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On a novel semigroup of enriched Chatterjea-type mappings in Banach spaces with a numerical experiment*

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Abstract. Recently in [V. Berinde and M. Păcurar, Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces, *J. Fixed Point Theory Appl.*, 23(4):66, 2021], using the technique of enrichment of contractive mappings by Krasnoselskij averaging, Berinde and Păcurar introduced a new type of mappings, called enriched Chatterjea-type mappings. The main aim of this article is to introduce a new semigroup of enriched Chatterjea-type mappings. We also establish weak and strong convergence results for enriched Chatterjea-type semigroups using a novel iterative process in uniformly convex Banach spaces. To support the theoretical results, we conduct numerical experiments demonstrating the convergence behavior of the iterative scheme under various initial conditions and control sequences. The findings confirm exponential convergence and highlight the effectiveness and robustness of the proposed method for common fixed point approximation.

Keywords: enriched Chatterjea-type mappings, semigroups, strong convergence, weak convergence.

1 Introduction

As well known, many real-world problems can be expressed in terms of nonlinear equations. Since these equations can be translated into equivalent fixed point problems, the fixed point theory has emerged as an important tool for solving many kinds of real-world problems. It brings to the fact that the fixed point theory is a fascinating subject with

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a variety of wide applications in pure and applied sciences. One of the famous results in the fixed point theory, which is known as Banach's contraction principle, was firstly given by Banach [2] in 1922. In a metric space setting, it can be briefly stated as follows:

Theorem 1. (*See* [2].) *Let (X, d) be a complete metric space, and let $T : X \to X$ be a Banach contraction mapping, i.e., there exists $b \in [0, 1)$ such that for each $x, y \in X$,*

$$d(Tx, Ty) \leqslant bd(x, y). \tag{1}$$

Then T has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

This principle is a very popular and powerful tool for solving problems arising in various fields of applied mathematical analysis and its applications. So far, according to importance and simplicity, the Banach's contraction principle have been improved, extended, and generalized in various directions by several authors. An interesting direction of research is the generalization of the Banach's contraction principle by defining contractive-type mappings on metric spaces. In the Rhoades' classification of contractive conditions in [22], there are many interesting types of contractive definitions on a metric space (X, d), for example: let C be a nonempty subset of a metric space X, and let T be a self-mapping on C.

(A) (Kannan [12]): A mapping T is called a Kannan contraction mapping if there exists $a_1 \in [0, 1/2)$ such that for each $x, y \in C$,

$$d(Tx, Ty) \leqslant a_1\big[d(x, Tx) + d(y, Ty)\big]; \tag{2}$$

(B) (Bianchini [5]): A mapping T is called a Bianchini contraction mapping if there exists $a_2 \in [0, 1)$ such that for each $x, y \in C$,

$$d(Tx, Ty) \leqslant a_2 \max\big\{d(x, Tx), d(y, Ty)\big\}; \tag{3}$$

(C) (Chatterjea [8]): A mapping T is called a Chatterjea contraction mapping if there exists $a_3 \in [0, 1/2)$ such that for each $x, y \in C$,

$$d(Tx, Ty) \leqslant a_3\big[d(x, Ty) + d(y, Tx)\big]; \tag{4}$$

(D) A mapping T is called a Chatterjea-type contraction mapping if there exists $a_4 \in [0, 1)$ such that for each $x, y \in C$,

$$d(Tx, Ty) \leqslant a_4 \max\big\{d(x, Ty), d(y, Tx)\big\}. \tag{5}$$

In 2021, using the technique of enrichment of contractive-type mappings by Krasnoselskii averaging, first introduced in [3], Berinde and Păcurar [4] introduced a new class of generalized Chatterjea contraction mappings in the setting of a Banach space as follows.

Definition 1. (See [4].) Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is called an enriched Chatterjea mapping if there exist $c \in [0, 1/2)$ and $k \in [0, \infty)$ such that for each $x, y \in C$,

$$\begin{aligned}&\big\|k(x-y)+Tx-Ty\big\|\\&\quad\leqslant c\big[\big\|(k+1)(x-y)+y-Ty\big\|+\big\|(k+1)(y-x)+x-Tx\big\|\big].\end{aligned}\tag{6}$$

To indicate two constants involved in (6), we shall also call T as a (k, c)-enriched Chatterjea mapping.

Definition 2. (See [4].) Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is called an enriched Chatterjea-type mapping if there exist $\tilde{c} \in [0, 1)$ and $k \in [0, \infty)$ such that for each $x, y \in C$,

$$\begin{aligned}&\big\|k(x-y)+Tx-Ty\big\|\\&\quad\leqslant \tilde{c}\max\big\{\big\|(k+1)(x-y)+y-Ty\big\|,\ \big\|(k+1)(y-x)+x-Tx\big\|\big\}.\end{aligned}\tag{7}$$

To indicate two constants involved in (7), we shall also call T as a $(k, \tilde{c})$-enriched Chatterjea-type mapping.

Moreover, they presented several examples to illustrate the richness of the new class of enriched Chatterjea contractions. The reader can see in the following remark.

Remark 1.

(i) A Chatterjea contraction mapping on a normed space satisfying (4) is a $(0, c)$-enriched Chatterjea mapping with $c = a_3$, but the converse is not true as shown by the next example.

(ii) Let $C = [0, 1]$ be a subset of a usual norm $(\mathbb{R}, \|\cdot\|)$, and let $T : C \to C$ be defined by $Tx = 1 - x$ for all $x \in C$. Then for any $a \in [0, 1/2)$, T is a $(1 - 2a, a)$-enriched Chatterjea mapping but T is not a Chatterjea contraction mapping (see [4] for more details).

(iii) By (1), each Banach contraction mapping with $b < 1/3$ is a $(0, b)$-enriched Chatterjea mapping, and also by (2), each Kannan contraction mapping with $a_1 < 1/4$ is a $(0, a_1)$-enriched Chatterjea mapping.

(iv) A Chatterjea-type contraction mapping on a normed space satisfying (5) is a $(0, \tilde{c})$-enriched Chatterjea-type mapping with $\tilde{c} = a_4$.

(v) Any (k, c)-enriched Chatterjea mapping satisfying (6) is a $(k, \tilde{c})$-enriched Chatterjea-type mapping with $\tilde{c} = 2c$.

To investigate correlations between the newly proposed mappings and the well-known classical mappings in the setting of normed spaces, let C be a nonempty subset of a normed space $(X, \|\cdot\|)$, T be a self-mapping on C, and $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C\colon Tx = x\}$. A mapping T is called a nonexpansive mapping if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$, and a mapping T is called a strictly quasicontractive mapping if $F(T) \neq \emptyset$ and there exists $a_5 \in [0, 1)$ such that for each $x \in C$ and $z \in F(T)$, we get

$$\|Tx - z\| \leqslant a_5\|x - z\|. \tag{8}$$

It is well known that the nonexpansive mapping is necessarily continuous on the whole domain, while the enriched Chatterjea mapping is, in general, not continuous (see the mapping T in Examples 1.3.4, 1.3.5, and 1.3.7 of [20]). In particular, there exist a nonexpansive mapping that is not enriched Chatterjea mapping and an enriched Chatterjea mapping that is not nonexpansive, so the class of enriched Chatterjea mappings is independent of that of nonexpansive. If $F(T) \neq \emptyset$ and we take $k = 0$ and $y = z \in F(T)$ in (6), then we obtain (8). This implies that the class of $(0, c)$-enriched Chatterjea mappings possessing a fixed point is contained within the class of strictly quasicontractive mappings. Moreover, the enriched Chatterjea mapping T in Remark 1(ii) is an enriched Chatterjea-type mapping and is also a nonexpansive, which is not a strictly quasicontractive mapping (see [4] for more details).

On the other hand, the study of common fixed points for a family of nonlinear mappings has been considered by many authors via several directions. One of the interest directions is to establish common fixed point results for semigroups of nonlinear mappings because these results are important subjects in nonlinear operator theory and their applications. In particular, nontrivial applications can reduce in the form of semigroups such as differential equations, integral equations, and dynamical systems. Due to the importance of common fixed point results for semigroups, many researchers have been investigated common fixed point results for semigroups of several types of nonlinear mappings such as nonexpansive semigroups [18, 24], Lipschitzian semigroups [7, 17], pseudocontraction semigroups [1,9], weak contraction semigroups [13,15], enriched nonexpansive semigroups [14], enriched Kannan semigroups [16,21], and so on (see [25–27] and the references therein). For instance, Kozlowski [17] presented the existence theorems for common fixed points of monotone contractive and monotone nonexpansive semigroups in ordered Banach spaces and discussed some applications to differential equations and dynamical systems. Very recently, a new semigroup of enriched Kannan contraction mappings was introduced by Kesahorm and Sintunavarat in [16]. They also provided some weak and strong convergence theorems for the Mann iterative process to approximate common fixed points of enriched Kannan semigroups in uniformly convex Banach spaces.

