss. We generalize the conditional tail risk measure of interest by incorporating a loss function into the conditional tail expectation. In the terminology of [18], our generalized conditional tail risk measure can be regarded as a specialization of the *extreme Wang distortion risk measure*. By specifying the loss function, various commonly-used risk measures are included, such as the *conditional tail expectation*, the *conditional tail variance*, and even more general the *conditional tail moments* of aggregate discounted net losses. These tail risk measures are often used as alternatives of the initial reserve for insurers to measure the expectation of downside risk and hence avoid insolvency with a given level of risk tolerance in practice.

The rest of the paper is organized as follows. In Section 2, we prepare some preliminaries needed for our study, in Section 3, we present the main results. In Section 4, we conduct some numerical studies to illustrate our main findings, and in Section 5, we complete all proofs.

2 Preliminaries

2.1 Notational convention

Throughout this paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, as usual, we write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$; write $f(x) = o(1)g(x)$ if $\lim f(x)/g(x) = 0$; write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \leqslant 1$; and write $f(x) \asymp g(x)$ if $0 < \liminf f(x)/g(x) \leqslant \limsup f(x)/g(x) < \infty$. Furthermore, for two positive bivariate function $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$, we write $f(x,t) \sim g(x,t)$ uniformly for all t in a nonempty set A if

$$\lim_{x\to\infty} \sup_{t\in A} \left| \frac{f(x,t)}{g(x,t)} - 1 \right| = 0.$$

For any set A, denote its indicator function by $\mathbf{1}_A$. We use $\mathbb{R}_+$ to represent the interval $[0,\infty)$ and use $\lfloor\cdot\rfloor$ to represent the floor function. For a nondecreasing function $f : \mathbb{R} \mapsto \mathbb{R}$, denote by $f^{\leftarrow}$ its càglàd inverse defined as

$$f^{\leftarrow}(y) = \inf\bigl\{x \in \mathbb{R}\colon\ f(x) \geqslant y\bigr\}, \quad y \in \mathbb{R},$$

where $\inf \emptyset = \infty$. For any $x, y \in \mathbb{R}$, write $x \vee y = \max\{x, y\}$ and $x^+ = x \vee 0$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, we write $[\boldsymbol{x}, \boldsymbol{y}] = [x_1, y_1] \times [x_2, y_2]$, $[\boldsymbol{x}, \boldsymbol{\infty}) = [x_1, \infty) \times [x_2, \infty)$, and so on.

2.2 Heavy-tailed distributions

In this paper, we shall assume that all insurance and financial risks are heavy-tailed. For this purpose, let us recall several important classes of heavy-tailed distributions. Necessarily, a distribution F discussed in this subsection is assumed to possess an ultimate right tail in the sense that $\overline{F}(x) = 1 - F(x) > 0$ for all $x > 0$.

A distribution F on $\mathbb{R}$ is said to be long tailed, written as $F \in \mathscr{L}$, if it holds for any $y \in \mathbb{R}$ that

$$\overline{F}(x+y) \sim \overline{F}(x). \tag{2}$$

Automatically, relation (2) holds uniformly on every compact set of y. This implies that there exists a nonnegative auxiliary function $l(\cdot)$ with $l(x) = o(x)$ and $l(x) \uparrow \infty$ such that (2) holds uniformly for all $-l(x) \leqslant y \leqslant l(x)$. A distribution F on $\mathbb{R}$ is said to be dominatedly varying tailed, written as $F \in \mathscr{D}$, if $\overline{F}(xy) = O(1)\overline{F}(x)$ holds for any $0 < y < 1$. The intersection $\mathscr{L} \cap \mathscr{D}$ forms a useful class of heavy-tailed distributions; see the monographs [2, 10, 11] for related discussions. An important subclass of $\mathscr{L} \cap \mathscr{D}$ is the class $\mathscr{R}$ of regularly varying tailed distributions, which contains almost all practically useful distributions in $\mathscr{L} \cap \mathscr{D}$. A distribution F on $\mathbb{R}$ is said to be regularly varying tailed with index $0 < \alpha < \infty$, written as $F \in \mathscr{R}_{-\alpha}$, if $\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$ holds for any $y > 0$. Generally, a positive function f on $\mathbb{R}$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $f \in \mathrm{RV}_{\alpha}$, if $f(xy) \sim y^{\alpha} f(x)$ holds for any $y > 0$. When $\alpha = 0$, this defines a slowly varying function at ∞. See [2] and [22] for textbook treatments of regular variation. Trivially, for a distribution function $F \in \mathscr{R}_{-\alpha}$ for some $\alpha > 0$, we can also write $\overline{F} \in \mathrm{RV}_{-\alpha}$.

In the terminology of [2], for a distribution F, the upper Matuszewska index of the function $(\overline{F}(x))^{-1}$ is defined as

$$J_F^+ = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y}\colon\ y > 1\right\} \quad \text{with } \overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)},\ y > 0.$$

This index is very useful in describing the tail behavior of a distribution function, especially that in the class $\mathscr{D}$. It is known that, for a distribution F, its upper Matuszewska index $0 \leqslant J_F^+ \leqslant \infty$, $F \in \mathscr{D}$ if and only if $J_F^+ < \infty$, and if $\overline{F} \in \mathrm{RV}_{-\alpha}$, then $J_F^+ = \alpha$. In addition, Proposition 2.2.1 of [2] gives that if $F \in \mathscr{D}$, then for any $p > J_F^+$, there exist two positive constants C and x_1 such that

$$\frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant Cy^{-p} \vee 1 \tag{3}$$

holds for all $x \geqslant x_1$ and $xy \geqslant x_1$. It is easy to see from (3) that the relation

$$x^{-p} = o(1)\overline{F}(x)$$

holds for any $p > J_F^+$. Moreover, Karamata's theorem gives that if $\overline{F} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 1$, then

$$\int_x^\infty \overline{F}(y)\,\mathrm{d}y \sim \frac{x\overline{F}(x)}{\alpha - 1}. \tag{4}$$

Initiated by [7], the concept of MRV substantially extends that of univariate regular variation and provides us with a rather flexible framework for modeling multivariate heavy-tailed risks and their tail dependence structure in a unified manner. The MRV has found its applications in insurance, finance, and risk management that involve extreme risks. For example, [12, 28] and [31] apply MRV in insurance and financial risk modeling, and [29] and [24] find its application in credit risk management. In this paper, to characterize each pair of insurance and financial risks, we briefly introduce here the two-dimensional version of MRV.

For simplicity, for a distribution F, we abbreviate the càglàd inverse of $1/\overline{F}$ to χ_F, i.e.,

$$\chi_F(x) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(x), \quad x \in \mathbb{R}_+.$$

A nonnegative random vector (X, Y) is said to possess a bivariate regularly varying (BRV) tail if there exist two representative distribution functions F, G and a nondegenerate (i.e., not identically 0) limit measure ν such that

$$\lim x\,\mathbf{P}\left(\left(\frac{X}{\chi_F(x)}, \frac{Y}{\chi_G(x)}\right) \in B\right) = \nu(B) \tag{5}$$

holds for every Borel set $B \subset [\mathbf{0}, \boldsymbol{\infty}]$ that is away from $\mathbf{0}$ and ν-continuous (i.e., ν assigns zero mass to the boundary ∂B). The definition of BRV in (5) implies that both $\overline{F}$ and $\overline{G}$ are regularly varying. Assume that $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$, for which case we write $(X, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(F, G, \nu)$. In (5), due to the nondegeneracy of the limit measure on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$, specifying B to $((1, 0), \boldsymbol{\infty}]$ and $((0, 1), \boldsymbol{\infty}]$, respectively, we can obtain the tail behaviors of X and Y

$$\lim \frac{\mathbf{P}(X > x)}{\overline{F}(x)} = \nu\big(((1, 0), \boldsymbol{\infty}]\big)$$

and

$$\lim \frac{\mathbf{P}(Y > x)}{\overline{G}(x)} = \nu\big(((0, 1), \boldsymbol{\infty}]\big).$$

Thus, the representative tails $\overline{F}$ and $\overline{G}$ in (5) are proportional, though not necessarily identical, to the marginal tails of (X, Y) if the two limits are positive. The reader is referred to [7] and [22, 23] for the introduction and comprehensive treatments of BRV or even MRV.

Remarkably, the information of tail dependence in the upper-right tail of (X, Y) is contained in the limit measure ν. It allows a variety of tail dependence structures for (X, Y). Under (5), if $\nu((\mathbf{1}, \boldsymbol{\infty}]) > 0$, then

$$\lim \frac{\mathbf{P}(X > \chi_F(x),\ Y > \chi_G(x))}{\mathbf{P}(X > \chi_F(x))} = \frac{\nu((\mathbf{1}, \boldsymbol{\infty}])}{\nu(((1, 0), \boldsymbol{\infty}])} > 0.$$

This means that (X, Y) exhibits large joint movements, and thus, X and Y are *asymptotically dependent*. In the case that the limit measure ν is concentrated on a first-quadrant ray from the origin, X and Y are so-called *fully tail dependent*. In contrast, if $\nu((\mathbf{1}, \boldsymbol{\infty}]) = 0$, then (X, Y) does not exhibit large joint movements indicating that its components are *asymptotically independent*.

3 Main results

This section collects two main results on the tail behavior of the aggregate discounted net loss S_n in the discrete-time risk model (1) in which the insurance and financial risks (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of i.i.d. copies of a generic random vector (X, Y) possessing a BRV structure. For brevity, we define

$$\kappa(x) = \frac{1}{(\chi_F \chi_G)^{\leftarrow}(x)}, \quad x \in \mathbb{R}. \tag{6}$$

If $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ for some $\alpha, \beta > 0$, then by Proposition 0.8(v) of [22] we have that $\chi_F \in \mathrm{RV}_{1/\alpha}$ and $\chi_G \in \mathrm{RV}_{1/\beta}$, and hence $\kappa \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$.

The following first result establishes an asymptotic formula for the tail probability of the aggregate discounted net loss.

Theorem 1. *Consider the discrete-time risk model* (1) *in which the generic pair* (X^+, Y) *possesses* $\mathrm{BRV}_{-\alpha,-\beta}(F, G, \nu)$ *for some nondegenerate measure* ν *and two representing tail distributions* $\overline{F} \in \mathrm{RV}_{-\alpha}$ *and* $\overline{G} \in \mathrm{RV}_{-\beta}$, $\alpha, \beta > 0$. *If* $\nu((\mathbf{1}, \boldsymbol{\infty}]) > 0$, *then it holds for every* $n \in \mathbb{N}$ *that as* $x \to \infty$,

$$\mathbf{P}(S_n > x) \sim \frac{1 - (\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1 - \mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]} \nu(A) \kappa(x), \tag{7}$$

where the set $A = \{(u, v) \in [\mathbf{0}, \boldsymbol{\infty}]\colon\ uv > 1\}$, *and* $\kappa(x)$ *is defined in* (6).

Remark 1. Theorem 1 indicates that the tail probability of the aggregate discounted net loss converges to 0 with a power-like function of order $-\alpha\beta/(\alpha + \beta)$. This is in sharp contrast to some recent results on *asymptotically independent* X and Y such as [32] in which the asymptotic behavior of $\mathbf{P}(S_n > x)$ is another power-like function of order $-\alpha$. The heavier tail of the aggregate discounted net loss obtained by Theorem 1 manifests the strong impact of the tail dependence between the insurance and financial risks.

The coefficient $\nu(A)$ on the right-hand side of (7) captures the interplay of the insurance and financial risks, which is explicit and computable, though a bit intricate.

Although [28] derives a similar result on the finite-time ruin probability, it is worth to remark that our Theorem 1 cannot be deduced from theirs due to the lower bound of $\mathbf{P}(S_n > x)$. Indeed, we adopt a different approach in random difference equations and mathematical induction to prove Theorem 1, whose idea comes from [32]; see Section 5.1 for a detailed discussion.

The second result studies the asymptotic behavior of the conditional loss-based tail expectation $\mathbf{E}[\varphi(S_n) \mid S_n > x]$, where $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ is a nondecreasing loss/cost function with $\varphi(\infty) = \infty$. A similar concept can be found in behavior economics; see, e.g., [1, 8, 13]. As pointed out by [15], for a nonnegative random variable ξ, an unconditional loss-based expectation $\mathbf{E}[\varphi(\xi)]$ coincides with the well-known expected utility in insurance, which covers many commonly-used risk measures such as the *expected value principle*, the *mean-variance measure*, the *Dutch risk measure*, and the *Esscher premium principle*.

Our second result aims to measure the risk of the aggregate discounted insurance net loss S_n under the condition that it has located in an extreme region. The purpose of incorporating a loss function φ into the conditional tail expectation is to better align the risk measure with the actual preferences of decision-makers, in a similar spirit of the use of utility functions in portfolio theory. In practice, the threshold x in $\mathbf{E}[\varphi(S_n) \mid S_n > x]$ is usually chosen as the *value at risk* (VaR) of S_n under a confidence level $q \in (0,1)$, i.e.,

$$\mathrm{VaR}_q(S_n) = F_{S_n}^{\leftarrow}(q) = \inf\bigl\{x\colon \mathbf{P}(S_n \leqslant x) \geqslant q\bigr\}. \tag{8}$$

In this scenario, if $\varphi(x) = x$, then $\mathbf{E}[\varphi(S_n) \mid S_n > \mathrm{VaR}_q(S_n)]$ reduces to the *conditional tail expectation* $(\mathrm{CTE}_q(S_n))$; and if $\varphi(x) = x^2$, then $\mathbf{E}[\varphi(S_n)|S_n > \mathrm{VaR}_q(S_n)] - (\mathrm{CTE}_q(S_n))^2$ becomes the so-called *tail variance* risk measure of S_n. If the loss function φ is chosen to be a power function of order three or four, then the conditional loss-based tail expectation is related to *tail skewness* or *tail kurtosis* of S_n.

In the terminology of a recent work of [18], they propose a general risk measure, called an *extreme Wang distortion risk measure*,

$$\rho_q(S_n) = \int_0^1 \varphi \circ F_{S_n}^{\leftarrow}\bigl(1 - p(1-q)\bigr)\,\mathrm{d}g(p),$$

where $g : [0,1] \mapsto [0,1]$ is a nondecreasing distortion function with $g(0) = 0$ and $g(1) = 1$. It can be verified that if $g(x) = x$, then for the continuous S_n, our exploring $\mathbf{E}[\varphi(S_n) \mid S_n > \mathrm{VaR}_q(S_n)]$ echoes a specialization of *extreme Wang distortion risk measures* for the aggregate discounted net loss.

To generalize a power loss function φ in a logical and useful manner, we introduce the basic technical assumption below, which comes from [9].

Assumption 1. Let $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ be a nondecreasing function satisfying:

- φ is eventually differentiable, i.e., it is differentiable for sufficiently large x;
- φ is eventually subhomogeneous, i.e., $\varphi(2x) \leqslant M\varphi(x)$ for some $M > 0$ and all sufficiently large x.

Theorem 2. *Consider the discrete-time risk model* (1). *Under the conditions of Theorem* 1 *and Assumption* 1, *if for all $i \in \mathbb{N}$ and all sufficiently large x,*

$$\mathbf{E}\left[\varphi\left(X_i \prod_{j=1}^{i} Y_j\right) \mathbf{1}_{(X_i \prod_{j=1}^{i} Y_j > x)}\right] < \infty, \tag{9}$$

then it holds for every $n \in \mathbb{N}$ that as $x \to \infty$,

$$\mathbf{E}\bigl[\varphi(S_n) \bigm| S_n > x\bigr] \sim \varphi(x) + \frac{1}{\kappa(x)} \int_x^{\infty} \varphi'(u)\kappa(u)\,\mathrm{d}u, \tag{10}$$

where the function κ is defined in (6).

By choosing a power loss function, a more explicit asymptotic formula for the conditional tail moment can be obtained from Theorem 2.