This paper is motivated by several interesting contractive types of mappings in [4]. We begin by introducing a novel semigroup framework based on enriched Chatterjea-type mappings. Next, we prove two types of convergence theorems for these semigroups using the novel iterative process in uniformly convex Banach spaces. To establish the weak convergence result, we employ Opial's condition together with the uniform convexity of the underlying Banach space. By analyzing the asymptotic behavior of the Krasnoselskii iterative sequence and applying properties of enriched Chatterjea-type mappings, we demonstrate that the generated sequence converges weakly to the unique common fixed point of the semigroup. For the strong convergence theorem, we combine the geometric properties of uniformly convex Banach spaces with key characteristics of enriched Chatterjea-type semigroups. By constructing a suitable error estimate and exploiting the convexity structure of the space, we demonstrate that the sequence converges strongly to the unique common fixed point. To support and show our theoretical results, we present numerical tests that highlight how well the proposed method works and how it converges with different starting points and parameter selections.

2 Preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of all real numbers and the set of all positive integers, respectively. Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X, and G be an unbounded subset of $[0,\infty)$ such that for all $s,t \in G$, we have

$$s+t \in G \quad \text{and} \quad \text{if } s>t, \text{ then } s-t \in G, \tag{9}$$

e.g., $G=[0,\infty)$, $G=\mathbb{N}$, or $G=\mathbb{N}\cup\{0\}$.

Recall that a Banach space X is said to satisfy the Opial's condition, as stated in [19], if for each sequence $\{x_n\}$ in X weakly convergent to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. In addition, for a bounded sequence $\{x_n\}$ in X, the set $A_C(\{x_n\})$ defined by

$$A_C\big(\{x_n\}\big) = \Big\{ y \in C\colon \limsup_{n\to\infty} \|x_n - y\| = \inf_{y_0\in C} \limsup_{n\to\infty} \|x_n - y_0\| \Big\}$$

is called the asymptotic center of $\{x_n\}$ relative to C. It is well known that in a uniformly convex Banach space, $A_C(\{x_n\})$ is a singleton set.

Next, we give the following facts that will be used in the proof of our results.

Lemma 1. (*See* [10].) *Let C be a closed convex subset of a uniformly convex Banach space X satisfying the Opial's condition. Suppose that $\{x_n\}$ is a sequence in C. If $\{x_n\}$ converges weakly to a point p, then $p \in A_C(\{x_n\})$.*

Lemma 2. (*See* [23].) *Let X be a uniformly convex Banach space and $0<\alpha_n<1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X. If there is a nonnegative number d such that $\limsup_{n\to\infty}\|x_n\| \leqslant d$, $\limsup_{n\to\infty}\|y_n\| \leqslant d$, and*

$$\lim_{n\to\infty} \big\|\alpha_n x_n + (1-\alpha_n) y_n\big\| = d,$$

then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

The end of this section presents a fundamental fact about the class of Chatterjea-type mappings, which will be used in our proof later in this paper.

Proposition 1. *Let C be a nonempty subset of a normed space X, and let $T: C \to C$ be a Chatterjea-type mapping on C, i.e., there exists $c \in [0,1)$ such that*

$$\|Tx - Ty\| \leqslant c \max\big\{\|x - Ty\|,\ \|y - Tx\|\big\}$$

for all $x,y \in C$. Then T is a mapping satisfying the following condition:

$$\|Tx - y\| \leqslant c\|x-y\| + \frac{1}{1-c}\|Ty - y\|$$

for all $x,y \in C$.

Proof. For each $x, y \in C$, we have

$$\begin{aligned}\|Tx - y\| &\leqslant \|Tx - Ty\| + \|Ty - y\| \\ &\leqslant c \max\bigl\{\|x - Ty\|, \|y - Tx\|\bigr\} + \|Ty - y\|. \end{aligned} \tag{10}$$

If $\|x - Ty\| \geqslant \|y - Tx\|$, then (10) implies

$$\begin{aligned}\|Tx - y\| &\leqslant c\|x - Ty\| + \|Ty - y\| \\ &\leqslant c\|x - y\| + c\|y - Ty\| + \|Ty - y\| \\ &\leqslant c\|x - y\| + (1 + c)\|Ty - y\|. \end{aligned} \tag{11}$$

If $\|x - Ty\| < \|y - Tx\|$, then (10) implies

$$\|Tx - y\| \leqslant c\|y - Tx\| + \|Ty - y\|$$

and so

$$\|Tx - y\| \leqslant \frac{1}{1 - c}\|Ty - y\|. \tag{12}$$

Since $1/(1 - c) \geqslant 1 + c$, from (11) and (12), for each $x, y \in C$, we have

$$\|Tx - y\| \leqslant c\|x - y\| + \frac{1}{1 - c}\|Ty - y\|. \qquad \square$$

3 Common fixed point results for enriched Chatterjea-type semigroups

The first purpose of this section is to introduce a new semigroup of enriched Chatterjea-type mappings, called an enriched Chatterjea-type semigroup, with the following definition.

Definition 3. Let C be a nonempty subset of a normed space X and G be an unbounded subset of $[0, \infty)$ satisfying condition (9). Then the family $\tau = \{T_t : C \to C,\ t \in G\}$ is called an enriched Chatterjea-type semigroup on C if the following conditions are satisfied:

- (C_1) For each $t \in G$, $T_t : C \to C$ is an enriched Chatterjea-type mapping on C, i.e., there are constants $k_t \in [0, \infty)$ and $h_t \in [0, 1)$ such that for each $x, y \in C$,

$$\begin{aligned}&\bigl\|k_t(x-y) + T_t x - T_t y\bigr\| \\ &\quad\leqslant h_t \max\bigl\{\bigl\|(k_t+1)(x-y) + y - T_t y\bigr\|, \bigl\|(k_t+1)(y-x) + x - T_t x\bigr\|\bigr\}; \end{aligned} \tag{13}$$

- (C_2) $T_{s+t}x = T_s T_t x$ for all $s, t \in G$ and $x \in C$;
- (C_3) For all $x \in C$, the mapping $G \ni t \mapsto T_t x$ is continuous.

If $k_t = k \in [0, \infty)$ and $h_t = h \in [0, 1)$ for all $t \in G$ in (13), the family $\tau = \{T_t : C \to C,\ t \in G\}$ will be called a (k, h)-enriched Chatterjea-type semigroup on C. The set of all common fixed points of τ is denoted by $\mathrm{Fix}(\tau)$, that is,

$$\mathrm{Fix}(\tau) = \{x \in C\colon\ T_t x = x \text{ for all } T_t \in \tau\}.$$

Remark 2. It is easy to show that if $\tau = \{T_t : C \to C,\ t \in G\}$ is an enriched Chatterjea-type semigroup on C and $\mathrm{Fix}(\tau) \neq \emptyset$, then $\mathrm{Fix}(\tau)$ is a singleton set. Indeed, let $\mathrm{Fix}(\tau) \neq \emptyset$. Suppose that $z_0, z_1 \in \mathrm{Fix}(\tau)$ with $z_0 \neq z_1$. For each $t \in G$, we get

$$\begin{aligned}
&(k_t + 1)\|z_0 - z_1\| \\
&\quad = \|k_t(z_0 - z_1) + T_t z_0 - T_t z_1\| \\
&\quad \leqslant h_t \max\bigl\{\bigl\|(k_t + 1)(z_0 - z_1) + z_0 - T_t z_0\bigr\|, \bigl\|(k_t + 1)(z_1 - z_0) + z_1 - T_t z_1\bigr\|\bigr\} \\
&\quad \leqslant h_t(k_t + 1)\|z_0 - z_1\|.
\end{aligned}$$

Obviously, this is a contradiction. Hence, τ has a unique common fixed point.

Now, we give some examples of enriched Chatterjea-type semigroups, which show that there exist an enriched Chatterjea-type semigroup in the sense of Definition 3.

Example 1. Let $C = [-1, 1]$ be a nonempty closed convex subset of a usual normed space $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ and $\tau = \{T_t : C \to C,\ t \in \mathbb{N}\}$ such that for each $t \in \mathbb{N}$, a mapping $T_t : C \to C$ defined by

$$T_t x = \begin{cases} -x\mathrm{e}^{1-t} & \text{if } x \in [-1, 0), \\ x\mathrm{e}^{-t} & \text{if } x \in [0, 1]. \end{cases}$$

Firstly, we claim that τ is a (k, h)-enriched Chatterjea-type semigroup on C with $k = 1$ and $h = 0.9$. In this claim, let $t \in \mathbb{N}$. For each $x, y \in C$ with $x = y$, it is clearly that $\|k(x - y) + T_t x - T_t y\| = 0$, and so (13) holds. Now we assume that $x \neq y$. In the first case when $x, y \in [-1, 0)$ with $x > y$, we obtain

$$\begin{aligned}
&\bigl\|k(x - y) + T_t x - T_t y\bigr\| \\
&\quad = \bigl|x - y - x\mathrm{e}^{1-t} + y\mathrm{e}^{1-t}\bigr| = \bigl|\bigl(1 - \mathrm{e}^{1-t}\bigr)(x - y)\bigr| \\
&\quad = \bigl(1 - \mathrm{e}^{1-t}\bigr)(x - y) \leqslant -1.8y + \bigl(1 - \mathrm{e}^{1-t}\bigr)x.
\end{aligned}$$

It follows from $(1 - \mathrm{e}^{1-t})x < 0.9(1 - \mathrm{e}^{1-t})x \leqslant 0$ together with the above inequality that

$$\begin{aligned}
&\bigl\|k(x - y) + T_t x - T_t y\bigr\| \\
&\quad \leqslant -1.8y + 0.9\bigl(1 - \mathrm{e}^{1-t}\bigr)x = 0.9\bigl(-2y + x - x\mathrm{e}^{1-t}\bigr) \\
&\quad = 0.9\bigl\|(1 + 1)(y - x) + x + x\mathrm{e}^{1-t}\bigr\| = h\bigl\|(k + 1)(y - x) + x - T_t x\bigr\| \\
&\quad \leqslant h \max\bigl\{\bigl\|(k + 1)(x - y) + y - T_t y\bigr\|, \bigl\|(k + 1)(y - x) + x - T_t x\bigr\|\bigr\}.
\end{aligned}$$

Similarly, for $x, y \in [-1, 0)$ with $x < y$, we can show that (13) holds. In the next case, let $x, y \in [0, 1]$. Without loss of generality, we may assume $x > y$. Then we get

$$\begin{aligned}
&\bigl\|k(x-y) + T_t x - T_t y\bigr\| \\
&\quad = \bigl|x - y + x\mathrm{e}^{-t} - y\mathrm{e}^{-t}\bigr| = \bigl|(1+\mathrm{e}^{-t})(x-y)\bigr| \\
&\quad = \bigl(1+\mathrm{e}^{-t}\bigr)(x-y) \leqslant 1.8x - \bigl(1+\mathrm{e}^{-t}\bigr)y.
\end{aligned} \tag{14}$$

Since $0 \leqslant 0.9(1+\mathrm{e}^{-t})y \leqslant (1+\mathrm{e}^{-t})y$, from Eq. (14) we obtain

$$\begin{aligned}
&\bigl\|k(x-y) + T_t x - T_t y\bigr\| \\
&\quad \leqslant 1.8x - 0.9\bigl(1+\mathrm{e}^{-t}\bigr)y = 0.9\bigl(2x - y - y\mathrm{e}^{-t}\bigr) \\
&\quad = 0.9\bigl\|(1+1)(x-y) + y - y\mathrm{e}^{-t}\bigr\| = h\bigl\|(k+1)(x-y) + y - T_t y\bigr\| \\
&\quad \leqslant h \max\bigl\{\bigl\|(k+1)(x-y) + y - T_t y\bigr\|, \bigl\|(k+1)(y-x) + x - T_t x\bigr\|\bigr\}.
\end{aligned}$$

In the last case, let $(x, y) \in ([-1, 0)\times[0, 1]) \cup ((0, 1]\times[-1, 0])$. Without loss of generality, we can assume that $x \in [-1, 0)$ and $y \in [0, 1]$. Hence, we get

$$\begin{aligned}
&\bigl\|k(x-y) + T_t x - T_t y\bigr\| \\
&\quad = \bigl|x - y - x\mathrm{e}^{1-t} - y\mathrm{e}^{-t}\bigr| = \bigl|\bigl(1-\mathrm{e}^{1-t}\bigr)x - \bigl(1+\mathrm{e}^{-t}\bigr)y\bigr| \\
&\quad = \bigl(1+\mathrm{e}^{-t}\bigr)y - \bigl(1-\mathrm{e}^{1-t}\bigr)x \leqslant 0.45\bigl(3+\mathrm{e}^{-t}\bigr)y - 0.45\bigl(3-\mathrm{e}^{1-t}\bigr)x \\
&\quad = 0.45\bigl[3y + \mathrm{e}^{-t}y - 3x + \mathrm{e}^{1-t}x\bigr] = 0.45\bigl[\bigl\|3x - \mathrm{e}^{1-t}x - 3y - \mathrm{e}^{-t}y\bigr\|\bigr] \\
&\quad = 0.45\bigl[\bigl\|2(x-y) + y - \mathrm{e}^{-t}y - 2(y-x) - x - \mathrm{e}^{1-t}x\bigr\|\bigr] \\
&\quad \leqslant 0.45\bigl[\bigl\|2(x-y) + y - \mathrm{e}^{-t}y\bigr\| + \bigl\|2(y-x) + x + \mathrm{e}^{1-t}x\bigr\|\bigr] \\
&\quad \leqslant 0.45(M+M) = 0.9M,
\end{aligned}$$

where $M := \max\{\|(k+1)(x-y) + y - T_t y\|, \|(k+1)(y-x) + x - T_t x\|\}$. From all possible cases we conclude that for each $t \in \mathbb{N}$, T_t is an enriched Chatterjea-type mapping on C satisfying (13) with $k = 1$ and $h = 0.9$. This implies that condition (C_1) holds.

Next, we will show that $T_{s+t}x = T_s T_t x$ for all $s, t \in G$ and for all $x \in C$. For fixed $s, t \in \mathbb{N}$ and for each $x \in [-1, 0)$, we have

$$T_s T_t x = T_s\bigl(-x\mathrm{e}^{1-t}\bigr) = \bigl(-x\mathrm{e}^{1-t}\bigr)\mathrm{e}^{-s} = -x\mathrm{e}^{1-(s+t)} = T_{s+t}x.$$

Similarly, we can verify $T_{s+t}x = T_s T_t x$ in the other case. Hence, the family τ satisfies (C_2).

Finally, it is clear that for each $x \in C$, the mapping $\mathbb{N} \ni t \mapsto T_t x \in C$ is continuous, which implies that the family τ satisfies (C_3).

Based on the aforementioned, we can conclude from Definition 3 that τ is a (k,h)-enriched Chatterjea-type semigroup on C with $k=1$ and $h=0.9$.

Example 2. Let $X = L^2([0,1])$ be equipped with the square-integrable norm $\|\cdot\|\colon X\times X\to\mathbb{R}$ defined by

$$\|f\| = \left(\int_0^1 |f(x)|^2\,\mathrm{d}x\right)^{1/2}$$

for all $f\in X$. Then $(\|\cdot\|, X)$ is a normed space. Let

$$C = \bigl\{f\in X\colon\ 0\leqslant f(x)\leqslant 1 \text{ almost everywhere (a.e.) on } [0,1]\bigr\}$$

be a nonempty closed convex subset of X and $\tau=\{T_t: C\to C,\ t\in\mathbb{N}\}$ such that for each $t\in\mathbb{N}$, a mapping $T_t\colon C\to C$ is defined for each $f\in C$ by

$$(T_t f)(x) = a^{-bt} f(x)$$

for all $x\in[0,1]$, where $a,b\in(1,\infty)$ such that $a^b\in(3,\infty)$. Then the family τ satisfies conditions (C_2) and (C_3) in Definition 3. Indeed, fix $s,t\in\mathbb{N}$, and for each $f\in C$, we have

$$\begin{aligned} T_sT_tf(x) &= T_s\bigl(a^{-bt}f(x)\bigr) = \bigl(a^{-bs}\bigr)\bigl(a^{-bt}f(x)\bigr)\\ &= a^{-b(s+t)}f(x) = T_{s+t}f(x)\end{aligned}$$

for all $x\in[0,1]$, and the mapping $\mathbb{N}\ni t\mapsto T_tf$ is continuous.