Corollary 1. *Under the conditions of Theorem* 1*, it holds for any* $r \in [0, \alpha\beta/(\alpha+\beta))$ *and every* $n \in \mathbb{N}$ *that as* $x \to \infty$*,*

$$\mathbf{E}\big[S_n^r \mid S_n > x\big] \sim \frac{\alpha\beta x^r}{\alpha\beta - (\alpha+\beta)r}. \tag{11}$$

Further, if both F *and* G *are eventually strictly increasing, then it holds that as* $q \to 1$*,*

$$\begin{aligned}&\mathbf{E}\big[S_n^r \mid S_n > \mathrm{VaR}_q(S_n)\big]\\ &\quad\sim \frac{\alpha\beta(F^{\leftarrow}(q)G^{\leftarrow}(q))^r}{\alpha\beta-(\alpha+\beta)r}\left(\frac{1-(\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1-\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\nu(A)\right)^{(\alpha+\beta)r/(\alpha\beta)}.\end{aligned} \tag{12}$$

An explicit asymptotic formula for the popular risk measure $\mathrm{CTE}_q(S_n)$ can be derived from (12) with $r = 1$.

4 Numerical studies

In this section, we perform some numerical studies. Firstly, we check the accuracy of approximations obtained from Theorem 1 and Corollary 1 for the tail probability $\mathbf{P}(S_n > x)$ and the conditional tail moment $\mathbf{E}[S_n^r \mid S_n > x]$ by Monte Carlo simulation; secondly, we adopt a new measurement to assess the insolvency risk of the underestimation of tail dependence; finally, we conduct a sensitivity analysis on two risk measures $\mathrm{VaR}_q(S_n)$ and $\mathrm{CTE}_q(S_n)$ to key model parameters including the regular variation indices and the tail dependence parameter.

Consider the aggregate discounted net loss S_n in (1). The dependence of the generic insurance and financial risk vector (X, Y) is described by a Gumbel copula

$$C(u,v) = \exp\big\{-\big((-\ln u)^\gamma + (-\ln v)^\gamma\big)^{1/\gamma}\big\}, \quad (u,v) \in (0,1)^2, \tag{13}$$

where $\gamma \in [1, \infty]$ determines the strength of upper tail dependence. Specifically, $\gamma = 1$ corresponds to the independence case, $\gamma > 1$ corresponds to the asymptotic dependence case with the upper tail dependence coefficient $2 - 2^{1/\gamma} > 0$, while $\gamma = \infty$ corresponds to the comonotonicity case; see, e.g., Example 7.37 of [17] for more details.

In our numerical studies, we use the Pareto II distribution to model both the insurance risk and the financial risk. Recall that a Pareto II distribution with tail index $\alpha > 0$, minimum parameter $\mu \in \mathbb{R}$, and scale parameter $\sigma > 0$, denoted by $\mathrm{Pareto}(\alpha, \mu, \sigma)$, is of the form

$$F(x; \alpha, \mu, \sigma) = 1 - \left(1 + \frac{x-\mu}{\sigma}\right)^{-\alpha}, \quad x \geqslant \mu.$$

Concrete model specifications are listed below:

- X follows $\mathrm{Pareto}(\alpha, \mu_1, \sigma_1)$ with $\alpha = 3$, $\mu_1 = -1$, $\sigma_1 = 4$, and Y follows $\mathrm{Pareto}(\beta, \mu_2, \sigma_2)$ with $\beta = 10.85$, $\mu_2 = 0$, $\sigma_2 = 6.5$;
- (X, Y) possesses the Gumbel copula (13) with $\gamma \in (1, \infty)$.

4.1 Accuracy of the asymptotic estimates

We first check the accuracy of the asymptotic estimate obtained from Theorem 1. All model specifications are listed above. Remark that the Gumbel copula (13) possessed by (X, Y) is a special Archimedean copula with its generator $\phi(u) = (-\ln u)^\gamma$, which satisfies $\lim_{s\downarrow 0} \phi(1-st)/\phi(1-s) = t^\gamma$ for any $t > 0$. Then, since (X, Y) has Pareto-distributed margins, Lemma 4.1 of [28] ensures that (X, Y) possesses the BRV structure with $\alpha = 3$, $\beta = 10.85$, and ν described by

$$\nu\big(\big[\mathbf{0},(s,t)\big]^c\big) = \big(s^{-\alpha\gamma} + t^{-\beta\gamma}\big)^{1/\gamma}, \quad (s,t) > \mathbf{0}, \tag{14}$$

where $[\mathbf{0},(s,t)]^c = \mathbb{R}_+^2 \setminus [\mathbf{0},(s,t)]$ denotes the complement of $[\mathbf{0},(s,t)]$. It is easy to verify that all conditions of Theorem 1 are satisfied, and thus, the asymptotic formula (7) is applicable. The accuracy of formula (7) is checked by comparing the tail probability $\mathbf{P}(S_n > x)$ on the left-hand side, computed by simulation, and its asymptotic estimate on the right-hand side.

To calculate the value of the right-hand side of (7), we need to determine $\nu(A)$ and $\kappa(x)$. Under the above-mentioned specifications, by (14), $\nu(A)$ can be calculated as

$$\nu(A) = \alpha\beta(\gamma-1) \iint_{st>1} \big(s^{-\alpha\gamma} + t^{-\beta\gamma}\big)^{1/\gamma-2} s^{-\alpha\gamma-1} t^{-\beta\gamma-1}\,\mathrm{d}s\,\mathrm{d}t, \tag{15}$$

and by (6), the function $\kappa(x)$ can be further approximated by

$$\kappa(x) \sim \left(\frac{x}{\sigma_1\sigma_2}\right)^{-\alpha\beta/(\alpha+\beta)}. \tag{16}$$

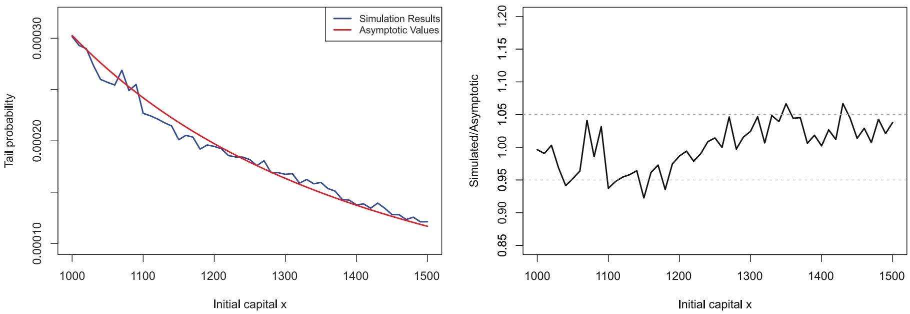
Indeed, since X and Y follow $\mathrm{Pareto}(\alpha, \mu_1, \sigma_1)$ and $\mathrm{Pareto}(\beta, \mu_2, \sigma_2)$, respectively, we have that $\overline{F_X}(x) \sim \sigma_1^{-\alpha} x^{-\alpha}$ and $\overline{G_Y}(x) \sim \sigma_2^{-\beta} x^{-\beta}$. Applying Proposition 2.13(iii) of [3] leads to $(\chi_F \chi_G)(x) \sim \sigma_1\sigma_2 x^{(\alpha+\beta)/(\alpha\beta)}$. Again by Proposition 2.13(iii) of [3], relation (16) follows. Therefore, the right-hand side of (7) becomes easily computable.

Now we assign $n = 3$ to the period and $\gamma = 5$ to the parameter value of the Gumbel copula (13). We simulate the left-hand side of (7) with a sample of size $N = 10^7$ but directly compute the right-hand side of (7) illustrated in Fig. 1(a). Figure 1(b) shows the ratio of the simulated probability $\mathbf{P}(S_n > x)$ to its estimated value by (7). We observe that almost all the ratios are close to 1 with estimation errors well within 5%.