Now, we will verify that for each $t\in\mathbb{N}$, T_t is an enriched Chatterjea-type mapping on C with $k_t\in[0,\,(1-3a^{-b})/2)$ and $h_t\in[2(k_t+a^{-b})/(1-a^{-b}),\,1)$. For fixed $t\in\mathbb{N}$ and for each $f_1,f_2\in C$, we get

$$\begin{aligned} &\bigl\|k_t(x-y)+T_tf_1-T_tf_2\bigr\|\\ &\quad= \bigl\|k_t(f_1-f_2)+a^{-bt}f_1-a^{-bt}f_2\bigr\| = \bigl\|\bigl(k_t+a^{-bt}\bigr)(f_1-f_2)\bigr\|\\ &\quad\leqslant \bigl(k_t+a^{-b}\bigr)\|f_1-f_2\|. \end{aligned} \tag{15}$$

Since $a^{-b}<1/3$, we obtain $0<1-a^{-b}<1$. Then we have

$$\begin{aligned} &\bigl(1-a^{-b}\bigr)\|f_1+f_2\|\\ &\quad= \bigl\|\bigl(1-a^{-b}\bigr)f_1+\bigl(1-a^{-b}\bigr)f_2\bigr\|\\ &\quad= \bigl\|f_1-a^{-b}f_1+f_2-a^{-b}f_2\bigr\| = \|f_1-T_tf_1+f_2-T_tf_2\|\\ &\quad= \bigl\|(k_t+1)(f_1-f_2)+f_2-T_tf_2+(k_t+1)(f_2-f_1)+f_1-T_tf_1\bigr\|\\ &\quad\leqslant \bigl\|(k_t+1)(f_1-f_2)+f_2-T_tf_2\bigr\|+\bigl\|(k_t+1)(f_2-f_1)+f_1-T_tf_1\bigr\|\\ &\quad\leqslant 2\max\bigl\{\bigl\|(k_t+1)(f_1-f_2)+f_2-T_tf_2\bigr\|,\bigl\|(k_t+1)(f_2-f_1)+f_1-T_tf_1\bigr\|\bigr\}. \end{aligned}$$

Multiplying both sides of the above inequality by $0 < (k_t + a^{-b})/(1 - a^{-b})$, we obtain

$$\begin{aligned}
&\big(k_t + a^{-b}\big)\|f_1 + f_2\| \\
&\quad\leqslant \frac{2(k_t + a^{-b})}{1 - a^{-b}} \max\big\{\big\|(k_t+1)(f_1-f_2) + f_2 - T_t f_2\big\|, \\
&\qquad \big\|(k_t + 1)(f_2 - f_1) + f_1 - T_t f_1\big\|\big\} \\
&\quad\leqslant h_t \max\big\{\big\|(k_t+1)(f_1-f_2) + f_2 - T_t f_2\big\|, \big\|(k_t+1)(f_2-f_1) + f_1 - T_t f_1\big\|\big\}, \quad (16)
\end{aligned}$$

where $k_t \in [0,\, (1 - 3a^{-b})/2)$ and $h_t \in [2(k_t + a^{-b})/(1 - a^{-b}),\, 1)$. From the fact that $\|f_1 - f_2\| \leqslant \|f_1 + f_2\|$ for all $f_1, f_2 \in C$ it follows together with (15) and (16) that

$$\begin{aligned}
&\big\|k_t(x - y) + T_t f_1 - T_t f_2\big\| \\
&\quad\leqslant \big(k_t + a^{-b}\big)\|f_1 - f_2\| \leqslant \big(k_t + a^{-b}\big)\|f_1 + f_2\| \\
&\quad\leqslant h_t \max\big\{\big\|(k_t+1)(f_1-f_2) + f_2-T_t f_2\big\|, \big\|(k_t+1)(f_2-f_1) + f_1 - T_t f_1\big\|\big\}.
\end{aligned}$$

This implies that condition (C_1) holds. Therefore, we conclude that τ is an enriched Chatterjea-type semigroup on C with $k_t \in [0,\, (1 - 3a^{-b})/2)$ and $h_t \in [2(k_t + a^{-b})/(1 - a^{-b}),\, 1)$.

Let C be a nonempty closed convex subset of a uniformly convex Banach space X and G be an unbounded subset of $[0, \infty)$ satisfying condition (9). Throughout this paper, unless otherwise specified, we will use the notation τ to denote an enriched Chatterjea-type semigroup on C, that is,

$$\tau = \{T_t\colon C \to C,\ t \in G\},$$

and we will also use the notation

$$\overline{\tau} := \big\{\overline{T}_t^{\mu_t} : C \to C,\ t \in G\big\},$$

which represents a family of the averaged mappings $\overline{T}_t^{\mu_t}$ associated with $T_t \in \tau$ and is defined for each $t \in G$ by

$$\overline{T}_t^{\mu_t} = (1 - \mu_t)I + \mu_t T_t,$$

where I is the identity mapping, and $\mu_t := 1/(k_t + 1)$ such that k_t is a constant in (13) of an enriched Chatterjea-type mapping T_t.

It is straightforward to prove that T_t and $\overline{T}_t^{\mu_t}$ share the same fixed points for all $t \in G$, which implies that $\mathrm{Fix}(\tau) = \mathrm{Fix}(\overline{\tau})$.

Moreover, for each $T_t \in \tau$, we obtain that the averaged operator $\overline{T}_t^{\mu_t}$ is a Chatterjea-type contraction mapping. Indeed, since T_t is an enriched Chatterjea-type mapping on C, there are $k_t \in [0, \infty)$ and $h_t \in [0, 1)$ such that

$$\begin{aligned}
&\big\|k_t(x - y) + T_t x - T_t y\big\| \\
&\quad\leqslant h_t \max\big\{\big\|(k_t + 1)(x - y) + y - T_t y\big\|, \big\|(k_t + 1)(y - x) + x - T_t x\big\|\big\}
\end{aligned}$$

for all $x, y \in C$. Putting $k_t = 1/(\mu_t - 1)$ in the above inequality, for each $x, y \in C$, we have

$$\begin{aligned}&\left\|\left(\frac{1}{\mu_t} - 1\right)(x-y) + T_t x - T_t y\right\| \\ &\quad\leqslant h_t \max\left\{\left\|\frac{1}{\mu_t}(x-y) + y - T_t y\right\|, \left\|\frac{1}{\mu_t}(y-x) + x - T_t x\right\|\right\},\end{aligned}$$

which yields

$$\begin{aligned}&\|(1-\mu_t)(x-y) + \mu_t T_t x - \mu_t T_t y\| \\ &\quad\leqslant h_t \max\{\|(x-y) + \mu_t y - \mu_t T_t y\|, \|(y-x) + \mu_t x - \mu_t T_t x\|\}.\end{aligned}$$

From the above inequality we obtain

$$\|\overline{T}_t^{\mu_t} x - \overline{T}_t^{\mu_t} y\| \leqslant h_t \max\{\|x - \overline{T}_t^{\mu_t} y\|, \|y - \overline{T}_t^{\mu_t} x\|\}$$

for all $x, y \in C$. This implies that $\overline{T}_t^{\mu_t}$ is a Chatterjea-type contraction mapping with a contractive constant h_t.

Now, we introduce a new iterative process $\{x_n\}$ in C within the context of a semigroup τ, inspired by the Krasnoselskii iterative process in the context of a single operator, as follows:

$$x_0 \in C, \qquad x_{n+1} = (1 - \lambda_{t_n}) x_n + \lambda_{t_n} T_{t_n} x_n \tag{17}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{t_n\}_{n \in \mathbb{N} \cup \{0\}} \subseteq G$, $T_{t_n} \in \tau$, and $\lambda_{t_n} \in (0, 1]$ for all $t_n \in G$.

Next, we introduce the definition needed to prove our main results in this section, which are motivated by the concept of modular contraction and the behavior of enriched Chatterjea-type semigroups.

Definition 4. Let C be a nonempty subset of a normed space X, G be an unbounded subset of $[0, \infty)$ satisfying condition (9), $\tau = \{T_t : C \to C,\ t \in G\}$ be an enriched Chatterjea-type semigroup, and $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ be a sequence defined by the iterative scheme (17) with a sequence $\{t_n\}_{n \in \mathbb{N} \cup \{0\}} \subseteq G$.

(i) τ is said to satisfy condition (L) if $\lim_{s \to \infty}(\sup_{x \in C} \|T_s x - x\|) = 0$.

(ii) $\{t_n\}_{n \in \mathbb{N} \cup \{0\}}$ is said to satisfy condition (K) if the following condition holds:

$$\lim_{n \to \infty} \|x_n - T_{t_n} x_n\| \quad \Longrightarrow \quad \lim_{n \to \infty} \|x_n - T_{s_n} x_n\|$$

whenever $\{s_n\}_{n \in \mathbb{N} \cup \{0\}} \subseteq G$ with $s_n \geqslant t_n$ for all $n \in \mathbb{N} \cup \{0\}$.

(iii) $\{t_n\}_{n \in \mathbb{N} \cup \{0\}}$ is said to satisfy condition (H) if $\prod_{i=0}^{n} h_{t_i} \to 0$ as $n \to \infty$, where $k_{t_n} \in [0, \infty)$ and $h_{t_n} \in (0, 1]$ are constants involving a (k_{t_n}, h_{t_n})-enriched Chatterjea-type mapping in τ.

Remark 3. From the above definition we get the following assertions:

(i) If τ satisfies condition (L) and $\lim_{n \to \infty} t_n = \infty$, then $\{t_n\}_{n \in \mathbb{N} \cup \{0\}}$ satisfies condition (K).

(ii) If G is a singleton set, then $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies condition (K).
(iii) In the case of (k,h)-enriched Chatterjea-type semigroups, it is evident that the sequence $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies both conditions (K) and (H).

In the sequel, we establish some weak and strong convergence theorems for enriched Chatterjea-type semigroups in uniformly convex Banach spaces.

Theorem 2. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C be a nonempty closed convex subset of X, G be an unbounded subset of $[0,\infty)$ satisfying condition* (9)*, and $\tau=\{T_t\colon C\to C,\ t\in G\}$ be a (k,h)-enriched Chatterjea-type semigroup with $\mathrm{Fix}(\tau)\neq\emptyset$. Suppose that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a sequence defined by the iterative scheme* (17) *with $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}\subseteq G$ satisfying condition* (K) *and $\lambda_{t_n}:=m\mu_{t_n}\in(0,1]$, where $m\in(0,1)$ and $\mu_{t_n}:=1/(k+1)$. Then $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges weakly to the unique common fixed point of τ.*

Proof. Assume that $z\in\mathrm{Fix}(\tau)=\mathrm{Fix}(\overline{\tau})$. We divide the proof into three steps.

Step 1. We will show that $\lim_{n\to\infty}\|x_n-T_{t_n}x_n\|=0$. For each $n\in\mathbb{N}\cup\{0\}$, we get

$$\begin{aligned}x_{n+1}&=(1-\lambda_{t_n})x_n+\lambda_{t_n}T_{t_n}x_n=(1-m\mu_{t_n})x_n+m\mu_{t_n}T_{t_n}x_n\\&=(1-m)x_n+m(1-\mu_{t_n})x_n+m\mu_{t_n}T_{t_n}x_n\\&=(1-m)x_n+m\bigl[(1-\mu_{t_n})x_n+\mu_{t_n}T_{t_n}x_n\bigr]\\&=(1-m)x_n+m\overline{T}_{t_n}^{\mu_{t_n}}x_n\end{aligned}\tag{18}$$

and then

$$\begin{aligned}\|x_{n+1}-z\|&=\bigl\|(1-m)x_n+m\overline{T}_{t_n}^{\mu_{t_n}}x_n-z\bigr\|\\&\leqslant(1-m)\|x_n-z\|+m\bigl\|\overline{T}_{t_n}^{\mu_{t_n}}x_n-z\bigr\|.\end{aligned}\tag{19}$$

By Proposition 1 and (19), we get

$$\begin{aligned}\|x_{n+1}-z\|&\leqslant(1-m)\|x_n-z\|+m\left(h\|x_n-z\|+\frac{1}{1-h}\|\overline{T}_{t_n}^{\mu_{t_n}}z-z\|\right)\\&\leqslant(1-m)\|x_n-z\|+m\|x_n-z\|=\|x_n-z\|.\end{aligned}\tag{20}$$

From inequality (20) we obtain that the sequence $\{\|x_n-z\|\}$ is nonincreasing and bounded below. Hence, $\lim_{n\to\infty}\|x_n-z\|$ exists. Now, we may assume that there is $d\geqslant 0$ such that

$$\lim_{n\to\infty}\|x_n-z\|=d.\tag{21}$$

From (18) and (21) we have

$$d=\lim_{n\to\infty}\|x_{n+1}-z\|=\lim_{n\to\infty}\bigl\|(1-m)(x_n-z)+m\bigl(\overline{T}_{t_n}^{\mu_{t_n}}x_n-z\bigr)\bigr\|.\tag{22}$$

Since $\overline{T}_{t_n}^{\mu_{t_n}}$ is a Chatterjea-type mapping for each $n\in\mathbb{N}\cup\{0\}$, we obtain

$$\begin{aligned}\|\overline{T}_{t_n}^{\mu_{t_n}}x_n-z\|&=\|\overline{T}_{t_n}^{\mu_{t_n}}x_n-Tz\|\leqslant h_{t_n}\max\bigl\{\|x_n-Tz\|,\|z-\overline{T}_{t_n}^{\mu_{t_n}}x_n\|\bigr\}\\&=h_{t_n}\max\bigl\{\|x_n-z\|,\|z-\overline{T}_{t_n}^{\mu_{t_n}}x_n\|\bigr\}\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Both cases of the maximum value imply that

$$\big\|\overline{T}_{t_n}^{\mu_{t_n}} x_n - z\big\| \leqslant \|x_n - z\|$$

for all $n \in \mathbb{N} \cup \{0\}$. It yields that

$$\limsup_{n\to\infty} \big\|\overline{T}_{t_n}^{\mu_{t_n}} x_n - z\big\| \leqslant \limsup_{n\to\infty} \|x_n - z\| = d. \tag{23}$$

It follows from Lemma 2 together with (21), (22), and (23) that

$$\lim_{n\to\infty} \big\|x_n - \overline{T}_{t_n}^{\mu_{t_n}} x_n\big\| = 0. \tag{24}$$

For each $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned}
\big\|x_n - \overline{T}_{t_n}^{\mu_{t_n}} x_n\big\| &= \big\|x_n - (1-\mu_{t_n})x_n - \mu_{t_n} T_{t_n} x_n\big\| \\
&= \|\mu_{t_n} x_n - \mu_{t_n} T_{t_n} x_n\| = \mu_{t_n} \|x_n - T_{t_n} x_n\|
\end{aligned}$$

and then

$$\|x_n - T_{t_n} x_n\| = (k+1)\big\|x_n - \overline{T}_{t_n}^{\mu_{t_n}} x_n\big\|. \tag{25}$$

From (24) and (25) we conclude that

$$\lim_{n\to\infty} \|x_n - T_{t_n} x_n\| = 0. \tag{26}$$

Step 2. We will verify that $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$ for all $t \in G$. Let $t \in G$. For each $n \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned}
&\|x_n - T_t x_n\| \\
&\quad \leqslant \|x_n - T_{t+t_n} x_n\| + \|T_{t+t_n} x_n - T_t x_n\| \\
&\quad = \|x_n - T_{t+t_n} x_n\| + \|T_t T_{t_n} x_n - T_t x_n\| \\
&\quad = \|x_n - T_{t+t_n} x_n\| + \big\|k(T_{t_n} x_n - x_n) + T_t T_{t_n} x_n - T_t x_n - k(T_{t_n} x_n - x_n)\big\| \\
&\quad \leqslant \|x_n - T_{t+t_n} x_n\| + \big\|k(T_{t_n} x_n - x_n) + T_t T_{t_n} x_n - T_t x_n\big\| + \big\|k(T_{t_n} x_n - x_n)\big\| \\
&\quad \leqslant \|x_n - T_{t+t_n} x_n\| \\
&\qquad + h \max\big\{\big\|(k+1)(T_{t_n} x_n - x_n) + x_n - T_t x_n\big\|, \\
&\qquad\qquad \big\|(k+1)(x_n - T_{t_n} x_n) + T_{t_n} x_n - T_{t+t_n} x_n\big\|\big\} + k\|T_{t_n} x_n - x_n\| \\
&\quad = \|x_n - T_{t+t_n} x_n\| \\
&\qquad + h \max\big\{\big\|(k+1)(T_{t_n} x_n - x_n) + (x_n - T_t x_n)\big\|, \\
&\qquad\qquad \big\|(k+1)(x_n - T_{t_n} x_n) + (x_n - T_{t+t_n} x_n)\big\|\big\} + k\|T_{t_n} x_n - x_n\| \\
&\quad \leqslant \|x_n - T_{t+t_n} x_n\| + h\big[(k+1)\|x_n - T_{t_n} x_n\| + \|x_n - T_t x_n\| + \|x_n - T_{t+t_n} x_n\|\big] \\
&\qquad + k\|x_n - T_{t_n} x_n\| \\
&\quad = (1+h)\|x_n - T_{t+t_n} x_n\| + \big(h(k+1) + k\big)\|x_n - T_{t_n} x_n\| + h\|x_n - T_t x_n\|,
\end{aligned}$$

which implies that

$$\|x_n - T_t x_n\| \leqslant \frac{1+h}{1-h}\|x_n - T_{t+t_n} x_n\| + \frac{h(k+1)+k}{1-h}\|x_n - T_{t_n} x_n\|.$$

Taking the limit as $n \to \infty$ in the above inequality, by (26) and condition (K), we have

$$\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$$

for all $t \in G$.