We next turn to the asymptotic estimate obtained by formula (11). Besides the above specifications and parameter setting, we assign $r = 2$ to the order of the conditional tail moment. The simulated conditional tail moment $\mathbf{E}[S_n^r \mid S_n > x]$ in (11) is constructed by

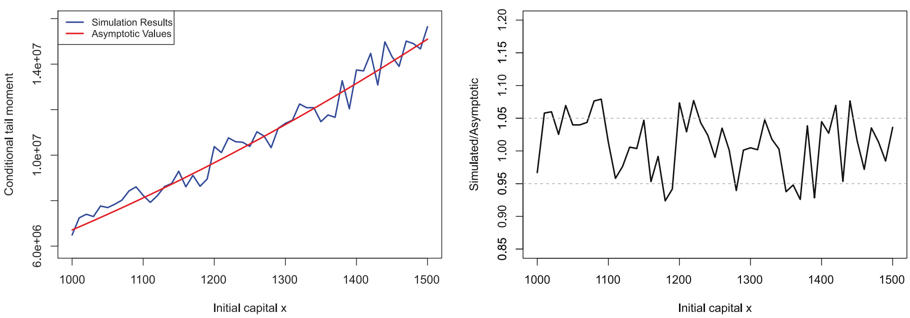
$$\frac{\sum_{k=1}^N \widehat{S}_{n,k}^r \mathbf{1}_{(\widehat{S}_{n,k} > x)}}{\sum_{k=1}^N \mathbf{1}_{(\widehat{S}_{n,k} > x)}},$$

where $\widehat{S}_{n,k} = \sum_{i=1}^n X_{i,k} \prod_{j=1}^i Y_{j,k}$, and $\{(X_{i,k}, Y_{i,k}),\ i = 1, \dots, n;\ k = 1, \dots, N\}$ is the sample of size N for n pairs of (X, Y). The simulated and estimated values of $\mathbf{E}[S_n^r \mid S_n > x]$, as well as their ratio, are shown in Fig. 2.



(a) (b)

Figure 1. Comparison between the simulated and the asymptotic values of $\mathbf{P}(S_n > x)$ (a) and their ratio (b) via the Gumbel copula.



(a) (b)

Figure 2. Comparison between the simulated and the asymptotic values of $\mathbf{E}[S_n^r \mid S_n > x]$ (a) and their ratio (b) via the Gumbel copula.

4.2 Consequences of underestimating the tail dependence

We are concerned about the consequences of a poor risk management that underestimates the tail dependence between X and Y. Now that we have verified the high accuracy of the asymptotic formula (7) in Section 4.1, we shall resort to the right-hand side of (7) when conducting the numerical studies in this section.

In this section, we continue to use the same model specifications and keep most of the parameter settings as those in Section 4.1, and rewrite the aggregate discounted net loss in (1) as $S_n(\gamma)$ and $\nu(A)$ in (15) as $\nu(A;\gamma)$ to reflect the tail dependence parameter γ. Suppose that the total economic capital prepared by the manager of an insurer is x, then $\mathbf{P}(S_n(\gamma) > x)$ represents the probability that the insurer becomes insolvent. As usual, in order to attain a solvency level $q \in (0,1)$, the economic capital requirement is $\mathrm{VaR}_q(S_n(\gamma))$ according to the internal ratings-based approach, which, recalling (8), is defined as

$$\mathrm{VaR}_q\big(S_n(\gamma)\big) = \inf\big\{x \in \mathbb{R}\colon \mathbf{P}\big(S_n(\gamma) \leqslant x\big) \geqslant q\big\}.$$

Table 1. The asymptotic estimate of $\mathrm{RPI}(\gamma; 1.1)$ defined by (17) with $q = 99.9\%$.

γ	1.1	1.2	1.3	1.4	1.5
$\mathrm{RPI}(\gamma; 1.1)$	0.1000%	0.1527%	0.1850%	0.2066%	0.2219%

Suppose that the manager models the aggregate discounted net loss $S_n(\gamma)$ in which the true parameter value $\gamma \in (1, \infty]$ of the Gumbel copula (13) underlying (X, Y) is unknown and is estimated to be $\hat{\gamma}$. If the manager underestimates the tail dependence of (X, Y), i.e., $\hat{\gamma} < \gamma$, then the economic capital $\mathrm{VaR}_q(S_n(\hat{\gamma}))$ may be underprepared, and hence the *realized probability of insolvency* will be higher than allowed.

To measure the consequences of this underestimation of the tail dependence, we define the *realized probability of insolvency* as

$$\mathrm{RPI}(\gamma; \hat{\gamma}) = \mathbf{P}\big(S_n(\gamma) > \mathrm{VaR}_q\big(S_n(\hat{\gamma})\big)\big), \quad q \in (0, 1). \tag{17}$$

In the case of $\hat{\gamma} < \gamma$, we shall see that $\mathrm{RPI}(\gamma; \hat{\gamma}) > 1 - q$, indicating a more acute exacerbation of the insolvency risk.

We conduct numerical studies of the quantity $\mathrm{RPI}(\gamma; \hat{\gamma})$ in (17) to visualize the consequences of the underestimation of the tail dependence. To this end, by assuming solvency level $q \uparrow 1$, we first apply formulas (7) and (16) to derive the asymptotic estimate for $\mathrm{VaR}_q(S_n(\hat{\gamma}))$

$$\mathrm{VaR}_q\big(S_n(\hat{\gamma})\big) \sim \sigma_1 \sigma_2 \left(\frac{(1 - (\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n) \nu(A; \hat{\gamma})}{(1 - q)(1 - \mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])} \right)^{(\alpha+\beta)/(\alpha\beta)}, \quad \hat{\gamma} > 1, \tag{18}$$

where we also used Proposition 2.13(iii) of [3]. By applying (7) and (16) again, we have that as $q \uparrow 1$, $\mathrm{RPI}(\gamma; \hat{\gamma}) \sim \nu(A; \gamma)/\nu(A; \hat{\gamma}) \cdot (1 - q)$.

Consider a scenario that the tail dependence of (X, Y) is underestimated, i.e., $1 < \hat{\gamma} < \gamma$. Specifically, suppose that the manager estimates the tail dependence parameter γ to be $\hat{\gamma} = 1.1$, but its true value is higher. Assume a solvency level 99.9%, and thus, the allowed probability of insolvency is 0.1%. Then the manager sets the economic capital to be $\mathrm{VaR}_{99.9\%}(S_n(1.1))$, which apparently becomes inadequate to attain the 99.9% solvency level since the true value of γ is larger than 1.1. We numerically compute $\mathrm{RPI}(\gamma; 1.1)$ in Table 1.

Table 1 tabulates $\mathrm{RPI}(\gamma; 1.1)$ with the true value of γ assumed to be larger than 1.1. Under the above-mentioned model specifications, when $\gamma = 1.2$, which is slightly larger than $\hat{\gamma} = 1.1$, the *realized probability of insolvency*, in comparison to the allowed level 0.1%, is increased by more than one half. It becomes even worse when a larger value of γ such as $\gamma = 1.4$ is underestimated as $\hat{\gamma} = 1.1$, which would cause the *realized probability of insolvency* to be higher than twice of the allowed level 0.1%. Thus, underestimating the tail dependence of (X, Y) makes the corresponding economic capital significantly inadequate to attain the desired solvency level, leading to serious consequences.

4.3 Sensitivity analysis

In Section 4.1, we have conducted an accuracy check for the asymptotic estimates of the tail probability $\mathbf{P}(S_n > x)$ and the conditional tail moment $\mathbf{E}[S_n^r \mid S_n > x]$. Building on

Table 2. The sensitivity analysis of $\mathrm{VaR}_q(S_n)$ with respect to α, β, and γ.

Model parameters	$\mathrm{VaR}_q(S_n)$		
	$q = 0.95$	$q = 0.99$	$q = 0.999$
% change in α and β			
+15%	−44.846%	−47.819%	−53.070%
+10%	−33.310%	−35.646%	−40.530%
+5%	−18.609%	−20.337%	−23.175%
$(\alpha = 3,\ \beta = 10.85)$	(11.362)	(47.073)	(251.146)
−5%	+26.160%	+27.913%	+30.674%
−10%	+59.434%	+66.511%	+76.148%
−15%	+105.888%	+121.135%	+147.752%
% change in γ			
+15%	+10.628%	+14.098%	+14.682%
+10%	+7.606%	+9.625%	+8.639%
+5%	+4.003%	+4.958%	+4.391%
$(\gamma = 1.5)$	(11.362)	(47.073)	(251.146)
−5%	−3.817%	−5.719%	−6.759%
−10%	−8.396%	−12.435%	−15.863%
−15%	−13.626%	−20.169%	−25.756%

Table 3. The sensitivity analysis of $\mathrm{CTE}_q(S_n)$ with respect to α, β, and γ.

Model parameters	$\mathrm{CTE}_q(S_n)$		
	$q = 0.95$	$q = 0.99$	$q = 0.999$
% change in α and β			
+15%	−52.172%	−52.356%	−55.448%
+10%	−40.243%	−39.413%	−41.853%
+5%	−24.989%	−22.988%	−24.702%
$(\alpha = 3,\ \beta = 10.85)$	(46.387)	(146.006)	(638.482)
−5%	+25.973	+34.568%	+40.645%
−10%	+64.554	+77.382%	+86.376%
−15%	+131.700%	+159.206%	+193.695%
% change in γ			
+15%	+8.221%	+13.615%	+16.162%
+10%	+6.873%	+13.329%	+16.610%
+5%	+1.244%	+6.214%	+7.327%
$(\gamma = 1.5)$	(46.387)	(146.006)	(638.482)
−5%	−10.298%	−7.142%	−7.771%
−10%	−16.455%	−13.920%	−13.391%
−15%	−25.040%	−24.532%	−26.911%

the previous specifications used in the simulations for Theorem 1 and Corollary 1, we further explore the impact of three parameters on the two commonly used risk measures $\mathrm{VaR}_q(S_n)$ and $\mathrm{CTE}_q(S_n)$. These three parameters under consideration are the regular variation indices α and β of the insurance and financial risks and their tail dependence parameter γ of the Gumbel copula (13). As a benchmark, we set $\alpha=3$, $\beta=10.85$, and $\gamma=5$.

We introduce small percentage changes to α, β, and γ, and then document how much $\mathrm{VaR}_q(S_n)$ and $\mathrm{CTE}_q(S_n)$ alter through the right-hand sides of (18) and (12). Tables 2 and 3 summarize the percentage changes in the simulated values of $\mathrm{VaR}_q(S_n)$ and $\mathrm{CTE}_q(S_n)$ for different large values of confidence level q. As expected, the two tables display that the smaller the values of α and β are, which correspond to heavier tails for the insurance and financial risks, the larger the two risk measures become. As the findings suggest, the $\mathrm{VaR}_q(S_n)$ and $\mathrm{CTE}_q(S_n)$ both increase as the dependence between the insurance and financial risks strengthens, and exhibits a relatively high level of sensitivity although significantly lower than that caused by the tail indices α and β. This indicates that the heavy tails of the insurance and financial risks play a dominant role in the effect on the two risk measures, and the dependence plays a second role.

5 Proofs

5.1 Proof of Theorem 1

We start this section with a series of lemmas. The first three lemmas collect some elementary but useful properties of heavy-tailed distributions. The first lemma is a refinement of long-tailed distribution, which can be found in Lemma 2.19 of [11].

Lemma 1. *For a distribution* $V \in \mathscr{L}$*, there exists an auxiliary function* l *satisfying* $l(x) \uparrow \infty$ *and* $l(x) = o(1)x^s$ *for any* $s > 0$ *such that* $\overline{V}(x-u) \sim \overline{V}(x)$ *holds uniformly for all* $u \in [-l(x), l(x)]$.

The second lemma shows an elementary result regarding dominatedly varying-tailed distribution, which is due to Proposition 3.1 of [34].

Lemma 2. *A distribution* $V \in \mathscr{D}$ *if and only if for any distribution* W *with* $\overline{W}(x) = o(1)\overline{V}(x)$*, there exists a positive function* g *such that* $g(x) \downarrow 0$*,* $xg(x) \uparrow \infty$*, and* $\overline{W}(xg(x)) = o(1)\overline{V}(x)$.

The third lemma is a variation of Lemma 7 of [30], and we can prove it along the line of Tang and Yuan's proof.

Lemma 3. *Let* ξ *be a random variable with distribution* $V \in \mathscr{D}$*, let* η *be a nonnegative random variable with* $\mathbf{E}[\eta^p] < \infty$ *for some* $p > J_V^+$*, and let* $\{\Delta_x,\ x \in \mathbb{R}\}$ *be a set of random events satisfying* $\lim_{x\to\infty} \mathbf{P}(\Delta_x) = 0$*. If* $\{\eta, \{\Delta_x, x \in \mathbb{R}\}\}$ *is independent of* ξ*, then it holds that*

$$\mathbf{P}(\xi\eta > x,\ \Delta_x) = o(1)\overline{V}(x) = o(1)\mathbf{P}(\xi\eta > x).$$

Remark that in Lemma 3, the arbitrary dependence is allowed both between η and $\{\Delta_x,\ x \in \mathbb{R}\}$ and among random events Δ_x for all $x \in \mathbb{R}$. The fourth lemma restates the result of (22) in [28].

Lemma 4. *Let* X *be a real-valued random variable, and let* Y *be a nonnegative random variable. Assume that* $(X^+, Y) \in \mathrm{BRV}_{-\alpha,-\beta}(F, G, \nu)$ *for a nondegenerate measure* ν *and that two representing tails* $\overline{F} \in \mathrm{RV}_{-\alpha}$ *and* $\overline{G} \in \mathrm{RV}_{-\beta}$ *for some* $\alpha, \beta > 0$*. If* $\nu((\mathbf{1}, \boldsymbol{\infty}]) > 0$*, then it holds that*

$$\mathbf{P}(XY > x) \sim \nu(A)\kappa(x),$$

where the set A *is defined in* (7)*, and the function* κ *is defined in* (6).

The following lemma is crucially important in the proof of Theorem 1.

Lemma 5. *Under the conditions of Lemma* 4*, if Z is a real-valued random variable, independent of (X,Y), with tail $\overline{H}\in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$, then it holds that*

$$\mathbf{P}\big((X+Z)Y>x\big)\sim \mathbf{P}(XY>x)+\mathbf{P}(ZY>x). \tag{19}$$

Proof. Clearly, Lemma 4 implies that $\mathbf{P}(XY>x)\in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$ because of $\nu(A)\geqslant \nu((\mathbf{1},\boldsymbol{\infty}])>0$. By $\overline{H}\in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$, $\overline{G}\in \mathrm{RV}_{-\beta}$, and Lemma 2, there exists a positive function g such that

$$g(x)\downarrow 0,\quad xg(x)\uparrow\infty,\quad \text{and}\quad \overline{G}\big(xg(x)\big)=o(1)\overline{H}(x). \tag{20}$$

Again by $\overline{H}\in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$, Lemma 1 implies that for the above g, there exists a function h such that $h(x)\uparrow\infty$, $1/g(x)-h(x)\to\infty$, $h(x)(g(x))^s\to 0$ for any $s>0$, and

$$\overline{H}\biggl(\frac{1}{g(x)}-u\biggr)\sim \overline{H}\biggl(\frac{1}{g(x)}\biggr) \tag{21}$$

holds uniformly for all $u\in[-h(x),h(x)]$.

We split the tail probability $\mathbf{P}((X+Z)Y>x)$ into three parts:

$$\begin{aligned}\mathbf{P}\big((X+Z)Y>x\big)&=\mathbf{P}\big((X+Z)Y>x,\ |X|\leqslant h(x),\ Y\leqslant xg(x)\big)\\&\quad+\mathbf{P}\big((X+Z)Y>x,\ |X|\leqslant h(x),\ Y>xg(x)\big)\\&\quad+\mathbf{P}\big((X+Z)Y>x,\ |X|>h(x)\big)\\&=I_1+I_2+I_3.\end{aligned} \tag{22}$$

Since Z is independent of (X,Y), we have that

$$\begin{aligned}I_1&=\int_{-h(x)}^{h(x)}\int_0^{xg(x)}\mathbf{P}\biggl(Z>\frac{x}{v}-u\biggr)\mathbf{P}(X\in\mathrm{d}u,\ Y\in\mathrm{d}v)\\&\sim\int_{-h(x)}^{h(x)}\int_0^{xg(x)}\mathbf{P}\biggl(Z>\frac{x}{v}\biggr)\mathbf{P}(X\in\mathrm{d}u,\ Y\in\mathrm{d}v)\\&=\mathbf{P}\big(ZY>x,\ |X|\leqslant h(x),\ Y\leqslant xg(x)\big)\\&=\mathbf{P}(ZY>x)-\mathbf{P}\big(ZY>x,\ \big(|X|>h(x)\big)\cup\big(Y>xg(x)\big)\big)\\&\sim\mathbf{P}(ZY>x),\end{aligned} \tag{23}$$

where we used the uniformity of (21) in $[-h(x),h(x)]$ in the second step and Lemma 3 in the last step by noticing $\overline{H}\in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$ and $\overline{G}\in \mathrm{RV}_{-\beta}$.