Step 3. We will verify that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges weakly to the unique common fixed point of τ. As X is uniformly convex and $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $\{x_{n_l}\}$ converges weakly to $z_0 \in X$. Set $u_n := x_{n_l}$, then it follows from Lemma 1 that $z_0 \in A_C(\{u_n\})$. For each $t \in G$, by Proposition 1, we have

$$\left\|\overline{T}_t^{\mu_t} z_0 - u_n\right\| \leqslant h\|u_n - z_0\| + \frac{1}{1-h}\left\|\overline{T}_t^{\mu_t} u_n - u_n\right\|$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \to \infty$ in the above inequality, we get

$$\limsup_{n\to\infty} \left\|\overline{T}_t^{\mu_t} z_0 - u_n\right\| \leqslant h \limsup_{n\to\infty} \|u_n - z_0\| + \frac{1}{1-h} \limsup_{n\to\infty} \left\|\overline{T}_t^{\mu_t} u_n - u_n\right\|.$$

It follows from the fact in Step 2 that

$$\limsup_{n\to\infty} \left\|\overline{T}_t^{\mu_t} z_0 - u_n\right\| \leqslant \limsup_{n\to\infty} \|z_0 - u_n\|. \tag{27}$$

Since $z_0 \in A_C(\{u_n\})$ and $\overline{T}_t^{\mu_t} z_0 \in C$, we get

$$\limsup_{n\to\infty} \|z_0 - u_n\| = \inf_{y\in C} \limsup_{n\to\infty} \|y - u_n\| \leqslant \limsup_{n\to\infty} \left\|\overline{T}_t^{\mu_t} z_0 - u_n\right\|. \tag{28}$$

Inequalities (27) and (28) imply that

$$\limsup_{n\to\infty} \|z_0 - u_n\| = \limsup_{n\to\infty} \left\|\overline{T}_t^{\mu_t} z_0 - u_n\right\|$$

and so $\overline{T}_t^{\mu_t} z_0 \in A_C(\{u_n\})$. Since X is uniformly convex, $A_C(\{u_n\})$ is a singleton set. Hence, $\overline{T}_t^{\mu_t} z_0 = z_0$ for all $t \in G$, that is, $z_0 \in \mathrm{Fix}(\overline{\tau}) = \mathrm{Fix}(\tau)$. Assume that another subsequence $\{x_{n_k}\}$ of $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is such that $\{x_{n_k}\}$ converges weakly to $z_1 \in X$. Similarly, we can show that $z_1 \in \mathrm{Fix}(\tau)$. Since $\mathrm{Fix}(\tau) \neq \emptyset$, we get that τ has a unique common fixed point. So $z_0 = z_1$, and it follows that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ has unique weak subsequential limit in $\mathrm{Fix}(\tau)$. Therefore, $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges weakly to the unique common fixed point of τ. $\square$

Based on Remark 3(i), we can derive the following result from Theorem 2.

Corollary 1. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C be a nonempty closed convex subset of X, G be an unbounded subset of $[0,\infty)$ satisfying condition* (9)*, $\tau=\{T_t\colon C\to C,\ t\in G\}$ be a (k,h)-enriched Chatterjea-type semigroup with $\operatorname{Fix}(\tau)\neq\emptyset$, and condition* (L) *is satisfied. Suppose that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a sequence defined by the iterative scheme* (17) *with $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}\subseteq G$ satisfying $\lim_{n\to\infty}t_n=\infty$ and $\lambda_{t_n}:=m\mu_{t_n}\in(0,1]$, where $m\in(0,1)$ and $\mu_{t_n}:=1/(k+1)$. Then $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges weakly to the unique common fixed point of τ.*

Theorem 3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, G be an unbounded subset of $[0,\infty)$ satisfying condition* (9)*, $\tau=\{T_t\colon C\to C,\ t\in G\}$ be an enriched Chatterjea-type semigroup, and $\operatorname{Fix}(\tau)\neq\emptyset$. Suppose that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a sequence defined by the iterative scheme* (17) *with $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}\subseteq G$ satisfying condition* (H) *and $\lambda_{t_n}:=\mu_{t_n}=1/(k_{t_n}+1)\in(0,1]$ such that k_{t_n} is a constant involving a (k_{t_n},h_{t_n})-enriched Chatterjea-type mapping. Then $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges strongly to the unique common fixed point of τ.*

Proof. For each $n\in\mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}\|x_{n+1}-z\| &= \big\|(1-\mu_{t_n})x_n+\mu_{t_n}T_{t_n}x_n-z\big\| = \big\|\overline{T}^{\mu_{t_n}}_{t_n}x_n-z\big\|\\ &\leqslant h_{t_n}\|z-x_n\|+\frac{1}{1-h_{t_n}}\big\|\overline{T}^{\mu_{t_n}}_{t_n}z-z\big\| = h_{t_n}\|z-x_n\|.\end{aligned}$$

By induction, we get

$$\|x_{n+1}-z\|\leqslant\prod_{i=0}^{n}h_{t_i}\|z-x_0\|$$

for all $n\in\mathbb{N}$. Taking the limit as $n\to\infty$ in the above inequality and using condition (H), we get

$$\lim_{n\to\infty}\|x_{n+1}-z\|=0.$$

Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges strongly to the unique common fixed point of τ. □

Finally, we will discuss the case of semigroups of enriched Chatterjea mappings. We begin with the following definition.

Definition 5. Let C be a nonempty closed convex subset of a uniformly convex normed space X, and let G be an unbounded subset of $[0,\infty)$ satisfying condition (9). Then the family $\tau=\{T_t\colon C\to C,\ t\in G\}$ is called an enriched Chatterjea semigroup on C if the following conditions are satisfied:

($\overline{\mathrm{C}}_1$) For each $t\in G$, $T_t\colon C\to C$, there is an enriched Chatterjea mapping on C, i.e., there are constants $k_t\in[0,\infty)$ and $c_t\in[0,1/2)$ such that for each $x,y\in C$,

$$\begin{aligned}&\big\|k_t(x-y)+T_tx-T_ty\big\|\\ &\quad\leqslant c_t\big[\big\|(k_t+1)(x-y)+y-T_ty\big\|+\big\|(k_t+1)(y-x)+x-T_tx\big\|\big];\end{aligned}\qquad(29)$$

($\overline{\mathrm{C}}_2$) $T_{s+t}x=T_sT_tx$ for all $s,t\in G$ and $x\in C$;

($\overline{\mathrm{C}}_3$) For all $x\in C$, the mapping $G\ni t\mapsto T_tx$ is continuous.

If $k_t = k \in [0, \infty)$ and $c_t = c \in [0, 1/2)$ for all $t \in G$ in (29), the family $\tau = \{T_t : C \to C, t \in G\}$ will be called a (k, c)-enriched Chatterjea semigroup on C.

Remark 4. It is easy to verify that any enriched Chatterjea semigroup on C is an enriched Chatterjea-type semigroup on C. Indeed, let $\tau = \{T_t : C \to C, t \in G\}$ be an enriched Chatterjea semigroup on C. Then, for each $t \in G$, there are constants $k_t \in [0, \infty)$ and $c_t \in [0, 1/2)$ such that

$$\begin{aligned}&\big\|k_t(x-y) + T_t x - T_t y\big\| \\ &\quad \leqslant c_t\big[\big\|(k_t+1)(x-y) + y - T_t y\big\| + \big\|(k_t+1)(y-x) + x - T_t x\big\|\big] \\ &\quad \leqslant 2c_t M_t,\end{aligned}$$

where

$$M_t := \max\big\{\big\|(k_t+1)(x-y) + y - T_t y\big\|, \big\|(k_t+1)(y-x) + x - T_t x\big\|\big\}.$$

Therefore, τ is an enriched Chatterjea-type semigroup on C. Obviously, any (k, c)-enriched Chatterjea semigroup is a (k, h)-enriched Chatterjea-type semigroup with $h = 2c$.

As a direct consequence of the above fact, we can derive the following corollaries from Theorems 2 and 3.

Corollary 2. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C be a nonempty closed convex subset of X, G be an unbounded subset of $[0, \infty)$ satisfying condition* (9)*, and $\tau = \{T_t : C \to C, t \in G\}$ be a (k, c)-enriched Chatterjea semigroup with $\mathrm{Fix}(\tau) \neq \emptyset$. Suppose that $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a sequence defined by the iterative scheme* (17) *with $\{t_n\}_{n \in \mathbb{N} \cup \{0\}} \subseteq G$ satisfying condition* (K) *such that $\lim_{n \to \infty} t_n = \infty$ and $\lambda_{t_n} := m\mu_{t_n} \in (0, 1]$, where $m \in (0, 1)$ and $\mu_{t_n} := 1/(k+1)$. Then $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ converges weakly to the unique common fixed point of τ.*

Corollary 3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, G be an unbounded subset of $[0, \infty)$ satisfying condition* (9)*, $\tau = \{T_t : C \to C, t \in G\}$ be an enriched Chatterjea semigroup, and $\mathrm{Fix}(\tau) \neq \emptyset$. Suppose that $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a sequence defined by the iterative scheme* (17) *with $\{t_n\}_{n \in \mathbb{N} \cup \{0\}} \subseteq G$ satisfying condition* (H) *and $\lambda_{t_n} := \mu_{t_n} = 1/(k_{t_n} + 1) \in (0, 1]$ such that k_{t_n} is a constant involving a (k_{t_n}, c_{t_n})-enriched Chatterjea mapping. Then $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ converges strongly to the unique common fixed point of τ.*

Proof. By Remark 4, τ is an enriched Chatterjea-type semigroup. Hence, by Theorem 3, we can conclude that $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ converges strongly to the unique common fixed point of τ. □

4 A numerical experiment

In this section, we give a numerical example to support Theorem 3 and present some results of numerical experiments for approximating common fixed points of such a semigroup by using the iterative process presented in (17). Additionally, we analyze the rate of convergence of the iterative scheme in our numerical example. By computing the error at each iteration and estimating the order of convergence, we assess how efficiently the iterative process approximates the common fixed points and verify the theoretical results. The concept of rate of convergence provides a quantitative measure of the efficiency of an iterative process. The following definition formalizes this concept in the context of common fixed point iteration.

Definition 6. Let C be a nonempty subset of a Banach space X, G be an unbounded subset of $[0,\infty)$ satisfying condition (9), $\tau = \{T_t : C \to C,\ t \in G\}$ be a family of mappings, and $\mathrm{Fix}(\tau) \neq \emptyset$. Suppose that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a sequence generated by an iterative process such that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges to $z \in \mathrm{Fix}(\tau)$. We say that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges to z with order of convergence $p \geqslant 1$ and rate constant $\mathcal{D} > 0$ if and only if

$$\lim_{n\to\infty} \frac{\|x_{n+1} - z\|}{\|x_n - z\|^p} = \mathcal{D}.$$

However, other more technical rate definitions are needed if the limit does not exist.

Integer powers of p are commonly encountered and are typically referred to by the following specific names:

- If $p = 1$ and $0 < \mathcal{D} < 1$, the convergence is called linear.
- If $p = 2$, the convergence is called quadratic.
- If $p = 3$, the convergence is called cubic.

In general, the convergence is said to be superlinear when $p > 1$, and the convergence is called sublinear when $0 < p < 1$.

In particular, the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is said to converge to z exponentially if there exist constants $\mathcal{M} > 0$ and $0 < \rho < 1$ such that $\|x_n - z\| \leqslant \mathcal{M}\rho^n$ for all $n \in \mathbb{N}$. Define $\omega_n = \|x_n - z\|$; this quantity measures the distance between the nth iterate and the common fixed point z. The behavior of the sequence $\{\omega_n\}$ provides important insight into the convergence rate of the iterative process. In this case, the error ω_n decays exponentially as $n \to \infty$.

Example 3. Consider again the family τ of Example 1, that is, $\tau = \{T_t : C \to C,\ t \in \mathbb{N}\}$ is defined for all $t \in \mathbb{N}$ by

$$T_t x = \begin{cases} -x\mathrm{e}^{1-t} & \text{if } x \in [-1, 0), \\ x\mathrm{e}^{-t} & \text{if } x \in [0, 1]. \end{cases}$$

Following Example 1, the family τ is a (k,h)-enriched Chatterjea-type semigroup on C with $k = 1$ and $h = 0.9$. Set $\lambda_{t_n} := \mu_{t_n} = 1/(k+1) = 1/2$ in the iterative scheme (17) with $\{t_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq \mathbb{N}$ such that $\lim_{n\to\infty} t_n = \infty$ and $t_n > 1$ for all $n \in \mathbb{N}\cup\{0\}$. Now, all conditions of Theorem 3 are satisfied. By Theorem 3, we can conclude that the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges strongly to a common fixed point z of τ. In this case, $z = 0$.

For each $x_0 \in [-1,1] \setminus \{0\}$, the iterative scheme (17) haves the following form:

$$x_1 = \begin{cases} \frac{1}{2}x_0(1-\mathrm{e}^{1-t_0}) & \text{if } x_0 \in [-1,0), \\ \frac{1}{2}x_0(1+\mathrm{e}^{-t_0}) & \text{if } x_0 \in (0,1], \end{cases}$$

$$x_2 = \begin{cases} \frac{1}{2}x_1(1-\mathrm{e}^{1-t_1}) = (\frac{1}{2})^2 x_0(1-\mathrm{e}^{1-t_0})(1-\mathrm{e}^{1-t_1}) & \text{if } x_0 \in [-1,0), \\ \frac{1}{2}x_1(1+\mathrm{e}^{-t_1}) = (\frac{1}{2})^2 x_0(1+\mathrm{e}^{-t_0})(1+\mathrm{e}^{-t_1}) & \text{if } x_0 \in (0,1], \end{cases}$$

$$\vdots$$

$$x_n = \begin{cases} (\frac{1}{2})^n x_0 \prod_{i=0}^{n-1}(1-\mathrm{e}^{1-t_i}) & \text{if } x_0 \in [-1,0), \\ (\frac{1}{2})^n x_0 \prod_{i=0}^{n-1}(1+\mathrm{e}^{-t_i}) & \text{if } x_0 \in (0,1] \end{cases} \tag{30}$$

for all $n \in \mathbb{N} \cup \{0\}$. To define $\omega_n = |x_n - z|$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain the following ratio for each $n \in \mathbb{N} \cup \{0\}$:

$$\frac{\omega_{n+1}}{\omega_n} = \frac{|x_{n+1}-0|}{|x_n-0|} = \begin{cases} \frac{1}{2}(1-\mathrm{e}^{1-t_n}) & \text{if } x_0 \in [-1,0), \\ \frac{1}{2}(1+\mathrm{e}^{-t_n}) & \text{if } x_0 \in (0,1], \end{cases}$$

and then $\lim_{n\to\infty} \omega_{n+1}/\omega_n = 1/2 \in (0,1)$. This implies that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges to 0 with order of convergence $p = 1$ and the rate constant $1/2 \in (0,1)$. Hence, the convergence is linear. In addition, from (30) we obtain

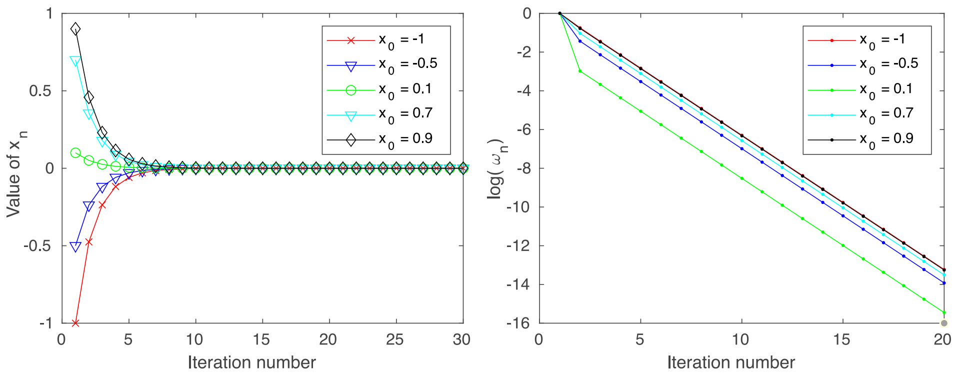
$$\begin{aligned} \omega_n &= |x_n - 0| \\ &= \begin{cases} |(\frac{1}{2})^n x_0 \prod_{i=0}^{n-1}(1-\mathrm{e}^{1-t_i})| & \text{if } x_0 \in [-1,0), \\ |(\frac{1}{2})^n x_0 \prod_{i=0}^{n-1}(1+\mathrm{e}^{-t_i})| & \text{if } x_0 \in (0,1] \end{cases} \\ &\leqslant \begin{cases} (\frac{1}{2})^n |x_0| & \text{if } x_0 \in [-1,0), \\ (\frac{1}{2})^n |x_0| (1+\mathrm{e}^{-1})^{n-1} & \text{if } x_0 \in (0,1] \end{cases} \\ &\leqslant \begin{cases} (\frac{1}{2})^n |x_0| & \text{if } x_0 \in [-1,0), \\ (\frac{1+\mathrm{e}^{-1}}{2})^n |x_0| & \text{if } x_0 \in (0,1] \end{cases} \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges to 0 exponentially and also refers to the error ω_n, which decays exponentially as $n \to \infty$.

Next, we present the results of the numerical experiment for approximating the common fixed point of τ by using the iterative process presented in (17). Here, each test is terminated when $\omega_n \leqslant 10^{-14}$.

In the first experiment, we choose $\{t_n\}_{n\in\mathbb{N}\cup\{0\}} = \{2(n+1)\}_{n\in\mathbb{N}\cup\{0\}}$. In this experiment, there are five initial guesses $x_0 \in \{-1, -0.5, 0.1, 0.7, 0.9\}$ for testing. Table 1 and Fig. 1 illustrate the performance of the iteration with various initial guesses x_0. The convergence behavior is consistent across all choices, but some trends emerge:

- As seen in Fig. 1(a), the iteration converges to the same fixed point regardless of the initial guess. However, the speed of convergence varies. For instance, starting



(a) (b)

Figure 1. Convergence behavior of Krasnoselskii iterative process $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ with different initial guesses x_0.