As for I_2, by (20) we have that

$$I_2\leqslant \overline{G}\big(xg(x)\big)=o(1)\overline{H}(x)=o(1)\mathbf{P}(ZY>x), \tag{24}$$

where the last step is due to $\mathbf{P}(ZY>x)\asymp\overline{H}(x)$ by $\overline{H}\in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$ and $\overline{G}\in \mathrm{RV}_{-\beta}$.

We mainly deal with I_3. First, consider the upper bound of I_3. For any $0 < \varepsilon < 1$,

$$\begin{aligned} I_3 &\leqslant \mathbf{P}\big(\big(XY > (1-\varepsilon)x\big) \cup (ZY > \varepsilon x),\ |X| > h(x)\big) \\ &= \mathbf{P}\big(XY > (1-\varepsilon)x,\ |X| > h(x)\big) \\ &\quad + \mathbf{P}\big(XY \leqslant (1-\varepsilon)x,\ ZY > \varepsilon x,\ |X| > h(x)\big) \\ &= I_{31} + I_{32}. \end{aligned} \tag{25}$$

By $\overline{H} \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$, $\overline{G} \in \mathrm{RV}_{-\beta}$, and Lemma 3, we have that

$$I_{32} \leqslant \mathbf{P}\big(ZY > \varepsilon x,\ |X| > h(x)\big) = o(1)\mathbf{P}(ZY > x). \tag{26}$$

As for I_{31}, by (4) and $\kappa \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$, we have that

$$I_{31} \leqslant \mathbf{P}\big(XY > (1-\varepsilon)x\big) \sim (1-\varepsilon)^{-\alpha\beta/(\alpha+\beta)}\mathbf{P}(XY > x). \tag{27}$$

Plugging (26) and (27) into (25) and by the arbitrariness of $\varepsilon > 0$, we obtain that

$$I_3 \lesssim \mathbf{P}(XY > x) + o(1)\mathbf{P}(ZY > x). \tag{28}$$

Next, consider the lower bound of I_3. For the above $0 < \varepsilon < 1$,

$$\begin{aligned} I_3 &\geqslant \mathbf{P}\big(XY > (1+\varepsilon)x,\ ZY > -\varepsilon x,\ X > h(x)\big) \\ &\geqslant \mathbf{P}\big(XY > (1+\varepsilon)x,\ X > h(x)\big) \\ &\quad - \mathbf{P}\big(XY > (1+\varepsilon)x,\ ZY \leqslant -\varepsilon x\big) \\ &= I_{33} - I_{34}. \end{aligned} \tag{29}$$

For brevity, we write $y(x) = 1/\kappa(x) \in \mathrm{RV}_{\alpha\beta/(\alpha+\beta)}$. Then, by $\chi_F \in \mathrm{RV}_{1/\alpha}$, we know $\chi_F \circ y \in \mathrm{RV}_{\beta/(\alpha+\beta)}$. Note that $h(x) = o(1)(g(x))^{-s} = o(1)x^s$ for any $s > 0$ by (20). This further implies that $h(x) = o(1)\chi_F \circ y(x)$. By $(\chi_F \cdot \chi_G) \circ y(x) \sim x$, we have that $x \leqslant (\chi_F \cdot \chi_G) \circ y((1+\varepsilon)x)$ for the above $0 < \varepsilon < 1$ and all large x. Thus, for the above $0 < \varepsilon < 1$ and all large x,

$$\begin{aligned} I_{33} &\geqslant \mathbf{P}\big(X^+Y > (\chi_F \cdot \chi_G) \circ y\big((1+\varepsilon)^2 x\big),\ X^+ > h(x)\big) \\ &\geqslant \mathbf{P}\bigg(\frac{X^+}{\chi_F \circ y((1+\varepsilon)^2 x)} \cdot \frac{Y}{\chi_G \circ y((1+\varepsilon)^2 x)} > 1,\ \frac{X^+}{\chi_F \circ y\big((1+\varepsilon)^2 x\big)} > \varepsilon\bigg) \\ &\sim \nu(A_\varepsilon)\kappa\big((1+\varepsilon)^2 x\big) \\ &\sim (1+\varepsilon)^{-2\alpha\beta/(\alpha+\beta)}\nu(A_\varepsilon)\kappa(x), \end{aligned} \tag{30}$$

where the set $A_\varepsilon = \{(u,v) \in [\mathbf{0},\boldsymbol{\infty}]\colon uv > 1, u > \varepsilon\}$ tends to the set A as $\varepsilon \downarrow 0$, and $\nu(\partial A_\varepsilon) = 0$ is verified by Lemma 5.2 of [28]. As for I_{34}, by $\overline{F_{XY}} \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$

and (24), we have that

$$\begin{aligned} I_{34} &= \mathbf{P}\big(XY > (1+\varepsilon)x,\ ZY \leqslant -\varepsilon x,\ 0 < Y \leqslant xg(x)\big) \\ &\quad + \mathbf{P}\big(XY > (1+\varepsilon)x,\ ZY \leqslant -\varepsilon x,\ Y > xg(x)\big) \\ &\leqslant \mathbf{P}(XY > (1+\varepsilon)x) H\left(-\frac{\varepsilon}{g(x)}\right) + \overline{G}\big(xg(x)\big) \\ &= o(1)\big(\mathbf{P}(XY > x) + \mathbf{P}(ZY > x)\big). \end{aligned} \tag{31}$$

Combining (29)–(31) with the arbitrariness of $\varepsilon > 0$ and (4) leads to

$$I_3 \gtrsim \big(1 + o(1)\big)\mathbf{P}(XY > x) + o(1)\mathbf{P}(ZY > x). \tag{32}$$

Therefore, (19) follows from (22)–(24), (28), and (32). □

Proof of Theorem 1. Since $\{(X_i, Y_i),\ i \in \mathbb{N}\}$ is a sequence of i.i.d. random vectors, it holds that

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \overset{\mathrm{d}}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j := T_n, \quad n \in \mathbb{N},$$

where $\overset{\mathrm{d}}{=}$ stands for equality in distribution. Thus, for (7), it suffices to prove that

$$\mathbf{P}(T_n > x) \sim \frac{1 - (\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1 - (\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])} \nu(A)\kappa(x). \tag{33}$$

Trivially, (33) holds for $n = 1$. Now we assume by induction that it holds for $n \in \mathbb{N}$, and we prove it for $n+1$. By the induction assumption and $\nu((\mathbf{1}, \boldsymbol{\infty}]) > 0$ (thus, $\nu(A) > 0$), we have that $\overline{F_{T_n}} \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$. Note that for every $n \in \mathbb{N}$,

$$T_{n+1} = (X_{n+1} + T_n)Y_{n+1}.$$

By using Lemma 5, (4), Breiman's theorem, and the induction assumption in turn, we derive that

$$\begin{aligned} \mathbf{P}(T_{n+1} > x) &\sim \mathbf{P}(X_{n+1}Y_{n+1} > x) + \mathbf{P}(T_n Y_{n+1} > x) \\ &\sim \nu(A)\kappa(x) + \mathbf{E}\big[Y^{\alpha\beta/(\alpha+\beta)}\big]\mathbf{P}(T_n > x) \\ &\sim \left(1 + \frac{\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}] - (\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^{n+1}}{1 - \mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\right)\nu(A)\kappa(x) \\ &= \frac{1 - (\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^{n+1}}{1 - \mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\nu(A)\kappa(x). \end{aligned}$$

This completes the proof. □

5.2 Proof of Theorem 2

In this subsection, we first introduce a lemma related to an alternative expected formula, which is due to Lemma 3 of [9]. For more discussions on alternative expected formulas, the reader can be referred to [16, 21, 25], among others.

Lemma 6. *Let ξ be a real-valued random variable with distribution V, and let $\varphi : [a,\infty) \mapsto \mathbb{R}$ be a differentiable function such that $\int_{a-}^{\infty}(\int_a^x |\varphi'(u)|\,\mathrm{d}u)\,V(\mathrm{d}x) < \infty$ holds for some $a > 0$. Then*

$$\mathbf{E}\big[\varphi(\xi)\mathbf{1}_{(\xi>x)}\big] = \varphi(x)\overline{V}(x) + \int_x^{\infty} \varphi'(u)\overline{V}(u)\,\mathrm{d}u$$

holds for all $x \geqslant a$.

The second lemma comes from [9], and we slightly modify its proof.