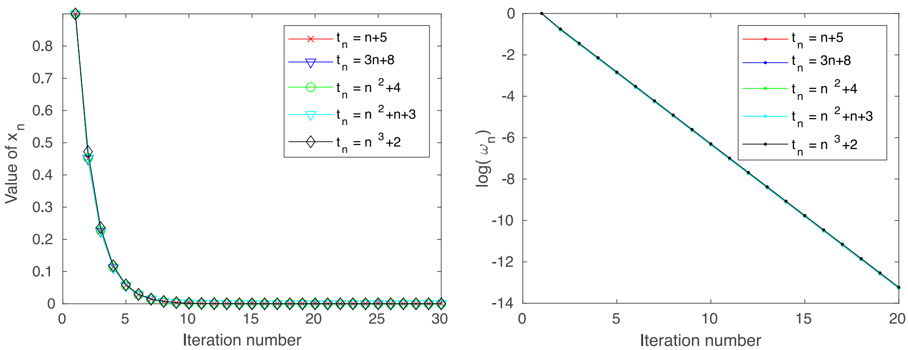
Table 1. Numerical results for Krasnoselskii iterative process $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ with different initial guesses x_0.

n	The value of ω_n for different initial guesses x_0				
	$x_0=-1$	$x_0=-0.5$	$x_0=0.1$	$x_0=0.7$	$x_0=0.9$
1	0.4751	0.2376	0.0509	0.3564	0.4582
5	0.0295	0.0147	0.0032	0.0223	0.0287
10	$9.2072\cdot10^{-4}$	$4.6036\cdot10^{-4}$	$9.9730\cdot10^{-5}$	$6.9811\cdot10^{-4}$	$8.9757\cdot10^{-4}$
20	$8.9914\cdot10^{-7}$	$4.4957\cdot10^{-7}$	$9.7393\cdot10^{-8}$	$6.8175\cdot10^{-7}$	$8.7653\cdot10^{-7}$
30	$8.7807\cdot10^{-10}$	$4.3903\cdot10^{-10}$	$9.5110\cdot10^{-11}$	$6.6577\cdot10^{-10}$	$8.5599\cdot10^{-10}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
43	$1.0719\cdot10^{-13}$	$5.3593\cdot10^{-14}$	$1.1601\cdot10^{-14}$	$8.1271\cdot10^{-14}$	$1.0449\cdot10^{-13}$
44	$5.3593\cdot10^{-14}$	$2.6796\cdot10^{-14}$	$\leqslant 10^{-14}$	$4.0635\cdot10^{-14}$	$5.2245\cdot10^{-14}$
45	$2.6796\cdot10^{-14}$	$1.3398\cdot10^{-14}$		$2.0318\cdot10^{-14}$	$2.6123\cdot10^{-14}$
46	$1.3398\cdot10^{-14}$	$\leqslant 10^{-14}$		$1.0159\cdot10^{-14}$	$1.3061\cdot10^{-14}$
47	$\leqslant 10^{-14}$			$\leqslant 10^{-14}$	$\leqslant 10^{-14}$

from $x_0 = 0.1$ results in the fastest convergence, reaching near-zero values within fewer iterations.

- Table 1 records the error ω_n, which confirms that all sequences converge to the common fixed point with high precision (within 10^{-14} by the 47th iteration).
- As illustrated in Fig. 1(b), which presents the plot of $\log(\omega_n)$ against the iteration number, the convergence behavior of the iterative process is clearly linear as indicated by the straight line pattern. This implies that the error decreases exponentially with each iteration.

In the second experiment, we examine the impact of different sequences $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$ defined for each $t \in \mathbb{N}\cup\{0\}$ by $n+5$, $3n+8$, n^2+4, n^2+n+3, and n^3+2, respectively, with fixed $x_0 = 0.9$ for testing. To evaluate the efficiency and convergence behavior of the proposed iterative process, we consider the performance under different sequences $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$ as shown in Table 2 and Fig. 2. The observations are summarized as follows.



(a) (b)

Figure 2. Convergence behavior of the Krasnoselskii iterative process $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ with different sequence $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$.

Table 2. Numerical results for the Krasnoselskii iterative process $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ with different sequence $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$.

n	The value of ω_n for different sequence $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$				
	$t_n = n + 5$	$t_n = 3n + 8$	$t_n = n^2 + 4$	$t_n = n^2 + n + 3$	$t_n = n^3 + 2$
1	0.4511	0.4500	0.4530	0.4530	0.4724
5	0.0282	0.0281	0.0283	0.0283	0.0295
10	$8.8236 \cdot 10^{-4}$	$8.7892 \cdot 10^{-4}$	$8.8513 \cdot 10^{-4}$	$8.8494 \cdot 10^{-4}$	$9.2271 \cdot 10^{-4}$
20	$8.6168 \cdot 10^{-7}$	$8.5832 \cdot 10^{-7}$	$8.6438 \cdot 10^{-7}$	$8.6420 \cdot 10^{-7}$	$9.0108 \cdot 10^{-7}$
30	$8.4148 \cdot 10^{-10}$	$8.3821 \cdot 10^{-10}$	$8.4412 \cdot 10^{-10}$	$8.4394 \cdot 10^{-10}$	$8.7996 \cdot 10^{-10}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
43	$1.0272 \cdot 10^{-13}$	$1.0232 \cdot 10^{-13}$	$1.0304 \cdot 10^{-13}$	$1.0302 \cdot 10^{-13}$	$1.0742 \cdot 10^{-13}$
44	$5.1360 \cdot 10^{-14}$	$5.1160 \cdot 10^{-14}$	$5.1521 \cdot 10^{-14}$	$5.1510 \cdot 10^{-14}$	$5.3709 \cdot 10^{-14}$
45	$2.5680 \cdot 10^{-14}$	$2.5580 \cdot 10^{-14}$	$2.5761 \cdot 10^{-14}$	$2.5755 \cdot 10^{-14}$	$2.6854 \cdot 10^{-14}$
46	$1.2840 \cdot 10^{-14}$	$1.2790 \cdot 10^{-14}$	$1.2880 \cdot 10^{-14}$	$1.2878 \cdot 10^{-14}$	$1.3427 \cdot 10^{-14}$
47	$\leqslant 10^{-14}$	$\leqslant 10^{-14}$	$\leqslant 10^{-14}$	$\leqslant 10^{-14}$	$\leqslant 10^{-14}$

- As shown in Fig. 2(a), the values of x_n approach the same common fixed point regardless of the form of t_n. The convergence curves nearly overlap, indicating that the iteration is not sensitive to the choice of t_n.
- Table 2 confirms uniform error reduction across all sequences. By the 47th iteration, w_n has dropped below 10^{-14} in every case, indicating high accuracy.
- As shown in Fig. 2(b), the plot of $\log(\omega_n)$ versus the iteration number demonstrates that the convergence is linear with the error magnitude decreasing exponentially at each step.

From the numerical results it is clearly demonstrated that the iterative process converges strongly to the common fixed point $z = 0$ for a wide range of initial guesses $x_0 \in [-1, 1] \setminus \{0\}$ and for various choices of the sequence $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$. The error values ω_n consistently decrease with each iteration, confirming the theoretical convergence results. From the tabulated values and graphical plots of $\log(\omega_n)$ it is evident that the iterative process exhibits exponential convergence regardless of the starting point or the form of t_n. Moreover, the rate of convergence remains stable and efficient across all considered scenarios, indicating the robustness and reliability of the proposed scheme. These results validate that the Krasnoselskii process, under appropriate conditions, is not only theoretically sound but also practically effective for common fixed point approximation in numerical settings.

5 Conclusions and future work

We first introduced an enriched Chatterjea-type semigroup by defining it in normed spaces. Also, we established weak and strong convergence theorems for enriched Chatterjea-type semigroups using the proposed iterative process in the setting of uniformly convex Banach spaces. Numerical examples are given to support that there exists an enriched Chatterjea-type semigroup in the sense of Definition 3, and the numerical experiment is presented to illustrate that the proposed iterative process converge strongly to the unique common fixed point of an enriched Chatterjea-type semigroup under certain assumptions in Theorem 3.

In light of the findings of this study, several promising directions for future research can be pursued. It would be of particular interest to explore the applicability of the proposed enriched Chatterjea-type mappings in broader mathematical settings such as smooth Banach spaces, CAT(0) spaces, or modular function spaces. Additionally, we could look into how these mappings work in b-metric spaces, particularly by including ideas of weakly T-Kannan and weakly T-Chatterjea contractions as mentioned by Kadelburg et al. in [11]. Such generalizations would contribute to extending the semigroup framework to more general.

Another direction involves analyzing the convergence properties of the proposed iteration process under weaker contractive conditions or through the use of hybrid iterative schemes that combine various types of contractive mappings. Inspired by the work of Borcut et al. in [6], investigating tripled fixed point theorems or multi-tupled fixed point problems for enriched Chatterjea-type mappings, particularly in partially ordered Banach spaces, represents an intriguing theoretical expansion.

From a computational perspective, extending the numerical experiments to higher-dimensional problems or applying the method to real-world scenarios, such as image processing, machine learning, or nonlinear systems modeling, may offer valuable insights. Additionally, conducting a comparative analysis of the proposed algorithm with other existing iterative methods, focusing on the rate of convergence and computational efficiency, could enhance its practical relevance and applicability in applied mathematics and engineering.

Author contributions. The authors contributed equally to this paper. The final version of the manuscript has been read and approved by both authors.

Conflicts of interest. The authors declare no conflicts of interest.

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