Lemma 7. *Let $\xi_1,\dots,\xi_n$ be n arbitrarily dependent real-valued random variables, and let $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ be a nondecreasing function satisfying Assumption 1. Then*

$$\mathbf{E}\left[\varphi\left(\sum_{i=1}^{n}\xi_i\right)\mathbf{1}_{(\sum_{i=1}^{n}\xi_i>nK)}\right] \leqslant M^{\lfloor \log_2 n\rfloor+1}\sum_{i=1}^{n}\mathbf{E}\big[\varphi(\xi_i)\mathbf{1}_{(\xi_i>K)}\big]$$

holds for every $n \in \mathbb{N}$, some $M > 0$, and some sufficiently large $K > 0$.

Proof. Since $\varphi(\cdot)$ is subhomogeneous by Assumption 1, it holds that $\varphi(nx) = O(\varphi(x))$. Indeed, by $\varphi(2x) \leqslant M(\varphi(x))$ for some $M > 0$, some large $K > 0$, and all $x \geqslant K$, it follows for every fined $m \in \mathbb{N}$ and all $x \geqslant 2^m K/n$ that

$$\varphi(nx) \leqslant M\varphi\left(\frac{nx}{2}\right) \leqslant \cdots \leqslant M^{m-1}\varphi\left(\frac{nx}{2^{m-1}}\right) \leqslant M^m\varphi\left(\frac{nx}{2^m}\right).$$

Choosing $m = \lfloor \log_2 n\rfloor + 1$ and noticing that $\varphi(\cdot)$ is nondecreasing, we have that for all $x \geqslant K$,

$$\varphi(nx) \leqslant M^m\varphi\left(\frac{nx}{2^m}\right) \leqslant M^m\varphi(x) = M^{\lfloor \log_2 n\rfloor+1}\varphi(x). \tag{34}$$

Since φ is a nondecreasing and nonnegative function, it follows from (34) that

$$\begin{aligned}
\mathbf{E}\left[\varphi\left(\sum_{i=1}^{n}\xi_i\right)\mathbf{1}_{(\sum_{i=1}^{n}\xi_i>nK)}\right] &\leqslant \mathbf{E}\left[\varphi\left(n\bigvee_{i=1}^{n}\xi_i\right)\mathbf{1}_{(\bigvee_{i=1}^{n}\xi_i>K)}\right] \\
&\leqslant M^{\lfloor \log_2 n\rfloor+1}\mathbf{E}\left[\varphi\left(\bigvee_{i=1}^{n}\xi_i\right)\mathbf{1}_{(\bigvee_{i=1}^{n}\xi_i>K)}\right] \\
&\leqslant M^{\lfloor \log_2 n\rfloor+1}\sum_{i=1}^{n}\mathbf{E}\big[\varphi(\xi_i)\mathbf{1}_{(\xi_i>K)}\big].
\end{aligned}$$

In the last step above, we used the method of mathematical induction. Indeed, it holds for $n = 1$ that

$$\mathbf{E}\left[\varphi\left(\bigvee_{i=1}^{n}\xi_i\right)\mathbf{1}_{(\bigvee_{i=1}^{n}\xi_i>K)}\right] \leqslant \sum_{i=1}^{n}\mathbf{E}\big[\varphi(\xi_i)\mathbf{1}_{(\xi_i>K)}\big].$$

Assume by induction that this inequality holds for $n \in \mathbb{N}$. Thus, it follows from the induction assumption that

$$\begin{aligned}
&\mathbf{E}\left[\varphi\left(\bigvee_{i=1}^{n+1}\xi_i\right)\mathbf{1}_{(\bigvee_{i=1}^{n+1}\xi_i>K)}\right]\\
&\quad= \mathbf{E}\left[\varphi\left(\bigvee_{i=1}^{n+1}\xi_i\right)\mathbf{1}_{(\bigvee_{i=1}^{n+1}\xi_i>K)}\big(\mathbf{1}_{(\bigvee_{i=1}^{n}\xi_i\geqslant\xi_{n+1})} + \mathbf{1}_{(\bigvee_{i=1}^{n}\xi_i<\xi_{n+1})}\big)\right]\\
&\quad\leqslant \mathbf{E}\left[\varphi\left(\bigvee_{i=1}^{n}\xi_i\right)\mathbf{1}_{(\bigvee_{i=1}^{n}\xi_i>K)}\right] + \mathbf{E}\big[\varphi(\xi_{n+1})\mathbf{1}_{(\xi_{n+1}>K)}\big] \leqslant \sum_{i=1}^{n+1}\mathbf{E}\big[\varphi(\xi_i)\mathbf{1}_{(\xi_i>K)}\big].
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 2. By condition (9), there exists some large $K > 0$ such that

$$\mathbf{E}\left[\varphi\left(X_i\prod_{j=1}^{i}Y_j\right)\mathbf{1}_{(X_i\prod_{j=1}^{i}Y_j>K)}\right] < \infty$$

for all $i = 1, \dots, n$. Since φ is differential and nondecreasing, by Lemma 7 we have that

$$\begin{aligned}
&\int_{nK}^{\infty}\left(\int_{nK}^{x}\left|\varphi'(u)\right|\mathrm{d}u\right)\mathbf{P}(S_n\in\mathrm{d}x)\\
&\quad= \int_{nK}^{\infty}\left(\int_{nK}^{x}\varphi'(u)\,\mathrm{d}u\right)\mathbf{P}(S_n\in\mathrm{d}x) = \mathbf{E}\big[\big(\varphi(S_n)-\varphi(nK)\big)\mathbf{1}_{(S_n>nK)}\big]\\
&\quad\leqslant M^{\lfloor\log_2 n\rfloor+1}\sum_{i=1}^{n}\mathbf{E}\left[\varphi\left(X_i\prod_{j=1}^{i}Y_j\right)\mathbf{1}_{(X_i\prod_{j=1}^{i}Y_j>K)}\right] < \infty.
\end{aligned}$$

Hence, Lemma 6 is applicable. Since φ is a nondecreasing function, we obtain from Theorem 1 that

$$\begin{aligned}
\mathbf{E}\big[\varphi(S_n)\mathbf{1}_{(S_n>x)}\big] &= \varphi(x)\mathbf{P}(S_n>x) + \int_{x}^{\infty}\varphi'(u)\mathbf{P}(S_n>u)\,\mathrm{d}u\\
&\sim \frac{1-(\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1-\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\nu(A)\left(\varphi(x)\kappa(x) + \int_{x}^{\infty}\varphi'(u)\kappa(u)\,\mathrm{d}u\right).
\end{aligned}$$

This, together with (7), gives relation (10). □

Proof of Corollary 1. By Theorem 2, $\kappa \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$, $r < \alpha\beta/(\alpha+\beta)$, and (4), we can obtain that

$$\mathbf{E}\big[S_n^r \mid S_n > x\big] \sim x^r + \frac{r}{\kappa(x)} \int\limits_x^\infty u^{r-1}\kappa(u)\,\mathrm{d}u \sim \frac{\alpha\beta x^r}{\alpha\beta - (\alpha+\beta)r}.$$

In the above deviation, the applicability of Theorem 2 is justified by the fact that both XY and Y have finite moments of order $r < \alpha\beta/(\alpha+\beta)$ because of $\overline{F_{XY}} \in \mathrm{RV}_{-\alpha\beta/(\alpha+\beta)}$ and $\overline{F_Y} \in \mathrm{RV}_{-\beta}$, and thus, for each $i \in \mathbb{N}$,

$$\mathbf{E}\Bigg[\Bigg(X_i \prod_{j=1}^i Y_j\Bigg)^r\Bigg] = \mathbf{E}\big[(X_iY_i)^r\big]\big(\mathbf{E}\big[Y^r\big]\big)^{i-1} < \infty,$$

implying (9) is satisfied.

Furthermore, our desired relation (12) follows from (11) and the fact that as $q \to 1$,

$$\begin{aligned}
\mathrm{VaR}_q(S_n) &= \left(\frac{1}{\overline{F_{S_n}}}\right)^{\leftarrow} \frac{1}{1-q} \\
&\sim \left(\frac{1}{\kappa}\right)^{\leftarrow} \left(\frac{1-(\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1-\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\nu(A)\frac{1}{1-q}\right) \\
&\sim \left(\frac{1-(\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1-\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\nu(A)\right)^{(\alpha+\beta)/(\alpha\beta)} \big((\chi_F\chi_G)^{\leftarrow}\big)^{\leftarrow} \frac{1}{1-q} \\
&= \left(\frac{1-(\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}])^n}{1-\mathbf{E}[Y^{\alpha\beta/(\alpha+\beta)}]}\nu(A)\right)^{(\alpha+\beta)/(\alpha\beta)} F^{\leftarrow}(q)G^{\leftarrow}(q),
\end{aligned}$$

where we used Theorem 1 and Proposition 0.8(vi) of [22] in the second step, $(1/\kappa)^{\leftarrow} \in \mathrm{RV}_{(\alpha+\beta)/(\alpha\beta)}$ in the third step, and the last equality holds because both F and G are eventually strictly increasing. □

References

1. J.Y. Ahn, N.D. Shyamalkumar, Asymptotic theory for the empirical Haezendonck-Goovaerts risk measure, *Insur. Math. Econ.*, **55**:78–90, 2014, https://doi.org/10.1016/j.insmatheco.2013.12.003.
2. N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Cambridge Univ. Press, Cambridge, 1987, https://doi.org/10.1017/CBO9780511721434.
3. K. Böcker, C. Klüppelberg, Multivariate models for operational risk, *Quant. Finance*, **10**(8): 855–869, 2010, https://doi.org/10.1080/14697680903358222.
4. Y. Chen, The finite-time ruin probability with dependent insurance and financial risks, *J. Appl. Probab.*, **48**(4):1035–1048, 2011, https://doi.org/10.1239/jap/1324046017.

5. Y. Chen, J. Liu, F. Liu, Ruin with insurance and financial risks following the least risky FGM dependence structure, *Insur. Math. Econ.*, **62**:98–106, 2015, https://doi.org/10.1016/j.insmatheco.2015.03.007.
6. Y. Chen, Z. Yuan, A revisit to ruin probabilities in the presence of heavy-tailed insurance and financial risks, *Insur. Math. Econ.*, **73**:75–81, 2017, https://doi.org/10.1016/j.insmatheco.2017.01.005.
7. L. de Haan, S.I. Resnick, On the observation closest to the origin, *Stoch. Proc. Appl.*, **11**:301–308, 1981, https://doi.org/10.1016/0304-4149(81)90032-6.
8. J. Dhaene, S. Vanduffel, M. Goovaerts, R. Kaas, Q. Tang, D. Vyncke, Risk measures and comonotonicity: A review, *Stoch. Models*, **22**:573–606, 2006, https://doi.org/10.1080/15326340600878016.
9. M. Dirma, N. Nakliuda, J. Šiaulys, Generalized moments of sums with heavy-tailed random summands, *Lith. Math. J.*, **63**(3):254–271, 2023, https://doi.org/10.1007/s10986-023-09606-y.
10. P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Stoch. Model. Appl. Probab., Vol. 33, Springer, Berlin, Heidelberg, 1997, https://doi.org/10.1007/978-3-642-33483-2.
11. S. Foss, D. Korshunov, S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions,* 2nd ed., Springer Ser. Oper. Res. Financ. Eng., Springer, New York, 2011, https://doi.org/10.1007/978-1-4614-7101-1.
12. A.-L. Fougères, C. Mercadier, Risk measures and multivariate extensions of Breiman's theorem, *J. Appl. Probab.*, **49**(2):364–384, 2012, https://doi.org/10.1239/jap/1339878792.
13. E. Furman, R. Zitikis, Weighted premium calculation principles, *Insur. Math. Econ.*, **42**(1):459–465, 2008, https://doi.org/10.1016/j.insmatheco.2007.10.006.
14. J. Li, Q. Tang, Interplay of insurance and financial risks in a discrete-time model with strongly regular variation, *Bernoulli*, **21**(3):1800–1823, 2015, https://doi.org/10.3150/14-BEJ625.
15. J. Liu, T. Shushi, Asymptotics of the loss-based tail risk measures in the presence of extreme risks, *Eur. Actuar. J.*, **14**:205–224, 2024, https://doi.org/10.1007/s13385-023-00361-5.
16. Y. Liu, A general treatment of alternative expectation formulae, *Stat. Probab. Lett.*, **166**:108894, 2020, https://doi.org/10.1016/j.spl.2020.108863.
17. A.J. McNeil, R. Frey, P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools,* revised ed., Princeton Ser. Finance, Princeton Univ. Press, Princeton, NJ, 2015.
18. J. El Methni, G. Stupfer, Extreme versions of Wang risk measures and their estimation for heavy-tailed distributions, *Stat. Sin.*, **27**:907–930, 2017, https://doi.org/10.5705/ss.202015.0345.
19. H. Nyrhinen, On the ruin probabilities in a general economic environment, *Stoch. Proc. Appl.*, **83**(2):319–330, 1999, https://doi.org/10.1016/S0304-4149(99)00030-7.
20. H. Nyrhinen, Finite and infinite time ruin probabilities in a stochastic economic environment, *Stoch. Proc. Appl.*, **92**(2):265–285, 2001, https://doi.org/10.1016/S0304-4149(00)00083-1.

21. H. Ogasawara, Alternative expectation formulas for real-valued random vectors, *Commun. Stat., Theory Methods*, **49**(2):454–470, 2020, https://doi.org/10.1080/03610926.2018.1543773.
22. S.I. Resnick, *Extreme Values, Regular Variation, and Point Processes*, Springer Ser. Oper. Res. Financ. Eng., Springer, New York, 1987, https://doi.org/10.1007/978-0-387-75953-1.
23. S.I. Resnick, *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*, Springer Ser. Oper. Res. Financ. Eng., Springer, New York, 2007, https://doi.org/10.1007/978-0-387-45024-7.
24. X. Shi, Q. Tang, Z. Yuan, A limit distribution of credit portfolio losses with low default probabilities, *Insur. Math. Econ.*, **73**:156–167, 2017, https://doi.org/10.1016/j.insmatheco.2017.02.003.
25. P. Song, S. Wang, A further remark on the alternative expectation formula, *Commun. Stat., Theory Methods*, **50**(11):2586–2591, 2021, https://doi.org/10.1080/03610926.2019.1672743.
26. Q. Tang, G. Tsitsiashvili, Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks, *Stoch. Proc. Appl.*, **108**(2): 299–325, 2003, https://doi.org/10.1016/j.spa.2003.07.001.
27. Q. Tang, G. Tsitsiashvili, Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments, *Adv. Appl. Probab.*, **36**(4):1278–1299, 2004, https://doi.org/10.1239/aap/1103662967.
28. Q. Tang, Y. Yang, Interplay of insurance and financial risks in a stochastic environment, *Scand. Actuar. J.*, **2019**(5):432–451, 2019, https://doi.org/10.1080/03461238.2019.1573753.
29. Q. Tang, Z. Yuan, Asymptotic analysis of the loss given default in the presence of multivariate regular variation, *N. Am. Actuar. J.*, **17**(3):253–271, 2013, https://doi.org/10.1080/10920277.2013.830557.
30. Q. Tang, Z. Yuan, Randomly weighted sums of subexponential random variables with application to capital allocation, *Extremes*, **17**:467–493, 2014, https://doi.org/10.1007/s10687-014-0191-z.
31. Q. Tang, Z. Yuan, Interplay of insurance and financial risks with bivariate regular variation, in D.K. Dey, J. Yan (Eds.), *Extreme Value Modeling and Risk Analysis: Methods and Applications*, CRC Press, Boca Raton, FL, 2016, pp. 419–438, https://doi.org/10.1201/b1972121.
32. Q. Tang, Z. Yuan, Random difference equations with subexponential innovations, *Science China Mathematics*, **59**(2):2411–2426, 2016, https://doi.org/10.1007/s11425-016-0146-0.
33. Y. Yang, D. G. Konstantinides, Asymptotics for ruin probabilities in a discrete-time risk model with dependent financial and insurance risks, *Scand. Actuar. J.*, **8**:641–659, 2015, https://doi.org/10.1080/03461238.2013.878853.
34. M. Zhou, K. Wang, Y. Wang, Estimates for the finite-time ruin probability with insurance and financial risks, *Acta Math. Appl. Sin., Engl. Ser.*, **28**:795–806, 2012, https://doi.org/10.1007/s10255-012-0189-8.